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COURSE: MATH464**:** FUNCTIONAL ANALYSIS

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Exercise 1.1

Let X be a normed linear space. Prove that for any $x, y \in X$,

(a). $\left| \|x\| - \|y\| \right| \le \|x - y\|$.

Solution:

Let $x = x - y + y$,

Taking norm on both sides,

$$\|x\| = \|x - y + y\|$$

By the triangle inequality,

$\|x\| \le \|x - y\| + \|y\|$ and so

$$\|x\| - \|y\| \le \|x - y\| \text{..} (1.1)$$

Interchanging x and y in (1.1) we obtain,

$$\|y\| - \|x\| \le \|y - x\| \text{..} (1.2)$$

However, $\|x - y\| = \|(-1)(y - x)\| = \|y - x\|$

Replacing the r.h.s of (1.2) by $\|x - y\|$ and multiplying through by -1 results in

$$-\|y\| + \|x\| \ge -\|x - y\|$$

$$\Rightarrow \|x\| - \|y\| \ge -\|x - y\|$$

$$\Rightarrow -\|x - y\| \le \|x\| - \|y\| \text{......................................} (1.3)$$

Thus from (1.1) and (1.3) we have,

$$-\|x - y\| \le \|x\| - \|y\| \le \|x - y\|$$

Hence,

$$\left| \|x\| - \|y\| \right| \le \|x - y\|$$

(b).

Since $x_n \to x$ as $n \to \infty$ and from results in (a), $\left|\|x\| - \|x_n\|\right| \le \|x - x_n\|$ for all positive integers *n*.
Thus it follows that $\|x_n\| \to \|x\|$ as n $\to \infty$.

(c).

Since $x_n \to x$ and $y_n \to y$ as $n \to \infty$ and

$$\begin{aligned}\|(x_n + y_n) - (x + y)\| &= \|(x_n - x) + (y_n - y)\| \\ &\le \|x_n - x\| + \|y_n - y\| \text{ for all positive integers } n.\end{aligned}$$

Hence it follows that $x_n + y_n \to x + y$ as $n \to \infty$

(d).

Since a_n is convergent, it's bounded and so there exist $K > 0$ such that $|a_n| \le K$ for all positive integer *n*.
Also,

$$\begin{aligned}\|a_n x_n - ax\| &= \|a_n x_n - a_n x + a_n x - ax\| \\ &= \|a_n (x_n - x) + x(a_n - a)\| \\ &\le \|a_n (x_n - x)\| + \|x(a_n - a)\| \\ &= |a_n| \|x_n - x\| + |a_n - a| \|x\| \\ &\le K \|x_n - x\| + |a_n - a| \|x\| \text{ for all positive integer } n.\end{aligned}$$

Hence, $a_n x_n \to ax$ as $n \to \infty$

Exercise 3.3

The linear space $\mathbb{C}^n$ with the with the function $<.,.>$, defined for arbitrary

$z=(z_1,z_2,...,z_n), w=(w_1,w_2,...,w_n)$ in $\mathbb{C}^n$ where $z_i, w_i \in \mathbb{C}$ by $\langle z,w\rangle = \sum_{i=1}^{n} z_i \overline{w_i}$,

is an inner product space.

I_1 : a. $\langle z,z\rangle = \sum_{i=1}^{n} z_i \overline{z_i} = \sum_{i=1}^{n} |z_i|^2 \geq 0$ since the absolute value of a complex number is non-negative.

b. Assume $\langle z,z\rangle = 0 \Rightarrow \sum_{i=1}^{n} z_i \overline{z_i} = \sum_{i=1}^{n} |z_i|^2 = 0 \Rightarrow |z_i|^2 = 0$ for $1 \leq i \leq n$ $\Rightarrow z_i = 0$ for $1 \leq i \leq n$

$\Rightarrow z = 0$

Next asume $z = 0 \Rightarrow z_i = 0$ for $1 \leq i \leq n \Rightarrow 0 = \sum_{i=1}^{n} |z_i|^2$

$\Rightarrow \sum_{i=1}^{n} z_i \overline{z_i} = \langle z,z\rangle = 0$

I_2 : For any $z=(z_1,z_2,...,z_n), w=(w_1,w_2,...,w_n) \in \mathbb{C}^n$

$$\langle z,w\rangle = \sum_{i=1}^{n} z_i \overline{w_i} = \sum_{i=1}^{n} \overline{w_i} z_i = \overline{\sum_{i=1}^{n} w_i \overline{z_i}} = \overline{\langle w,z\rangle}$$

I_3 : For any $z=(z_1,z_2,...,z_n), y=(y_1,y_2,...,y_n), w=(w_1,w_2,...,w_n) \in \mathbb{C}^n$ and $\alpha,\beta \in \mathbb{C}$

$$\begin{aligned}
\langle \alpha z + \beta y, w\rangle &= \sum_{i=1}^{n} (\alpha z_i + \beta y_i)\overline{w_i} \\
&= \sum_{i=1}^{n} \left(\alpha z_i \overline{w_i} + \beta y_i \overline{w_i}\right) \\
&= \sum_{i=1}^{n} \alpha z_i \overline{w_i} + \sum_{i=1}^{n} \beta y_i \overline{w_i} \\
&= \alpha \sum_{i=1}^{n} z_i \overline{w_i} + \beta \sum_{i=1}^{n} y_i \overline{w_i} \\
&= \alpha \langle z,w\rangle + \beta \langle y,w\rangle
\end{aligned}$$

Hence the linear space $\mathbb{C}^n$ with the function $\langle .,.\rangle$, defined by $\langle z,w\rangle = \sum_{i=1}^{n} z_i \overline{w_i}$ is an inner product space.

Exercise 3.4

The linear space l_2 with the with the function $<.,.>$, defined for arbitrary

$x = (x_1, x_2, ...), y = (y_1, y_2, ...)$ in l_2 Where $\langle x, y\rangle = \sum_{i=1}^{\infty} x_i \overline{y_i}$, is an inner product space

I_1 : a. $\langle x, x\rangle = \sum_{i=1}^{\infty} x_i \overline{x_i} = \sum_{i=1}^{\infty} |x_i|^2 \geq 0$ since the absolute value of a complex number is non-negative.

b. Assume $\langle x, x\rangle = 0 \Rightarrow \sum_{i=1}^{\infty} x_i \overline{x_i} = \sum_{i=1}^{\infty} |x_i|^2 = 0 \Rightarrow |x_i|^2$ for $1 \leq i \leq \infty \Rightarrow x_i = 0$ for $1 \leq i \leq \infty$

$\Rightarrow x = 0$

Next asume $x = 0 \Rightarrow x_i = 0$ for $1 \leq i \leq \infty \Rightarrow 0 = \sum_{i=1}^{\infty} |x_i|^2$

$\Rightarrow \sum_{i=1}^{\infty} x_i \overline{x_i} = \langle x, x\rangle = 0$

I_2 : For any $x = (x_1, x_2, ...), y = (y_1, y_2, ...,) \in l_2$

$$\langle x, y\rangle = \sum_{i=1}^{\infty} x_i \overline{y_i} = \sum_{i=1}^{\infty} \overline{y_i} x_i = \overline{\sum_{i=1}^{\infty} y_i \overline{x_i}} = \overline{\langle y, x\rangle}$$

I_3 : For any $x = (x_1, x_2, ...), y = (y_1, y_2, ...), z = (z_1, z_2, ...,) \in l_2$ and $\alpha, \beta \in l_2$

$$\begin{aligned}
\langle \alpha x + \beta y, z\rangle &= \sum_{i=1}^{\infty} (\alpha x_i + \beta y_i)\overline{z_i} \\
&= \sum_{i=1}^{\infty} \left(\alpha x_i \overline{z_i} + \beta y_i \overline{z_i}\right) \\
&= \sum_{i=1}^{\infty} \alpha x_i \overline{z_i} + \sum_{i=1}^{\infty} \beta y_i \overline{z_i} \\
&= \alpha \sum_{i=1}^{\infty} x_i \overline{z_i} + \beta \sum_{i=1}^{\infty} y_i \overline{z_i} \\
&= \alpha \langle x, z\rangle + \beta \langle y, z\rangle
\end{aligned}$$

Hence the linear space l_2 with the function $\langle .,.\rangle$, defined by $\langle x, y\rangle = \sum_{i=1}^{\infty} x_i \overline{y_i}$, is an inner product space.

Exercise 3.5

The linear space $C[0,1]$ with the function $\langle .,. \rangle$ defined for arbitrary $f,g \in C[0,1]$ by

$\langle f,g \rangle := \int_0^1 f(t)\overline{g(t)}dt$ is inner product space.

I_1 a.: For any $f \in C[0,1]$

$$\langle f, f \rangle = \int_0^1 f(t)\overline{f(t)}\,dt = \int_0^1 |f(t)|^2\,dt \geq 0$$ since the absolute value of any number is non-negative.

b. Assume that $\langle f, f \rangle = 0 \Rightarrow \int_0^1 |f(t)|^2\,dt = 0 \Rightarrow |f(t)|^2 = 0 \Rightarrow |f(t)| = 0 \Rightarrow f(t) = 0 \Rightarrow f = 0$

Next assume that $f = 0 \Rightarrow |f(t)| = 0 \Rightarrow |f(t)|^2 = 0 \Rightarrow \int_0^1 |f(t)|^2\,dt = 0$

Hence $\langle f, f \rangle = \int_0^1 |f(t)|^2\,dt = 0$

I_2 : For any $f,g \in C[0,1]$

$$\begin{aligned}
\langle f,g \rangle = \int_0^1 f(t)\overline{g(t)}\,dt &= \int_0^1 \overline{g(t)}f(t)\,dt \\
&= \int_0^1 \overline{g(t)\overline{f(t)}}\,dt \\
&= \overline{\langle g, f \rangle}
\end{aligned}$$

I_3: For any $f,g,h \in C[0,1]$ and $\lambda,\mu \in C[0,1]$

$$\begin{aligned}
\langle \lambda f + \mu g, h \rangle &= \int_0^1 (\lambda f(t) + \mu g(t))\overline{h(t)}\,dt \\
&= \int_0^1 \left(\lambda f(t)\overline{h(t)} + \mu g(t)\overline{h(t)}\right) dt \\
&= \int_0^1 \lambda f(t)\overline{h(t)}\,dt + \int_0^1 \mu g(t)\overline{h(t)}\,dt \\
&= \lambda \langle f,h \rangle + \mu \langle g,h \rangle
\end{aligned}$$

Hence the linear space $C[0,1]$ with the function $\langle .,. \rangle$, defined by $\langle f,g \rangle = \int_0^1 f(t)\overline{g(t)}\,dt$ is an inner product space.

Exercise 3.6

The linear space $L_2[0,1]$ with the function $\langle .,. \rangle$ defined for arbitrary $f,g \in L_2[0,1]$ by

$\langle f,g \rangle := \int_0^T f(t)\overline{g(t)}dt$ is inner product space.

I_1 a.: For any $f \in L_2[0,1]$

$$\langle f,f \rangle = \int_0^T f(t)\overline{f(t)}\,dt = \int_0^T |f(t)|^2\,dt \geq 0$$ since the absolute value of any number is non-negative.

b. Assume that $\langle f,f \rangle = 0 \Rightarrow \int_0^T |f(t)|^2\,dt = 0 \Rightarrow |f(t)|^2 = 0 \Rightarrow |f(t)| = 0 \Rightarrow f(t) = 0 \Rightarrow f = 0$

Next assume that $f = 0 \Rightarrow |f(t)| = 0 \Rightarrow |f(t)|^2 = 0 \Rightarrow \int_0^T |f(t)|^2\,dt = 0$

Hence $\langle f,f \rangle = \int_0^T |f(t)|^2\,dt = 0$

I_2 : For any $f,g \in L_2[0,1]$

$$\begin{aligned}\langle f,g \rangle &= \int_0^T f(t)\overline{g(t)}\,dt = \int_0^T \overline{g(t)}f(t)\,dt \\ &= \int_0^T \overline{g(t)\,\overline{f(t)}}\,dt \\ &= \overline{\langle g,f \rangle}\end{aligned}$$

I_3: For any $f,g,h \in L_2[0,1]$ and $\lambda,\mu \in L_2[0,1]$

$$\begin{aligned}\langle \lambda f + \mu g, h \rangle &= \int_0^T (\lambda f(t) + \mu g(t))\overline{h(t)}\,dt \\ &= \int_0^T \left(\lambda f(t)\overline{h(t)} + \mu g(t)\overline{h(t)}\right)dt \\ &= \int_0^T \lambda f(t)\overline{h(t)}\,dt + \int_0^T \mu g(t)\overline{h(t)}\,dt \\ &= \lambda\langle f,h \rangle + \mu\langle g,h \rangle\end{aligned}$$

Hence the linear space $L_2[0,1]$ with the function $\langle .,. \rangle$, defined by $\langle f,g \rangle = \int_0^T f(t)\overline{g(t)}\,dt$ is an inner product space.

Lemma 3.10

The inner product is continuous on $E \times E$.

Proof:

Subtracting and adding a term using the triangle inequality, we obtain.

$$\begin{aligned}
\left|\langle x_n, y_n\rangle - \langle x, y\rangle\right| &= \left|\langle x_n, y_n\rangle - \langle x_n, y\rangle + \langle x_n, y\rangle - \langle x, y\rangle\right| \\
&\le \left|\langle x_n, y_n\rangle - \langle x_n, y\rangle\right| + \left|\langle x_n, y\rangle - \langle x, y\rangle\right| \\
&= \left|\langle x_n, y_n - y\rangle\right| + \left|\langle x_n - x, y\rangle\right|
\end{aligned}$$

Using the Cauchy-Schwartz Inequality

$$\left|\langle x_n, y_n - y\rangle\right| + \left|\langle x_n - x, y\rangle\right| \le \|x_n\|\|y_n - y\| + \|x_n - x\|\|y\| \to 0$$

Since $y_n - y \to 0$ and $x_n - x \to 0$ as $n \to \infty$