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A B

2x2 2x2 2x2 2x1 ⇒ 2x1

$$\begin{bmatrix} 1 & 2 \\ 3 & 7 \end{bmatrix} \begin{bmatrix} 2 & 4 \\ 8 & 3 \end{bmatrix}$$

$$\begin{matrix} 2+16 & 4+6 \\ 6 \quad 56 & 12 \quad 21 \end{matrix}$$

$$\begin{bmatrix} 18 & 10 \\ 62 & 33 \end{bmatrix}$$

$$1\begin{bmatrix} 2 \\ 8 \end{bmatrix} + 3\begin{bmatrix} 4 \\ 3 \end{bmatrix}$$

C

$$2\begin{bmatrix} 2 \\ 8 \end{bmatrix} + 7\begin{bmatrix} 4 \\ 3 \end{bmatrix}$$

1

$$\begin{bmatrix} 14 & 32 \\ 17 & 37 \end{bmatrix}$$

$$\begin{matrix} 2 \\ 8 \end{matrix}\begin{bmatrix} 1 & 2 \\ 3 & 7 \end{bmatrix} + \begin{matrix} 4 \\ 3 \end{matrix}\begin{bmatrix} 1 & 2 \\ 3 & 7 \end{bmatrix}$$

2x2 2x1 (2x1)

$$\begin{bmatrix} 1 & 2 \\ 3 & 7 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$

A x

$$x_1\begin{bmatrix} 1 \\ 3 \end{bmatrix} + x_2\begin{bmatrix} 2 \\ 7 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 \\ 3 & 7 \end{bmatrix}$$

Columns of C are linear comb^{n}s of columns of A

2

A B C

$$\begin{bmatrix} 1 & 2 \\ 3 & 7 \end{bmatrix} \begin{bmatrix} 2 & 4 \\ 8 & 3 \end{bmatrix} \Rightarrow \begin{bmatrix} 18 & 10 \\ 62 & 33 \end{bmatrix}$$

$$\begin{matrix} 2 & 4 \\ 16 + & 6 \end{matrix} \Rightarrow \begin{matrix} 8 & 12 \\ 56 & 21 \end{matrix} \qquad \begin{bmatrix} 18 & 10 \\ 62 & 33 \end{bmatrix}$$

~~Note~~ Rows of C are linear comb^{n}s of ~~B~~ rows of B

3

$$\begin{array}{ccc} m \times 1 & 1 \times p & m \times p \\ \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix} & \begin{bmatrix} 1 & 6 \end{bmatrix} = & \begin{bmatrix} 2 & 12 \\ 3 & 18 \\ 4 & 24 \end{bmatrix} \end{array}$$

AB = Sum of (cols of A) x (rows of B)

$$\begin{array}{ccccc} A & B & & & \\ \begin{bmatrix} 2 & 7 \\ 3 & 8 \\ 4 & 9 \end{bmatrix} & \begin{bmatrix} 1 & 6 \\ 0 & 0 \end{bmatrix} = & \cancel{\begin{bmatrix} 2 & 56 \\ 3 & 48 \\ 4 & 54 \end{bmatrix}} & \begin{bmatrix} 2 & 12 \\ 3 & 18 \\ 4 & 24 \end{bmatrix} \\ (3\times 2) & (2\times 2) & \cancel{(3\times 2)} & (3\times 2) \end{array}$$

4

$$\begin{bmatrix} 2 & 7 \\ 3 & 8 \\ 4 & 9 \end{bmatrix} \begin{bmatrix} 1 & 6 \end{bmatrix} = \begin{bmatrix} 2 & 12 \\ 3 & 18 \\ 4 & 24 \end{bmatrix}$$

$$\boxed{AA^{-1} = I = A^{-1}A}$$

(S1.)

$$\boxed{(AB)(B^{-1}A^{-1}) = I}$$

$$\boxed{\begin{aligned} AA^{-1} &= I \\ (A^{-1})^T A^T &= I \end{aligned}}$$

Inverse of A^T is $(A^{-1})^T = (A^T)^{-1}$

Transpose & Inverse order does not matter

$$\underset{A}{\begin{bmatrix} 2 & 1 \\ 8 & 7 \end{bmatrix}} = \underset{L}{\begin{bmatrix} 1 & 0 \\ 4 & 1 \end{bmatrix}} \underset{U}{\begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix}}$$

1's on diagonal — pivots on diagonal

$$= \underset{L}{\begin{bmatrix} 1 & 0 \\ 4 & 1 \end{bmatrix}} \underset{D}{\begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}} \underset{U}{\begin{bmatrix} 1 & 1/2 \\ 0 & 1 \end{bmatrix}}$$

$E_{32}E_{31}E_{21}A = U$ (no row exchanges)

$A = E_{21}^{-1}E_{31}^{-1}E_{32}^{-1}U$

$A = LU$

$$\underset{E_{32}}{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -5 & 1 \end{bmatrix}} \underset{E_{21}}{\begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}} = \underset{E}{\begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 10 & -5 & 1 \end{bmatrix}}$$

$$\underset{E_{32}^{-1}}{\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}} \underset{E_{21}^{-1}}{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 5 & 1 \end{bmatrix}} = \underset{L}{\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 5 & 1 \end{bmatrix}}$$

$$\boxed{\begin{aligned} EA &= U \\ A &= LU \end{aligned}}$$

If no row exchanges, the multipliers go directly into L

Cost of Elimination :→ No. of operations for $n \times n$ matrix

$L = n^2 + \dots 1^2$

$\approx \frac{1}{3}n^3$

cost for Augmented side $\approx n^2$

Permutation Matrices

└→ $\boxed{P^{-1} = P^T}$

3×3 →	$6\ P_s$
4×4 →	$24\ P_s$

$P \to$ identity matrix with reordered rows $\to$ Permutations $\to$ $n!$

Elimination with row exchanges :-

$\boxed{PA = LU}$ // any invertible A

$\boxed{P^{-1} = P^T}$

$\boxed{P^T P = I}$

$\boxed{\text{Symmetric matrices} \to A^T = A}$

$$\begin{bmatrix} 1 & 3 \\ 2 & 3 \\ 4 & 1 \end{bmatrix} \leftarrow R^T \qquad \begin{bmatrix} 1 & 2 & 4 \\ 3 & 3 & 1 \end{bmatrix} \to R$$

$\boxed{R^T R \text{ is always symmetric}}$

$$(R^T R)^T = R^T R^{TT}$$
$$= R^T R$$

Vector Spaces

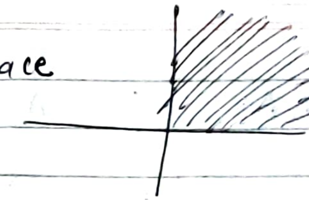
R^2 = all 2 dim real vectors

R^3 = all vectors with 3 real components.

R^n = all column vectors with n real components

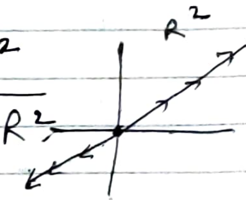
Adding & multiplying must land us in same space

Not a vector space



A vector space inside R^2

→ subspace of R^2



line passing through origin

Subspaces of R^2

→ ① all of R^2 — P

② any line through $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$ — L

③ Zero vector only — Z

$$A = \begin{bmatrix} 1 & 3 \\ 2 & 3 \\ 4 & 1 \end{bmatrix}$$

→ columns in R^3 & all their comb^n's form a subspace

$\boxed{\text{Column space } C(A)}$

Column Space and Null space

Vec Space requirements →
1) $v+w$ and cv are in the space
2) All comb^{n}s of $cv + dw$ are in the space.

(L)
line through origin.

Plane through 'O'
(Subspace) (P)

$\boxed{P \cup L \rightarrow \text{Not a subspace}}$

$\boxed{P \cap L \rightarrow \text{subspace}}$

Subspaces S & T → $S \cap T$ is a subspace

$$A = \begin{bmatrix} 1 & 1 & 2 \\ 2 & 1 & 3 \\ 3 & 1 & 4 \\ 4 & 1 & 5 \end{bmatrix}$$

C(A) is subspace of R^4 ↳ all linear comb^n of columns

Can solve $Ax=b$ exactly when 'b' is in the C(A)

Nullspace of A = all sol^{n}s $X = \begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix}$ to $AX=0$ → R^3

$$A = \begin{bmatrix} 1 & 1 & 2 \\ 2 & 1 & 3 \\ 3 & 1 & 4 \\ 4 & 1 & 5 \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Null space is a subspace

sol^{n}s are not a subspace
[line or plane not passing through origin]

$$AX = \begin{bmatrix} 1 & 1 & 2 \\ 2 & 1 & 3 \\ 3 & 1 & 4 \\ 4 & 1 & 5 \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$$

Solving $Ax = 0$

$$A = \begin{bmatrix} 1 & 2 & 2 & 2 \\ 2 & 4 & 6 & 8 \\ 3 & 6 & 8 & 10 \end{bmatrix}$$

Doing elimination does not ~~the~~ change the Null space but it does change the column space

echeleon form → Pivots → (Pivot columns) → Free columns

$$U = \begin{bmatrix} 1 & 2 & 2 & 2 \\ 0 & 0 & 2 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Rank of A = No. of Pivots = 2

special sol^ns →

$$\text{Sol}^n \text{ to } Ax = 0 \rightarrow x = c\begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + d\begin{bmatrix} 2 \\ 0 \\ -2 \\ 1 \end{bmatrix}$$

→ free variables →

Nullspace contains all the comb^ns of special sol^ns

Rank $= r = 2$ → Pivot variables

$n - r = 2$ → Free variables

R → Reduced row echelon form → zeros above & below pivots

$$U \rightarrow R \rightarrow \begin{bmatrix} 1 & 2 & 0 & -2 \\ 0 & 0 & 2 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 & 0 & -2 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \boxed{R}$$

↳ notice I in Pivot r&c

Sol^ns to $Ax=0$, $Ux=0$ & $Rx=0$ are still the same

$$I \leftarrow \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \begin{bmatrix} 2 & -2 \\ 0 & 2 \end{bmatrix} \rightarrow F$$

Pivot col's — Free col's

0 0 0 0

$$R = \begin{bmatrix} I & F \\ 0 & 0 \end{bmatrix}$$ r pivot rows

r pivot col's — n−r free col^ns

N → nullspace matrix whose columns are the special sol^ns

$$RN = 0 \qquad N = \begin{bmatrix} -F \\ I \end{bmatrix}$$

$$Rx = 0$$

$$\begin{bmatrix} I & F \end{bmatrix}\begin{bmatrix} x_{pivot} \\ x_{free} \end{bmatrix} = 0 \Rightarrow \boxed{x_{pivot} = -F x_{free}}$$

P_c P_c → $F_c = n - r = 1$

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 2 & 6 & 8 \\ 2 & 8 & 10 \end{bmatrix} \Rightarrow U = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$r = 2$ again!

$$c\begin{bmatrix} -F \\ I \end{bmatrix} = x = c\begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}$$ → nullspace

$$R = \begin{array}{c} \begin{array}{cc} I & F \end{array} \\ \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \end{array}$$

Solving $Ax = b$

$\rightarrow$ Augmented matrix $= [A\ b]$

$$A = \left[\begin{array}{cccc|c} 1 & 2 & 2 & 2 & b_1 \\ 2 & 4 & 6 & 8 & b_2 \\ 3 & 6 & 8 & 10 & b_3 \end{array}\right]$$

$$b = \begin{bmatrix} 1 \\ 5 \\ 6 \end{bmatrix}$$

$$u = \left[\begin{array}{cccc|c} 1 & 2 & 2 & 2 & b_1 \\ 0 & 0 & 2 & 4 & b_2 - 2b_1 \\ 0 & 0 & 0 & 0 & b_3 - b_2 - b_1 \end{array}\right]$$

Solvability cond^n on b =>
$Ax = b$ is solvable when b is in $C(A)$

If comb^n of rows of A gives zero row then same comb^n of entries of b must give 0

To find complete sol^n to $Ax = b$ =>

① $X_{particular}$ (X_p) : set all free var. to zero
Solve ~~Ax = b~~ for pivot var.
↳ X_2 and $X_4 = 0$

+

② $X_{nullspace}$ (X_n)

$$X_p = \begin{bmatrix} -2 \\ 0 \\ 3/2 \\ 0 \end{bmatrix}$$

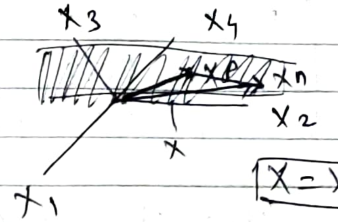
$r = m = n$	$r = n < m$	$r = m < n$
$R = I$	$R = \begin{bmatrix} I \\ 0 \end{bmatrix}$	$R = [I\ F]$
one sol^n	zero or one sol^n	~~0 or~~ ∞ sol^n

Complete sol^n
$$X = X_p + X_n$$

$$\begin{aligned} AX_p &= b \\ AX_n &= 0 \\ \hline A(X_p + X_n) &= b \end{aligned} \Rightarrow X_{complete} = \begin{bmatrix} -2 \\ 0 \\ 3/2 \\ 0 \end{bmatrix} + c_1 \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 2 \\ 0 \\ -2 \\ 1 \end{bmatrix}$$

Nullspace → 2 dim subspace in R^4
↓
Plot all sol^n x in R^4

doesn't pass through origin



$$\boxed{X = X_p + X_n}$$

m by n matrix A of rank r
(Know $r \le m$ and $r \le n$)

- Full col^n rank means $r = n$ — Unique sol^n If exists
  $N(A) = \{\text{zero vector}\}$ No free variables
  Sol^n to $Ax = b$: $\boxed{X = X_p}$
  zero or one sol^n

eg. $\begin{bmatrix} 1 & 3 \\ 2 & 1 \\ 6 & 1 \\ 5 & 1 \end{bmatrix} \rightarrow r = 2 \Rightarrow R = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$

$N(A) = 0$

- Full row rank means $r = m$ — Sol^n exists
  ↳ can solve $Ax = b$ for every b
  ↳ left with $(n - r)$ free variables
  ↳ $(n - m)$

I F

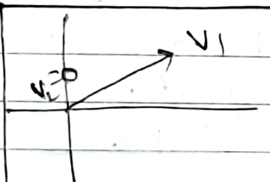
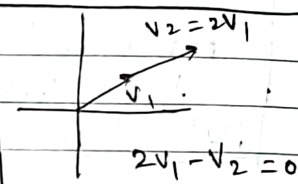
eg. $A = \begin{bmatrix} 1 & 2 & 6 & 5 \\ 3 & 1 & 1 & 1 \end{bmatrix} \Rightarrow R = \begin{bmatrix} 1 & 0 & - & - \\ 0 & 1 & - & - \end{bmatrix}$

- $r = m = n$ eg. $A = \begin{bmatrix} 1 & 2 \\ 3 & 1 \end{bmatrix}$ — Invertible
  ↳ $R = [I]$ → One sol^n to $Ax = b$

Independence, Basis and Dimension

Cond^n → $m < n$ [more unknowns than eq^{n}s]
Nullspace has more than zero vector
↳ Reason: There will be free variables!!

Vectors $X_1, X_2, \ldots X_n$ are independent if no comb^n gives zero vector (except the zero comb^n)
$c_1 X_1 + c_2 X_2 + \ldots c_n X_n \neq 0$
↳ all $c_i = 0$

If one of vector is zero vector Independence is dead		
	Dependent	Dependent

No free variables ← $\boxed{r = n}$

when $V_1, \ldots V_n$ are columns of A : →
They are independent if $N(A) = \{\text{zero vector}\}$
They are dependent if $AC = 0$ for some non-zero C
$\boxed{\text{rank} < n}$ → Free variables

Vectors $V_1, \ldots V_\ell$ span a space means: The space consists of all comb^{n}s of those vectors

Basis for a vector space is a seq. of vectors $V_1, \ldots V_d$ with two properties.
1) They are independent
2) They span the space.

Eg. Space is R^3
one basis is $\begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\end{bmatrix}$

for R^n → n vectors give basis if the $n \times n$ matrix with those columns is invertible

$\boxed{\text{Every basis for the space has the same no. of vectors}}$
↓
$\boxed{\text{Dimension of the space}}$

$$\text{eg. } A = \begin{array}{cccc} \uparrow P_c & \uparrow P_c & \uparrow F_c & \uparrow F_c \\ \end{array}$$
$$A = \begin{bmatrix} 1 & 2 & 3 & 1 \\ 1 & 1 & 2 & 1 \\ 1 & 2 & 3 & 1 \end{bmatrix}$$

Space is $C(A)$
$N(A) = \begin{bmatrix}-1\\-1\\1\\0\end{bmatrix}$

These 4 vectors span $C(A)$ & are dependent

$\boxed{2 = \text{rank}(A) = \text{no. of Pivot columns} = \text{dimension of } C(A)}$

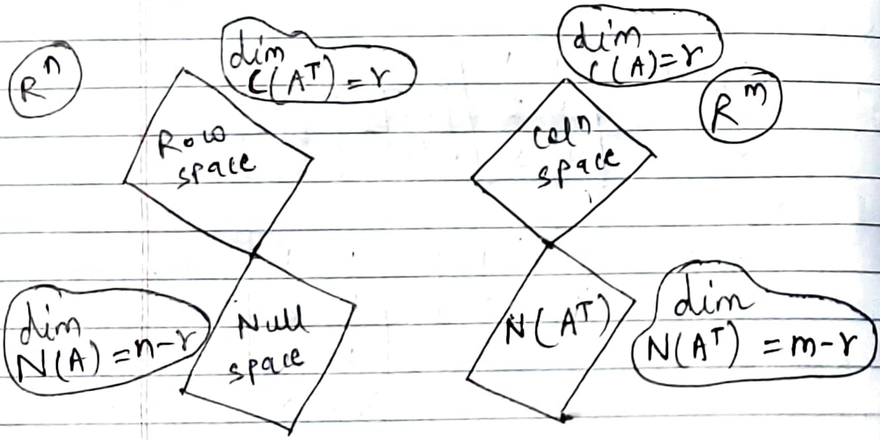
$\boxed{\dim[N(A)] = \text{no. of free variables}}$

$\boxed{\dim[N(A)] = n - r}$

The Four Fundamental Subspaces

✱ 4 Subspaces

① Column space = $C(A)$ in R^m
② Nullspace = $N(A)$ in R^n
③ row space = all comb's of rows
= all comb's of columns of A^T
row space = $C(A^T)$ in R^n
④ Null space of A^T = $N(A^T)$ in R^m
↳ $\boxed{\text{left Null space of } A}$



Basis :-

$C(A)$	$N(A)$	$C(A^T)$	$N(A^T)$
Pivot col^ns	Special sol^ns	first 'r' rows of R	

$$A \rightarrow \begin{bmatrix} 1 & 2 & 3 & 1 \\ 1 & 1 & 2 & 1 \\ 1 & 2 & 3 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = R$$

$C(R) \neq C(A)$
↳ Different col^n space but same row space

When You are doing row operations From A to R → You are still in the rowspace.
But basis changes.

✱ 4th space : $N(A^T)$

$A^T y = 0$
↳ then y is in the $N(A^T)$

$$\begin{bmatrix} A^T \end{bmatrix} \begin{bmatrix} \ \end{bmatrix} = \begin{bmatrix} 0 \end{bmatrix}$$

$$y^T A = 0^T$$

$$\begin{bmatrix} \ \end{bmatrix} \begin{bmatrix} A \end{bmatrix} = \begin{bmatrix} 0 \end{bmatrix}$$

$$\text{rref} \begin{bmatrix} A_{m\times n} & I_{m\times m} \end{bmatrix} \rightarrow \begin{bmatrix} R_{m\times n} & \overset{(E)}{E} \end{bmatrix}$$

R was I

$EA = R$

In Case of sq. Invertible Matrix
↳ then E was A^{-1}

Basis for $N(A^T)$

$$E \begin{bmatrix} -1 & 2 & 0 \\ 1 & -1 & 0 \\ -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} A \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = R$$

Orthogonal Vectors & Subspaces

dot Product :- $x^T y$

x and y are orthogonal if $x^T y = 0$

Subspace S is orthogonal to subspace T means:
every vector in S is orthogonal to every vector in T

Row space is orthogonal to Null space.

$$Ax = 0 \Rightarrow \begin{bmatrix} \cancel{\text{row}}\ 1 \text{ of } A \\ \vdots \\ \text{row } m \text{ of } A \end{bmatrix} \begin{bmatrix} | \\ x \\ | \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

Nullspace and rowspace are orthogonal complements in $\mathbb{R}^n$

Nullspace contains all vectors that are perpendicular to row space

Goal: Solve $Ax = b$ when there is no solution

(when b is not in the $C(A)$)

$m > n$

(Too many Eqns)

$N(A^TA) = N(A)$
rank of A^TA = rank of A

$\boxed{A^TA}$ → Square → Invertible if col^n s of A are independent

→ Symmetric

$(A^TA)^T = A^T A^{TT} = A^TA$

~~At~~, $Ax = b$ → no solution

$A^TA\hat{x} = A^Tb$ → best possible sol^n

eg, $A = \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$ $r = 2$

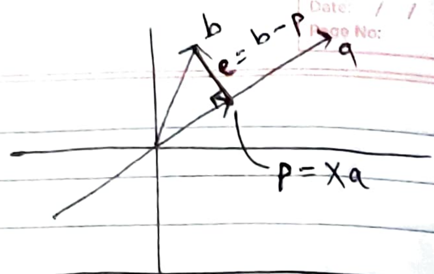
$$A^TA = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 5 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 5 \end{bmatrix} = \begin{bmatrix} 3 & 8 \\ 8 & 30 \end{bmatrix}$$

(2×3) (3×2) (2×2)

Projections onto Subspaces

①

$$\boxed{a^T(b - xa) = 0}$$



$$\boxed{x\,a^Ta = a^Tb}$$

③

$$\boxed{x = \frac{a^Tb}{a^Ta} \qquad p = ax \qquad p = a\frac{a^Tb}{a^Ta}}$$

(projection matrix)

projection $p = Pb$ → $\boxed{P = \frac{aa^T}{a^Ta}}$

⑤ ← $\boxed{p = \frac{aa^T}{a^Ta}\cdot b}$ ⟹ $p = Pb$

$C(P)$ = line through a

$\text{rank}(P) = 1$

Note: when you multiply any vector to a matrix, you land in the col^n space of that matrix

$\boxed{P^T = P}$ → Projection matrix is symmetric

$\boxed{P^2 = P}$ → If we project second time, we get the same result.

④ <u>Why Project?</u>

Bcoz $Ax = b$ may have no sol^n.

↳ b is not in $C(A)$. So find the closest vector to b which is in $C(A)$

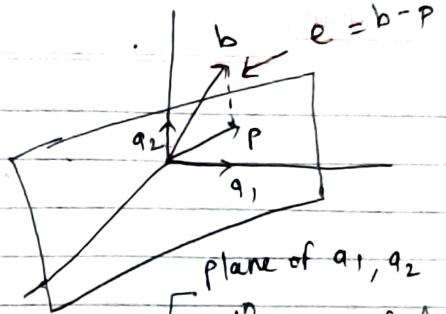
↳ solve $Ax = p$ → Projection of b onto col^n space

e is ⊥^{r} to plane

$p = \hat{x}_1 a_1 + \hat{x}_2 a_2$

$p = A\hat{x}$

$$A = \begin{bmatrix} | & | \\ a_1 & a_2 \\ | & | \end{bmatrix}$$



$p = A\hat{x}$ find $\hat{x}$

Key: $b - A\hat{x}$ is ⊥^{r} to plane

$a_1^T(b - A\hat{x}) = 0$ and $a_2^T(b - A\hat{x}) = 0$

e is ⊥^{r} to $C(A)$

e is in $N(A^T)$ ← $A^T(b - A\hat{x}) = 0$ ②

$$\begin{bmatrix} a_1^T \\ a_2^T \end{bmatrix}(b - A\hat{x}) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

④ $\boxed{A^TA\hat{x} = A^Tb}$ → $\hat{x} = (A^TA)^{-1}A^Tb$

$p = A\hat{x}$ ⟹ $\boxed{p = A(A^TA)^{-1}A^Tb}$ → ⑥

Projection matrix $P = A(A^TA)^{-1}A^T$ ← Not sq. mat

Note: ⟹ $(A^TA)^{-1} \neq A^{-1}(A^T)^{-1}$ → A is not invertible

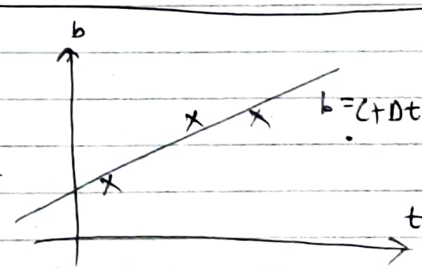
If A is sq. invertible matrix, projection is identity

<u>Appl^n → Least Squares</u>

$C + D = 1$

$C + 2D = 2$

$C + 3D = 2$



$$\overset{A}{\begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{bmatrix}} \overset{x}{\begin{bmatrix} C \\ D \end{bmatrix}} = \overset{b}{\begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}}$$

Too many eqn^s but few unknowns

No soln ⟹ $\boxed{A^TA\hat{x} = A^Tb}$

Projection Matrix & least sq.

Note: least squares is sensitive to outliers

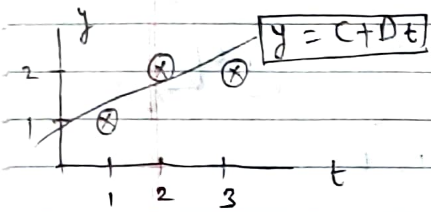
$$P = A(A^TA)^{-1}A^T$$

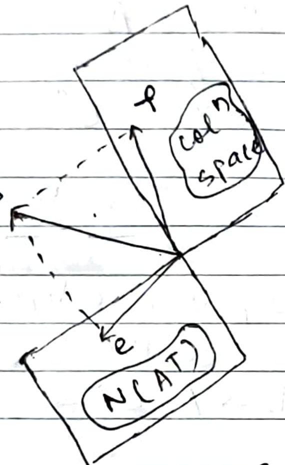
If b is in $C(A)$, $Pb = b$

If b is $\perp^r$ to $C(A)$, $Pb = 0$

$p + e = b$

$Pb + (I-P)b = b$





No sol^n

$C+D=1$

$C+2D=2$

$C+3D=2$

Minimize $\|Ax-b\|^2 = \|e\|^2$

Find $\hat{x} = \begin{bmatrix}\hat{C}\\ \hat{D}\end{bmatrix}$, p { best possible sol^n

$A^TA\hat{x} = A^Tb$

$A^TA = \begin{bmatrix}1&1&1\\1&2&3\end{bmatrix}\begin{bmatrix}1&1\\1&2\\1&3\end{bmatrix} = \begin{bmatrix}3&6\\6&14\end{bmatrix}$ $\quad b = \begin{bmatrix}1\\2\\2\end{bmatrix}$, $A^Tb = \begin{bmatrix}5\\11\end{bmatrix}$

$e = (C+D-1)^2 + (C+2D-2)^2 + (C+3D-2)^2$

↳ Take partial derivatives

$3C+6D=5$

$6C+14D=11$

↓

$2D=1$

$\boxed{D = 1/2,\quad C = 2/3}$

Best line is $\frac{2}{3} + \frac{1}{2}t$

$e_1 = -1/6$, $e_2 = 2/6$, $e_3 = -1/6$

$b = p + e$ $\qquad$ $p \perp^r e$

$$\begin{bmatrix}1\\2\\2\end{bmatrix} = \begin{bmatrix}7/6\\5/3\\13/6\end{bmatrix} + \begin{bmatrix}-1/6\\2/6\\-1/6\end{bmatrix}$$

e is $\perp^r$ to every vector in $C(A)$

$A^TA\hat{x} = A^Tb$

$p = A\hat{x}$

If A has independent columns, then A^TA is invertible.

Suppose $A^TAx = 0$ ← { Prove A^TA is invertible

Idea → $x^TA^TAx = 0$ $\qquad$ $\boxed{x \text{ must be } 0}$

↳ $(Ax)^TAx = 0$ } → $(\text{length of } Ax)^2 = 0$

$\boxed{Ax = 0}$

If A has independent col^{n}s and $Ax = 0$ then $x = 0$

Columns are definitely independent if they are perpendicular unit vectors

↓

Orthonormal Vectors

Orthogonal Matrices and Gram-Schmidt

- orthonormal vectors : $q_i^T q_j = \begin{cases} 0 & \text{if } i \neq j \\ 1 & \text{if } i = j \end{cases}$

$$Q = \begin{bmatrix} | & & | \\ q_1 & \cdots & q_n \\ | & & | \end{bmatrix} \qquad Q^TQ = \begin{bmatrix} -q_1^T- \\ \vdots \\ -q_n^T- \end{bmatrix} \begin{bmatrix} | & & | \\ q_1 & \cdots & q_n \\ | & & | \end{bmatrix} = \begin{bmatrix} 1 & 0 & \cdots \\ 0 & 1 & \cdots \\ 0 & \cdots & 1 \end{bmatrix}$$

$$Q^T Q = I$$

If Q is square matrix then
$\boxed{Q^TQ = I}$ tells us $\boxed{Q^T = Q^{-1}}$

eg. $Q = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$ or $Q = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$

eg. $Q = \frac{1}{2} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix}$

eg. $Q = \frac{1}{3} \begin{bmatrix} 1 & -2 \\ 2 & -1 \\ 2 & 2 \end{bmatrix}$

- Q has orthonormal columns
→ Project onto its column space

Proj. Matrix $P = Q(Q^TQ)^{-1}Q^T$

$\boxed{P = I}$ $\boxed{P = QQ^T}$

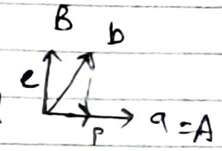
- If matrix Q is square, then col^n space is the whole space & Projection Matrix onto whole space is Identity Matrix

$b = e + p$
$e = b - p \Rightarrow B = b - Pb$

$$P^2 = Q\underbrace{Q^TQ}_{I}Q^T = QQ^T$$

- $A^TA\hat{x} = A^Tb$ $\boxed{\hat{x}_j = q_j^T b}$
Now A is Q
$Q^TQ\hat{x} = Q^Tb \Rightarrow \boxed{\hat{x} = Q^Tb}$

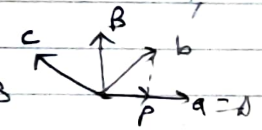
- Gram-Schmidt



→ independent vectors a & b
↓ orthogonal orthonormal
Goal: $a, b \rightarrow A, B \rightarrow q_1, q_2$

$$B = b - \frac{A^Tb}{A^TA}A \Rightarrow A^TB = A^T\left(b - \frac{A^Tb}{A^TA}A\right) = 0$$

- → If there are three independent vectors a, b & c



$$C = c - \frac{A^Tc}{A^TA}A - \frac{B^Tc}{B^TB}B$$
$\|C\|$

- $A = a = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$, $b = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$ $\Rightarrow B = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} - \frac{3}{3}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

$\boxed{A \perp^r B}$ $B = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$
B

$$Q = \begin{bmatrix} 1/\sqrt{3} & 0 \\ 1/\sqrt{3} & -1/\sqrt{2} \\ 1/\sqrt{3} & 1/\sqrt{2} \end{bmatrix}$$

$\boxed{\text{col}^n \text{ space of } Q \text{ \& } A \text{ are same}}$

$\boxed{A = QR}$ → R is upper triangular

Properties of Determinants

If determinant is zero, matrix is singular. (not invertible)

① $\det I = 1$

② Exchange rows: reverse sign of det

(3a) $\begin{vmatrix} ta & tb \\ c & d \end{vmatrix} = t \begin{vmatrix} a & b \\ c & d \end{vmatrix}$

(3b) $\begin{vmatrix} a+a' & b+b' \\ c & d \end{vmatrix} = \begin{vmatrix} a & b \\ c & d \end{vmatrix} + \begin{vmatrix} a' & b' \\ c & d \end{vmatrix}$

$\boxed{\det(A+B) \neq \det(A) + \det(B)}$

④ 2 equal rows $\rightarrow \det = 0$

$\rightarrow$ exchange those rows $\rightarrow$ same matrix

⑤ Subtract $l \times$ row i from row k

$\rightarrow$ det doesn't change

$\begin{vmatrix} a & b \\ c-la & d-lb \end{vmatrix} = \begin{vmatrix} a & b \\ c & d \end{vmatrix} - l \begin{vmatrix} a & b \\ a & b \end{vmatrix} = \begin{vmatrix} a & b \\ c & d \end{vmatrix}$

⑥ Row of zeroes $\longrightarrow \det(A) = 0$

⑦ $U = \begin{vmatrix} d_1 & & * & * & * \\ 0 & d_2 & & * & * \\ 0 & 0 & d_3 & * & * \\ 0 & 0 & 0 & \ddots & \\ 0 & 0 & 0 & & d_n \end{vmatrix} \rightarrow \det(U) = d_1 . d_2 \cdots d_n$

$\underline{= \text{product of pivots}}$

$\det(\text{diagonal mat.}) = d_1 . d_2 . d_3 \cdots d_n$

⑧ $\det A = 0$ when A is singular

Any software performs elimination from A to U and then calculates determinant as product of pivots.

$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \rightarrow \begin{bmatrix} a & b \\ 0 & d - \frac{c}{a} b \end{bmatrix} \rightarrow \det = ad - bc$

⑨ $\det(AB) = (\det A)(\det B)$

$\boxed{\det(A^{-1}) = 1/\det A}$

$A^{-1}A = I \Rightarrow \det A^{-1} . \det A = 1$

$\boxed{\det A^2 = (\det A)^2}$

$\boxed{\det 2A = 2^n \det A}$

⑩ $\det A^T = \det A$

$|A^T| = |A|$

$|U^T L^T| = |LU|$

$|U^T||L^T| = |L||U| \rightarrow$ product of diagonals

Determinant Formulas & Cofactors

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = \begin{vmatrix} a & 0 \\ c & d \end{vmatrix} + \begin{vmatrix} 0 & b \\ c & d \end{vmatrix} = \begin{vmatrix} a & 0 \\ c & 0 \end{vmatrix} + \begin{vmatrix} a & 0 \\ 0 & d \end{vmatrix} + \begin{vmatrix} 0 & b \\ c & 0 \end{vmatrix} + \begin{vmatrix} 0 & b \\ 0 & d \end{vmatrix}$$

$\boxed{ad - bc} \leftarrow$

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = \begin{vmatrix} a_{11} & 0 & 0 \\ 0 & a_{22} & 0 \\ 0 & 0 & a_{33} \end{vmatrix} + \begin{vmatrix} a_{11} & 0 & 0 \\ 0 & 0 & a_{23} \\ 0 & a_{32} & 0 \end{vmatrix} + \cdots$$

↳ one entry from each row & column

$$= a_{11}a_{22}a_{33} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33} + a_{13}a_{21}a_{32} + a_{12}a_{23}a_{31} - a_{13}a_{22}a_{31}$$

- BIG FORMULA ($n \times n$ det)

$$\boxed{\det A = \sum_{n!\ \text{terms}} \pm\, a_{1\alpha} a_{2\beta} a_{3\gamma} \cdots a_{n\omega} \qquad (\alpha, \beta, \gamma \ldots \omega) = \text{Permutation of } (1, 2 \ldots n)}$$

eg. $\begin{vmatrix} 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{vmatrix}$ → Permutations

$(4,3,2,1) \to +1$
$(3,2,1,4) \to -1$ } rest are 0

det = 0

Cofactors 3×3 (In parenthesis)

$$\det = a_{11}(a_{22}a_{33} - a_{23}a_{32}) - a_{12}(a_{21}a_{33} - a_{23}a_{31}) + a_{13}(a_{32}a_{21} - a_{22}a_{31})$$

$$= \begin{vmatrix} a_{11} & 0 & 0 \\ 0 & a_{22} & a_{23} \\ 0 & a_{32} & a_{33} \end{vmatrix} + \begin{vmatrix} 0 & a_{12} & 0 \\ a_{21} & 0 & a_{23} \\ a_{31} & 0 & a_{33} \end{vmatrix} + \begin{vmatrix} 0 & 0 & a_{13} \\ a_{21} & a_{22} & 0 \\ a_{31} & a_{32} & 0 \end{vmatrix}$$

Cofactor of a_{ij} $= \pm \det \begin{pmatrix} n-1 \text{ matrix} \\ \text{with row } i \text{ erased} \\ \text{col } j \end{pmatrix} = C_{ij}$

(+) if $i+j$ even
(−) if $i+j$ odd

Cofactor formula (along row 1)

$$\boxed{\det A = a_{11}C_{11} + a_{12}C_{12} + \cdots a_{1n}C_{1n}}$$

$\begin{vmatrix} \cancel{a} & \cancel{b} \\ c & d \end{vmatrix}$ $-bc$, $\begin{vmatrix} \cancel{a} & \cancel{b} \\ \cancel{c} & d \end{vmatrix}$ ad $\rightarrow \boxed{ad - bc}$

Eg. $A_4 = \begin{vmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{vmatrix}$ $|A_1| = 1$ $|A_2| = 0$ $|A_3| = -1$

$|A_4| = -1 \longleftarrow |A_4| = 1 \cdot |A_3| - 1 \cdot |A_2|$

$$|A_n| = |A_{n-1}| - |A_{n-2}|$$

$|A_5| = 0,\ |A_6| = 1,\ |A_7| = 1$

Cramer's Rule, Inverse Matrix & Volume

$$A^{-1} = \frac{1}{\det A} C^T \rightarrow \text{products of } n-1 \text{ entries}$$

$C \rightarrow$ Co-factor Matrix

$\det A \rightarrow$ products of n entries

check $AC^T = (\det A) I$

$$\begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \\ a_{n1} & \cdots & a_{nn} \end{bmatrix} \begin{bmatrix} c_{11} & & c_{n1} \\ c_{12} & \cdots & \vdots \\ \vdots & & \vdots \\ c_{1n} & & c_{nn} \end{bmatrix} = \begin{bmatrix} \det A & & \\ & \det A & \\ & & \det A \end{bmatrix}$$

A Row times the row of cofactor of another row gives zero.

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

- $Ax = b$

cramer's rule

$$x = A^{-1} b = \frac{1}{\det A} C^T b$$

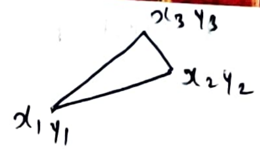
number times cofactor

det of some matrix

$$x_1 = \frac{\det B_1}{\det A}, \quad x_2 = \frac{\det B_2}{\det A}$$

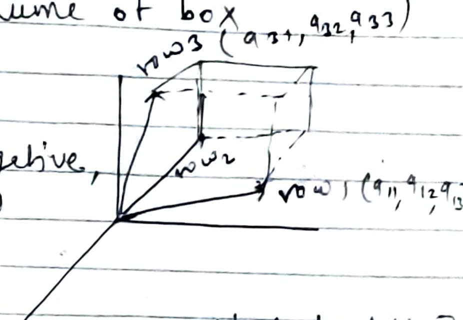
$$B_1 = \begin{bmatrix} | & n-1 \\ b & \text{col}^{n}\text{s} \\ | & \text{of } A \end{bmatrix} \rightarrow \text{A with column 1 replaced by RHS } b$$

B_j = A with column j replaced by b



$$\text{area} = \frac{1}{2} \begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$$

$\det A$ = Volume of box



- If det is negative, there are row exchanges involved and the sides (edges) are switched

[Right handed box or Left handed box]

If $A = Q$, it is a cube.

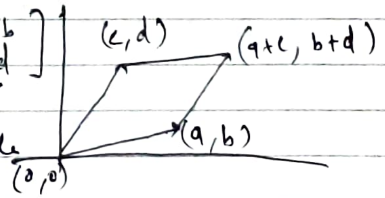
$$Q^T Q = I$$
$$\det(Q^T Q) = \det(I)$$
$$\det Q^T \det Q = 1$$
$$(\det Q)^2 = 1$$
$$\boxed{\text{So } \det Q = \pm 1}$$

3A → If we double the one edge, volume doubles and so does the determinant.

3B → $\begin{vmatrix} a+a' & b+b' \\ c & d \end{vmatrix} = \begin{vmatrix} a & b \\ c & d \end{vmatrix} + \begin{vmatrix} a' & b' \\ c & d \end{vmatrix}$

$\text{area} = \det \begin{bmatrix} a & b \\ c & d \end{bmatrix} = ad - bc$

$\frac{1}{2}(ad-bc) \rightarrow$ for triangle



Eigen Vectors and Eigen values

If after a linear transformation, a vector only streches or squishes itself and remains on its span then it is an Eigen vector

Ax parallel to x → Eigen Vectors

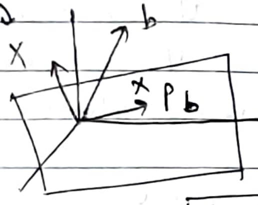
$$\boxed{Ax = \lambda x}$$

If A is singular, then $\lambda = 0$ is eigen value.

- what are x's & λ's for Projection Matrix?

$\lambda = 1 \rightarrow$

Any x in plane: $Px = x$



Any x $\perp^r$ to plane: $Px = 0x$, $\lambda = 0$

$\boxed{\lambda = 1, -1}$

Eg. $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ $\quad x = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ $\quad Ax = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ $\boxed{\lambda = 1}$

$x_1 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$, $Ax_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ $\boxed{\lambda = -1}$

x & x_1 are $\perp^r$

$\boxed{n \times n \text{ matrices can have } n \text{ eigen values}}$

$\boxed{\text{Sum of } \lambda\text{'s} = a_{11} + a_{22} + \cdots a_{nn}}$

- Solving $Ax = \lambda x$

$(A - \lambda I)x = 0 \rightarrow$ Singular

$$\boxed{\det(A - \lambda I) = 0}$$

Trace = sum of λ
determinant = product of λ

$\lambda^2 - \text{trace}\,\lambda + \det = 0$

Eg. $A = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}$ $\quad \det(A - \lambda I) = \begin{vmatrix} 3-\lambda & 1 \\ 1 & 3-\lambda \end{vmatrix}$

$(3-\lambda)^2 - 1 = 0 \Rightarrow \lambda^2 - 6\lambda + 8 = 0$

$A + 3I \rightarrow$ $\boxed{\lambda_1 = 4 \;\&\; \lambda_2 = 2}$

for $\lambda_1 = 4 \rightarrow$ $A - 4I = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} \xrightarrow{\text{solve for } N(A)} x_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

for $\lambda_2 = 2 \rightarrow x_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$

If $Ax = \lambda x$, then $(A + 3I)x = \lambda x + 3x$

Same eigen vectors with changed λ ← $(A + 3I)x = (\lambda + 3)x$

$Ax = \lambda x$, $Bx = \alpha x$
$(A + B)x \neq (\lambda + \alpha)x$
Not same eigen vectors

Eg. 90° rotation

$Q = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$

$\det(Q - \lambda I) = \begin{vmatrix} -\lambda & -1 \\ 1 & -\lambda \end{vmatrix} = \lambda^2 + 1 = 0$

↳ $\lambda_1 = i$, $\lambda_2 = -i$

If matrices are symmetric or close to symmetric, eigen values stay real.

If mat. is triangular λ's are on diagonal

$A = \begin{bmatrix} 3 & 1 \\ 0 & 3 \end{bmatrix}$ $\quad \det(A - \lambda I) = \begin{vmatrix} 3-\lambda & 1 \\ 0 & 3-\lambda \end{vmatrix} = 0$

↳ $(3 - \lambda)(3 - \lambda) = 0$ $\quad \lambda = 3, 3$

$(A - \lambda I)x = 0 \Rightarrow \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow x_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $x_2 = ?$

(No 2nd λ)

Diagonalization and Powers of A

Suppose A has n independent eigen vectors.
↳ Put them in columns of S

$\boxed{Ax=\lambda x} \to AS = A\begin{bmatrix} | & | & & | \\ x_1 & x_2 & \cdots & x_n \\ | & | & & | \end{bmatrix} = \begin{bmatrix} | & & | \\ \lambda_1 x_1 & \cdots & \lambda_n x_n \\ | & & | \end{bmatrix}$

$$S\Lambda = \underbrace{\begin{bmatrix} | & & | \\ x_1 & \cdots & x_n \\ | & & | \end{bmatrix}}_{S}\underbrace{\begin{bmatrix} \lambda_1 & \cdots & 0 \\ 0 & \ddots & 0 \\ \vdots & & 0 \\ & & \lambda_n \end{bmatrix}}_{\Lambda} =$$

$AS = S\Lambda$

$\boxed{S^{-1}AS = \Lambda} \to$ Diagonalization

$\boxed{A = S\Lambda S^{-1}}$ $\boxed{\text{E.value of } A^2 \text{ are } \lambda^2 \text{ \& E.vectors are same}}$

If $Ax=\lambda x \Rightarrow A^2x = \lambda Ax \Rightarrow \boxed{A^2x = \lambda^2 x}$

$\to A^2 = S\Lambda S^{-1}S\Lambda S^{-1} = S\Lambda^2 S^{-1}$

$\boxed{A^k = S\Lambda^k S^{-1}}$

(abs. value)

$\boxed{\text{Th}^m\text{:- } A^k \to 0 \text{ as } k\to\infty \text{ if all } |\lambda_i| < 1}$

> A is sure to have n independent e.vectors
> (and be diagonalizable)
> if all the λ's are different.
> (No repeated λ's)

If repeated λ's $\to$ may or may not have n indep. e.vectors

Repetition of λ's $\to$ Algebraic multiplicity increases

- If A is I $\to \Lambda = S^{-1}AS = I$
  (A is already diagonalized)
- If A is triangular $\to$
  $A = \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix} \Rightarrow \lambda = 2, 2 \to A - 2I = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$
  Algebraic Multiplicity = 2
  $x_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \to$ No 2 indep. e.vectors
  ↳ Not diagonalizable

Start with given vector u_0

Eq^n: $u_{k+1} = Au_k$

$u_1 = Au_0,\ u_2 = A^2u_0 \Rightarrow \boxed{u_k = A^k u_0}$

To really solve:

$u_0 = c_1x_1 + c_2x_2 + \cdots c_nx_n$

$Au_0 = c_1\lambda_1x_1 + \cdots c_n\lambda_nx_n$

↳ $A^{100}u_0 = c_1\lambda_1^{100}x_1 + \cdots c_n\lambda_n^{100}x_n$ — (1)

- Fibonacci example $\Rightarrow 0, 1, 1, 2, 3, 5, 8, 13 \ldots$
  $F_{k+2} = F_{k+1} + F_k,\ F_{k+1} = F_{k+1}$ $\boxed{F_{100} = ?}$

TRICK: $u_k = \begin{bmatrix} F_{k+1} \\ F_k \end{bmatrix} \to u_{k+1} = \underbrace{\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}}_{A}\begin{bmatrix} F_{k+1} \\ F_k \end{bmatrix}$

$A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \to A - \lambda I = \begin{vmatrix} 1-\lambda & 1 \\ 1 & -\lambda \end{vmatrix}$

$\lambda_1 = \frac{1}{2}(1+\sqrt{5}),\ \lambda_2 = \frac{1}{2}(1-\sqrt{5})$
$\approx 1.6 \qquad \approx -0.6$

Eigen values are controlling growth of Fibonacci seq.

$F_{100} \approx c_1\left(\frac{1+\sqrt{5}}{2}\right)^{100} \to \boxed{\text{look (1)}}$

$x_1 = \begin{bmatrix} \lambda_1 \\ 1 \end{bmatrix},\ x_2 = \begin{bmatrix} \lambda_2 \\ 1 \end{bmatrix} \Rightarrow u_0 \begin{bmatrix} F_1 \\ F_0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$c_1x_1 + c_2x_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$\boxed{A = A^T}$ Symmetric Matrices & Positive Definiteness

$A = A^T$

① Eigen values are Real. → can be chosen

② Eigen vectors are perpendicular.

Usual case → $A = S\Lambda S^{-1}$

Symmetric case → $A = Q\Lambda Q^{-1}$ { orthonormal eigen vectors (col^ns of Q)

$\boxed{Q^{-1} = Q^T}$

$\boxed{A = Q\Lambda Q^T}$

• Why real eigen values?

$Ax = \lambda x \xRightarrow{\text{always}} \bar{A}\bar{x} = \bar{\lambda}\bar{x} \Rightarrow \bar{x}^T A^T = \bar{x}^T\bar{\lambda}$ → conjugates

→ $A^T = A$ → symmetric → $\bar{x}^T A x = \bar{x}^T \bar{\lambda} x$

~~$\bar{x}^T A = \bar{x}^T \bar{\lambda}$~~

~~$A\bar{x} = \bar{\lambda}\bar{x} \Rightarrow \bar{x}^T A\bar{x} = \bar{\lambda}\bar{x}^T\bar{x}$~~

$Ax = \lambda x \rightarrow \bar{x}^T A x = \lambda \bar{x}^T x$

→ $\lambda \bar{x}^T x = \bar{\lambda}\bar{x}^T x \Rightarrow \boxed{\lambda = \bar{\lambda}}$ → λ is real

$\bar{x}^T x$ → (length)^2

$\bar{x}^T x = [\bar{x}_1\ \bar{x}_2 \cdots \bar{x}_n]\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = \bar{x}_1 x_1 + \cdots + \bar{x}_n x_n$

→ $(a - ib)(a + ib)$

→ $a^2 + b^2$

Good Matrices

{ Real λ's and ⊥r x's

→ $A = A^T$

$A = Q\Lambda Q^T$

$= \lambda_1 q_1 q_1^T + \lambda_2 q_2 q_2^T + \cdots$

Every symmetric matrix is combination of perp^r projection matrices

For symmetric matrices,
Signs of pivots are same as signs of λ's
No. of (+)ve pivots = No of (+)ve λ's

For symmetric matrices,
product (pivots) = product (λ's) = det

• Positive definite Symmetric Matrix

→ All the eigen values are positive

→ All the pivots are positive

eg, $A = \begin{bmatrix} 5 & 2 \\ 2 & 3 \end{bmatrix} \rightarrow u = \begin{bmatrix} 5 & 2 \\ 0 & 11/5 \end{bmatrix} \rightarrow \lambda^2 - 8\lambda + 11 = 0$

$\lambda = 4 \pm \sqrt{5}$

→ All sub-determinants are positive

Positive definite Matrices & Minima

Tests for Positive definiteness →

$A = \begin{bmatrix} a & b \\ b & c \end{bmatrix}$

① $\lambda_1 > 0, \lambda_2 > 0$
② $a > 0$, $ac - b^2 > 0$
③ Pivots $a > 0$, $\frac{ac - b^2}{a} > 0$
④ $x^T A x > 0$

eg. $A_1 = \begin{bmatrix} 2 & 6 \\ 6 & 18 \end{bmatrix}$ → det = 0 → Positive Semidefinite

$\lambda = 0, 20$ → only one pivot → 2

$\lambda \geq 0$

$[x_1 \; x_2] \begin{bmatrix} 2 & 6 \\ 6 & 18 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 2x_1^2 + 12x_1x_2 + 18x_2^2$

$x^T A x = ax^2 + 2bxy + cy^2$

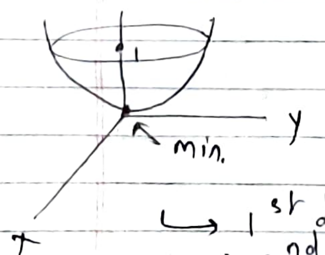
(Quadratic form)

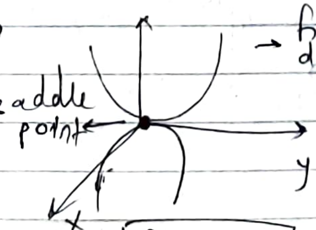
If $A_2 = \begin{bmatrix} 2 & 6 \\ 6 & 7 \end{bmatrix}$ → Pivots 2 & −11
→ det is negative

→ $x^T A x < 0 \Rightarrow 2x^2 + 12xy + 7y^2$

Graphs of $f(x,y) = x^T A x = ax^2 + bxy + cy^2$

If $A_3 = \begin{bmatrix} 2 & 6 \\ 6 & 20 \end{bmatrix}$ → $x^T A x > 0$





→ first deriv. = 0

fig A_2

↳ 1st derivatives = 0
↳ 2nd derivative > 0

fig. A_3 → $\boxed{\begin{array}{l}\text{If cut through height} = 1 \Rightarrow x^T A x = 1 \text{ (ellipse)} \\ \text{If cut through saddle point} \Rightarrow \text{hyperbola}\end{array}}$

Calculus: MIN ~ $\frac{dy}{dx} = 0$, $\frac{d^2y}{dx^2} > 0$

Lin Alg: MIN ~ $f(x_1, x_2 \dots x_n)$ $\boxed{\text{Matrix of second derv.}}$ → is $\boxed{\text{Positive Definite}}$

$A_3 \rightarrow f(x,y) = 2x^2 + 12xy + 20y^2$
$= \boxed{2}(x + \boxed{3}y)^2 + \boxed{2}y^2$

↑ Elimination

$A_3 \rightarrow \begin{bmatrix} 2 & 6 \\ 6 & 20 \end{bmatrix} \xrightarrow{R_2 - 3R_1} \begin{bmatrix} \boxed{2} & 6 \\ 0 & \boxed{2} \end{bmatrix}$ → Positive pivots

Matrix of 2nd derv. → $A_4 = \begin{bmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{bmatrix}$

• 3x3 example

$A_5 = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}$ → dets = 2, 3, 4
→ pivots = 2, $\frac{3}{2}$, $\frac{4}{3}$
λ's → $2 - \sqrt{2}$, 2, $2 + \sqrt{2}$

$\boxed{\text{A can be written as } Q \Lambda Q^T}$

$x^T A x = 2x_1^2 + 2x_2^2 + 2x_3^2 - 2x_1x_2 - 2x_2x_3 > 0$

→ If cut through height = 1, ⇒ eqⁿ of ellipsoid (football)

will have 3 axes and the length of axes will be determined by eigen values.

Similar Matrices and Jordan Form

- If A is symmetric positive definite:

  $A \to \lambda_1, \lambda_2 \ldots$ $\qquad A^{-1} \to 1/\lambda_1, \frac{1}{\lambda_2} \ldots$

  λ's are positive $\qquad \frac{1}{\lambda}$'s are positive

- If A & B are positive definite:

  $x^T A x > 0$, $x^T B x > 0$, $x^T (A+B) x > 0$

  $(A+B)$ is also positive definite. (with rank 'n')

- If A is $m \times n$ matrix, $A^T A$ is square symmetric matrix & pos. definite.

  $\to x^T A^T A x = (Ax)^T (Ax)$

  $= \|Ax\|^2 \geq 0$

  $= \|Ax\|^2 > 0$ (except zero vec)

  $\boxed{\text{Rank}(A) = n}$

- Similar Matrices

  If square $n \times n$ matrices A and B are similar then for some M

  $\boxed{B = M^{-1} A M}$

- If A has n independent eigen vectors

  $\Lambda = S^{-1} A S$

  $\to \boxed{A \text{ is similar to } \Lambda}$

eg. $A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$ $\qquad \Lambda = \begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix}$

Every square A is similar to a Jordan Matrix J | $J = \begin{bmatrix} J_1 & & \\ & J_2 & \\ & & \ddots J_d \end{bmatrix}$ no. of blocks = no. of e. vectors

Any $M = \begin{bmatrix} 1 & 4 \\ 0 & 1 \end{bmatrix}$

$$\underbrace{M^{-1} A M}_{} = \overset{M^{-1}}{\begin{bmatrix} 1 & -4 \\ 0 & 1 \end{bmatrix}} \overset{A}{\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}} \overset{M}{\begin{bmatrix} 1 & 4 \\ 0 & 1 \end{bmatrix}} = \overset{B}{\begin{bmatrix} -2 & -15 \\ 1 & 6 \end{bmatrix}}$$

$\boxed{A \text{ \& } B \text{ has same eigen values}}$ (and same no. of eigen vectors)

$\boxed{\text{Similar matrices have same eigen values.}}$

$\to \boxed{Ax = \lambda x} \to M^{-1} A M M^{-1} x = \lambda M^{-1} x$

$\to B M^{-1} x = \lambda M^{-1} x$

$\to \boxed{B \hat{x} = \lambda \hat{x}}$

$\boxed{\text{Eigen vector of } B \;\; M^{-1} \text{ times e.vector of } A}$

- BAD case: $\lambda_1 = \lambda_2 = 4$

  $\to$ one small family of mat. has only $\begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix}$

  $\to$ big family has $\to$ $\begin{bmatrix} 4 & 1 \\ 0 & 4 \end{bmatrix} \leftarrow$ Jordan form
  & other matrices

  (Not diagonalizable) $\qquad$ Closest to diagonalizable

eg. $A = \left[\begin{array}{cc|cc} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \end{array}\right]$ $\to$ Rank = 2 $\to \lambda = 0, 0, 0, 0$

$\boxed{\text{Not similar}}$ $\to$ 2 eigen vectors. | $\dim(N(A)) = 2$

eg. $A = \left[\begin{array}{cc|cc} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right]$ $\to \lambda = 0, 0, 0, 0$ $\to$ Rank = 2 $\to$ 2 eigen vectors

$\boxed{\text{Jordan Block}} \to J_i = \begin{bmatrix} \lambda_i & 1 & & 0 \\ & \lambda_i & 1 & \\ & & \ddots & 1 \\ 0 & & & \lambda_i \end{bmatrix}$

(1 eigen vector)

$\boxed{\text{Good } J \text{ is } \Lambda}$

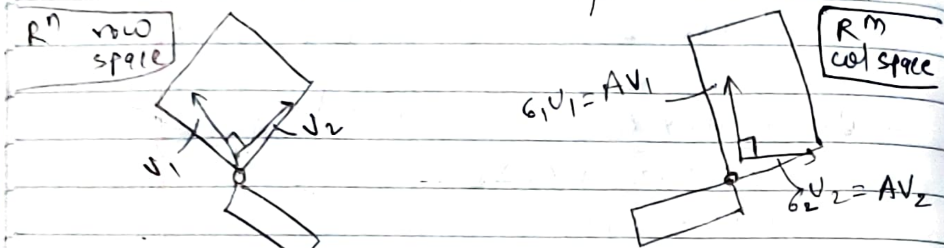
SVD [Singular Value Decomposition]

$A = U\Sigma V^T$ // $\Sigma \to$ diagonal; $U, V \to$ orthogonal

$A = S\Lambda S^{-1}$

$\boxed{A = Q\Lambda Q^T}$ → Symmetric Pos. definite

↳ SVD in case of Symm. Pos. definite



→ looking for an orthogonal basis which stays orthogonal even after getting knocked over to another space.

$$A\begin{bmatrix} v_1 & v_2 & \cdots & v_r \end{bmatrix} = \begin{bmatrix} u_1 & u_2 & \cdots & u_r \end{bmatrix}\begin{bmatrix} \sigma_1 & & 0 \\ & \sigma_2 & \\ 0 & & \sigma_r \end{bmatrix}$$

$AV = U\Sigma$

$$\boxed{A = U\Sigma V^{-1} = U\Sigma V^T}$$

$$A^TA = V\Sigma^T \underbrace{U^TU}_{I}\Sigma V^T$$

$$A^TA = V\begin{bmatrix} \sigma_1^2 & & \\ & \sigma_2^2 & \\ & & \ddots \end{bmatrix}V^T$$

σ's (positive λ's)

col^n's of V → orthonormal eigen vectors of A^TA

col^n's of U → orthonormal eigen vectors of AA^T

• $A = \begin{bmatrix} 4 & 4 \\ -3 & 3 \end{bmatrix} \Rightarrow A^TA = \begin{bmatrix} 4 & -3 \\ 4 & 3 \end{bmatrix}\begin{bmatrix} 4 & 4 \\ -3 & 3 \end{bmatrix}$

$A^TA = \begin{bmatrix} 25 & 7 \\ 7 & 25 \end{bmatrix}$ → e.vectors → $32\begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$ & $18\begin{bmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{bmatrix}$

$$\begin{bmatrix} 4 & 4 \\ -3 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} \sqrt{32} & 0 \\ 0 & \sqrt{18} \end{bmatrix}\begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix}$$

$A \quad = \quad U \quad \Sigma \quad V^T$

$AA^T = U\Sigma\Sigma^TU^T$ $\qquad AA^T = \begin{bmatrix} 32 & 0 \\ 0 & 18 \end{bmatrix}$

$\boxed{\text{Eigen values of } A^TA \text{ \& } AA^T \text{ stay same}}$

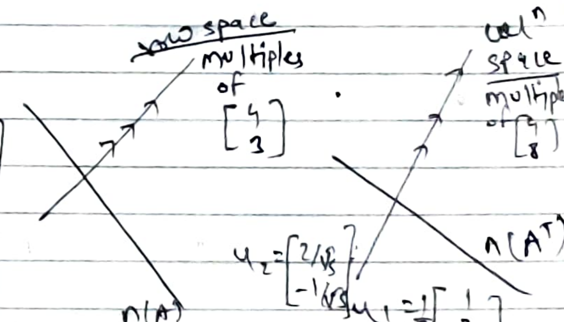
$U = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

• $A = \begin{bmatrix} 4 & 6 \\ 8 & 3 \end{bmatrix}$

$v_1 = \begin{bmatrix} 0.8 \\ 0.6 \end{bmatrix}$

$v_2 = \begin{bmatrix} 0.6 \\ -0.8 \end{bmatrix}$

$u_2 = \begin{bmatrix} 2/\sqrt{5} \\ -1/\sqrt{5} \end{bmatrix}$ $\quad u_1 = \frac{1}{\sqrt{5}}\begin{bmatrix} 1 \\ 2 \end{bmatrix}$



$$\begin{array}{cccc} A & U & \Sigma & V^T \\ \begin{bmatrix} 4 & 3 \\ 8 & 6 \end{bmatrix} = & \frac{1}{\sqrt{5}}\begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix} & \begin{bmatrix} \sqrt{125} & 0 \\ 0 & 0 \end{bmatrix} & \begin{bmatrix} 0.8 & 0.6 \\ 0.6 & -0.8 \end{bmatrix} \end{array}$$

$A^TA = \begin{bmatrix} 80 & 60 \\ 60 & 45 \end{bmatrix}$ → σ's → 0, 125

and $Av_i = \sigma_i u_i$

• $v_1, \ldots, v_r$ → orthonormal basis for row space

$u_1, \ldots, u_r$ → " column space

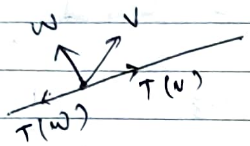
$v_{r+1}, \ldots, v_n$ → " null space

$u_{r+1}, \ldots, u_m$ → " $N(A^T)$

Linear Transformations

$$\boxed{\begin{aligned} T(v+w) &= T(v) + T(w) \\ T(cv) &= c\,T(v) \end{aligned}} \rightarrow \text{Linear } Tr^n$$

Eg. 1. Projection $T: R^2 \rightarrow R^2$ $\rightarrow$ linear



2. $T(v) = \|v\|$ $\quad T: R^3 \rightarrow R^1 \rightarrow$ Not linear

3. Rotation by $45°$ $\quad T: R^2 \rightarrow R^2 \rightarrow$ linear

4. $T(v) = Av$

$A(v+w) = Av + Aw$, $\quad A(cv) = cAv$

- Start: $T: R^3 \rightarrow R^2$

$A \rightarrow 2 \times 3$ matrix

eg. $T(v) = Av$

o/p in R^2 ↑ $\quad$ ↑ i/p in R^3

- Information needed to know $T(v)$ for all i/p

Basis spans the whole space $\rightarrow$ $T(v_1), T(v_2), \dots T(v_n)$ for any i/p basis $v_1, \dots v_n$,

Every $v = c_1 v_1 + \dots c_n v_n$

Then we know, $T(v) = c_1 T(v_1) + \dots c_n T(v_n)$

- Co-ordinates come from a basis

Coordinates of $v = c_1 v_1 + \dots c_n v_n$

$$A \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} c_2 \\ 2c_3 \end{bmatrix} \Rightarrow A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

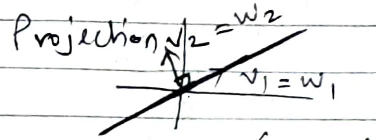
- Construct a matrix A that represents linear Transformation

$T: R^n \rightarrow R^m$

$\rightarrow$ Choose basis $v_1, \dots v_n$ for i/p R^n

$\rightarrow$ choose basis $w_1 \dots w_n$ for o/ps R^m

Goal $\rightarrow$ Matrix A



$v = c_1 v_1 + c_2 v_2$

$T(v) = c_1 v_1$

(c_1, c_2) ↓ $(c_1, 0)$

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} c_1 \\ 0 \end{bmatrix}$$

A $\quad$ i/p coodr $\quad$ o/P cod'rs

Eigen vector basis leads to diagonal matrix A

- Projection onto $45°$ line using std. basis

$v_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} = w_1$, $\quad v_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} = w_2$

$$P = \frac{aa^T}{a^Ta} = \begin{bmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{bmatrix}$$

- Rule to find A $\quad$ Given bases $v_1 — v_n$, $w_1 — w_m$

linear

1^{st} col^n of A: Apply L.Tr. $T(v_1) = a_{11} w_1 + a_{21} w_2 + \dots a_{m1} w_m$

2^{nd} col^n of A: $T(v_2) = a_{12} + \dots a_{m2} w_m$

$$\boxed{A \left(\text{i/p coords}\right) = \text{o/p coords}}$$

$T = \frac{d}{dx}$ $\quad$ i/p $\rightarrow c_1 + c_2 x + c_3 x^2$, basis: $1, x, x^2$: R_3

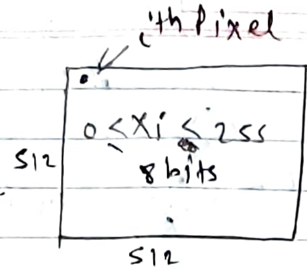
o/p $\rightarrow c_2 + 2c_3 x$, basis: $1, x$: R_2

$\boxed{\text{Change of Basis}}$

$X \in R^n$

$n = (512)^2$

$$X = \begin{bmatrix} \vdots \\ 121 \\ 224 \\ 126 \\ \vdots \end{bmatrix} \rightarrow \text{length} = (512)^2$$

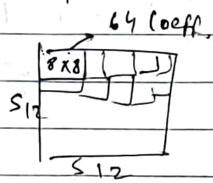


Standard Basis

$$\begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \end{bmatrix} \cdots \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}$$

Better Basis

$$\begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \\ -1 \\ -1 \\ -1 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \\ \vdots \end{bmatrix}$$



Fourier basis 8x8

w → complex no.

$$\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \begin{bmatrix} 1 \\ w \\ w^2 \\ \vdots \\ w^{n-1} \end{bmatrix}$$

lossless ← Signal P

↓ change Basis

Coeffs. c

lossy ← ↓ compression

Coeffs. $\hat{c}$ (many zeroes)

↓

$$\hat{X} = \sum \hat{c}_i v_i$$

Video → sequence of images that are highly co-related.

Eigenvector Basis

$$T(v_i) = \lambda_i v_i \Rightarrow A = \begin{bmatrix} \lambda_1 & & 0 \\ & \lambda_2 & \\ 0 & & \ddots \ \lambda_n \end{bmatrix}$$

Wavelet basis (8x8) (W)

$$\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ -1 \\ -1 \\ -1 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ -1 \\ -1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 1 \\ -1 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \cdots$$

orthogonal

std basis

$$\begin{bmatrix} p_1 \\ p_2 \\ \vdots \\ p_8 \end{bmatrix}$$

$$p = c_1 w_1 + \cdots c_8 w_8$$

$$P = \begin{bmatrix} \vdots & \vdots & W \\ \vdots & \vdots & \cdots \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_8 \end{bmatrix}$$

$$p = Wc \Rightarrow c = W^{-1} p$$

Good basis → ① fast FFT, FWT
② FEW is enough

- change of basis

col's of W = new basis vectors

$$[x]_{\text{old basis}} \longrightarrow [c]_{\text{new basis}}$$

$$\boxed{x = Wc}$$

- T with respect to $v_1, \dots v_8$ has matrix A
— to $w_1, \dots w_8$ has matrix B

$\boxed{\text{A \& B are similar}} \Rightarrow \boxed{B = M^{-1} A M}$

A = ? using basis $v_1, \dots v_8$

$T \rightarrow T(v_1), T(v_2) \dots T(v_8)$

$$A = \begin{bmatrix} a_{11} & a_{21} & \cdots \\ \vdots & \vdots & \\ a_{18} & a_{28} & \end{bmatrix}$$

$$T(v_1) = a_{11} v_1 + a_{21} v_2 + \cdots a_{81} v_8$$

Left & Right Inverses / Pseudoinverse

2 sided Inverse: $AA^{-1} = I = A^{-1}A$ } → $r = m = n$ → full rank

Left Inverse: } → full col^n rank. $r = n < m$
→ $N(A) = \{0\}$ indep. cols
→ 0 or 1 sol^n to $Ax = b$

→ A^TA is invertible

$$\underbrace{(A^TA)^{-1}A^T}_{A^{-1}_{left}}A = I$$

$$\underset{n\times m}{A^{-1}_{left}}\ \underset{m\times n}{A} = \underset{n\times n}{I}$$

$$\boxed{\begin{array}{l} A \cdot A^{-1}_{left} = P \\ A(A^TA)^{-1}A^T = P \\ \text{Proj}^n \text{ onto col}^n \text{ space} \end{array}}$$

Right Inverse: } full row rank $r = m < n$
→ $N(A^T) = \{0\}$ indep. rows
→ ∞ many sol^{n}s to $Ax = b$
→ $n - m$ free variables

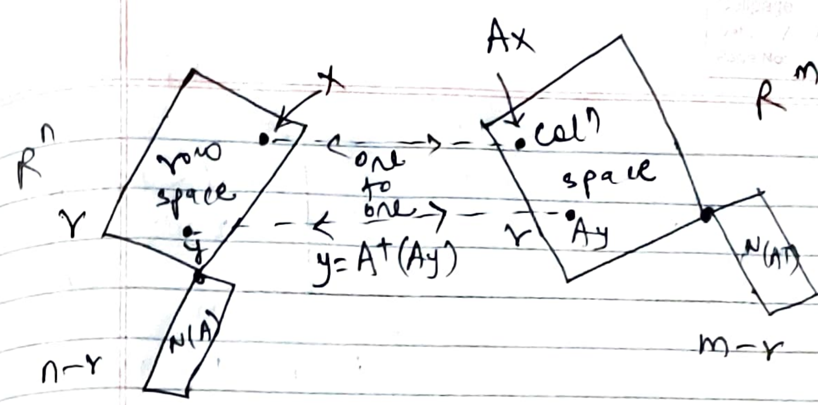
$$A\underbrace{A^T(AA^T)^{-1}}_{A^{-1}_{right}} = I$$

$$\boxed{\begin{array}{l} A^{-1}_{right}A \\ A^T(AA^T)^{-1}A = \text{Proj}^n \text{ onto row space} \end{array}}$$

If $x \neq y$ both in row space then col^n space
$Ax \neq Ay$

Proof → Suppose $Ax = Ay$
$A(x - y) = 0$

In Nullspace also in row-space → $\boxed{\text{only zero vector}}$



- Find the Pseudo Inverse A^+

(1) Start from SVD: $A = U\Sigma V^T$

$$\Sigma = \begin{bmatrix} \sigma_1 & & & 0 & 0 \\ 0 & \sigma_2 & & & \\ 0 & 0 & \ddots \ \sigma_r & 0 & 0 \end{bmatrix} \begin{array}{l} m \text{ rows} \\ \text{rank } r \end{array} \quad m \times n$$
n cols

$$\Sigma^+ = \begin{bmatrix} 1/\sigma_1 & & 0 & & \\ & 1/\sigma_2 & & 0 & 0 \\ 0 & 0 & 1/\sigma_r & & \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}_{n \times m}$$

$$\Sigma\Sigma^+ = \begin{bmatrix} 1 & & 0 & 0 \\ 0 & 1 & & 0 \\ & & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}_{m \times m}$$ Proj^n matrix onto col^n space

$$\underset{n\times m}{\Sigma^+}\underset{m\times n}{\Sigma} = \begin{bmatrix} 1 & 0 & 0 & \\ & 1 & & 0 \\ 0 & & 1 & 0 \\ 0 & & 1 & 0 \\ 0 & & 0 & 0 \end{bmatrix}_{n \times n}$$ Proj^n matrix onto row space

$A = U\Sigma V^T$
$A^+ = V\Sigma^+ U^T$