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Exercise 1

3D Computer Vision

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Part I
Theory

1 Properties of Rotation Matrices

1. Let $\mathbf{R}_{i\cdot}, \mathbf{R}_{\cdot j}$ denote i-th row and j-th column of $\mathbf{R}$ separately, where $i \neq j$.

   $\mathbf{R}$ is orthogonal matrix
   $\Rightarrow \mathbf{R} \cdot \mathbf{R}^T = \mathbf{I}$
   $\Rightarrow \mathbf{R}_{i\cdot} \cdot \mathbf{R}_{\cdot j}^T = 0, (i \neq j)$
   $\Rightarrow \mathbf{R}_{i\cdot}, \mathbf{R}_{\cdot j}^T$ are orthogonal
   since $\mathbf{R}_{\cdot j}^T = \mathbf{R}_{j\cdot}$,
   therefore, $\mathbf{R}_{i\cdot} \cdot \mathbf{R}_{j\cdot} = 0, (i \neq j)$ (rows orthogonal)
   same, based on $\mathbf{R}^T \cdot \mathbf{R} = \mathbf{I}$ we have columns orthogonal.

   $\mathbf{R} \cdot \mathbf{R}^T = \mathbf{I}$
   $\Rightarrow \mathbf{R}_{i\cdot} \cdot \mathbf{R}_{\cdot i}^T = 1$
   since $\mathbf{R}_{\cdot i}^T = \mathbf{R}_{i\cdot}$,
   thus $\mathbf{R}_{i\cdot} \cdot \mathbf{R}_{i\cdot} = 1$
   $\Rightarrow \|\mathbf{R}_{i\cdot}\| = 1 \Rightarrow$ each row of $\mathbf{R}$ has length 1.
   same, based on $\mathbf{R}^T \cdot \mathbf{R} = \mathbf{I}$ we can prove each column of $\mathbf{R}$ has length 1. $\qquad \square$

2. • Rotation matrix by ψ around z axis is

$$\mathbf{R}_z(\psi) = \begin{bmatrix} \cos\psi & -\sin\psi & 0 \\ \sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

   the inverse is

$$\mathbf{R}_z^{-1}(\psi) = \begin{bmatrix} \cos\psi & \sin\psi & 0 \\ -\sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{bmatrix} = \mathbf{R}_z^T(\psi),$$

   And also,

$$\det(\mathbf{R}_z) = \cos\psi \cdot \cos\psi - \sin\psi \cdot (-\sin\psi) + 0 = 1$$

   Thus $\mathbf{R}_z(\psi)$ fulfills the properties of rotation matrices for any angles ψ.

   • Rotation matrix by θ around y axis is

$$\mathbf{R}_y(\theta) = \begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix},$$

   the inverse is

$$\mathbf{R}_y^T(\theta) = \begin{bmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{bmatrix} = \mathbf{R}_y^{-1}(\theta),$$

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And also,
$$\det(\mathbf{R}_y) = \cos\theta \cdot \cos\theta + 0 + \sin\theta \cdot (0 - (-\sin\theta)) = 1$$

Thus $\mathbf{R}_y(\theta)$ fulfills the properties of rotation matrices for any angles θ.

- Rotation matrix by ϕ around x axis is
$$\mathbf{R}_x(\phi) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & -\sin\phi \\ 0 & \sin\phi & \cos\phi \end{bmatrix},$$

the inverse is
$$\mathbf{R}_x(\phi) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & \sin\phi \\ 0 & -\sin\phi & \cos\phi \end{bmatrix} = \mathbf{R}_x^{-1}(\phi),$$

And also,
$$\det(\mathbf{R}_x) = \cos\phi \cdot \cos\phi - (-\sin\phi) \cdot \sin\phi + 0 + 0 = 1$$

Thus $\mathbf{R}_x(\phi)$ fulfills the properties of rotation matrices for any angles ϕ.

- Since
$$\mathbf{R} = \mathbf{R}_z(\psi)\mathbf{R}_y(\theta)\mathbf{R}_x(\phi)$$

Thus,
$$\begin{aligned} \mathbf{R}^{-1} &= (\mathbf{R}_z(\psi)\mathbf{R}_y(\theta)\mathbf{R}_x(\phi))^{-1} \\ &= \mathbf{R}_x^{-1}\mathbf{R}_y^{-1}\mathbf{R}_z^{-1} \\ &= \mathbf{R}_x^T\mathbf{R}_y^T\mathbf{R}_z^T \\ &= (\mathbf{R}_z\mathbf{R}_y\mathbf{R}_x)^T \\ &= \mathbf{R}^T \end{aligned}$$

$$\det(\mathbf{R}) = \det(\mathbf{R}_z\mathbf{R}_y\mathbf{R}_x) = \det(\mathbf{R}_z)\det(\mathbf{R}_y)\det(\mathbf{R}_x) = 1 \cdot 1 \cdot 1 = 1$$

$\square$

3. The geometric interpretation of the determinant of a square 3×3 matrix is the scale of the matrix. Rotation matrix have to have determinant 1, because the rotation operation is performed by multiple original object with rotation matrix, which should not change the scale of the object, thus it has to be with determinant 1.

2   Transformation Chain

The transformation chain is

$$\text{World coordinate} \begin{pmatrix} X_s \\ Y_s \\ Z_s \end{pmatrix} \rightarrow \text{Camera coordinate} \begin{pmatrix} x_s \\ y_s \\ z_s \end{pmatrix} \rightarrow \text{pixel coordinate} \begin{pmatrix} x_{pix} \\ y_{pix} \end{pmatrix}$$

$$\begin{pmatrix} u\prime \\ v\prime \\ w\prime \end{pmatrix} = K(I_3|0_3) \begin{pmatrix} x_s \\ y_s \\ z_s \\ 1 \end{pmatrix},$$

where

$$\begin{pmatrix} x_s \\ y_s \\ z_s \end{pmatrix} = \begin{pmatrix} t_x \\ t_y \\ t_z \end{pmatrix} + \mathbf{R} \begin{pmatrix} X_s \\ Y_s \\ Z_s \end{pmatrix}$$

Pixel coordinate

$$\begin{cases} x_{pix} = \frac{u\prime}{w\prime} \\ y_{pix} = \frac{v\prime}{w\prime} \end{cases}$$

Part II
Implementation