Staggered fracture-elasticity solver

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1 Introduction

This document describes the weak form and associated linearized equations for the staggered fracture-elasticity solver implemented in *IFEM*, based on the papers by Borden *et al.* [1], and Gerasimov and De Lorenzis [2].

2 Energy functionals

The energy functional for the quasi-static brittle fracture problem, using a phase-field to represent the crack geometry is in [1] given as follows (see Equation (17) therein):

$$\begin{aligned} E_c(\boldsymbol{u}, c) \;&=\; \int\limits_\Omega \left[g(c)\Psi_0^+(\boldsymbol{\epsilon}(\boldsymbol{u})) + \Psi_0^-(\boldsymbol{\epsilon}(\boldsymbol{u}))\right] \,\mathrm{d}V \\ &+ G_c \int\limits_\Omega \left(\frac{1}{4\ell_0}(1-c)^2 + \ell_0 \boldsymbol{\nabla} c \cdot \boldsymbol{\nabla} c\right) \mathrm{d}V \end{aligned} \tag{1}$$

where c is the phase field describing the crack, i.e., it has the value 1.0 where the material is undamaged and equal to 0.0 for fully cracked material. $g(c) = (1-k)c^2 + k$ is the stress degradation function used to scale down the tensile part of the strain energy density in the elasticity equation. $k \geq 0.0$ is a (small) stability parameter that can be used to improve the conditioning of the resulting linear equation system.

In [2], an alternative formulation is used, where $d = 1 - c$ is used as the unknown phase field instead of c, i.e., it has the value 0.0 in the undamaged material and 1.0 in the fully cracked material. Moreover, a penalty term is introduced to ensure crack irreversibility, and the length scale parameter ℓ_0 is replaced by $\ell = 2\ell_0$. The energy functional therefore reads as follows (see also Equation (3) in [2]):

$$\begin{aligned} E_d(\boldsymbol{u}, d) \;&=\; \int\limits_\Omega \left[g(d)\Psi_0^+(\boldsymbol{\epsilon}(\boldsymbol{u})) + \Psi_0^-(\boldsymbol{\epsilon}(\boldsymbol{u}))\right] \,\mathrm{d}V \\ &+ G_c \int\limits_\Omega \left(\frac{1}{2\ell}d^2 + \frac{\ell}{2}\boldsymbol{\nabla} d \cdot \boldsymbol{\nabla} d\right) \mathrm{d}V + \frac{1}{2\gamma}\int\limits_{\mathrm{CR}_{l-1}} (1-d)^2 \,\mathrm{d}V \end{aligned} \tag{2}$$

where the stress degradation function now reads $g(d) = (1-k)(1-d)^2 + k$.

The directional derivatives of E_c and E_d read, respectively:

$$\begin{aligned} E_c'(\boldsymbol{u}, c; \boldsymbol{v}, w) \;&:=\; \frac{\partial E_c}{\partial \boldsymbol{u}} \cdot \boldsymbol{v} \;+\; \frac{\partial E_c}{\partial c} w \\ &= \int\limits_\Omega \left[g(c)\frac{\partial \Psi_0^+}{\partial \boldsymbol{\epsilon}}(\boldsymbol{\epsilon}(\boldsymbol{u})) + \frac{\partial \Psi_0^-}{\partial \boldsymbol{\epsilon}}(\boldsymbol{\epsilon}(\boldsymbol{u}))\right] : \boldsymbol{\epsilon}(\boldsymbol{v}) \,\mathrm{d}V \\ &+ \int\limits_\Omega g'(c)\Psi_0^+(\boldsymbol{\epsilon}(\boldsymbol{u}))w \,\mathrm{d}V \;+\; G_c \int\limits_\Omega \left(\frac{1}{2\ell_0}(c-1)w + 2\ell_0 \boldsymbol{\nabla} c \cdot \boldsymbol{\nabla} w\right) \mathrm{d}V \end{aligned} \tag{3}$$

with $g'(c) = 2(1-k)c$, and

$$
\begin{aligned}
E_d'(\boldsymbol{u}, d; \boldsymbol{v}, w) &:= \frac{\partial E_d}{\partial \boldsymbol{u}} \cdot \boldsymbol{v} \; + \; \frac{\partial E_d}{\partial d} w \\
&= \int\limits_\Omega \left[g(d) \frac{\partial \Psi_0^+}{\partial \boldsymbol{\epsilon}}(\boldsymbol{\epsilon}(\boldsymbol{u})) + \frac{\partial \Psi_0^-}{\partial \boldsymbol{\epsilon}}(\boldsymbol{\epsilon}(\boldsymbol{u})) \right] : \boldsymbol{\epsilon}(\boldsymbol{v}) \, \mathrm{d}V \\
&+ \int\limits_\Omega g'(d) \Psi_0^+(\boldsymbol{\epsilon}(\boldsymbol{u})) w \, \mathrm{d}V \; + \; G_c \int\limits_\Omega \left(\frac{1}{\ell} d w + \ell \boldsymbol{\nabla} d \cdot \boldsymbol{\nabla} w \right) \mathrm{d}V + \frac{1}{\gamma} \int\limits_{\mathrm{CR}_{l-1}} (d-1) w \, \mathrm{d}V
\end{aligned} \tag{4}
$$

with $g'(d) = 2(1-k)(d-1)$. The problem is then solved by seeking a solution $(\boldsymbol{u}, c) \in \boldsymbol{V}_1 \times H^1(\Omega, [0,1])$, such that

$$
E_c'(\boldsymbol{u}, c; \boldsymbol{v}, w) = 0 \quad \forall \, (\boldsymbol{v}, w) \in \boldsymbol{V}_0 \times H^1(\Omega, [0,1]) \tag{5}
$$

or alternatively, a solution $(\boldsymbol{u}, d) \in \boldsymbol{V}_1 \times H^1(\Omega, [0,1])$, such that

$$
E_d'(\boldsymbol{u}, d; \boldsymbol{v}, w) = 0 \quad \forall \, (\boldsymbol{v}, w) \in \boldsymbol{V}_0 \times H^1(\Omega, [0,1]) \tag{6}
$$

3 Residual and tangent for the Borden formulation

Equation (5) is rewritten into a system of coupled equations:

$$
Q_1(\boldsymbol{u}, c; \boldsymbol{v}) \;=\; \int\limits_\Omega \left[g(c) \frac{\partial \Psi_0^+}{\partial \boldsymbol{\epsilon}}(\boldsymbol{\epsilon}(\boldsymbol{u})) + \frac{\partial \Psi_0^-}{\partial \boldsymbol{\epsilon}}(\boldsymbol{\epsilon}(\boldsymbol{u})) \right] : \boldsymbol{\epsilon}(\boldsymbol{v}) \, \mathrm{d}V = 0 \tag{7}
$$

$$
Q_2(\boldsymbol{u}, c; w) \;=\; \int\limits_\Omega \left[g'(c) \Psi_0^+(\boldsymbol{\epsilon}(\boldsymbol{u})) w + G_c \left(\frac{1}{2\ell_0}(c-1) w + 2\ell_0 \boldsymbol{\nabla} c \cdot \boldsymbol{\nabla} w \right) \right] \mathrm{d}V \;=\; 0 \tag{8}
$$

To ensure crack irreversibility, the tensile strain energy density Ψ_0^+ in Equation (8) is replaced by a history field $\mathcal{H}(\boldsymbol{x}, t)$, with the property

$$
\mathcal{H}(\boldsymbol{x}, t_0) \;=\; \frac{G_c}{4\ell_0} \left(\frac{1}{C} - 1 \right) \left(1 - \max\left\{ \frac{\delta(\boldsymbol{x}, \Gamma)}{\ell_0}, 1 \right\} \right) \tag{9}
$$

$$
\mathcal{H}(\boldsymbol{x}, t_n) \;=\; \max\left\{ \mathcal{H}(\boldsymbol{x}, t_{n-1}), \Psi_0^+(\boldsymbol{\epsilon}(\boldsymbol{u}(\boldsymbol{x}, t_n))) \right\} \;, \; n > 0 \tag{10}
$$

where C is a constant and $\delta(\boldsymbol{x}, \Gamma)$ denotes the shortest distance from the point $\boldsymbol{x}$ to the initial crack Γ. Moreover, Equation (8) is scaled by $\frac{\ell}{G_c} = \frac{2\ell_0}{G_c}$ to obtain

$$
Q_2(\boldsymbol{u}, c; w) \;=\; \int\limits_\Omega \left[\frac{g'(c)\ell}{G_c} \mathcal{H}(\boldsymbol{x}, t) w + (c-1) w + \ell^2 \boldsymbol{\nabla} c \cdot \boldsymbol{\nabla} w \right] \mathrm{d}V \;=\; 0 \tag{11}
$$

The Equations (7) and (11) are solved in a staggered manner, where in the former the phase field c is assumed known and in the latter the displacement field $\boldsymbol{u}$ is assumed known. We then do a linearization of the functionals Q_1 and Q_2 about a certain known configuration $(\boldsymbol{u}_0, c_0)$ for the unknown variables, i.e.

$$
Q_1(\boldsymbol{u}_0 + \Delta \boldsymbol{u}, c_0; \boldsymbol{v}) \;\approx\; Q_1(\boldsymbol{u}_0, c_0; \boldsymbol{v}) \;+\; \frac{\partial Q_1}{\partial \boldsymbol{u}} \cdot \Delta \boldsymbol{u} \;=\; 0 \tag{12}
$$

$$
Q_2(\boldsymbol{u}_0, c_0 + \Delta c; w) \;\approx\; Q_2(\boldsymbol{u}_0, c_0; \boldsymbol{v}) \;+\; \frac{\partial Q_2}{\partial c} \Delta c \;=\; 0 \tag{13}
$$

from which we obtain the tangent operators

$$
\frac{\partial Q_1}{\partial \boldsymbol{u}} \cdot \Delta \boldsymbol{u} \;=\; \int\limits_\Omega \boldsymbol{\epsilon}(\Delta \boldsymbol{u}) : \left[g(c) \frac{\partial^2 \Psi_0^+}{\partial \boldsymbol{\epsilon}^2}(\boldsymbol{\epsilon}(\boldsymbol{u})) + \frac{\partial^2 \Psi_0^-}{\partial \boldsymbol{\epsilon}^2}(\boldsymbol{\epsilon}(\boldsymbol{u})) \right] : \boldsymbol{\epsilon}(\boldsymbol{v}) \, \mathrm{d}V \tag{14}
$$

$$
\frac{\partial Q_2}{\partial c} \Delta c \;=\; \int\limits_\Omega \left[\Delta c \left(\frac{g''(c)\ell}{G_c} \mathcal{H}(\boldsymbol{x}, t) + 1 \right) w \;+\; \ell^2 \boldsymbol{\nabla}(\Delta c) \cdot \boldsymbol{\nabla} w \right] \mathrm{d}V \tag{15}
$$

Since $g''(c) = 2(1-k)$ is a constant, the latter is linear in Δc and w, and independent of c, hence a linear solve is sufficient for Equation (13).

Alternatively to using the history field, we may include a penalty term to enforce the crack irreversibility as in Equation (2), i.e.:

$$\begin{aligned} E_c(\boldsymbol{u}, c) \;=\; & \int\limits_\Omega \left[g(c)\Psi_0^+(\boldsymbol{\epsilon}(\boldsymbol{u})) + \Psi_0^-(\boldsymbol{\epsilon}(\boldsymbol{u})) \right] \, \mathrm{d}V \\ & + G_c \int\limits_\Omega \left(\frac{1}{4\ell_0}(1-c)^2 + \ell_0 \boldsymbol{\nabla} c \cdot \boldsymbol{\nabla} c \right) \mathrm{d}V \;+\; \frac{1}{2\gamma} \int\limits_{\mathrm{CR}_{l-1}} c^2 \, \mathrm{d}V \end{aligned} \tag{16}$$

such that the alternatives to Equations (11) and (15) become, respectively

$$Q_2(\boldsymbol{u}, c; w) \;=\; \int\limits_\Omega \left[\frac{g'(c)\ell}{G_c} \Psi_0^+(\boldsymbol{\epsilon}(\boldsymbol{u}))w + (c-1)w + \ell^2 \boldsymbol{\nabla} c \cdot \boldsymbol{\nabla} w \right] \mathrm{d}V \;+\; \frac{1}{\gamma} \int\limits_{\mathrm{CR}_{l-1}} \frac{\ell}{G_c} c w \, \mathrm{d}V \;=\; 0 \tag{17}$$

and

$$\frac{\partial Q_2}{\partial c}\Delta c \;=\; \int\limits_\Omega \left[\Delta c \left(\frac{g''(c)\ell}{G_c} \Psi_0^+(\boldsymbol{\epsilon}(\boldsymbol{u})) + 1 \right) w \;+\; \ell^2 \boldsymbol{\nabla}(\Delta c) \cdot \boldsymbol{\nabla} w \right] \mathrm{d}V \;+\; \frac{1}{\gamma} \int\limits_{\mathrm{CR}_{l-1}} \Delta c \frac{\ell}{G_c} w \, \mathrm{d}V \tag{18}$$

4 Residual and tangent for the penalty formulation

Equation (6) is rewritten into a system of coupled equations:

$$Q_1(\boldsymbol{u}, d; \boldsymbol{v}) \;=\; \int\limits_\Omega \left[g(d) \frac{\partial \Psi_0^+}{\partial \boldsymbol{\epsilon}}(\boldsymbol{\epsilon}(\boldsymbol{u})) + \frac{\partial \Psi_0^-}{\partial \boldsymbol{\epsilon}}(\boldsymbol{\epsilon}(\boldsymbol{u})) \right] : \boldsymbol{\epsilon}(\boldsymbol{v}) \, \mathrm{d}V \;=\; 0 \tag{19}$$

$$Q_2(\boldsymbol{u}, d; w) \;=\; \int\limits_\Omega \left[g'(d)\Psi_0^+(\boldsymbol{\epsilon}(\boldsymbol{u}))w + G_c \left(\frac{1}{\ell} d w + \ell \boldsymbol{\nabla} d \cdot \boldsymbol{\nabla} w \right) \right] \mathrm{d}V + \frac{1}{\gamma} \int\limits_{\mathrm{CR}_{l-1}} (d-1) w \, \mathrm{d}V \;=\; 0 \tag{20}$$

These two equations are solved in a staggered manner, where in Equation (19) the phase field d is assumed known and in Equation (20) the displacement field $\boldsymbol{u}$ is assumed known. We then do a linearization of the functionals Q_1 and Q_2 about a certain known configuration $(\boldsymbol{u}_0, d_0)$ for the unknown variables, i.e.

$$Q_1(\boldsymbol{u}_0 + \Delta \boldsymbol{u}, d_0; \boldsymbol{v}) \;\approx\; Q_1(\boldsymbol{u}_0, d_0; \boldsymbol{v}) \;+\; \frac{\partial Q_1}{\partial \boldsymbol{u}} \cdot \Delta \boldsymbol{u} \;=\; 0 \tag{21}$$

$$Q_2(\boldsymbol{u}_0, d_0 + \Delta d; w) \;\approx\; Q_2(\boldsymbol{u}_0, d_0; \boldsymbol{v}) \;+\; \frac{\partial Q_2}{\partial d} \Delta d \;=\; 0 \tag{22}$$

from which we obtain the tangent operators

$$\frac{\partial Q_1}{\partial \boldsymbol{u}} \cdot \Delta \boldsymbol{u} \;=\; \int\limits_\Omega \boldsymbol{\epsilon}(\Delta \boldsymbol{u}) : \left[g(d) \frac{\partial^2 \Psi_0^+}{\partial \boldsymbol{\epsilon}^2}(\boldsymbol{\epsilon}(\boldsymbol{u})) + \frac{\partial^2 \Psi_0^-}{\partial \boldsymbol{\epsilon}^2}(\boldsymbol{\epsilon}(\boldsymbol{u})) \right] : \boldsymbol{\epsilon}(\boldsymbol{v}) \, \mathrm{d}V \tag{23}$$

$$\frac{\partial Q_2}{\partial d} \Delta d \;=\; \int\limits_\Omega \left[\Delta d \left(g''(d) \Psi_0^+(\boldsymbol{\epsilon}(\boldsymbol{u})) + \frac{G_c}{\ell} \right) w \;+\; G_c \ell \boldsymbol{\nabla}(\Delta d) \cdot \boldsymbol{\nabla} w \right] \mathrm{d}V \;+\; \frac{1}{\gamma} \int\limits_{\mathrm{CR}_{l-1}} \Delta d w \, \mathrm{d}V \tag{24}$$

Since $g''(d) = 2(1-k)$ is a constant, the latter is linear in Δd and w, and independent of d, hence a linear solve is sufficient for Equation (22).

References

[1] M. J. Borden, C. V. Verhoosel, M. A. Scott, T. J. R. Hughes, and C. M. Landis. A Phase-field Description of Dynamic Brittle Fracture. *Computer Methods in Applied Mechanics and Engineering*, 217–220:77–95, 2012.

[2] T. Gerasimov and L. De Lorenzis. A Line Search Assisted Monolithic Approach for Phase-field Computing of Brittle Fracture. *Computer Methods in Applied Mechanics and Engineering*, 312:276–303, 2016.