NUMERICAL SOLUTIONS FOR A CLASS OF NONLINEAR VOLTERRA INTEGRAL EQUATION

by

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Abstract

Numerous studies on linear and nonlinear Volterra integral equations (VIEs), have been performed. These studies mainly considered the existence and uniqueness of the solution, and numerical solutions of these equations. In this work, a class of nonlinear (nonstandard) Volterra integral equation that has received very little attention in the literature is considered. The existence and uniqueness of the solution for the nonlinear VIE is proved using the contraction mapping theorem in the space $C[0, d]$.

Collocation methods, repeated trapezoidal rule and repeated Simpson's rule are used to solve the nonlinear (nonstandard) VIE. For the collocation solutions we considered two cases: implicit Euler method and implicit midpoint method. Examples are used to compare the performance of these methods and the results show that the repeated Simpson's rule performs better than the other methods. An analysis of the collocation solution and the solution by the repeated trapezoidal rule is performed. Sufficient conditions for existence and uniqueness of the numerical solution are given. The collocation methods and repeated trapezoidal rule yield convergence of order one.

Dedication

This work is dedicated to my loving husband Liyandza Meluleki Mamba as well as my son and mother.

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Publications

- H.S.Malindzisa and M.Khumalo, Numerical Solutions of a class of Nonlinear Volterra Integral Equations. *Abstract and applied analysis*, Volume 2014(2014), Article ID 652631, 8 pages.

- H.S.Malindzisa and M.Khumalo, Collocation Solutions of a class of Nonlinear Volterra Integral Equations. *Proceedings of the 14th International Conferencce on Computational and Mathematical Methods in Science and Engineering*, Volume 3 (2014), 7 pages.

- H.S.Mamba and M.Khumalo, On the Analysis of Numerical Methods for non-standard Volterra Integral Equation. *Abstract and applied analysis*,Volume 2014(2014), Article ID 763160, 7 pages.

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Chapter 1

Preliminaries

1.1 Introduction

Many methods have been used to find exact solutions to integral equations such as adomian decomposition method (ADM), modified ADM, succesive approximations method, series solution method, variational iteration method, and Laplace transform method [26]. A solution is called exact if it can be written in closed form such as a trigonometric fuction, exponential function, polynomial or a combination of these functions. However, it is often difficult or impossible to find exact solutions for some integral equations, especially if they are nonlinear. Therefore the need for numerical methods cannot be over stated.

In this preliminary chapter we

- classify and define the types of integral equations,

- describe the VIE considered in this dissertation,

- and introduce the numerical methods used in this study.

1.2 Integral equations

Integral equations are equations in which the unknown function $u(t)$ to be determined appears under an integral sign, Wazwaz [26]. There are two basic types of integral equations, namely *Fredholm integral equations* and *Volterra integral equations(VIEs).*

An integral equation of the form

$$u(t) = g(t) + \int_a^b k(t,s) f(u(s))\, ds, \tag{1.1}$$

where $a, b \in \mathbb{R}$ is a Fredholm integral equation of the second kind.

On the other hand an integral equation of the form

$$u(t) = g(t) + \int_a^t k(t,s) f(u(s))\, ds, \tag{1.2}$$

where $a \in \mathbb{R}$ is a Volterra integral equation of the second kind.

If in (1.1) or (1.2), the unknown $u(t)$ appears only in the integrand, then the corresponding integral equation is of the first kind, see Wazwaz [26]. Moreover, if in (1.1) or (1.2) f is a nonlinear function of u, then the integral equation is nonlinear, otherwise it is linear.

When solving integral equations, the choice of method depends on the nature of the kernel. The types of kernel often encountered in practice include:

- Convolution kernel: $k(t,s) = k(t-s)$
- Degenerate kernel: $k(t,s) = \sum_{i=1}^{r} A_i(t)B_i(s)$
- Singular kernel: $k(t,s) = \frac{s^{\mu-1}}{t^{\mu}}M(t,s)$
- Weakly singular kernel: $k(t,s) = (t-s)^{-\alpha}M(t,s),\ 0 < -\alpha < 1$, where $M(t,s)$ is a smooth fucntion
- Symmetric kernel: $k(t,s) = k(s,t)$
- Centro-symmetric: $k(t,s) = k(a+b-t, a+b-s)$.

Fredholm integral equations are usually easier to solve analytically than Volterra integral equations. Therefore, the main focus of numerical work is on both linear and nonlinear VIEs.

1.2.1 Volterra integral equation

Numerical methods often change the integral equation into a linear or nonlinear system of equations that may be solved using direct or iterative methods. Examples of numerical methods used in the literature include collocation methods [6, 8, 12–14, 22], Galerkin methods [21, 28], spectral methods [9, 25], and quadrature rules [2, 15, 19, 24].

Volterra integral equations play an important part in scientific and engineering problems such as population dynamics, spread of epidemics, semiconductor devices, wave progration, conservative systems, superfluidity and travelling wave analysis [4, 22]. Therefore, the ability to obtain reliable solutions for VIEs is of critical importance in science and engineering.

In our work we consider the nonlinear (nonstandard) VIE of the form

$$u(t) = \sum_{j=1}^{r} b_j \left(g_j(t) + \int_0^t k_j(t,s)u(s)\,ds \right)^j \qquad t \in [0,T], \tag{1.3}$$

where $r \in \mathbb{N}, r \geq 2$, with $b_j \in \mathbb{R}$ and g_j, k_j are continuous functions. In cases where the kernel is of convolution type ($k(t,s) = k(t-s)$) the solution to (1.3) include elliptic functions and natural generalizations of these functions which have wide applications in the fields of science and engineering [23]. Some examples of (1.3) arise from nonlinear ordinary differential equations used to model dynamical and conservative systems [20, 29]. When doing numerical computations we concentrate on the case where $r = 2$ and (1.3) becomes

$$u(t) = \sum_{j=1}^{2} b_j \left(g_j(t) + \int_0^t k_j(t,s)u(s)\,ds \right)^j \qquad t \in [0,T]. \tag{1.4}$$

1.3 Methods

In this work we study (1.3), and prove the existence and uniqueness of the solution to (1.3) in the space $C[0,d]$ using a procedure analogous to the one used in Sloss and Blyth [23]. We use properties of the norm applicable to $C[a,b]$ to establish the conditions for existence and uniqueness of the solutions.

To approximate the solution to (1.3) we use two approaches, collocation methods and quadrature rules. We first use one point collocation methods using a uniform mesh to obtain the solution u_h that belongs to the piecewise constant polynomial space $S_0^{-1}(\Pi_N)$. We then extend our discussion to iterated collocation corresponding to u_h.

We also use the repeated trapezodial rule whereby the integrating interval is divided into subintervals of equal length $h = (b-a)/n$. However, since the trapezoidal rule is known to be less accurate than other quadrature rules [11], we used a very small stepsize h inorder to obtain better results.

Another quadrature rule we use is Simpson's rule which is of higher order than the trapezoidal rule. However, with Simpson's rule the number of subintervals must be even. Consider Simpson's rule when applied to (1.3)

$$\begin{aligned}
u(t_0) &= \sum_{j=1}^{r} b_j (g_j(t_0))^j, \\
u(t_1) &= ?, \\
u(t_2) &= \sum_{j=1}^{r} b_j \left(g_j(t_2) + 1/3h[k_j(t_2,t_0)u(t_0) + 4k_j(t_2,t_1)u(t_1) + k_j(t_2,t_2)u(t_2)]\right)^j .
\end{aligned} \tag{1.5}$$

In (1.5) to obtain u_2, u_0 and u_1 must be used, but Simpson's rule cannot be used to define u_1 because not enough points are available for the rule [11]. Therefore, we use the repeated Simpson's rule with an even number of subintervals.

Chapter 2

Literature Review

2.1 Introduction

Much work has been done in the study of numerical methods for solving integral equations. Atkinson [1] discussed different types of numerical methods that are used in solving integral equations of the second kind depending on their nature. In their disscussion, degenerate kernel methods, Nystrom method and projection methods such as collocation methods and Galerkin's methods are considered. In Linz [18] numerical methods such as the trapezoidal rule, Simpson's rule, block by block method and Gregory method for solving linear and nonlinear Volterra integral equations are discussed.

2.1.1 Collocation Methods

Some authors used collocation methods to approximate solutions of Volterra integral equations in the literature. Benitez and Bolos [3] pointed out that collocation methods have proven to be a very suitable technique for approximating solutions to nonlinear integral equations, because of their stability and accuracy.

Brunner [7] constructed collocation solutions for linear Volterra integral equation of the second kind. In thier study they showed that iterated collocation approximations of these Volterra integral equations restore optimal local superconvergence. Brunner and Blom [8] used collocation and iterated collocation methods to solve general nonlinear Volterra integral equations. Their collocation solution was an element of the polynomial spline space $S_{m-1}^{(-1)}(\Pi_N)$. In their work the results of the superconvergence properties were used to obtain both local and global error estimates.

Brunner [6] and Horvat [14] also used collocation methods to approximate the solution for nonlinear Volterra integral equations with delay constant and continuous kernel. Brunner [6] used iterated collocation method corresponding to the collocation method in the polynomial space $S_{m-1}^{(-1)}(\Pi_N)$. They showed that $O(h^{2m})$ convergence at the mesh points Π_N could be attained by the use of the iterated approximation. Horvat [14] then constructed a collocation solution in the polynomial space $S_{m+d}^{(d)}(\Pi_N)$, and analysed the order of global and local convergence.

Solutions to Volterra integral equations with weakly singular kernels have also been approximeted using collocation methods in recent years. In Diogo and Lima [13] collocation methods are used to approximate the solution for Volterra integral equations with weakly singular kernels, whereby Gauss-Legendre points are used as collocation parameters. The discrete superconvergence properties of spline collocation solutions were also analysed by Diogo and Lima [13]. Diogo [12] used collocation methods and iterated collocation method, and further showed that piecewise polynomials of degree $m-1$ on a uniform mesh yield global convergence of order m. Both these studies dealt with Volterra integral equations with weakly singular kernel where $\mu > 0$, $k(t,s)$ is a smooth function and g is a given function.

2.1.2 Quadrature Methods

Quadrature rules have been used to solve linear and nonlinear VIEs over the years. Aigo [2] solved linear VIEs of the second kind using two quadrature rules. They used the trapezoidal rule and the repeated Simpson's rule, dividing the integration interval into subintervals with equal length, $h = (b-a)/n$. At each subinterval $[t_{2i}, t_{2i+1}, t_{2i+2}]$ they applied Simpson's rule for an even N. Their results showed that with a sufficiently small h a good accuracy is obtained.

Mirzae [19] on the other hand used the repeated Simpson's rule and the repeated modified Simpson's method to solve linear VIEs. In their work they compared these two approaches with other methods such as the repeated trapezoidal rule and concluded on the error of approximation. They found that the error of approximation when using the repeated modified Simpson's method is $O(h^7)$. When using the repeated Simpson's method or the repeated modified trapezoidal method the error is $O(h^5)$.

Saber-Nadjafi and Heidari [24] took a different approach in solving VIEs. In their study they divided the intervel of integration into n subintervals of different lengths such that

$$\begin{cases} h_1 = k\alpha, \\ h_i = k\alpha d^{i-1} = dh_{i-1}, \quad i = 2, 3, ..., n \end{cases}$$

where $\alpha = \frac{(b-a)(d-1)}{k(d^n-1)}$. They used the variable step with the repeated trapezoidal rule, repeated Simpson's rule and the block-by-block methods to solve linear VIEs. Their study showed that the variable step gives more accuracy and normalizes the errors at the last points.

Other authors such as Kilicman, Dehkordi and Kajani [16] and Katani and Shahmorad [15] have used quadrature rules to solve a system of nonlinear VIEs. Applying the quadrature rule to the system of nonlinear VIEs gave a nonlinear system of nonlinear equations. Kilicman, Dehkordi and Kajani [16] used Simpson's 3/8 rule and showed that the method has convergence order $O(h^4)$. On the other hand Katani and Shahmorad [15] solved the system of nonlinear VIEs using the block-by-block method and they proved that at least a fourth order of convergence could be archieved.

2.1.3 The nonlinear (nonstandard) Volterra integral equation

The nonlinear (nonstandard) VIE (1.3) was studied by Sloss and Blyth [23]. They proved a theorem which provided sufficient conditions for the existence of a solution to (1.3), using the Banach fixed point theorem in the space L^2. For the case when $b_1 = 0$ and $r = 2$ in (1.3), they proved existence and uniqueness of a solution in the space L^2. Sloss and Blyth [23], used

Corrington's walsh function method to solve the nonlinear VIE (1.3), and gave sufficient conditions for the convergence of the method. Futhermore, they derived error estimates for the method in L^2 norm. The method they used may be considered as a spectral method since walsh function methods are spectral methods.

Chapter 3

Existence and uniqueness of the solution

In this chapter we prove the existence and uniqueness of the local solution to (1.3) in the space $C[0,1]$ using the contraction mapping theorem. We first show that when $r = 2$ and $b_1 = 0$ the integral equation (1.3) has a unique solution in the space $C[0,1]$. In this case the VIE takes the form

$$z(t) = b\left(g(t) + \int_0^t k(t,s)z(s)\,ds\right)^2 \tag{3.1}$$

where $g(t), k(t,s)$ are continuous functions. The existence and uniqueness of the solution to the general equation (1.3) is proved in theorem 3.

We state a generalization of the fixed point theorem which we then use in proving the existence and uniqueness of the solution.

Theorem 1. *Given a continuous mapping T of a complete metric space R into itself, suppose T^n is a contraction mapping ($n > 1$ an integer). Then T*

has a unique fixed point.

See [17] for the proof of theorem 1.

Theorem 2. *Assume the data b, g and $K = \sup_{[0,1]\times[0,1]} |k(t,s)|$ are sufficientlly small in the sense that there exist a real number $d > 0$ such that $bK(2\|g\|_\infty + Kd) < 1$. Then the VIE (3.1) has a unique solution $z \in C[0,1]$ satisfying $\|z\|_\infty \leq d$.*

Proof. Let T be the mapping of $C[0,1]$ into itself defined by

$$Tz(t) = b\left(g(t) + \int_0^t k(t,s)z(s)\,ds\right)^2,$$

and let $z_1, z_2 \in C[0,1]$. Then

$$\begin{aligned}
Tz_1(t) - Tz_2(t) &= b\left[2g(t) + \int_0^t k(t,s)(z_1(s) + z_2(t))\,ds\right] \\
&\quad \cdot \int_0^t k(t,s)(z_1(s) - z_2(s))\,ds \\
&\stackrel{\text{def}}{=} bv(z_1,z_2)(t)\int_0^t k(t,s)(z_1(s) - z_2(s))\,ds. \\
|Tz_1(t) - Tz_2(t)| &\leq b\sup v(z_1,z_2)(t)\int_0^1 |k(t,s)||(z_1(s) - z_2(s))|\,ds \\
&\leq b\sup v(z_1,z_2)(t)K\|z_1(s) - z_2(s)\|_\infty,
\end{aligned}$$

But

$$v(z_1,z_2)(t) = 2g(t) + \int_0^t k(t,s)(z_1(s) + z_2(t))\,ds$$

$$\|v(z_1,z_2)(t)\|_\infty \leq 2\|g\|_\infty + K\|z_1 + z_2\|_\infty.$$

Therefore we have that

$$\|Tz_1 - Tz_2\|_\infty \leq |b|K(2\|g\|_\infty + Kd)\|z_1 - z_2\|_\infty. \tag{3.2}$$

If $bK(2\|g\|_\infty + Kd) < 1$ then T is a contraction on $C[0,1]$.

Then we consider the iterates of T

$$\begin{aligned}\left|T^2 z_1(t) - T^2 z_2(t)\right| &\leq b^2(\sup v(z_1, z_2)(t))^2 K^2 \|z_1(s) - z_2(s)\|_\infty \int_0^1 ds \\ &= b^2(\sup v(z_1, z_2)(t))^2 K^2 \|z_1(s) - z_2(s)\|_\infty.\end{aligned}$$

Continuing in this way, we get the general result that

$$|T^n z_1(t) - T^n z_2(t)| \leq b^n(\sup v(z_1, z_2)(t))^n K^n \|z_1(s) - z_2(s)\|_\infty,$$

which implies

$$\|T^n z_1 - T^n z_2\|_\infty \leq b^n(\sup v(z_1, z_2))^n K^n \|z_1 - z_2\|_\infty \tag{3.3}$$

When $n \gg 0$, $b^n K^n(2\|g\|_\infty + Kd)^n < 1$, so T^n is a contraction mapping on $C[0,1]$. It follows from Theorem 1 that the VIE (3.1) has a unique solution.

□

Theorem 3. *Assume the data b, g and $K = \sup_{[0,1]\times[0,1]} |k(t,s)|$ are sufficientlly small in the sense that there exist a real number $d > 0$ such that $N_b \sum_{j=1}^{r} j b_j K_j(\|g_j\|_\infty + K_j d)^{j-1} < 1$, where N_b is the number of nonzero b_j. Then the nonlinear VIE (1.3) has a unique solution $u \in C[0,1]$ satisfying $\|u\|_\infty \leq d$.*

Proof. Let T' be the mapping of $C[0,1]$ into itself defined by

$$T'u(t) = \sum_{j=1}^{r} b_j \left(g_j(t) + \int_0^t k_j(t,s)u(s)\,ds \right)^j$$

and $u_1, u_2 \in C[0,1]$. Consider

$$\begin{aligned}
T'u_1(t) - T'u_2(t) &= \sum_{j=1}^{r} b_j \Bigg[\left(g_j(t) + \int_0^t k_j(t,s)u_1(s)\,ds \right)^j \\
&\qquad\qquad - \left(g_j(t) + \int_0^t k_j(t,s)u_2(s)\,ds \right)^j \Bigg] \\
&= \sum_{j=1}^{r} b_j \Bigg[\Bigg(g_j(t) + \int_0^t k_j(t,s)u_1(s)\,ds - g_j(t) \\
&\qquad\qquad - \int_0^t k_j(t,s)u_2(s)\,ds \Bigg) \\
&\qquad \cdot \sum_{i=0}^{j-1} \left(g_j(t) + \int_0^t k_j(t,s)u_1(s)\,ds \right)^i \\
&\qquad \cdot \left(g_j(t) + \int_0^t k_j(t,s)u_2(s)\,ds \right)^{j-1-i} \Bigg] \\
&\stackrel{\text{def}}{=} \sum_{j=1}^{r} b_j F_j(t, u_1, u_2) \int_0^t k_j(t,s)(u_1(s) - u_2(s))\,ds.
\end{aligned}$$

So

$$
\begin{aligned}
|T'u_1(t) - T'u_2(t)| &\le N_b \sum_{j=1}^{r} b_j \left(\int_0^t |k_j(t,s)(u_1(t) - u_2(t))| \, ds \right) \\
&\qquad \cdot F_j(t, u_1, u_2) \\
&\le N_b \sum_{j=1}^{r} b_j \sup F_j(t, u_1, u_2) \\
&\qquad \cdot \left(\int_0^1 |k_j(t,s)||u_1(t) - u_2(t)| \, ds \right) \\
&\le N_b \sum_{j=1}^{r} b_j \sup F_j(t, u_1, u_2) K_j \|u_1(t) - u_2(t)\|_\infty .
\end{aligned}
$$

But

$$
\begin{aligned}
|F_j(t, u_1, u_2)| &\le \sum_{i=0}^{j-1} \left(|g_j(t)| + \int_0^t |k_j(t,s) u_1(s)| \, ds \right)^i \\
&\qquad \cdot \left(|g_j(t)| + \int_0^t |k_j(t,s) u_2(s)| \, ds \right)^{j-1-i} \\
\|F_j(t, u_1, u_2)\|_\infty &\le \sum_{i=0}^{j-1} \left(\|g_j\|_\infty + K_j \int_0^t |u_1(s)| \, ds \right)^i \\
&\qquad \cdot \left(\|g_j\|_\infty + K_j \int_0^t |u_2(s)| \, ds \right)^{j-1-i} \\
&\le \sum_{i=0}^{j-1} \left(\|g_j\|_\infty + K_j \int_0^1 |u_1(s)| \, ds \right)^i \\
&\qquad \cdot \left(\|g_j\|_\infty + K_j \int_0^1 |u_2(s)| \, ds \right)^{j-1-i}
\end{aligned}
$$

$$
\le \sum_{i=0}^{j-1} (\|g_j\|_\infty + K_j \|u_1\|_\infty)^i (\|g_j\|_\infty + K_j \|u_2\|_\infty)^{j-1-i}.
$$

Let $T' : [0,d] \to [0,1]$ where $[0,d] = \{w \in [0,1] : \|w\| \le d\}$ then we have

$$\begin{aligned} &\le \sum_{i=1}^{j-1} (\|g_j\|_\infty + K_j d)^i (\|g_j\|_\infty + K_j d)^{j-1-i} \\ &\le j(\|g_j\|_\infty + K_j d)^{j-1}, \end{aligned} \tag{3.4}$$

which gives

$$\|F_j(t,u_1,u_2)\|_\infty \le j(\|g_j\|_\infty + K_j d)^{j-1}.$$

Thus

$$\|T'u_1 - T'u_2\|_\infty \le N_b \sum_{j=1}^{r} j b_j (\|g_j\|_\infty + K_j d)^{j-1} K_j \|u_1 - u_2\|_\infty.$$

Consequently T' is a contraction mapping if

$$N_b \sum_{j=1}^{r} j b_j (\|g_j\|_\infty + K_j d)^{j-1} K_j < 1.$$

Then we consider the iterates of T'

$$\begin{aligned} \left|T'^2 u_1(t) - T'^2 u_2(t)\right| &\le N_b^2 \left(\sum_{j=1}^{r} b_j \sup F_j(t,u_1,u_2) K_j \right)^2 \|u_1(t) - u_2(t)\|_\infty \int_0^1 ds \\ &= N_b^2 \left(\sum_{j=1}^{r} b_j \sup F_j(t,u_1,u_2) K_j \right)^2 \|u_1(t) - u_2(t)\|_\infty. \end{aligned}$$

Continuing in this way, we get the general result that

$$|T'^n u_1(t) - T'^n u_2(t)| \le N_b^n \left(\sum_{j=1}^{r} b_j \sup F_j(t,u_1,u_2) K_j \right)^n \|u_1(t) - u_2(t)\|_\infty,$$

which implies

$$\|T'^n u_1 - T'^n u_2\|_\infty \leq N_b^n \left(\sum_{j=1}^{r} b_j \sup F_j(t, u_1, u_2) K_j \right)^n \|u_1 - u_2\|_\infty \qquad (3.5)$$

When $n \gg 0$, $N_b^n \left(\sum_{j=1}^{r} b_j \sup F_j(t, u_1, u_2) K_j \right)^n < 1$, so T'^n is a contraction mapping.

We need to show that $T' : C[0, d] \to C[0, d]$.

Observe that

$$\begin{aligned}
&\left\| \left(g_j(t) + \int_0^t k_j(t,s)u(s)\,ds \right)^j \right\|_\infty \\
&\qquad = \int_0^1 \left(|g_j(t)| + \int_0^t k_j(t,s)u(s)\,ds \right)^j \\
&\qquad \leq \int_0^1 \left(|g_j(t)| + \int_0^t |k_j(t,s)||u(s)|\,ds \right)^j \\
&\qquad \leq \int_0^1 \left(|g_j(t)| + \int_0^1 |k_j(t,s)||u(s)|\,ds \right)^j \\
&\qquad \leq \int_0^1 \left(\|g_j\|_\infty + K_j \int_0^1 |u(s)|\,ds \right)^j \\
&\qquad = \int_0^1 \left(\|g_j\|_\infty + K_j \|u(s)\|_\infty\,ds \right)^j \\
&\qquad \leq \left(\|g_j\|_\infty + K_j d \right)^j,
\end{aligned}$$

see [23].

Therefore

$$\|T'v\|_\infty \leq \sum_{j=1}^{r} |b_j| \left(\|g_j\|_\infty + K_j d \right)^j,$$

thus $T' : C[0,d] \to C[0,d]$ if

$$\sum_{j=1}^{r} |b_j| \left(\|g_j\|_\infty + K_j d\right)^j < d.$$

Hence the map T' is a contraction and maps [0,d] into itself provided

$$N_b \sum_{j=1}^{r} j b_j K_j (\|g_j\|_\infty + K_j d)^{j-1} < 1.$$

Therefore by Theorem 1 equation (1.3) has a unique solution in $C[0,1]$. $\square$

Chapter 4

The collocation methods

4.1 Introduction

In this chapter we consider one point collocation methods, see [5]. We construct the collocation solution to (1.3) using a uniform mesh which leads us to two special cases. For each case we present numerical computations for three examples using different values of the stepsize h.

4.1.1 One point collocation methods

Let $t_n := nh$ $(n = 0, 1, ..., N-1)$ define a uniform partition for $I = [0, T]$ and $I_0 := [t_0, t_1]$, $\quad I_h := (t_n, t_{n+1}]$ $(1 \leq n \leq N-1)$. Define $\Pi_N := t_0, ..., t_N$, $\bar{\Pi_N} \cup T$.

The solution to (1.3) will be approximated by using collocation in the piecewise constant polynomial space $S_0^{-1}(\Pi_N)$. For a given collocation parameter c_1, define the set $X_N := t_{n,1}$ of *collocation points* by

$$t_{n,1} = t_n + c_1 h \ (0 \leq c_1 \leq 1, \quad n = 0, ..., N-1). \tag{4.1}$$

The collocation solution $u_h \in S_0^{-1}(\Pi_N)$ is defined by the collocation equation

$$u_h(t) = \sum_{j=1}^{r} b_j \left(g_j(t) + \int_0^t k_j(t,s)u(s)\, ds \right)^j, \quad t \in X_N. \tag{4.2}$$

Since

$$u_h(t) = u_h(t_n + \nu h) = L_1(\nu)U_{n,1}, \quad \nu \in (0,1], \tag{4.3}$$

where $L_1(\nu) = 1$ and is a *Lagrange fundamental polynomial.* Thus for $t = t_{n,1} := t_n + c_1 h$ and $0 < c_1 \leq 1$ the collocation equation (6.19) assumes the form

$$u_h(t) = \sum_{j=1}^{r} b_j \left(g_j(t) + \int_0^{t_n} k_j(t,s)u_i(s) + h \int_0^{c_1} k_j(t, t_n + sh)u_h(t_n + sh)\, ds \right)^j.$$

Expressing the collocation equation in terms of the stage values $U_{n,1}$ we get

$$U_{n,1} = \sum_{j=1}^{r} b_j \left(g_j(t_{n,1}) + F_n(t_{n,1}) + h \left(\int_0^{c_1} k_j(t_{n,1}, t_n + sh)\, ds \right) U_{n,1} \right)^j, \tag{4.4}$$

where

$$\begin{aligned} F_n(t) &:= \int_0^{t_n} k_j(t,s)u_i(s)\, ds \\ &= h \sum_{i=0}^{n-1} \int_0^1 k_j(t, t_i + sh)u_i(t_i + sh)\, ds \end{aligned} \tag{4.5}$$

denotes the *lag term* corresponding to the collocation solution, [5]. Let $t = t_{n,1}$ and use (4.3) to get

$$F_n(t_{n,1}) = h \sum_{i=0}^{n-1} \left(\int_0^1 k_j(t_{n,1}, t_i + sh)\, ds \right) U_{i,1}. \tag{4.6}$$

We extend this work to the *iterated collocation* associated with the collocation solution for (1.3). The iterated approximation u^I corresponding to u is defined by

$$u^I(t) = \sum_{j=1}^{r} b_j \left(g_j(t) + \int_0^t k_j(t, s) u(s)\, ds \right)^j \quad t \in I \tag{4.7}$$

see [7, 8, 12]. Set $t = t_n \in \bar{\Pi}_N$ and use (4.3) we may write (4.7) in the form

$$u^I(t_n) = \sum_{j=1}^{r} b_j \left(g_j(t_n) + h \sum_{i=0}^{n-1} \int_0^1 k_j(t_n, t_i + sh)\, ds U_{i,1} \right)^j \tag{4.8}$$

4.2 Numerical Computations

Example 1 Consider the nonlinear VIE

$$u(t) = 2 \left(1 + \int_0^t (t - s) u(s)\, ds \right)^2 \qquad 0 \le t \le 1, \tag{4.9}$$

which arises from a nonlinear differential equation in [10] where $b_1 = 0$ and $b_2 = 2$. We use (4.4) to approximate the solutions considering two special cases: $c_1 = 1/2$ (implicit midpoint method) and $c_1 = 1$ (implicit Euler method). Using (4.4) the collocation solution of (4.9) becomes

$$U_{n,1} = 2\left[1 + F_n(t_{n,1}) + h\left(\int_0^{c_1}(t_{n,1} - t_n - sh)\,ds\right)U_{n,1}\right]^2, \tag{4.10}$$

where

$$F_n(t_{n,1}) = h\sum_{i=0}^{n-1}\left[\int_0^1 (t_{n,1} - t_i - sh)\,ds\right]U_{i,1}. \tag{4.11}$$

Substitute $t_{n,1} = t_n + c_1 h$ to (4.10) and (4.11) and integrate to get

$$U_{n,1} = 2\left[1 + F_n(t_{n,1}) + \frac{h^2}{2}c_1^2 U_{n,1}\right]^2 \tag{4.12}$$

where

$$F_n(t_{n,1}) = h\sum_{i=0}^{n-1}\left[t_n - t_i + (c_1 - \frac{1}{2})h\right]U_{i,1}. \tag{4.13}$$

We use an iterative procedure to obtain the solution to (4.12).

4.2.1 Implicit Euler

We substitute $c_1 = 1$ and $t_{n,1} = t_n + h$ to both (4.12) and (4.23) then the collocation solution to (4.9) becomes

$$U_{n,1} = 2\left(1 + F_n(t_{n,1}) + U_{n,1}\frac{h^2}{2}\right)^2, \tag{4.14}$$

where

$$F_n(t_{n,1}) = h\sum_{i=0}^{n-1}\left(t_n - t_i + \frac{h}{2}\right)U_{i,1}$$

.

4.2.2 Implicit midpoint method

For $c_1 = \frac{1}{2}$, $t_{n,1} = t_n + \frac{h}{2}$ the method is called implicit midpoint and the collocation solution (4.12) of (4.9) is given by

$$U_{n,1} = 2\left(1 + F_n(t_{n,1}) + U_{n,1}\frac{h^2}{8}\right)^2, \tag{4.15}$$

where

$$F_n(t_{n,1}) = h\sum_{i=0}^{n-1}(t_n - t_i)\,U_{i,1}.$$

4.2.3 Iterated collocation solution when $c_1 = \frac{1}{2}$

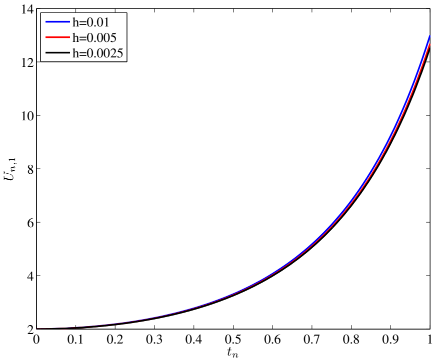
For $c_1 = \frac{1}{2}$ the iterated collocation solution of (4.9) is given as

$$u^I(t_n) = 2\left(1 + h\sum_{i=0}^{n-1}\int_0^1 (t_n - t_i - sh)\,dsU_{i1}\right)^2$$

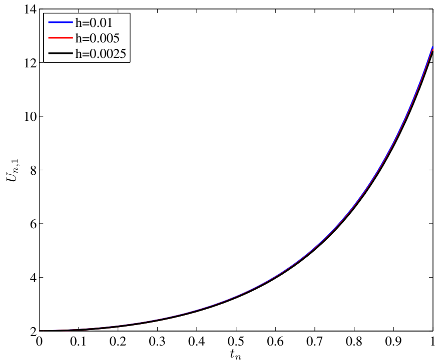
Integrate to obtain

$$u^I(t_n) = 2\left(1 + h\sum_{i=0}^{n-1}(t_n - t_i - \frac{h}{2})U_{i1}\right)^2. \tag{4.16}$$

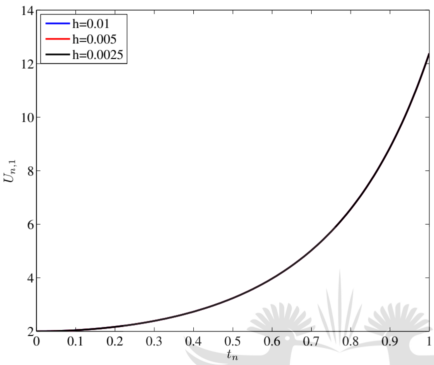
Figure 4.2.1 shows the graphs of solution to (4.9) computed using the implicit euler method, implicit midpoint methods and the iterated collocation method.



(a) by implicit Euler



(b) by implicit midpoint



(c) by iterated collocation

Figure 4.2.1: The collocation solution of (4.9)

Example 2

$$u(t) = \left(1 + \int_0^t (t-s)u(s)\,ds\right) + \frac{1}{2}\left(1 + \int_0^t (t-s)u(s)\,ds\right)^2 \qquad 0 \le t \le 1, \tag{4.17}$$

where $b_1 = 1$ and $b_2 = \frac{1}{2}$. The integral equation (4.17) arises from a nonlinear differential equation the represent conservative systems, see [20]. Using (4.4) the collocation solution of (4.17) becomes

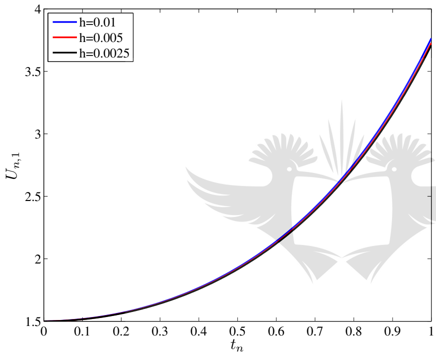
$$U_{n,1} = \left(1 + F_n(t_{n,1}) + \frac{h^2}{2}c_1^2 U_{n,1}\right) + \frac{1}{2}\left(1 + F_n(t_{n,1}) + \frac{h^2}{2}c_1^2 U_{n,1})\right)^2 \tag{4.18}$$

$$F_n(t_{n,1}) = h\sum_{i=0}^{n-1}\left[t_n - t_i + (c_1 - \frac{1}{2})h\right]U_{i,1}. \tag{4.19}$$

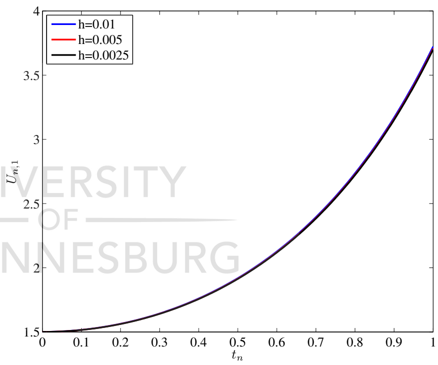
The iterated collocation solution for (4.17) is

$$u^I(t_n) = \left(1 + h\sum_{i=0}^{n-1}(t_n - t_i - \frac{h}{2})U_{i1}\right) + \frac{1}{2}\left(1 + h\sum_{i=0}^{n-1}(t_n - t_i - \frac{h}{2})U_{i1}\right)^2. \tag{4.20}$$

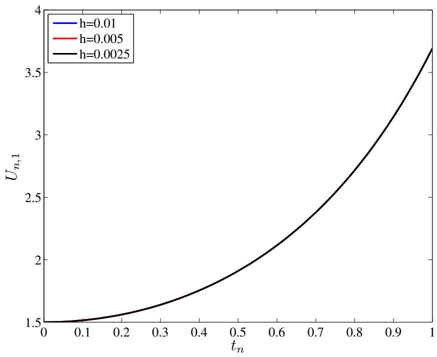
The expression for $F_n(t_{n,1})$ is the same as the one in example 1 since the kernel is of convolution type. Figure 4.2.2 shows the graphs of the collocation and iterated collocation solution to (4.17).



(a) by implicit Euler



(b) by implicit midpoint



(c) by iterated collocation

Figure 4.2.2: The collocation solution of (4.17)

Example 3 Consider the integral equation

$$u(t) = 2\left(1 + \int_0^t (t-s)u(s)\,ds\right) + \left(1 + \int_0^t (t-s)u(s)\,ds\right)^2 \qquad 0 \le t \le 1, \tag{4.21}$$

where $b_1 = 2$ and $b_2 = 1$. The nonlinear VIE arises from a nonlinear differential equation in [29]. Using (4.4) the collocation solution of (4.21) becomes

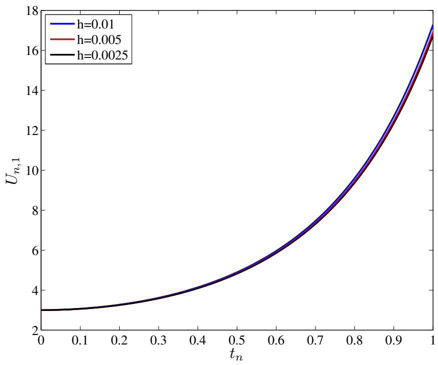
$$U_{n,1} = 2\left(1 + F_n(t_{n,1}) + \frac{h^2}{2}c_1^2 U_{n,1}\right) + \left(1 + F_n(t_{n,1}) + \frac{h^2}{2}c_1^2 U_{n,1})\right)^2 \tag{4.22}$$

$$F_n(t_{n,1}) = h\sum_{i=0}^{n-1}\left[t_n - t_i + (c_1 - \frac{1}{2})h\right] U_{i,1}. \tag{4.23}$$

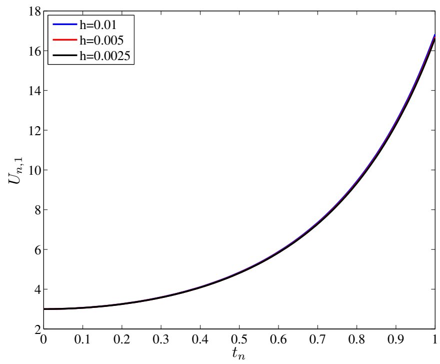
The iterated collocation solution for (4.21) is

$$u^I(t_n) = 2\left(1 + h\sum_{i=0}^{n-1}(t_n - t_i - \frac{h}{2})U_{i1}\right) + \left(1 + h\sum_{i=0}^{n-1}(t_n - t_i - \frac{h}{2})U_{i1}\right)^2. \tag{4.24}$$

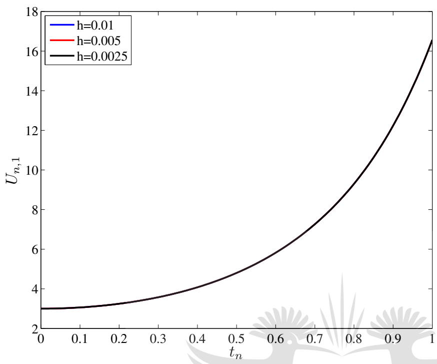
Figure 4.2.3 shows the graphs of the collocation and iterated collocation solution to (4.21).



(a) by implicit Euler



(b) by implicit midpoint



(c) by iterated collocation

Figure 4.2.3: The collocation solution of (4.21)

Table 4.1 shows the absolute errors for the solutions to example 1-3 by implicit Euler method and the absolute errors by implicit midpoint method are shown in table 4.2.

Table 4.1: Absolute errors by implicit Euler method when $h = 0.01$

t	**Example 1**	**Example 2**	**Example 3**
0.1	0.0077	0.0028	0.0116
0.2	0.0161	0.0057	0.0239
0.3	0.0266	0.0089	0.0389
0.4	0.0406	0.0127	0.0586
0.5	0.0607	0.0171	0.0857
0.6	0.0907	0.0226	0.1247
0.7	0.1371	0.0295	0.1829
0.8	0.2115	0.0385	0.2727
0.9	0.3360	0.0501	0.4163
1.0	0.5530	0.0659	0.6541

Table 4.2: Absolute errors by implicit midpoint method when $h = 0.01$

t	**Example 1**	**Example 2**	**Example 3**
0.1	0.0038	0.0014	0.0057
0.2	0.0078	0.0028	0.0116
0.3	0.0127	0.0043	0.0187
0.4	0.0190	0.0061	0.0274
0.5	0.0275	0.0081	0.0390
0.6	0.0397	0.0105	0.0549
0.7	0.0576	0.0135	0.0773
0.8	0.0849	0.0172	0.1104
0.9	0.1280	0.0219	0.1607
1.0	0.1992	0.0279	0.2399

4.3 Discussion

We constructed one-point collocation solutions for the nonlinear (nonstandard) VIE (1.3). We approximated the solutions to example 1-3 using the implicit euler method and implicit midpoint method, using various values of the stepsize, see figure 4.2.1-4.2.3. At $h = 0.001$ and below we obtained a similiar solution from both methods, hence we take that as our ‘exact’ solution. When the stepsize is greater that 0.001 we obtained different numerical solutions from each of the two methods. We use the ‘exact’ solution and absolute error to study the performance of each method when the stepsize is increased. From the results in table 4.1 and 4.2 we observe that the implicit midpoint method gave smaller absolute errors compared to implicit Euler method. Therefore, the implicit midpoint method performs better than implicit Euler method. We then found an iterated collocation solution for the implicit midpoint method and the accuracy of the method improved.

Chapter 5

Quadrature Methods

5.1 Introduction

In this chapter we solve (1.3) by approximating the integral using a qudrature rule which results in a system of equations. We use the repeated trapezoidal rule and repeated Simpson's rule and we solve the resulting equations using Matlab.

5.1.1 Using the repeated trapezoidal rule

Using the repeated trapezoidal rule we construct the solution to the integral equation (1.3), see [24]. Let the interval $[a, b]$ be finite and partitioned by n equally spaced points, we then have

$$\begin{cases} t_0 = a, t_n = b \\ t_i = t_0 + ih \quad i = 0, 1, 2, ..., n. \end{cases} \tag{5.1}$$

Let

$$
\begin{cases}
u(t_0) = \displaystyle\sum_{j=1}^{r} b_j \left(g_j(t_0)\right)^j \\
u(t_i) = \displaystyle\sum_{j=1}^{r} b_j \Big(g_j(t_i) + \sum_{l=1}^{i} \int_{t_{l-1}}^{t_i} k_j(t_i,s)u(s)\,ds\Big)^j \quad i = 1,2,...,n.
\end{cases}
\tag{5.2}
$$

The approximation of the integral in (1.3) by repeated trapezoidal will be given by the following system:

$$
\begin{cases}
u(t_0) = \displaystyle\sum_{j=1}^{r} b_j \left(g_j\right)^j, \\
u(t_1) = \displaystyle\sum_{j=1}^{r} b_j \left(g_j(t_1) + \frac{h}{2}(k_j(t_1,t_0)u(t_0) + k_j(t_1,t_1)u(t_1)\right)^j, \\
u(t_2) = \displaystyle\sum_{j=1}^{r} b_j \Big(g_j(t_2) + \frac{h}{2}k_j(t_2,t_0)u(t_0) + hk_j(t_2,t_1)u(t_1) \\
\qquad\qquad\qquad + \dfrac{h}{2}k_j(t_2,t_2)u(t_2)\Big)^j, \\
u(t_3) = \displaystyle\sum_{j=1}^{r} b_j \Big(g_j(t_3) + \frac{h}{2}k_j(t_3,t_0)u(t_0) + h(k_j(t_3,t_1)u(t_1) \\
\qquad\qquad\qquad + k_j(t_3,t_2)u(t_2) + \dfrac{h}{2}k_j(t_3,t_3)u(t_3)\Big)^j, \\
\quad\vdots \\
u(t_n) = \displaystyle\sum_{j=1}^{r} b_j \Big(g_j(t_n) + \frac{h}{2}k_j(t_n,t_0)u(t_0) + h\sum_{i=1}^{n-1} k_j(t_n,t_i)u(t_i) \\
\qquad\qquad\qquad + \dfrac{h}{2}k_j(t_n,t_n)u(t_n)\Big)^j.
\end{cases}
\tag{5.3}
$$

5.1.2 Using repeated Simpson's rule

We use the repeated Simpson's rule to construct the solution to the integral equation (1.3), see [2]. If n is even, then Simpson's rule may be applied to each subinterval $[t_{2i}, t_{2i+1}, t_{2i+2}]$. For $i = 0, 1, ..., \frac{N}{2} - 1$ we have

$$\int_{t_{2i}}^{t_{2i+2}} f(t)\,dt \simeq \frac{h}{3}\left[f(t_{2i}) + 4f(t_{2i+1}) + f(t_{2i+2})\right].$$

Summing up,

$$\int_a^b f(t)\,dt = \frac{h}{3}\sum_{l=0}^{N-1}[f(t_{2i}) + 4f(\frac{t_{2l}+t_{2l+2}}{2}) + f(t_{2l+2})]. \tag{5.4}$$

We use (5.4) to solve the nonlinear (nonstandard) VIE. The approximation of (1.3) in the even nodes t_{2m} is given by

$$u_{2m} = \sum_{j=1}^{r} b_j \left[g_j(t_{2m}) + \int_a^{t_{2m}} k_j(t_{2m}, s)u(s)\,ds\right]^j. \tag{5.5}$$

Using

$$u(t_{2l+1}) \simeq \frac{u(t_{2l}) + u(t_{2l+2})}{2}.$$

we obtain

$$\begin{aligned} u(t_{2m}) = \sum_{j=1}^{r} b_j \Bigg[g_j(t_{2m}) + \frac{h}{3}\sum_{l=0}^{m-1} k_j(t_{2m}, t_{2l})u(t_{2l}) \\ + 4k_j(t_{2m}, t_{2l+1})\frac{u(t_{2l}) + u(t_{2l+2})}{2} + k_j(t_{2m}, t_{2l+2})u(t_{2l+2})\Bigg]^j \end{aligned}$$

$$
\begin{aligned}
u(t_{2m}) = \sum_{j=1}^{r} b_j \Bigg[g_j(t_{2m}) + \frac{h}{3} \sum_{l=0}^{m-1} \Big(k_j(t_{2m}, t_{2l}) + 2k_j(t_{2m}, t_{2l+1}) \Big) u(t_{2l}) \\
+ \Big(k_j(t_{2m}, t_{2l+2}) + 2k_j(t_{2m}, t_{2l+1}) \Big) u(t_{2l+2}) \Bigg]^j
\end{aligned}
$$

$$
\begin{aligned}
u(t_{2m}) = \sum_{j=1}^{r} b_j \Bigg[g_j(t_{2m}) + \frac{h}{3} \sum_{l=0}^{m-1} \Big(k_j(t_{2m}, t_{2l}) + 2k_j(t_{2m}, t_{2l+1}) \Big) u(t_{2l}) \\
+ \sum_{l=1}^{m} \Big(k_j(t_{2m}, t_{2l}) + 2k_j(t_{2m}, t_{2l-1}) \Big) u(t_{2l}) \Bigg]^j
\end{aligned}
$$

$$
\begin{aligned}
u(t_{2m}) = \sum_{j=1}^{r} b_j \Bigg[g_j(t_{2m}) + \frac{h}{3} \Bigg[\Big(k_j(t_{2m}, t_0) + 2k_j(t_{2m}, t_1) \Big) u(t_0) \\
+ \Big(k_j(t_{2m}, t_{2m}) + k_j(t_{2m}, t_{2m-1}) \Big) u(t_{2m}) \Bigg] \\
+ \frac{h}{3} \sum_{l=1}^{m-1} \Big(2k_j(t_{2m}, t_{2l-1}) + 2k_j(t_{2m}, t_{2l}) + 2k_j(t_{2m}, t_{2l+1}) \Big) u(t_{2l}) \Bigg]^j
\end{aligned}
$$

$$
\begin{aligned}
u(t_{2m}) = \sum_{j=1}^{r} b_j \Bigg[g_j(t_{2m}) + \frac{h}{3} \Big(k_j(t_{2m}, t_0) + 2k_j(t_{2m}, t_1) \Big) u(t_0) \\
+ \frac{h}{3} \Big(k_j(t_{2m}, t_{2m}) + k_j(t_{2m}, t_{2m-1}) \Big) u(t_{2m}) \\
+ \frac{2h}{3} \sum_{l=1}^{m-1} \Big(k_j(t_{2m}, t_{2l-1}) + k_j(t_{2m}, t_{2l}) + k_j(t_{2m}, t_{2l+1}) \Big) u(t_{2l}) \Bigg]^j
\end{aligned}
\tag{5.6}
$$

with $u(a) = u(t_0) = \sum_{j=1}^{r} b_j \left(g_j(t_0)\right)^j$.

5.1.3 Numerical computation

Example 1 The solution for (4.9) by repeated trapezoidal rule (5.3) is

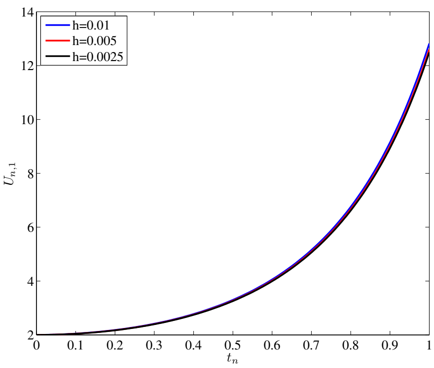
$$u(t_n) = 2\left(1 + \frac{h}{2}(t_n - t_0)u(0) + h\sum_{i=1}^{n-1}(t_n - t_i)u(t_i) + \frac{h}{2}(t_n - t_n)u(t_n)\right)^2, \tag{5.7}$$

where $u(0) = 2$.

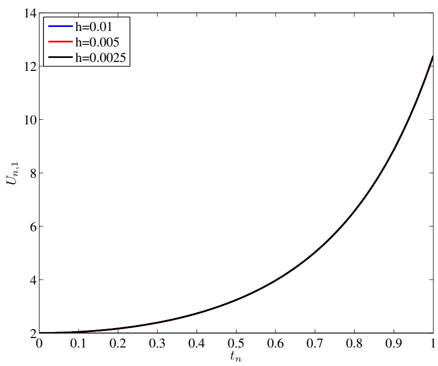
The solution for (4.9) by repeated Simpson's rule (5.6) is

$$\begin{aligned} u(t_{2m}) = 2\Bigg[1 &+ \frac{h}{3}\Big((t_{2m} - t_0) + 2(t_{2m} - t_1)\Big)u(0) \\ &+ \frac{h}{3}\Big((t_{2m} - t_{2m}) + (t_{2m} - t_{2m-1})\Big)u(t_{2m)} \\ &+ \frac{2h}{3}\sum_{l=1}^{m-1}\Big((t_{2m} - t_{2l-1}) + (t_{2m} - t_{2l}) + (t_{2m} - t_{2l+1})\Big)u(t_{2l})\Bigg]^2, \end{aligned} \tag{5.8}$$

where $u(0) = 2$. We use various values of h, and figure 5.1.1 show the graphs for the solutions to (4.9).



(a) by repeated trapezoidal rule



(b) by repeated Simpson's rule

Figure 5.1.1: The solution of (4.9)

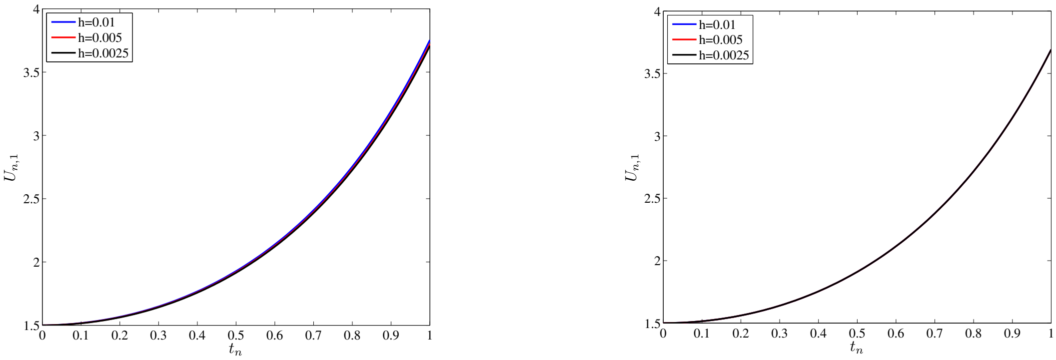
Example 2 The solution for (4.17) by repeated trapezoidal rule (5.3) is

$$
\begin{aligned}
u(t_n) = &\left(1 + \frac{h}{2}(t_n - t_0)u(0) + h\sum_{i=1}^{n-1}(t_n - t_i)u(t_i) + \frac{h}{2}(t_n - t_n)u(t_n)\right) \\
&+ \frac{1}{2}\left(1 + \frac{h}{2}(t_n - t_0)u(0) + h\sum_{i=1}^{n-1}(t_n - t_i)u(t_i) + \frac{h}{2}(t_n - t_n)u(t_n)\right)^2.
\end{aligned}
\tag{5.9}
$$

The solution for (4.17) by repeated Simpson's rule (5.6) is

$$
\begin{aligned}
u(t_{2m}) = &\Bigg[1 + \frac{h}{3}\Big((t_{2m} - t_0) + 2(t_{2m} - t_1)\Big)u(0) \\
&+ \frac{h}{3}\Big((t_{2m} - t_{2m}) + (t_{2m} - t_{2m-1})\Big)u(t_{2m)} \\
&+ \frac{2h}{3}\sum_{l=1}^{m-1}\Big((t_{2m} - t_{2l-1}) + (t_{2m} - t_{2l}) + (t_{2m} - t_{2l+1})\Big)u(t_{2l})\Bigg] \\
&+ \frac{1}{2}\Bigg[1 + \frac{h}{3}\Big((t_{2m} - t_0) + 2(t_{2m} - t_1)\Big)u(0) \\
&+ \frac{h}{3}\Big((t_{2m} - t_{2m}) + (t_{2m} - t_{2m-1})\Big)u(t_{2m)} \\
&+ \frac{2h}{3}\sum_{l=1}^{m-1}\Big((t_{2m} - t_{2l-1}) + (t_{2m} - t_{2l}) + (t_{2m} - t_{2l+1})\Big)u(t_{2l})\Bigg]^2,
\end{aligned}
\tag{5.10}
$$

where $u(0) = \frac{3}{2}$ in both cases. Figure 5.1.2 show the graphs for the solutions to (4.17).



(a) by repeated trapezoidal rule (b) by repeated Simpson's rule

Figure 5.1.2: The solution of (4.17)

Example 3 The solution for (4.21) by repeated trapezoidal rule (5.3) is

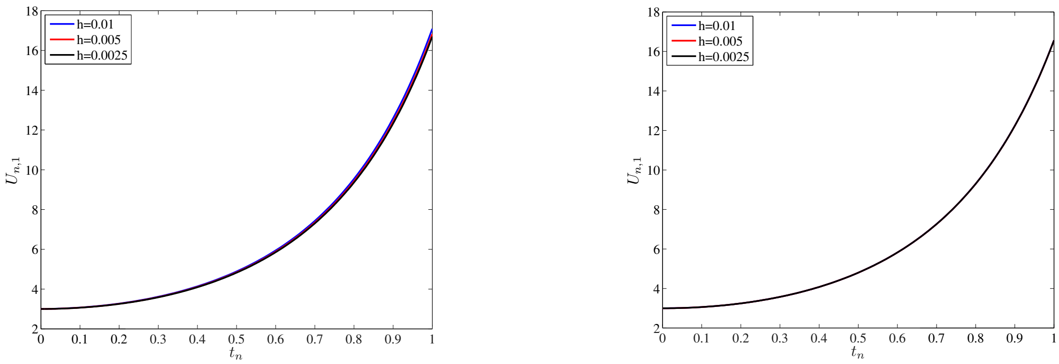
$$\begin{aligned} u(t_n) = {} & 2\left(1+\frac{h}{2}(t_n-t_0)u(0)+h\sum_{i=1}^{n-1}(t_n-t_i)u(t_i)+\frac{h}{2}(t_n-t_n)u(t_n)\right) \\ & +\left(1+\frac{h}{2}(t_n-t_0)u(0)+h\sum_{i=1}^{n-1}(t_n-t_i)u(t_i)+\frac{h}{2}(t_n-t_n)u(t_n)\right)^2. \end{aligned} \tag{5.11}$$

The solution for (4.21) by repeated Simpson's rule (5.6) is

$$\begin{aligned} u(t_{2m}) = {} & 2\Bigg[1+\frac{h}{3}\Big((t_{2m}-t_0)+2(t_{2m}-t_1)\Big)u(0) \\ & \quad +\frac{h}{3}\Big((t_{2m}-t_{2m})+(t_{2m}-t_{2m-1})\Big)u(t_{2m}) \\ & \quad +\frac{2h}{3}\sum_{l=1}^{m-1}\Big((t_{2m}-t_{2l-1})+(t_{2m}-t_{2l})+(t_{2m}-t_{2l+1})\Big)u(t_{2l})\Bigg] \\ & +\Bigg[1+\frac{h}{3}\Big((t_{2m}-t_0)+2(t_{2m}-t_1)\Big)u(0) \\ & \quad +\frac{h}{3}\Big((t_{2m}-t_{2m})+(t_{2m}-t_{2m-1})\Big)u(t_{2m}) \end{aligned}$$

$$+\frac{2h}{3}\sum_{l=1}^{m-1}\Big((t_{2m}-t_{2l-1})+(t_{2m}-t_{2l})+(t_{2m}-t_{2l+1})\Big)u(t_{2l})\Big]^2 \tag{5.12}$$

where $u(0) = 3$ in both cases. Figure 5.1.3 show the graphs for the solutions to (4.21).



(a) by repeated trapezoidal rule (b) by repeated Simpson's rule

Figure 5.1.3: The solution of (4.21)

Table 5.1 and 5.2 shows the absolute errors for example 1-3 by the repeated trapezoidal rule and the repeated Simpson's rule.

Table 5.1: Absolute errors by repeated trapezoidalrule when $h = 0.01$

t	**Example 1**	**Example 2**	**Example 3**
0.1	0.0074	0.0027	0.0110
0.2	0.0154	0.0055	0.0229
0.3	0.0252	0.0086	0.0369
0.4	0.0377	0.00121	0.0543
0.5	0.0487	0.0162	0.0774
0.6	0.0789	0.0209	0.1089
0.7	0.1146	0.0239	0.1537
0.8	0.1689	0.0342	0.2196
0.9	0.2549	0.0436	0.3196
1.0	0.3966	0.0533	0.4771

Table 5.2: Absolute errors by repeated Simpson's rule when $h = 0.01$

t	**Example 1**	**Example 2**	**Example 3**
0.1	-	-	-
0.2	0.0001	-	0.0001
0.3	-	-	-
0.4	0.0001	0.0001	0.0002
0.5	0.0005	0.0001	0.0003
0.6	0.0004	0.0001	0.0005
0.7	0.0007	-	0.0009
0.8	0.0013	0.0001	0.0016
0.9	0.0023	0.0001	0.0028
1.0	0.0044	0.0002	0.0049

5.2 Discussion

We solved examples of (1.3) using the repeated trapezoidal rule and repeated Simpson's rule. Table 5.1 and 5.2 show the absolute errors in the solution of the three examples, with the solution when $h = 0.001$ as our 'exact' solution. We observe that the repeated trapezoidal rule produces larger errors compared to repeated Simpson's rule. The repeated Simpson's rule performs better than repeated trapezoidal rule especially when a large value of the stepsize h is used.

From the results in the previous and current chapter we observe that the repeated Simpson's rule performs better followed by the implicit midpoint method then the repeated trapezoidal rule. Among the four methods used the implicit Euler method gives larger absolute errors as h increases. For all the methods we also observed that for sufficiently small h we get a good accuracy of the numerical solutions. According to our numerical results, we conclude that the repeated Simpson's rule performs well since it gives better solutions when a reasonably large value of the stepsize is used. These observations are consistent for all three examples used.

Chapter 6

Analysis of the numerical solutions

6.1 Introduction

In this chapter we consider the collocation solution and the solution by the repeated trapezoidal rule for (1.4). First we provide sufficient conditions for the existence and uniqueness of the collocation solution, as well as the solution by the repeated trapezoidal rule for (1.4). We then analyze the convergence of the collocation methods and the repeated trapezoidal rule. Numerical experiments are used to illustrate the theoretical results.

6.2 Existence and uniquenes of the numerical solution

Consider the nonlinear VIE

$$z(t) = b\left(g(t) + \int_0^t k(t,s)z(s)\,ds\right)^2 \tag{6.1}$$

where $k \in C(D)$, $(D := (t,s) : 0 \leq s \leq t \leq T)$ and $g \in C(I)$. We prove a theorem analogeous to the one in [27], establishing the existence and uniqueness of the collocation solution for (6.1). The collocation solution to (6.1) is given by

$$z_h(t) = b\Big(g(t) + (\gamma(z_h))(t)\Big)^2 \tag{6.2}$$

where

$$(\gamma(z_h))(t) = \int_0^t k(t,s)z(s)\,ds.$$

Then the collocation solution can be give as,

$$Z_{n,1} = b\left(g(t_{n,1}) + F_n(t_{n,1}) + h\left(\int_0^{c_1} k(t_{n,1}, t_n + sh)\,ds\right) Z_{n,1}\right)^2 \tag{6.3}$$

where

$$F_n(t_{n,1}) = h\sum_{i=0}^{n-1}\left(\int_0^1 k(t_{n,1}, t_i + sh)\,ds\right) Z_{i,1}. \tag{6.4}$$

Let

$$B_n = \int_0^{c_1} k(t_{n,1}, t_n + sh)\,ds \tag{6.5}$$

and $g = g(t_{n,1})$ and $F = F_n(t_{n,1})$, then

$$Z_{n,1} = b\left(g^2 + 2gF + F^2 + 2ghB_nZ_{n,1} + 2hFB_nZ_{n,1} + h^2B_n^2Z_{n,1}^2\right)$$

$$[1 - h(2bgB_n + 2bFB_n + hB_n^2Z_{n,1})]Z_{n,1} = b(g+F)^2. \tag{6.6}$$

To solve (6.6) we use an iterative procedure, thus we rewrite (6.6) in the form

$$[1 - h(2bgB_n + 2bFB_n + hB_n^2Z_{n,1}^{(s)})]Z_{n,1}^{(s+1)} = b(g+F)^2 \tag{6.7}$$

where $s = 1, 2....$ Let

$$\wp^{(s)} = (2bgB_n + 2bFB_n + hB_n^2Z_{n,1}^{(s)}).$$

Then (6.7) can be rewritten as

$$[1 - h\wp^{(s)}]Z_{n,1}^{(s+1)} = b(g+F)^2. \tag{6.8}$$

Theorem 4. *Assume that the given functions g and k in the nonlinear VIE (6.1) are continuous in their respective domains I and D. Then there exists a constant $\bar{h} > 0$, so that for any uniform mesh I_h with $h < \bar{h}$ each of the equations (6.6) has a unique solution $Z_{n,1}$. Hence equation (6.2) defines a unique collocation solution $z_h \in S_0^{(-1)}(I_h)$ for (6.1).*

Proof. Since the kernel k in (6.1) is continuous on D then $\wp^{(1)}$ in (6.8) is bounded. Whenever $h\left|\wp^{(1)}\right| < 1,\ 1 - h\wp^{(1)} \neq 0$. Hence, for $s = 1$ in (6.8) $Z_{n,1}^{(2)}$, exists and is bounded.

For some $s > 1$ assume that $Z_{n,1}^{(s)}$ is bounded. Then, arguing as above, $Z_{n,1}^{(s+1)}$ exists and is bounded if $h\left|\wp^{(s)}\right| < 1$.

The above holds if there is an $\bar{h} > 0$ such that for a uniform mesh I_h with $h < \bar{h}$, the condition $h\left|\wp^{(s)}\right| < 1$ holds for all $s \geq 1$.

Since $C[0,1]$ is a Banach space it suffices to show that the sequence $\{Z_{n,1}^{(s+1)}\}$ is convergent. Consider (6.7) and define

$$D_n = 2bgB_n + 2bFB_n,$$

then

$$[1 - h(D_n + hB_n^2 Z_{n,1}^{(s-1)})]Z_{n,1}^{(s)} = b(g+F)^2 \tag{6.9}$$

Subtracting (6.9) from (6.7) we obtain

$$\delta^{(s+1)} = \frac{h^2 B_n^2 Z_{n,1}^{(s)}}{(1-hD_n)}\delta^{(s+1)} + \frac{h^2 B_n^2 Z_{n,1}^{(s)}}{(1-hD_n)}\delta^{(s)}, \tag{6.10}$$

where $\delta^s = Z_{n,1}^{(s)} - Z_{n,1}^{(s-1)}$. Rearranging and simplfying gives

$$\delta^{(s+1)} = \frac{h^2 B_n^2 Z_{n,1}^{(s)}}{(1-h\wp^{(s)})}\delta^{(s)} \tag{6.11}$$

Therefore for each $j \geq 0$

$$\delta^{(s+1)} = \prod_{k=0}^{j} \frac{(h\wp^{(s-k)}hD_n)}{1-h\wp^{(s-k)}}\delta^{(s-j)}.$$

Hence

$$\left|\prod_{k=0}^{j} \frac{(h\wp^{(s-k)}hD_n)}{1-h\wp^{(s-k)}}\right| < 1$$

if for each k,

$$\left|\frac{(h\wp^{(s-k)}hD_n)}{1-h\wp^{(s-k)}}\right| < 1.$$

Then it follows that $\{Z_{n,1}^{(s+1)}\}$ is a Cauchy sequence provided $0 < 1-hD_n < 4$. Thus as $s \to \infty$, $\{Z_{n,1}^{(s+1)}\} \to Z_{n,1}$. Hence there exists a unique solution $Z_{n,1}$ for (6.6), and the assertion of the theorem follows. □

Corollary 1. *Under the same conditions as in Theorem 1, the equation*

$$z(t_n) = b\left(g(t_n) + \frac{h}{2}k(t_n,t_0)z(t_0) + h\sum_{i=1}^{n-1}k(t_n,t_i)z(t_i) + \frac{h}{2}k(t_n,t_n)z(t_n)\right)^2, \tag{6.12}$$

defines a unique solution to(6.1) *by the repeated trapezoidal rule.*

Proof. Expanding equation (6.12) we get

$$\begin{aligned} z_n - h\left(bg_nk_{n,n} + b\frac{h}{2}k_{n,0}z_0k_{n,n} + bh\sum_{i=1}^{n-1}k_{n,i}z_ik_{n,n} + b\frac{h}{2}(k_{n,n})^2z_n\right)z_n \\ = b\Bigg[g_n^2 + hg_nk_{n,0}z_0 + (\frac{h}{2}k_{n,0}z_0)^2 + \left(\sum_{i=1}^{n-1}k_{n,i}z_i\right)^2 + 2hg_n\sum_{i=1}^{n-1}k_{n,i}z_i \\ + h^2k_{n,0}z_0\sum_{i=1}^{n-1}k_{n,i}z_i\Bigg], \end{aligned} \tag{6.13}$$

where $z_n = z(t_n)$, $g_n = g(t_n)$ and $k_{n,i} = k(t_n,t_i)$ for $i = 0, ..., n$. Let

$$\wp^{(s)} = bg_nk_{n,n} + b\frac{h}{2}k_{n,0}z_0k_{n,n} + bh\sum_{i=1}^{n-1}k_{n,i}z_ik_{n,n} + b\frac{h}{2}(k_{n,n})^2z_n^{(s)}. \tag{6.14}$$

Then we have

$$[1-h\wp^{(s)}]z_n^{(s+1)} = b\Big[g_n^2 + hg_nk_{n,0}z_0 + (\frac{h}{2}k_{n,0}z_0)^2 + \Big(\sum_{i=1}^{n-1}k_{n,i}z_i\Big)^2 + 2hg_n\sum_{i=1}^{n-1}k_{n,i}z_i + h^2k_{n,0}z_0\sum_{i=1}^{n-1}k_{n,i}z_i\Big]. \tag{6.15}$$

Taking

$$z_n^{(s+1)} = Z_{n,1}^{(s+1)}$$

and following the same steps as above we have that as $s \to \infty$, $z_n^{(s+1)} \to z_n$. Hence there exists a unique solution for (6.12). □

Corollary 2. *Applying collocation methods or the repeated trapezoidal rule results in a unique solution for* (1.4).

Proof. Existence and uniqueness for the case $b_1 = 0$, $b_2 \neq 0$ follows from Theorem 1 and corollary 1. For the case $b_1 \neq 0$,$b_2 = 0$ existence and uniqueness for the collocation solution is established in [5]. For the case $b_1 \neq 0$,$b_2 = 0$ the existence and uniqueness of the solution by the repeated trapezoidal rule follows from continuity of g and k. Combining these two cases establishes the exitence and uniqueness of the collocation solution and the solution by the repeated trapezoidal rule for (1.4). □

6.3 Convergence for the numerical methods

6.3.1 Global convergence for the collocation methods

The global convergence for the case where $b_2 = 0$ in (1.4) has been analysed, see [5], thus we will study the case where $b_1 = 0$. We use a procedure analogeous to the one used in [27] to analyse the global convergence of the collocation solution $z_h \in S_0^{-1}(\Pi_N)$.

Theorem 5. *Assume that $k \in C(D)$, $g \in C(I)$ and $z_h \in S_0^{-1}(\Pi_N)$, defined by* (6.3)*, is the collocation solution for* (6.1)*. Then for all sufficiently small $h > 0$, we have*

$$\|z_h - z\|_\infty \leq C(\left\|(1 - \mathcal{P}_h)bg^2\right\|_\infty + \|(\kappa - \alpha\mathcal{P}_h)\beta z\|_\infty) \tag{6.16}$$

and

$$\|z_h - z\|_\infty := \sup_{t \in I}|z_h(t) - z(t)| \leq Ch\|z\|_\infty \tag{6.17}$$

where $\mathcal{P}_h$ is the Lagrange interpolation operator corresponding to the collocation parameter c_1 and the constant C does not depend on h.

Proof. Define βz as follows

$$\beta z = \int_0^t k(t,s)z(s)\,ds.$$

Then the operator formulations for the VIE (6.1) and its collocation equation are given by

$$\begin{cases} z = b\left(g + \beta z\right)^2 \\ z_h = b\mathcal{P}_h\left(g + \beta z_h\right)^2. \end{cases}$$

Based on the solvability of the VIE and its collocation equation, we implement an iterative procedure and obtain

$$\begin{cases} z^{(s+1)} = (1-\kappa\beta)^{-1}bg^2 \\ z_h^{(s+1)} = (1-\alpha\beta\mathcal{P}_h)^{-1}\mathcal{P}_h bg^2 \end{cases}$$

where $\kappa = 2bg\beta + b\beta^2 z^{(s)}$ and $\alpha = 2bg\beta\mathcal{P}_h + b\beta^2\mathcal{P}_h z_h^{(s)}$.

Then the error between z_h and z can be written as

$$\begin{aligned} e_h := z_h - z &= (1-\alpha\mathcal{P}_h\beta)^{-1}\mathcal{P}_h bg^2 - (1-\kappa\beta)^{-1}bg^2 \\ &= (1-\kappa\beta)^{-1}\mathcal{P}_h bg^2 + (1-\alpha\mathcal{P}_h\beta)^{-1}\mathcal{P}_h bg^2 - (1-\kappa\beta)^{-1}(bg^2 - \mathcal{P}_h bg^2) \\ &= (1-\alpha\mathcal{P}_h\beta)^{-1}(\kappa\beta - \alpha\mathcal{P}_h\beta)(1-\kappa\beta)^{-1}\mathcal{P}_h bg^2 + (1-\kappa\beta)^{-1}(bg^2 - \mathcal{P}_h bg^2) \\ &= (1-\alpha\mathcal{P}_h\beta)^{-1}(\kappa\beta - \alpha\mathcal{P}_h\beta)(1-\kappa\beta)^{-1}(\mathcal{P}_h bg^2 - bg^2) + (1-\alpha\mathcal{P}_h\beta)^{-1} \\ &\quad (\kappa\beta - \alpha\mathcal{P}_h\beta)(1-\kappa\beta)^{-1}bg^2 + (1-\kappa\beta)^{-1}(1-\mathcal{P}_h)bg^2 \\ &= (1-\alpha\mathcal{P}_h\beta)^{-1}(\kappa\beta - \alpha\mathcal{P}_h\beta)(1-\kappa\beta)^{-1}(\mathcal{P}_h - 1)bg^2 + (1-\alpha\mathcal{P}_h\beta)^{-1} \\ &\quad (\kappa - \alpha\mathcal{P}_h)\beta z + (1-\kappa\beta)^{-1}(1-\mathcal{P}_h)bg^2, \end{aligned} \tag{6.18}$$

which implies

$$\|z_h - z\|_\infty \leq C(\left\|(1-\mathcal{P}_h)bg^2\right\|_\infty + \|(\kappa - \alpha\mathcal{P}_h)\beta z\|_\infty).$$

From the error estimates of the interpolation $\mathcal{P}_h$, we have

$$\left\|(1-\mathcal{P}_h)bg^2\right\|_\infty \leq Ch\left\|bg^2\right\|_\infty \leq Ch\|z\|_\infty,$$

and with appropriate assumptions on α and κ,

$$\|(\kappa-\alpha\mathcal{P}_h)\beta z\|_\infty \leq Ch\|\beta z\|_\infty \leq Ch\|z\|_\infty,$$

which leads to

$$\|z_h - z\| \leq Ch\|z\|_\infty.$$

□

On the other hand, recall that the collocation solution to (1.4) is given by

$$u_h(t) = \sum_{j=1}^{2} b_j \left(g_j(t) + \int_0^t k_j(t,s)u(s)\,ds \right)^j, \quad t \in X_N. \tag{6.19}$$

Corollary 3. *If the solution* $u_h \in S_0^{-1}(\Pi_N)$ *defined by* (6.19) *is the collocation solution to* (1.4), *then for a sufficiently small* $h > 0$

$$\|u_h - u\|_\infty := \sup_{t\in I}|u_h(t) - u(t)| \leq Ch\|u\|_\infty, \tag{6.20}$$

holds for any set X_N *of collocation point with* $0 \leq c_1 \leq 1$. *The constant* C *depends on the* c_1 *but not on* h.

Proof. From [5] we know that the collocation solution for the VIE

$$y(t) = g(t) + \int_0^t k(t,s)y(s)\,ds, \tag{6.21}$$

satisfies

$$\|y_h - y\|_\infty := \sup_{t\in I}|y_h(t) - y(t)| \leq Ch\|y\|_\infty. \tag{6.22}$$

Hence

$$u_h - u = (y_h - y) + (z_h - z).$$

By the triangle inequality we have

$$\begin{aligned} \| u_h - u \|_\infty &\leq \| (y_h - y) \|_\infty + \| (z_h - z) \|_\infty \\ &\leq Ch\|y\|_\infty + Ch\|z\|_\infty \\ &\leq Ch(\|y\|_\infty + \|z\|_\infty). \end{aligned} \tag{6.23}$$

□

6.3.2 Repeated trapezoidal rule

Consider the solution to (6.1) by repeated trapezoidal rule

$$z(t_n) = b\left(g(t_n) + h\sum_{i=0}^{n} w_i k(t_n, t_i) z(t_i)\right)^2. \tag{6.24}$$

Theorem 6. *The approximate method given by* (6.24), *is convergent and its order of convergence is at least one.*

Proof. Putting $t = t_n$ in (6.1), we have

$$
\begin{aligned}
\mid \varepsilon_n \mid &= |v_h(t_n) - v(t_n)| \\
&= \Bigg| b \Bigg[\left(h \sum_{i=0}^{n} 2g(t_n) w_i k(t_n, t_i) v(t_i) \right) - \left(2g(t_n) \int_0^{t_n} k(t_n, s) v(s)\, ds \right) \\
&\qquad + \left(h \sum_{i=0}^{n} w_i k(t_n, t_i) v(t_i) \right)^2 - \left(\int_0^{t_n} k(t_n, s) v(s)\, ds \right)^2 \Bigg] \Bigg| \\
&= \Bigg| b \Bigg[\left(h \sum_{i=0}^{n} 2g(t_n) w_i k(t_n, t_i) v(t_i) \right) - \left(2g(t_n) \int_0^{t_n} k(t_n, s) v(s)\, ds \right) \\
&\qquad + \left(h \sum_{i=0}^{n} w_i k(t_n, t_i) v(t_i) - \int_0^{t_n} k(t_n, s) v(s)\, ds \right) \\
&\qquad \cdot \left(h \sum_{i=0}^{n} w_i k(t_n, t_i) v(t_i) + \int_0^{t_n} k(t_n, s) v(s)\, ds \right) \Bigg] \Bigg|
\end{aligned}
$$

$$
\begin{aligned}
= \Bigg| b \Bigg[& \left(h \sum_{i=0}^{n} 2g(t_n) w_i k(t_n, t_i) v(t_i) \right) - \left(2g(t_n) \int_0^{t_n} k(t_n, s) v(s)\, ds \right) \\
& + \left(h \sum_{i=0}^{n} w_i k(t_n, t_i) v(t_i) - \int_0^{t_n} k(t_n, s) v(s)\, ds \right) \\
& \cdot \Bigg(h \sum_{i=0}^{n} w_i k(t_n, t_i) v(t_i) - \int_0^{t_n} k(t_n, s) v(s)\, ds \\
& \qquad + 2 \int_0^{t_n} k(t_n, s) v(s)\, ds \Bigg) \Bigg] \Bigg|.
\end{aligned} \tag{6.25}
$$

Using the Lipschitz condition, (6.25) can be written as

$$
\mid \varepsilon_n \mid \leq hA_1 \sum_{i=0}^{n} \mid \varepsilon_i \mid + |R_{i,1}| + \left(hA_2 \sum_{i=0}^{n} \mid \varepsilon_i \mid + |R_{i,1}| \right) \left(hA_2 \sum_{i=0}^{n} \mid \varepsilon_i \mid + |R_{i,2}| \right), \tag{6.26}
$$

where $R_{i,1}$ and $R_{i,2}$ are the errors of the integration rule.

Then, let $R = \max_i[R_{i,1}, R_{i,2}]$ hence

$$
\begin{aligned}
\mid \varepsilon_n \mid \leq \frac{hA_1}{1-hA_1}\sum_{i=0}^{n-1}\mid \varepsilon_i \mid + \frac{R}{1-hA_1} + \left(\frac{hA_2}{1-hA_2}\sum_{i=0}^{n-1}\mid \varepsilon_i \mid + \frac{R}{1-hA_2}\right) \\
\cdot \left(\frac{hA_2}{1-hA_2}\sum_{i=0}^{n-1}\mid \varepsilon_i \mid + \frac{R}{1-hA_2}\right).
\end{aligned} \tag{6.27}
$$

Then we have

$$
\begin{aligned}
\mid \varepsilon_n \mid \leq \frac{1}{1-hA_1}\left\{R + hA_1\sum_{i=0}^{n-1}\mid \varepsilon_i \mid\right\} e^{(A_1 t_n/(1-A_1))} \\
+ \left(\frac{1}{1-hA_2}\left\{R + hA_2\sum_{i=0}^{n-1}\mid \varepsilon_i \mid\right\} e^{(A_2 t_n/(1-A_2))}\right) \\
\cdot \left(\frac{1}{1-hA_2}\left\{R + hA_2\sum_{i=0}^{n-1}\mid \varepsilon_i \mid\right\} e^{(A_2 t_n/(1-A_2))}\right).
\end{aligned} \tag{6.28}
$$

For the functions k and g with at least first order derivatives, we have $R = O(h)$. Hence we have

$$
\begin{aligned}
\mid \varepsilon_n \mid &= O(h) + O(h)O(h) \\
&= O(h) + O(h^2) \\
&= O(h).
\end{aligned} \tag{6.29}
$$

□

Corollary 4. *The repeated trapezoidal solution for* (1.4) *defined by*

$$
u(t_n) = \sum_{j=1}^{2} b\left(g(t_n) + h\sum_{i=0}^{n} w_i k(t_n, t_i)u(t_i)\right)^2. \tag{6.30}
$$

is convergent and its order of convergence is at least one.

Proof. Since $\| \varepsilon_n \|_\infty = O(h)$ for the repeated trapezoidal rule when used to solve (6.1) and (6.21), it follows that for (1.4) we have

$$\begin{aligned} \| \varepsilon_n \|_\infty &= O(h) + O(h) \\ &= O(h). \end{aligned} \tag{6.31}$$

□

6.4 Numerical results

In this section we present numerical results obtained from the collocation methods and the repeated trapezoidal rule. To obtain the estimates for the convergence rates we used the following quantity

$$p = \frac{\log \left| \frac{u^{h/2} - u^h}{u^{h/4} - u^{h/2}} \right|}{\log 2}, \tag{6.32}$$

where u^h, $u^{h/2}$ and $u^{h/4}$ denote approximations to $u(t)$ using the step sizes h, $h/2$ and $h/4$. The results are shown in table 6.1-6.3, which indicates first order convergence and this is in agreement with the results of the theorems in previous sections. The approximations for the convergence rates are done using results from the following examples

Example 1

$$u(t) = 2\left(1 + \int_0^t (t-s)u(s)\, ds\right)^2 \qquad 0 \le t \le 1. \tag{6.33}$$

Example 2

$$u(t) = \left(1 + \int_0^t (t-s)u(s)\,ds\right) + \frac{1}{2}\left(1 + \int_0^t (t-s)u(s)\,ds\right)^2 \qquad 0 \le t \le 1. \tag{6.34}$$

Table 6.1: Convergence rates for several values of h using example 1

	Implicit euler		Implicit midpoint	
t_n	0.1	0.02	0.1	0.02
0.2	1.2755	1.0544	1.2807	1.0495
0.4	1.2534	1.0498	1.0944	1.01355
0.6	1.3065	1.0583	1.0990	1.0162
0.8	1.4018	1.0721	1.1195	1.0211
1.0	1.4433	1.0919	1.1497	1.03141

Table 6.2: Convergence rates for several values of h using example 2

	Implicit euler		Implicit midpoint	
t_n	0.1	0.02	0.1	0.02
0.2	1.2475	1.0874	1.1220	1.0000
0.4	1.1765	1.0302	1.0870	1.0211
0.6	1.1711	0.9668	1.07625	1.0368
0.8	1.1869	1.0332	1.0693	1.0075
1.0	1.2179	1.0405	1.0711	1.0231

Table 6.3: Convergence rates for the repeated trapezoidal rule using several values of h

	Example 1		Example 2	
t_n	0.1	0.02	0.1	0.02
0.2	1.0067	1.0000	0.9953	1.0000
0.4	1.0148	1.0137	1.0042	1.0107
0.6	1.0320	1.0115	1.0098	1.0076
0.8	1.0463	1.0137	1.0178	1.0076
1.0	1.0627	1.0191	1.0254	1.0093

6.5 Discussion

In this chapter we provided sufficient conditions for the existence and uniqueness of the collocation solution and solution by the repeated trapezoidal rule for (6.1). In theorem 5 we proved that the collocation methods yield global convergence of order one. We also proved in theorem 6 that the repeated trapezoidal rule has convergence of order one. From results in table 6.1-6.3, we observe that the numerical approximations of the convergence orders of the implicit Euler, implicit midpoint and repeated trapezoidal rule are in agreement with the results from theorem 5 and 6. Brunner [5] showed that when collocation methods have convergence order of m when used to solve linear VIE's of the second kind.

Chapter 7

Conclusion

In this work, we studied a class of nonlinear (nonstandard) VIE. We proved the existence and uniqueness of the solution for the nonlinear VIE using the contraction mapping theorem. Approximate solutions to the nonlinear VIE (1.3) were computed using one-point collocation methods and quadrature rules.

When constructing the one-point collocation solution for the nonlinear VIE (1.3) we used a uniform mesh. We concentrated on two cases: $c_1 = 1$ (implicit Euler method) and $c_1 = \frac{1}{2}$ (implicit midpoint method), see [5]. We compared the results of these methods and conclude that the implicit midpoint method performs better than the implicit Euler method. The implicit midpoint method gives better solutions when a reasonably large value of the step size is used. We also constructed an iterated collocation solution for the case when $c_1 = \frac{1}{2}$, and we obtained better solutions compared to the implicit midpoint method.

We then used the repeated trapezoidal rule and repeated Simpson's rule to approximate the solution for (1.3), see [2,24]. From the results we conclude that the repeated Simpson's rule performs better than repeated trapezoidal rule as expected. Infact, the repeated Simpson's rule gave better results compared to all the other methods we used to solve the nonlinear VIE (1.3).

Futhermore, we performed a numerical analysis of the solution to the nonlinear VIE (1.4) obtained by the collocation methods and repeated trapezoidal rule. We provided sufficient conditions for the existence and uniqueness for the numerical solution to (1.4). We performed a convergence analysis, and proved that the one-point collocation methods yield global convergence of one; similarly and the repeated trapezoidal rule has convergence of order one. The computed orders of convergence in chapter 6 are in agreement with the theoretical results.

We only analysed the numerical solutions to the nonlinear VIE obtained by the one-point collocation methods and repeated trapezoidal rule. For future work a numerical analysis of the solution obtained by the repeated Simpson's rule can be done. Also other numerical methods can be used to solve the nonlinear VIE (1.3).

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Appendix A

Matlab codes for solving the nonlinear VIE

A Code for the implicit Euler method

```
h=0.0025;                  % h   the step size
c1=1;                      % c1  the collocation parameter
a=0;b=1;                   % a = t0 = tinitial and b = tfinal
n=(b-a)/h;                 % n   number of steps
t=a:h:b;
tol=10^(-4)                % tol is the desired accuracy
U(1)=2;                    % U(1) = U(t0) initial value
z=2;
for j=2:n+1
    s=0;
    % calculate the sum
```

```
    for i=1:j-1
        s=s + U(i)*(t(j)-t(i)+h*(c1- 1/2));
    end
    % calculate U(j) by performing iterations at each step
    w= 2*(1+h*s + z *(c1^2*h^2)/2)^2;
    while abs(w-z)>tol
        w=z;
        z =2*(1+h*s + w *(c1^2*h^2)/2)^2;
     end
    U(j)=w;
    disp([t(j)  U(j)]);
end
```


B Code for the implicit midpoint method

```
h=0.0025;                % h   the step size
c1=1/2;                  % c1  the collocation parameter
a=0;b=1;                 % a = tinitial and b = tfinal
n=(b-a)/h;               % n   number of steps
t=a:h:b;
tol=10^(-4);              % tol is the desired accuracy
U(1)=2;                   % U(1) = U(t0) initial value
z=2;
for j=2:n+1
```

```
    s=0;
    % calculate the sum
    for i=1:j-1
        s=s + U(i)*(t(j)-t(i)+h*(c1- 1/2));
    end
    % calculate U(j) by performing iterations at each
    %step
     w= 2*(1+h*s + z *((c1^2)*h^2)/2)^2;
    while abs(w-z)>tol
        w=z;
        z =2*(1+h*s + w *((c1^2)*h^2)/2)^2;
     end
    U(j)=w;
    disp([t(j)  U(j)]);
end
```

C Code for the repeated trapezoidal rule

```
h=0.0025;                 % h  the step size
a=0;b=1;                  % a = tinitial and b = tfinal
n=(b-a)/h;                % n   number of steps
t=a:h:b;
U(1)=2;                    % U(1) = U(t0) initial value
for i=2:n+1
```

```
    % calculate U(i) using the repeated trapezoidal rule
    % at each step
    v=0;
    for j=1:i-1
        v = v + (t(i)-t(j))*U(j);
    end
    U(i)= 2*( 1 + (h/2)*(t(i)-t(1))*U(1)+ h*v )^2;
    disp([t(i) U(i)]);
end
```

D Code for the repeated Simpson's rule

```
h=0.0025;               % h the step size
a=0;b=1;                % a = t0 = tinitial and b = tfinal
n=(b-a)/h;              % n   number of steps
t=a:h:b;
m(1)=t(1);              % t(1) = tinitial
tol=10^(-4);            % tol is the desired accuracy
U(1)=2;                 % U(1) = U(t0) initial value
s(1)=U(1);
z=2;
for j=1:n/2
    % calculate u(t_2m)=U(2*j+1) using repeated Simpson's
    % rule in the even nodes
       v=0;
```

```
    for i=1:j-1
     v= v + (3*t(2*j+1)-t(2*i+2)-t(2*i+1)-t(2*i))*U(2*i+1);
    end
    w=2*(1 + h/3 *U(1)*(3*t(2*j+1)-2*t(2)) + h/3 *(2*t(2*j+1)
           -2*t(2*j))* z +(2/3)*h*v)^2;
   while abs(w-z)>tol
     w=z;
     z = 2*(1 + h/3 *U(1)*(3*t(2*j+1)-2*t(2))+ h/3
              *(2*t(2*j+1)-2*t(2*j))* w +(2/3)*h*v)^2;
   end
   U(2*j+1)=w;
  s(j+1)= U(2*j+1);
  m(j+1)=t(2*j+1);
  disp([m(j+1) s(j+1)])
 end
```