Derivative of the Piola transform

Abdullah Abdulhaque[a,]

[a] *Department of Mathematical Sciences, Norwegian University of Science and Technology, Trondheim, Norway*

Abstract

We present the derivatives of the Piola transform.

1. First-order derivatives

If we have a mapping $\mathcal{F} : \boldsymbol{\xi} \to \mathbf{x}$, then its Jacobi-matrix is defined as

$$J = \begin{bmatrix} x_\xi & x_\eta \\ y_\xi & y_\eta \end{bmatrix} \qquad \text{or} \qquad J = \begin{bmatrix} x_\xi & x_\eta & x_\zeta \\ y_\xi & y_\eta & y_\zeta \\ z_\xi & z_\eta & z_\zeta \end{bmatrix} \tag{1}$$

We denote the determinant as $|J|$, and it is given by

$$|J| = \begin{cases} x_\xi y_\eta - x_\eta y_\xi & , \quad \boldsymbol{\xi} = (\xi, \eta) \\ x_\xi(y_\eta z_\zeta - y_\zeta z_\eta) - x_\eta(y_\xi z_\zeta - y_\zeta z_\xi) + x_\zeta(y_\xi z_\eta - y_\eta z_\xi) & , \quad \boldsymbol{\xi} = (\xi, \eta, \zeta) \end{cases} \tag{2}$$

From the general inversion formula, we get

$$J^{-1} = \begin{cases} \begin{bmatrix} \xi_x & \xi_y \\ \eta_x & \eta_y \end{bmatrix} = \frac{1}{|J|} \begin{bmatrix} y_\eta & -x_\eta \\ -y_\xi & x_\xi \end{bmatrix} & , \quad \boldsymbol{\xi} = (\xi, \eta) \\ \begin{bmatrix} \xi_x & \xi_y & \xi_z \\ \eta_x & \eta_y & \eta_z \\ \zeta_x & \zeta_y & \zeta_z \end{bmatrix} = \frac{1}{|J|} \begin{bmatrix} \begin{vmatrix} y_\eta & y_\zeta \\ z_\eta & z_\zeta \end{vmatrix} & -\begin{vmatrix} y_\xi & y_\zeta \\ z_\xi & z_\zeta \end{vmatrix} & \begin{vmatrix} y_\xi & y_\eta \\ z_\xi & z_\eta \end{vmatrix} \\ -\begin{vmatrix} x_\eta & x_\zeta \\ z_\eta & z_\zeta \end{vmatrix} & \begin{vmatrix} x_\xi & x_\zeta \\ z_\xi & z_\zeta \end{vmatrix} & -\begin{vmatrix} x_\xi & x_\eta \\ z_\xi & z_\eta \end{vmatrix} \\ \begin{vmatrix} x_\eta & x_\zeta \\ y_\eta & y_\zeta \end{vmatrix} & -\begin{vmatrix} x_\xi & x_\zeta \\ y_\xi & y_\zeta \end{vmatrix} & \begin{vmatrix} x_\xi & x_\eta \\ y_\xi & y_\eta \end{vmatrix} \end{bmatrix} & , \quad \boldsymbol{\xi} = (\xi, \eta, \zeta) \end{cases} \tag{3}$$

From the chain rule, we have an universal formula for the total derivative:

$$\nabla_x f = \begin{bmatrix} f_x \\ f_y \end{bmatrix} = \begin{bmatrix} f_\xi \xi_x + f_\eta \eta_x \\ f_\xi \xi_y + f_\eta \eta_y \end{bmatrix} = \begin{bmatrix} \xi_x & \eta_x \\ \xi_y & \eta_y \end{bmatrix} \begin{bmatrix} f_\xi \\ f_\eta \end{bmatrix} = J^{-T} \nabla_\xi f \quad , \quad \boldsymbol{\xi} = (\xi, \eta) \tag{4a}$$

$$\nabla_x f = \begin{bmatrix} f_x \\ f_y \\ f_z \end{bmatrix} = \begin{bmatrix} f_\xi \xi_x + f_\eta \eta_x + f_\zeta \zeta_x \\ f_\xi \xi_y + f_\eta \eta_y + f_\zeta \zeta_y \\ f_\xi \xi_z + f_\eta \eta_z + f_\zeta \zeta_z \end{bmatrix} = \begin{bmatrix} \xi_x & \eta_x & \zeta_x \\ \xi_y & \eta_y & \zeta_y \\ \xi_z & \eta_z & \zeta_z \end{bmatrix} \begin{bmatrix} f_\xi \\ f_\eta \\ f_\zeta \end{bmatrix} = J^{-T} \nabla_\xi f \quad , \quad \boldsymbol{\xi} = (\xi, \eta, \zeta) \tag{4b}$$

Email address: `abdullah.abdulhaque@ntnu.no` (Abdullah Abdulhaque)

The Piola transform is defined as $J/|J|$. Its first-order derivatives become

$$\frac{\partial}{\partial x_i}\left[\frac{1}{|J|}J\right] = -\frac{|J|_{x_i}}{|J|^2}J + \frac{1}{|J|}J_{x_i} \qquad , \qquad x_i = \{x, y, z\} \tag{5}$$

In 2D ($x_i = \{x, y\}$), the components of the first-order derivatives become

$$J_{x_i} = \begin{bmatrix} x_{\xi\xi} & x_{\xi\eta} \\ y_{\xi\xi} & y_{\xi\eta} \end{bmatrix} \xi_{x_i} + \begin{bmatrix} x_{\xi\eta} & x_{\eta\eta} \\ y_{\xi\eta} & y_{\eta\eta} \end{bmatrix} \eta_{x_i} = J_\xi \xi_{x_i} + J_\eta \eta_{x_i} \tag{6a}$$

$$|J|_{x_i} = |J|_\xi \xi_{x_i} + |J|_\eta \eta_{x_i} = \left(|J|_\xi, |J|_\eta\right) \cdot (J^{-1})_{:,i} \tag{6b}$$

$$|J|_\xi = (x_\xi y_\eta - x_\eta y_\xi)_\xi \tag{6c}$$

$$|J|_\eta = (x_\xi y_\eta - x_\eta y_\xi)_\eta \tag{6d}$$

Likewise, in 3D ($x_i = \{x, y, z\}$), the components of the first-order derivatives become

$$J_{x_i} = J_\xi \xi_{x_i} + J_\eta \eta_{x_i} + J_\zeta \zeta_{x_i} \tag{7a}$$

$$\begin{aligned} |J|_{x_i} &= |J|_\xi \xi_{x_i} + |J|_\eta \eta_{x_i} + |J|_\zeta \zeta_{x_i} \\ &= \left(|J|_\xi, |J|_\eta, |J|_\zeta\right) \cdot (J^{-1})_{:,i} \end{aligned} \tag{7b}$$

$$\begin{aligned} |J|_\xi = \; & x_{\xi\xi}(y_\eta z_\zeta - y_\zeta z_\eta) + x_\xi(y_\eta z_\zeta - y_\zeta z_\eta)_\xi - x_{\xi\eta}(y_\xi z_\zeta - y_\zeta z_\xi) \\ & - x_\eta(y_\xi z_\zeta - y_\zeta z_\xi)_\xi + x_{\xi\zeta}(y_\xi z_\eta - y_\eta z_\xi) + x_\zeta(y_\xi z_\eta - y_\eta z_\xi)_\xi \end{aligned} \tag{7c}$$

$$\begin{aligned} |J|_\eta = \; & x_{\xi\eta}(y_\eta z_\zeta - y_\zeta z_\eta) + x_\xi(y_\eta z_\zeta - y_\zeta z_\eta)_\eta - x_{\eta\eta}(y_\xi z_\zeta - y_\zeta z_\xi) \\ & - x_\eta(y_\xi z_\zeta - y_\zeta z_\xi)_\eta + x_{\eta\zeta}(y_\xi z_\eta - y_\eta z_\xi) + x_\zeta(y_\xi z_\eta - y_\eta z_\xi)_\eta \end{aligned} \tag{7d}$$

$$\begin{aligned} |J|_\zeta = \; & x_{\xi\zeta}(y_\eta z_\zeta - y_\zeta z_\eta) + x_\xi(y_\eta z_\zeta - y_\zeta z_\eta)_\zeta - x_{\eta\zeta}(y_\xi z_\zeta - y_\zeta z_\xi) \\ & - x_\eta(y_\xi z_\zeta - y_\zeta z_\xi)_\zeta + x_{\zeta\zeta}(y_\xi z_\eta - y_\eta z_\xi) + x_\zeta(y_\xi z_\eta - y_\eta z_\xi)_\zeta \end{aligned} \tag{7e}$$

Imagine now that we have a vector function on the form

$$\begin{bmatrix} u \\ v \end{bmatrix} = \frac{1}{|J|}J \begin{bmatrix} \widehat{u} \\ \widehat{v} \end{bmatrix} \qquad \text{or} \qquad \begin{bmatrix} u \\ v \\ w \end{bmatrix} = \frac{1}{|J|}J \begin{bmatrix} \widehat{u} \\ \widehat{v} \\ \widehat{w} \end{bmatrix}$$

By using the product rule, we obtain

$$\begin{bmatrix} u \\ v \end{bmatrix}_{x_i} = \left(-\frac{|J|_{x_i}}{|J|^2}J + \frac{1}{|J|}J_{x_i}\right) \begin{bmatrix} \widehat{u} \\ \widehat{v} \end{bmatrix} + \frac{1}{|J|}J \begin{bmatrix} \widehat{u}_\xi & \widehat{u}_\eta \\ \widehat{v}_\xi & \widehat{v}_\eta \end{bmatrix} \begin{bmatrix} \xi \\ \eta \end{bmatrix}_{x_i} \tag{8a}$$

$$\begin{bmatrix} u \\ v \\ w \end{bmatrix}_{x_i} = \left(-\frac{|J|_{x_i}}{|J|^2}J + \frac{1}{|J|}J_{x_i}\right) \begin{bmatrix} \widehat{u} \\ \widehat{v} \\ \widehat{w} \end{bmatrix} + \frac{1}{|J|}J \begin{bmatrix} \widehat{u}_\xi & \widehat{u}_\eta & \widehat{u}_\zeta \\ \widehat{v}_\xi & \widehat{v}_\eta & \widehat{v}_\zeta \\ \widehat{w}_\xi & \widehat{w}_\eta & \widehat{w}_\zeta \end{bmatrix} \begin{bmatrix} \xi \\ \eta \\ \zeta \end{bmatrix}_{x_i} \tag{8b}$$

2. Second-order derivatives

From the double product rule, we have second-order derivative is

$$\left[\frac{1}{|J|}J\right]_{x_ix_i} = \left[\frac{1}{|J|}\right]_{x_ix_i} J - 2\frac{|J|_{x_i}}{|J|^2}J_{x_i} + \frac{1}{|J|}J_{x_ix_i} \qquad , \qquad x_i = \{x, y, z\} \tag{9}$$

Applying the same rule on $1/|J|$ yields the similar fraction derivatives

$$\left[\frac{1}{|J|}\right]_{x_ix_i} = -\frac{|J|_{x_ix_i}}{|J|^2} + \frac{2|J|^2_{x_i}}{|J|^3} \qquad , \qquad x_i = \{x, y, z\} \tag{10}$$

As we see from these formulas, the real objective is finding $|J|_{xx}$, $|J|_{yy}$, $|J|_{zz}$, J_{xx}, J_{yy} and J_{zz}. We start with J and use the total derivative rule:

$$J_{x_ix_i} = (J_{\xi\xi}\xi_{x_i} + J_{\xi\eta}\eta_{x_i})\xi_{x_i} + J_\xi\xi_{x_ix_i} + (J_{\xi\eta}\xi_{x_i} + J_{\eta\eta}\eta_{x_i})\eta_{x_i} + J_\eta\eta_{x_ix_i} \tag{11a}$$

$$\begin{aligned} J_{x_ix_i} = {} & (J_{\xi\xi}\xi_{x_i} + J_{\xi\eta}\eta_{x_i} + J_{\xi\zeta}\zeta_{x_i})\xi_{x_i} + J_\xi\xi_{x_ix_i} + (J_{\xi\eta}\xi_{x_i} + J_{\eta\eta}\eta_{x_i} + J_{\eta\zeta}\zeta_{x_i})\eta_{x_i} \\ & + J_\eta\eta_{x_ix_i} + (J_{\xi\zeta}\xi_{x_i} + J_{\eta\zeta}\eta_{x_i} + J_{\zeta\zeta}\zeta_{x_i})\zeta_{x_i} + J_\zeta\zeta_{x_ix_i} \end{aligned} \tag{11b}$$

In 2D ($x_i = \{x, y\}$), the second-order derivative determinants become

$$|J|_{x_i} = \left(|J|_{\xi\xi}\xi_{x_i} + |J|_{\xi\eta}\eta_{x_i}\right)\xi_{x_i} + |J|_\xi\xi_{x_ix_i} + \left(|J|_{\xi\eta}\xi_{x_i} + |J|_{\eta\eta}\eta_{x_i}\right)\eta_{x_i} + |J|_\eta\eta_{x_ix_i} \tag{12a}$$

$$|J|_{\xi\xi} = (x_\xi y_\eta - x_\eta y_\xi)_{\xi\xi} \tag{12b}$$

$$|J|_{\xi\eta} = (x_\xi y_\eta - x_\eta y_\xi)_{\xi\eta} \tag{12c}$$

$$|J|_{\eta\eta} = (x_\xi y_\eta - x_\eta y_\xi)_{\eta\eta} \tag{12d}$$

Likewise, in 3D ($x_i = \{x, y, z\}$), the second-order derivative determinants become

$$\begin{aligned} |J|_{x_i} = {} & \left(|J|_{\xi\xi}\xi_{x_i} + |J|_{\xi\eta}\eta_{x_i} + |J|_{\xi\zeta}\zeta_{x_i}\right)\xi_{x_i} + |J|_\xi\xi_{x_ix_i} + \left(|J|_{\xi\eta}\xi_{x_i} + |J|_{\eta\eta}\eta_{x_i} + |J|_{\eta\zeta}\zeta_{x_i}\right)\eta_{x_i} \\ & + |J|_\eta\eta_{x_ix_i} + \left(|J|_{\xi\zeta}\xi_{x_i} + |J|_{\eta\zeta}\eta_{x_i} + |J|_{\zeta\zeta}\zeta_{x_i}\right)\zeta_{x_i} + |J|_\zeta\zeta_{x_ix_i} \end{aligned} \tag{13a}$$

$$\begin{aligned} |J|_{\xi\xi} = {} & x_{\xi\xi\xi}(y_\eta z_\zeta - y_\zeta z_\eta) + 2x_{\xi\xi}(y_\eta z_\zeta - y_\zeta z_\eta)_\xi + x_\xi(y_\eta z_\zeta - y_\zeta z_\eta)_{\xi\xi} \\ & - x_{\xi\xi\eta}(y_\xi z_\zeta - y_\zeta z_\xi) - 2x_{\xi\eta}(y_\xi z_\zeta - y_\zeta z_\xi)_\xi - x_\eta(y_\xi z_\zeta - y_\zeta z_\xi)_{\xi\xi} \\ & + x_{\xi\xi\zeta}(y_\xi z_\eta - y_\eta z_\xi) + 2x_{\xi\zeta}(y_\xi z_\eta - y_\eta z_\xi)_\xi + x_\zeta(y_\xi z_\eta - y_\eta z_\xi)_{\xi\xi} \end{aligned} \tag{13b}$$

$$\begin{aligned} |J|_{\eta\eta} = {} & x_{\xi\eta\eta}(y_\eta z_\zeta - y_\zeta z_\eta) + 2x_{\xi\eta}(y_\eta z_\zeta - y_\zeta z_\eta)_\eta + x_\xi(y_\eta z_\zeta - y_\zeta z_\eta)_{\eta\eta} \\ & - x_{\eta\eta\eta}(y_\xi z_\zeta - y_\zeta z_\xi) - 2x_{\eta\eta}(y_\xi z_\zeta - y_\zeta z_\xi)_\eta - x_\eta(y_\xi z_\zeta - y_\zeta z_\xi)_{\eta\eta} \\ & + x_{\eta\eta\zeta}(y_\xi z_\eta - y_\eta z_\xi) + 2x_{\eta\zeta}(y_\xi z_\eta - y_\eta z_\xi)_\eta + x_\zeta(y_\xi z_\eta - y_\eta z_\xi)_{\eta\eta} \end{aligned} \tag{13c}$$

$$\begin{aligned} |J|_{\zeta\zeta} = {} & x_{\xi\zeta\zeta}(y_\eta z_\zeta - y_\zeta z_\eta) + 2x_{\xi\zeta}(y_\eta z_\zeta - y_\zeta z_\eta)_\zeta + x_\xi(y_\eta z_\zeta - y_\zeta z_\eta)_{\zeta\zeta} \\ & - x_{\eta\zeta\zeta}(y_\xi z_\zeta - y_\zeta z_\xi) - 2x_{\eta\zeta}(y_\xi z_\zeta - y_\zeta z_\xi)_\zeta - x_\eta(y_\xi z_\zeta - y_\zeta z_\xi)_{\zeta\zeta} \\ & + x_{\zeta\zeta\zeta}(y_\xi z_\eta - y_\eta z_\xi) + 2x_{\zeta\zeta}(y_\xi z_\eta - y_\eta z_\xi)_\zeta + x_\zeta(y_\xi z_\eta - y_\eta z_\xi)_{\zeta\zeta} \end{aligned} \tag{13d}$$

$$\begin{aligned} |J|_{\xi\eta} = {} & x_{\xi\xi\eta}(y_\eta z_\zeta - y_\zeta z_\eta) + x_{\xi\xi}(y_\eta z_\zeta - y_\zeta z_\eta)_\eta + x_{\xi\eta}(y_\eta z_\zeta - y_\zeta z_\eta)_\xi + x_\xi(y_\eta z_\zeta - y_\zeta z_\eta)_{\eta\eta} \\ & - x_{\xi\eta\eta}(y_\xi z_\zeta - y_\zeta z_\xi) - x_{\xi\eta}(y_\xi z_\zeta - y_\zeta z_\xi)_\eta - x_{\eta\eta}(y_\xi z_\zeta - y_\zeta z_\xi)_\xi - x_\eta(y_\xi z_\zeta - y_\zeta z_\xi)_{\xi\eta} \\ & + x_{\xi\eta\zeta}(y_\xi z_\eta - y_\eta z_\xi) + x_{\xi\zeta}(y_\xi z_\eta - y_\eta z_\xi)_\eta + x_{\eta\zeta}(y_\xi z_\eta - y_\eta z_\xi)_\xi + x_\zeta(y_\xi z_\eta - y_\eta z_{\xi\xi\eta}) \end{aligned} \tag{13e}$$

$$\begin{aligned} |J|_{\xi\zeta} = {} & x_{\xi\xi\zeta}(y_\eta z_\zeta - y_\zeta z_\eta) + x_{\xi\xi}(y_\eta z_\zeta - y_\zeta z_\eta)_\zeta + x_{\xi\zeta}(y_\eta z_\zeta - y_\zeta z_\eta)_\xi + x_\xi(y_\eta z_\zeta - y_\zeta z_\eta)_{\xi\zeta} \\ & - x_{\xi\eta\zeta}(y_\xi z_\zeta - y_\zeta z_\xi) - x_{\xi\eta}(y_\xi z_\zeta - y_\zeta z_\xi)_\zeta - x_{\eta\zeta}(y_\xi z_\zeta - y_\zeta z_\xi)_\xi - x_\eta(y_\xi z_\zeta - y_\zeta z_\xi)_{\xi\zeta} \end{aligned}$$

$$
\begin{aligned}
&+ x_{\xi\zeta\zeta}(y_\xi z_\eta - y_\eta z_\xi) + x_{\xi\zeta}(y_\xi z_\eta - y_\eta z_\xi)_\zeta + x_{\zeta\zeta}(y_\xi z_\eta - y_\eta z_\xi)_\xi + x_\zeta(y_\xi z_\eta - y_\eta z_\xi)_{\xi\zeta} \qquad (13f)
\end{aligned}
$$

$$
\begin{aligned}
|J|_{\eta\zeta} = {} & x_{\xi\eta\zeta}(y_\eta z_\zeta - y_\zeta z_\eta) + x_{\xi\eta}(y_\eta z_\zeta - y_\zeta z_\eta)_\zeta + x_{\xi\zeta}(y_\eta z_\zeta - y_\zeta z_\eta)_\eta + x_\xi(y_\eta z_\zeta - y_\zeta z_\eta)_{\eta\zeta} \\
& - x_{\eta\eta\zeta}(y_\xi z_\zeta - y_\zeta z_\xi) - x_{\eta\eta}(y_\xi z_\zeta - y_\zeta z_\xi)_\zeta - x_{\eta\zeta}(y_\xi z_\zeta - y_\zeta z_\xi)_\eta - x_\eta(y_\xi z_\zeta - y_\zeta z_\xi)_{\eta\zeta} \\
& + x_{\eta\zeta\zeta}(y_\xi z_\eta - y_\eta z_\xi) + x_{\eta\zeta}(y_\xi z_\eta - y_\eta z_\xi)_\zeta + x_{\zeta\zeta}(y_\xi z_\eta - y_\eta z_\xi)_\eta + x_\zeta(y_\xi z_\eta - y_\eta z_\xi)_{\zeta\zeta} \qquad (13g)
\end{aligned}
$$

3. Application to the Stokes equation

3.1. Isoparametric mapping

If we are solving the Stokes equation with Taylor-Hood or Subgrid elements, we just assume that we have a mapping $\mathcal{F} : \widehat{\Omega} \to \Omega$ represented by $\boldsymbol{\xi} \to \mathbf{x} = f(\boldsymbol{\xi})$, and the Jacobian of f is J as shown in Equation (1). The mapping of the componentwise mass matrix is straightforward:

$$\int_{\Omega} u^{(i)}(\mathbf{x}) v^{(i)}(\mathbf{x})\, d\mathbf{x} = \int_{\widehat{\Omega}} \widehat{u}^{(i)}(\boldsymbol{\xi}) \widehat{v}^{(i)}(\boldsymbol{\xi}) \cdot |J|\, d\boldsymbol{\xi} \qquad , \qquad i = \{1, 2, 3\} \tag{14}$$

where $\widehat{u}^{(i)} \equiv u^{(i)} \circ f$ and $\widehat{v}^{(i)} \equiv v^{(i)} \circ f$, i.e. we have $\widehat{u}^{(i)}, \widehat{v}^{(i)} \in \widehat{\Omega}$ and $u^{(i)}, v^{(i)} \in \Omega$. From Equation (4), we know that $\nabla_x u = J^{-T} \nabla_\xi u$, so the mapping for the stiffness matrix becomes

$$\begin{aligned}
\int_{\Omega} \nabla u^{(i)} \cdot \nabla v^{(i)}\, d\mathbf{x} &= \int_{\widehat{\Omega}} (J^{-T} \nabla \widehat{u}^{(i)})^T (J^{-T} \nabla \widehat{v}^{(i)}) \cdot |J|\, d\boldsymbol{\xi} \\
&= \int_{\widehat{\Omega}} (\nabla \widehat{u}^{(i)})^T (J^{-1} J^{-T}) \nabla \widehat{v}^{(i)} \cdot |J|\, d\boldsymbol{\xi} \\
&= \int_{\widehat{\Omega}} (\nabla \widehat{u}^{(i)})^T (J^T J)^{-1} \nabla \widehat{v}^{(i)} \cdot |J|\, d\boldsymbol{\xi}
\end{aligned} \tag{15}$$

From the definition of total derivative, the divergence matrices become

$$\int_{\Omega} u_x^{(1)} q\, d\mathbf{x} = \int_{\widehat{\Omega}} \left[\xi_x \widehat{u}_\xi^{(1)} + \eta_x \widehat{u}_\eta^{(1)} + \zeta_x \widehat{u}_\zeta^{(1)} \right] q \cdot |J|\, d\boldsymbol{\xi} \tag{16a}$$

$$\int_{\Omega} u_y^{(2)} q\, d\mathbf{x} = \int_{\widehat{\Omega}} \left[\xi_y \widehat{u}_\xi^{(2)} + \eta_y \widehat{u}_\eta^{(2)} + \zeta_y \widehat{u}_\zeta^{(2)} \right] q \cdot |J|\, d\boldsymbol{\xi} \tag{16b}$$

$$\int_{\Omega} u_z^{(3)} q\, d\mathbf{x} = \int_{\widehat{\Omega}} \left[\xi_z \widehat{u}_\xi^{(3)} + \eta_z \widehat{u}_\eta^{(3)} + \zeta_z \widehat{u}_\zeta^{(3)} \right] q \cdot |J|\, d\boldsymbol{\xi} \tag{16c}$$

3.2. Piola mapping

If we use Nédélec or Raviart-Thomas elements instead, the transformation will depend whether we are dealing with scalars or vectors:

$$u(\mathbf{x}) = \frac{1}{|J|} \widehat{u}(\boldsymbol{\xi}) \qquad , \qquad \begin{bmatrix} u^{(1)}(\mathbf{x}) \\ u^{(2)}(\mathbf{x}) \end{bmatrix} = \frac{J}{|J|} \begin{bmatrix} \widehat{u}^{(1)}(\boldsymbol{\xi}) \\ \widehat{u}^{(2)}(\boldsymbol{\xi}) \end{bmatrix} \qquad , \qquad \begin{bmatrix} u^{(1)}(\mathbf{x}) \\ u^{(2)}(\mathbf{x}) \\ u^{(3)}(\mathbf{x}) \end{bmatrix} = \frac{J}{|J|} \begin{bmatrix} \widehat{u}^{(1)}(\boldsymbol{\xi}) \\ \widehat{u}^{(2)}(\boldsymbol{\xi}) \\ \widehat{u}^{(3)}(\boldsymbol{\xi}) \end{bmatrix} \tag{17}$$

In this way, we get a slightly modified version of the transformed mass matrix:

$$\begin{aligned}
\int_{\Omega} u(\mathbf{x}) v(\mathbf{x})\, d\mathbf{x} &= \int_{\widehat{\Omega}} \frac{1}{|J|} \widehat{u}(\boldsymbol{\xi}) \frac{1}{|J|} \widehat{v}(\boldsymbol{\xi}) \cdot |J|\, d\boldsymbol{\xi} \\
&= \int_{\widehat{\Omega}} \widehat{u}(\boldsymbol{\xi}) \widehat{v}(\boldsymbol{\xi}) \cdot \frac{1}{|J|}\, d\boldsymbol{\xi}
\end{aligned} \tag{18}$$

The gradient of a vector is formally defined as

$$\nabla \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} u_x & u_y \\ v_x & v_y \end{bmatrix} \qquad , \qquad \nabla \begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} u_x & u_y & u_z \\ v_x & v_y & v_z \\ w_x & w_y & w_z \end{bmatrix}$$

Due to domain transform, we get the following expression in 2D:

$$
\begin{aligned}
\nabla \begin{bmatrix} u^{(1)} \\ u^{(2)} \end{bmatrix} &= -\frac{1}{|J|^2} \begin{bmatrix} (\xi_x |J|_\xi + \eta_x |J|_\eta)(x_\xi \widehat{u}^{(1)} + x_\eta \widehat{u}^{(2)}) & (\xi_y |J|_\xi + \eta_y |J|_\eta)(x_\xi \widehat{u}^{(1)} + x_\eta \widehat{u}^{(2)}) \\ (\xi_x |J|_\xi + \eta_x |J|_\eta)(y_\xi \widehat{u}^{(1)} + y_\eta \widehat{u}^{(2)}) & (\xi_y |J|_\xi + \eta_y |J|_\eta)(y_\xi \widehat{u}^{(1)} + y_\eta \widehat{u}^{(2)}) \end{bmatrix} \\
&\quad + \frac{1}{|J|} \begin{bmatrix} \xi_x (x_\xi \widehat{u}^{(1)} + x_\eta \widehat{u}^{(2)})_\xi + \eta_x (x_\xi \widehat{u}^{(1)} + x_\eta \widehat{u}^{(2)})_\eta & \xi_y (x_\xi \widehat{u}^{(1)} + x_\eta \widehat{u}^{(2)})_\xi + \eta_y (x_\xi \widehat{u}^{(1)} + x_\eta \widehat{u}^{(2)})_\eta \\ \xi_x (y_\xi \widehat{u}^{(1)} + y_\eta \widehat{u}^{(2)})_\xi + \eta_x (y_\xi \widehat{u}^{(1)} + y_\eta \widehat{u}^{(2)})_\eta & \xi_y (y_\xi \widehat{u}^{(1)} + y_\eta \widehat{u}^{(2)})_\xi + \eta_y (y_\xi \widehat{u}^{(1)} + y_\eta \widehat{u}^{(2)})_\eta \end{bmatrix} \\
&= -\frac{1}{|J|^2} \begin{bmatrix} (\xi_x |J|_\xi + \eta_x |J|_\eta)(J\widehat{u})_{1,:} \\ (\xi_y |J|_\xi + \eta_y |J|_\eta)(J\widehat{u})_{2,:} \end{bmatrix} + \frac{1}{|J|} \begin{bmatrix} (J^{-1})_{:,1} \cdot \nabla (J\widehat{u})_1 & (J^{-1})_{:,2} \cdot \nabla (J\widehat{u})_1 \\ (J^{-1})_{:,1} \cdot \nabla (J\widehat{u})_2 & (J^{-1})_{:,2} \cdot \nabla (J\widehat{u})_2 \end{bmatrix} \qquad (19)
\end{aligned}
$$

The similar expression in 3D becomes

$$
\nabla \begin{bmatrix} u^{(1)} \\ u^{(2)} \\ u^{(3)} \end{bmatrix} = -\frac{1}{|J|^2} \begin{bmatrix} (\xi_x |J|_\xi + \eta_x |J|_\eta + \zeta_x |J|_\zeta)(J\widehat{u})_{1,:} \\ (\xi_y |J|_\xi + \eta_y |J|_\eta + \zeta_y |J|_\zeta)(J\widehat{u})_{2,:} \\ (\xi_z |J|_\xi + \eta_z |J|_\eta + \zeta_z |J|_\zeta)(J\widehat{u})_{3,:} \end{bmatrix} + \frac{1}{|J|} \begin{bmatrix} (J^{-1})_{:,1} \cdot \nabla (J\widehat{u})_1 & (J^{-1})_{:,2} \cdot \nabla (J\widehat{u})_1 & (J^{-1})_{:,3} \cdot \nabla (J\widehat{u})_1 \\ (J^{-1})_{:,1} \cdot \nabla (J\widehat{u})_2 & (J^{-1})_{:,2} \cdot \nabla (J\widehat{u})_2 & (J^{-1})_{:,3} \cdot \nabla (J\widehat{u})_2 \\ (J^{-1})_{:,1} \cdot \nabla (J\widehat{u})_3 & (J^{-1})_{:,2} \cdot \nabla (J\widehat{u})_3 & (J^{-1})_{:,3} \cdot \nabla (J\widehat{u})_3 \end{bmatrix} \qquad (20)
$$

When it comes to weak boundary conditions, we have

$$
\mathbf{u} \cdot \mathbf{n} = \begin{cases} \left(\frac{1}{|J|} J \widehat{\mathbf{u}} \right) \cdot \begin{bmatrix} \xi_x f_\xi^{(1)} + \eta_x f_\eta^{(1)} \\ \xi_y f_\xi^{(2)} + \eta_y f_\eta^{(2)} \end{bmatrix} & , \mathbf{x} \in \mathbb{R}^2 \\ \left(\frac{1}{|J|} J \widehat{\mathbf{u}} \right) \cdot \begin{bmatrix} \xi_x f_\xi^{(1)} + \eta_x f_\eta^{(1)} + \zeta_x f_\zeta^{(1)} \\ \xi_y f_\xi^{(2)} + \eta_y f_\eta^{(2)} + \zeta_y f_\zeta^{(2)} \\ \xi_z f_\xi^{(3)} + \eta_z f_\eta^{(3)} + \zeta_z f_\zeta^{(3)} \end{bmatrix} & , \mathbf{x} \in \mathbb{R}^3 \end{cases} \qquad (21)
$$