

MPPDC Derivation

Aleksis Kazubiernis Xenophon

November 19, 2018

Indices

i	Node index
n	Reference node index
k	Edge index
g	Generator index
t	Trading interval index
p	The first element (from node) for AC edge k
q	The last element (to node) for AC edge k

Sets

Ω^I	Nodes in the network
Ω^N	Set of reference nodes
Ω^K	AC network edges
Ω^M	HVDC network edges
Ω^T	Trading intervals
Ω^G	Generators
Ω_i^G	Generators connected to node i
Ω_i^K	AC edges connected to node i
Ω^Θ	Upper-level program primal variables
Ω^Λ	Lower-level program primal variables
Ω^Ξ	Lower-level program dual variables

Primal variables for upper-level program

τ	Permit price [\$/tCO ₂]
ϕ	EmiSubions intensity baseline [tCO ₂ /MWh]

Primal variables for lower-level program

$p_{g,t}$	Power output of generator g during interval t [MW]
$\theta_{i,t}$	Voltage angle (with respect to reference node) at node i for trading interval t [rad]
$h_{m,t}$	Power flow over HVDC link m during period t [MW]

Dual variables

$\alpha_{g,t}$	Minimum generation constraint dual variable
$\beta_{g,t}$	Maximum generation constraint dual variable
$\gamma_{pq,t}$	Lower voltage angle difference constraint dual variable for branch k connecting nodes i and j
$\delta_{pq,t}$	Upper voltage angle difference constraint dual variable for branch k connecting nodes i and j
$\zeta_{i,t}$	Reference node voltage angle constraint dual variable for $i \in \Omega^N$
$\lambda_{i,t}$	Power balance constraint dual variable at node i during period t
$\kappa_{pq,t}$	Minimum power flow constraint dual variable for flow between nodes i and j during period t
$\eta_{pq,t}$	Maximum power flow constraint dual variable for flow between nodes i and j during period t
$\omega_{m,t}$	Power flow lower bound dual variable for HVDC link m
$\psi_{m,t}$	Power flow upper bound dual variable for HVDC link m

Parameters

L_t	Duration of trading interval t [hrs]
$D_{i,t}$	Power demand at node i during period t [MW]
$Y_{i,t}$	Fixed power injection at node i during period t [MW]
Z	Target electricity price [\$/MWh]
E_g	Emissions intensity of generator g [tCO ₂ /MWh]
R	Scheme revenue constraint [\\$]
X	Emissions intensity constraint [tCO ₂ /MWh]
\bar{P}_g	Maximum output for generator g [MW]
\underline{P}_g	Minimum output for generator g [MW]
$\bar{\theta}$	Maximum voltage angle difference between buses i and j [rad]
$\underline{\theta}$	Minimum voltage angle difference between buses i and j [rad]

B_{pq}	Susceptance between nodes i and j [pu]
\underline{H}_m	Lower bound for power flow from node i to node j for HVDC link m [MW]
\overline{H}_m	Upper bound for power flow from node i to node j for HVDC link m [MW]
U_{in}	Matrix describing network reference nodes
C_{mi}	HVDC incidence matrix

Mapping functions

$f(g)$	Maps generator g to the node at which it is located
--------	---

Table 1: Model notation

1 Introduction

This document describes the reformulation procedure used to convert the policy maker’s bi-level program into a single level program. This is accomplished by converting the lower-level optimisation problem into a system of equations. The reformulation strategy involves constructing the dual of the primal problem, and then combining constraints of the primal and dual programs, and adding a strong duality constraint. The resulting system of equations is known as a Mathematical Program with Primal and Dual Constraints (MPPDC).

2 Bi-level formulation

The policy maker's bi-level optimisation program is as follows:

$$\begin{aligned}
& \min_{\Omega^\Theta, \Omega^\Lambda, \Omega^\Xi} \left\| \frac{\sum_{t \in \Omega^T} \sum_{i \in \Omega^I} \lambda_{i,t} L_t D_{i,t}}{\sum_{t \in \Omega^T} \sum_{i \in \Omega^I} L_t D_{i,t}} - Z \right\|_1 \\
& \text{s.t.} \\
& \sum_{t \in \Omega^T} \sum_{g \in \Omega^G} (E_g - \phi) \tau p_{g,t} L_t \geq R \\
& \frac{\sum_{t \in \Omega^T} \sum_{g \in \Omega^G} E_g p_{g,t} L_t}{\sum_{t \in \Omega^T} \sum_{g \in \Omega^G} D_{i,t} L_t} \leq X \\
& \min_{\Lambda_t} \sum_{g \in \Omega^G} [A_g + (E_g - \phi) \tau] p_{g,t} \\
& \text{s.t.} \\
& \underline{P}_g - p_{g,t} \leq 0 \quad (\alpha_{g,t}) \quad \forall g \in \Omega^G \\
& p_{g,t} - \bar{P}_g \leq 0 \quad (\beta_{g,t}) \quad \forall g \in \Omega^G \\
& \underline{\theta} - \theta_{p,t} + \theta_{q,t} \leq 0 \quad (\gamma_{pq,t}) \quad \forall k \in \Omega^K \\
& \theta_{p,t} - \theta_{q,t} - \bar{\theta} \leq 0 \quad (\delta_{pq,t}) \quad \forall k \in \Omega^K \\
& \theta_i = 0 \quad (\zeta_{i,t}) \quad \forall i \in \Omega^N \\
& S \sum_{j \in \Omega^I} B_{ij} (\theta_{i,t} - \theta_{j,t}) + \sum_{m \in \Omega^M} C_{mi} h_{m,t} + D_{i,t} - Y_{i,t} - \sum_{g \in \Omega_i^G} p_{g,t} = 0 \quad (\lambda_{i,t}) \quad \forall i \in \Omega^I \\
& \underline{F}_k - S B_{pq} (\theta_{p,t} - \theta_{q,t}) \leq 0 \quad (\kappa_{pq,t}) \quad \forall k \in \Omega^K \\
& S B_{pq} (\theta_{p,t} - \theta_{q,t}) - \bar{F}_k \leq 0 \quad (\eta_{pq,t}) \quad \forall k \in \Omega^K \\
& \underline{H}_m - h_{m,t} \leq 0 \quad (\omega_{m,t}) \quad \forall m \in \Omega^M \\
& h_{m,t} - \bar{H}_m \leq 0 \quad (\psi_{m,t}) \quad \forall m \in \Omega^M
\end{aligned} \quad \left. \vphantom{\begin{aligned} \min_{\Lambda_t} \sum_{g \in \Omega^G} [A_g + (E_g - \phi) \tau] p_{g,t} \\ \text{s.t.} \\ \underline{P}_g - p_{g,t} \leq 0 \quad (\alpha_{g,t}) \quad \forall g \in \Omega^G \\ p_{g,t} - \bar{P}_g \leq 0 \quad (\beta_{g,t}) \quad \forall g \in \Omega^G \\ \underline{\theta} - \theta_{p,t} + \theta_{q,t} \leq 0 \quad (\gamma_{pq,t}) \quad \forall k \in \Omega^K \\ \theta_{p,t} - \theta_{q,t} - \bar{\theta} \leq 0 \quad (\delta_{pq,t}) \quad \forall k \in \Omega^K \\ \theta_i = 0 \quad (\zeta_{i,t}) \quad \forall i \in \Omega^N \\ S \sum_{j \in \Omega^I} B_{ij} (\theta_{i,t} - \theta_{j,t}) + \sum_{m \in \Omega^M} C_{mi} h_{m,t} + D_{i,t} - Y_{i,t} - \sum_{g \in \Omega_i^G} p_{g,t} = 0 \quad (\lambda_{i,t}) \quad \forall i \in \Omega^I \\ \underline{F}_k - S B_{pq} (\theta_{p,t} - \theta_{q,t}) \leq 0 \quad (\kappa_{pq,t}) \quad \forall k \in \Omega^K \\ S B_{pq} (\theta_{p,t} - \theta_{q,t}) - \bar{F}_k \leq 0 \quad (\eta_{pq,t}) \quad \forall k \in \Omega^K \\ \underline{H}_m - h_{m,t} \leq 0 \quad (\omega_{m,t}) \quad \forall m \in \Omega^M \\ h_{m,t} - \bar{H}_m \leq 0 \quad (\psi_{m,t}) \quad \forall m \in \Omega^M \end{aligned}} \right\} \quad \forall t \in \Omega^T$$

3 Lagrangian for lower level problems

The Lagrangian for each lower-level optimisation problem is as follows:

$$\begin{aligned}
\mathcal{L}(\Omega^\Lambda, \Omega^\Xi) = & \sum_{g \in \Omega^G} [A_g + (E_g - \phi)\tau] p_{g,t} + \alpha_{g,t}(\underline{P}_g - p_{g,t}) + \beta_{g,t}(p_{g,t} - \overline{P}_g) \\
& + \sum_{k \in \Omega^K} \gamma_{pq,t}(\underline{\theta} - \theta_{p,t} + \theta_{q,t}) + \delta_{pq,t}(\theta_{p,t} - \theta_{q,t} - \overline{\theta}) + \kappa_{pq,t} [\underline{F}_k - SB_{pq}(\theta_{p,t} - \theta_{q,t})] + \eta_{pq,t} [SB_{pq}(\theta_{p,t} - \theta_{q,t}) - \overline{F}_k] \\
& + \sum_{m \in \Omega^M} \omega_{m,t} [\underline{H}_m - h_{m,t}] + \psi_{m,t} [h_{m,t} - \overline{H}_m] \\
& + \sum_{i \in \Omega^I} \lambda_{i,t} \left[S \sum_{j \in \Omega^I} B_{ij}(\theta_{i,t} - \theta_{j,t}) + \sum_{m \in \Omega^M} C_{mi} h_{m,t} + D_{i,t} - Y_{i,t} - \sum_{g \in \Omega_i^G} p_{g,t} \right] \\
& + \sum_{i \in \Omega^N} \zeta_{i,t} \theta_{i,t}
\end{aligned} \tag{1}$$

3.1 Re-writing terms

Some terms in the Lagrangian must be re-written, in particular voltage angles indexed by j and q . Using a four node network, we verify that the original and re-formulated terms match.

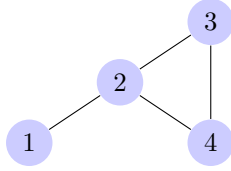


Figure 1: 4 node network

The set of edges is given by: $K = \{(1, 2), (2, 3), (2, 4), (3, 4)\}$

3.1.1 Re-writing: $\sum_{k \in \Omega^K} \gamma_{pq,t} [\underline{\theta} - \theta_{p,t} + \theta_{q,t}] = \sum_{i \in \Omega^I} \theta_{i,t} \sum_{k \in \Omega_i^K} \gamma_{pq,t} [\text{sign}(i - p) + \text{sign}(i - q)] + \underline{\theta} \sum_{k \in \Omega^K} \gamma_{pq,t}$

LHS (original) expression:

$$\begin{aligned}
& \sum_{k \in \Omega^K} \gamma_{pq,t} [\underline{\theta} - \theta_{p,t} + \theta_{q,t}] \\
& = \gamma_{12,t} [\underline{\theta} - \theta_{1,t} + \theta_{2,t}] + \gamma_{23,t} [\underline{\theta} - \theta_{2,t} + \theta_{3,t}] + \gamma_{24,t} [\underline{\theta} - \theta_{2,t} + \theta_{4,t}] + \gamma_{34,t} [\underline{\theta} - \theta_{3,t} + \theta_{4,t}] \\
& = \gamma_{12,t} \underline{\theta} - \gamma_{12,t} \theta_{1,t} + \gamma_{12,t} \theta_{2,t} + \gamma_{23,t} \underline{\theta} - \gamma_{23,t} \theta_{2,t} + \gamma_{23,t} \theta_{3,t} + \gamma_{24,t} \underline{\theta} - \gamma_{24,t} \theta_{2,t} + \gamma_{24,t} \theta_{4,t} + \gamma_{34,t} \underline{\theta} - \gamma_{34,t} \theta_{3,t} + \gamma_{34,t} \theta_{4,t} \\
& = \theta_{1,t} [-\gamma_{12,t}] + \theta_{2,t} [\gamma_{12,t} - \gamma_{23,t} - \gamma_{24,t}] + \theta_{3,t} [\gamma_{23,t} - \gamma_{34,t}] + \theta_{4,t} [\gamma_{24,t} + \gamma_{34,t}] + \underline{\theta} [\gamma_{12,t} + \gamma_{23,t} + \gamma_{24,t} + \gamma_{34,t}]
\end{aligned} \tag{2}$$

RHS (reformulated) expression:

$$\begin{aligned}
& \sum_{i \in \Omega^I} \theta_{i,t} \sum_{k \in \Omega_i^K} \gamma_{pq,t} [\text{sign}(i - p) + \text{sign}(i - q)] + \underline{\theta} \sum_{k \in \Omega^K} \gamma_{pq,t} \\
& = \theta_{1,t} [\gamma_{12,t} [\text{sign}(1 - 1) + \text{sign}(1 - 2)]] \\
& + \theta_{2,t} [\gamma_{12,t} [\text{sign}(2 - 1) + \text{sign}(2 - 2)] + \gamma_{23,t} [\text{sign}(2 - 2) + \text{sign}(2 - 3)] + \gamma_{24,t} [\text{sign}(2 - 2) + \text{sign}(2 - 4)]] \\
& + \theta_{3,t} [\gamma_{23,t} [\text{sign}(3 - 2) + \text{sign}(3 - 3)] + \gamma_{34,t} [\text{sign}(3 - 3) + \text{sign}(3 - 4)]] \\
& + \theta_{4,t} [\gamma_{24,t} [\text{sign}(4 - 2) + \text{sign}(4 - 4)] + \gamma_{34,t} [\text{sign}(4 - 3) + \text{sign}(4 - 4)]] \\
& + \underline{\theta} [\gamma_{12,t} + \gamma_{23,t} + \gamma_{24,t} + \gamma_{34,t}] \\
& = \theta_{1,t} [-\gamma_{12,t}] + \theta_{2,t} [\gamma_{12,t} - \gamma_{23,t} - \gamma_{24,t}] + \theta_{3,t} [\gamma_{23,t} - \gamma_{34,t}] + \theta_{4,t} [\gamma_{24,t} - \gamma_{34,t}] + \underline{\theta} [\gamma_{12,t} + \gamma_{23,t} + \gamma_{24,t} + \gamma_{34,t}]
\end{aligned} \tag{3}$$

Note that the expanded expressions for (2) and (3) match.

3.1.2 Re-writing: $\sum_{k \in \Omega^K} \delta_{pq,t} [\theta_{p,t} - \theta_{q,t} - \bar{\theta}] = \sum_{i \in \Omega^I} \theta_{i,t} \sum_{k \in \Omega_i^K} -\delta_{pq,t} [\text{sign}(i-p) + \text{sign}(i-q)] - \bar{\theta} \sum_{k \in \Omega^K} \delta_{pq,t}$

LHS (original) expression:

$$\begin{aligned}
& \sum_{k \in \Omega^K} \delta_{pq,t} [\theta_{p,t} - \theta_{q,t} - \bar{\theta}] \\
&= \delta_{12,t} [\theta_{1,t} - \theta_{2,t} - \bar{\theta}] + \delta_{23,t} [\theta_{2,t} - \theta_{3,t} - \bar{\theta}] + \delta_{24,t} [\theta_{2,t} - \theta_{4,t} - \bar{\theta}] + \delta_{34,t} [\theta_{3,t} - \theta_{4,t} - \bar{\theta}] \\
&= \delta_{12,t} \theta_{1,t} - \delta_{12,t} \theta_{2,t} - \delta_{12,t} \bar{\theta} + \delta_{23,t} \theta_{2,t} - \delta_{23,t} \theta_{3,t} - \delta_{23,t} \bar{\theta} + \delta_{24,t} \theta_{2,t} - \delta_{24,t} \theta_{4,t} - \delta_{24,t} \bar{\theta} + \delta_{34,t} \theta_{3,t} - \delta_{34,t} \theta_{4,t} - \delta_{34,t} \bar{\theta} \\
&= \theta_{1,t} \delta_{12,t} + \theta_{2,t} [-\delta_{12,t} + \delta_{23,t} + \delta_{24,t}] + \theta_{3,t} [-\delta_{23,t} + \delta_{34,t}] + \theta_{4,t} [-\delta_{24,t} - \delta_{34,t}] - \bar{\theta} [\delta_{12,t} + \delta_{23,t} + \delta_{24,t} + \delta_{34,t}]
\end{aligned} \tag{4}$$

RHS (reformulated) expression

$$\begin{aligned}
& \sum_{i \in \Omega^I} \theta_{i,t} \sum_{k \in \Omega_i^K} -\delta_{pq,t} [\text{sign}(i-p) + \text{sign}(i-q)] - \bar{\theta} \sum_{k \in \Omega^K} \delta_{pq,t} \\
&= \theta_{1,t} [-\delta_{12,t} [\text{sign}(1-1) + \text{sign}(1-2)]] \\
&+ \theta_{2,t} [-\delta_{12,t} [\text{sign}(2-1) + \text{sign}(2-2)] - \delta_{23,t} [\text{sign}(2-2) + \text{sign}(2-3)] - \delta_{24,t} [\text{sign}(2-2) + \text{sign}(2-4)]] \\
&+ \theta_{3,t} [-\delta_{23,t} [\text{sign}(3-2) + \text{sign}(3-3)] + -\delta_{34,t} [\text{sign}(3-3) + \text{sign}(3-4)]] \\
&+ \theta_{4,t} [-\delta_{24,t} [\text{sign}(4-2) + \text{sign}(4-4)] + -\delta_{34,t} [\text{sign}(4-3) + \text{sign}(4-4)]] \\
&- \bar{\theta} [\delta_{12,t} + \delta_{23,t} + \delta_{24,t} + \delta_{34,t}] \\
&= \theta_{1,t} \delta_{12,t} + \theta_{2,t} [-\delta_{12,t} + \delta_{23,t} + \delta_{24,t}] + \theta_{3,t} [-\delta_{23,t} + \delta_{34,t}] + \theta_{4,t} [-\delta_{24,t} - \delta_{34,t}] - \bar{\theta} [\delta_{12,t} + \delta_{23,t} + \delta_{24,t} + \delta_{34,t}]
\end{aligned} \tag{5}$$

Note that the expanded expressions for (4) and (5) match.

3.1.3 Re-writing: $\sum_{k \in \Omega^K} \kappa_{pq,t} [\underline{F}_k - SB_{pq}(\theta_{p,t} - \theta_{q,t})] = \sum_{i \in \Omega^I} \theta_{i,t} \sum_{k \in \Omega_i^K} \kappa_{pq,t} SB_{pq} [\text{sign}(i-p) + \text{sign}(i-q)] + \sum_{k \in \Omega^K} \kappa_{pq,t} \underline{F}_k$

LHS (original) expression:

$$\begin{aligned}
& \sum_{k \in \Omega^K} \kappa_{pq,t} [\underline{F}_k - SB_{pq}(\theta_{p,t} - \theta_{q,t})] \\
&= \kappa_{12,t} [\underline{F}_k - SB_{12}(\theta_{1,t} - \theta_{2,t})] + \kappa_{23,t} [\underline{F}_k - SB_{23}(\theta_{2,t} - \theta_{3,t})] \\
&+ \kappa_{24,t} [\underline{F}_k - SB_{24}(\theta_{2,t} - \theta_{4,t})] + \kappa_{34,t} [\underline{F}_k - SB_{34}(\theta_{3,t} - \theta_{4,t})] \\
&= \kappa_{12,t} \underline{F}_k - \kappa_{12,t} SB_{12} \theta_{1,t} - \kappa_{12,t} SB_{12} \theta_{2,t} + \kappa_{23,t} \underline{F}_k - \kappa_{23,t} SB_{23} \theta_{2,t} - \kappa_{23,t} SB_{23} \theta_{3,t} \\
&+ \kappa_{24,t} \underline{F}_k - \kappa_{24,t} SB_{24} \theta_{2,t} - \kappa_{24,t} SB_{24} \theta_{4,t} + \kappa_{34,t} \underline{F}_k - \kappa_{34,t} SB_{34} \theta_{3,t} - \kappa_{34,t} SB_{34} \theta_{4,t} \\
&= \theta_{1,t} [-\kappa_{12,t} SB_{12}] + \theta_{2,t} [\kappa_{12,t} SB_{12} - \kappa_{23,t} SB_{23} - \kappa_{24,t} SB_{24}] + \theta_{3,t} [\kappa_{23,t} SB_{23} - \kappa_{34,t} SB_{34}] \\
&+ \theta_{4,t} [\kappa_{24,t} SB_{24} + \kappa_{34,t} SB_{34}] + \kappa_{12,t} \underline{F}_{12} + \kappa_{23,t} \underline{F}_{23} + \kappa_{24,t} \underline{F}_{24} + \kappa_{34,t} \underline{F}_{34}
\end{aligned} \tag{6}$$

RHS (reformulated) expression:

$$\begin{aligned}
& \sum_{i \in \Omega^I} \theta_{i,t} \sum_{k \in \Omega_i^K} \kappa_{pq,t} SB_{pq} [\text{sign}(i-p) + \text{sign}(i-q)] + \sum_{k \in \Omega^K} \kappa_{pq,t} \underline{F}_k \\
&= \theta_{1,t} [\kappa_{12,t} SB_{12} [\text{sign}(1-1) + \text{sign}(1-2)]] \\
&+ \theta_{2,t} [\kappa_{12,t} SB_{12} [\text{sign}(2-1) + \text{sign}(2-2)] + \kappa_{23,t} SB_{23} [\text{sign}(2-2) + \text{sign}(2-3)] + \kappa_{24,t} SB_{24} [\text{sign}(2-2) + \text{sign}(2-4)]] \\
&+ \theta_{3,t} [\kappa_{23,t} SB_{23} [\text{sign}(3-2) + \text{sign}(3-3)] + \kappa_{34,t} SB_{34} [\text{sign}(3-3) + \text{sign}(3-4)]] \\
&+ \theta_{4,t} [\kappa_{24,t} SB_{24} [\text{sign}(4-2) + \text{sign}(4-4)] + \kappa_{34,t} SB_{34} [\text{sign}(4-3) + \text{sign}(4-4)]] \\
&= \theta_{1,t} [-\kappa_{12,t} SB_{12}] + \theta_{2,t} [\kappa_{12,t} SB_{12} - \kappa_{23,t} SB_{23} - \kappa_{24,t} SB_{24}] + \theta_{3,t} [\kappa_{23,t} SB_{23} - \kappa_{34,t} SB_{34}] \\
&+ \theta_{4,t} [\kappa_{24,t} SB_{24} + \kappa_{34,t} SB_{34}] + \kappa_{12,t} \underline{F}_{12} + \kappa_{23,t} \underline{F}_{23} + \kappa_{24,t} \underline{F}_{24} + \kappa_{34,t} \underline{F}_{34}
\end{aligned} \tag{7}$$

Note that the expanded expressions for (6) and (7) match.

3.1.4 Re-writing: $\sum_{k \in \Omega^K} \eta_{pq,t} [SB_{pq}(\theta_{p,t} - \theta_{q,t}) - \bar{F}_k] = \sum_{i \in \Omega^I} \theta_{i,t} \sum_{k \in \Omega_i^K} -\eta_{pq,t} SB_{pq} [\text{sign}(i - p) + \text{sign}(i - q)] - \sum_{k \in \Omega^K} \eta_{pq,t} \bar{F}_k$

LHS (original) expression:

$$\begin{aligned}
& \sum_{k \in \Omega^K} \eta_{pq,t} [SB_{pq}(\theta_{p,t} - \theta_{q,t}) - \bar{F}_k] \\
&= \eta_{12,t} [SB_{12}(\theta_{1,t} - \theta_{2,t}) - \bar{F}_k] + \eta_{23,t} [SB_{23}(\theta_{2,t} - \theta_{3,t}) - \bar{F}_k] \\
&\quad + \eta_{24,t} [SB_{24}(\theta_{2,t} - \theta_{4,t}) - \bar{F}_k] + \eta_{34,t} [SB_{34}(\theta_{3,t} - \theta_{4,t}) - \bar{F}_k] \\
&= \eta_{12,t} SB_{12} \theta_{1,t} - \eta_{12,t} SB_{12} \theta_{2,t} - \eta_{12,t} \bar{F}_k + \eta_{23,t} SB_{23} \theta_{2,t} - \eta_{23,t} SB_{23} \theta_{3,t} - \eta_{23,t} \bar{F}_k \\
&\quad + \eta_{24,t} SB_{24} \theta_{2,t} - \eta_{24,t} SB_{24} \theta_{4,t} - \eta_{24,t} \bar{F}_k + \eta_{34,t} SB_{34} \theta_{3,t} - \eta_{34,t} SB_{34} \theta_{4,t} - \eta_{34,t} \bar{F}_k \\
&= \theta_{1,t} [\eta_{12,t} SB_{12}] + \theta_{2,t} [-\eta_{12,t} SB_{12} + \eta_{23,t} SB_{23} + \eta_{24,t} SB_{24}] + \theta_{3,t} [-\eta_{23,t} SB_{23} + \eta_{34,t} SB_{34}] \\
&\quad + \theta_{4,t} [-\eta_{24,t} SB_{24} - \eta_{34,t} SB_{34}] - \eta_{12,t} \bar{F}_{12} - \eta_{23,t} \bar{F}_{23} - \eta_{24,t} \bar{F}_{24} - \eta_{34,t} \bar{F}_{34}
\end{aligned} \tag{8}$$

RHS (reformulated) expression:

$$\begin{aligned}
& \sum_{i \in \Omega^I} \theta_{i,t} \sum_{k \in \Omega_i^K} -\eta_{pq,t} SB_{pq} [\text{sign}(i - p) + \text{sign}(i - q)] - \sum_{k \in \Omega^K} \eta_{pq,t} \bar{F}_k \\
&= \theta_{1,t} [-\eta_{12,t} SB_{12} [\text{sign}(1 - 1) + \text{sign}(1 - 2)]] \\
&\quad + \theta_{2,t} [-\eta_{12,t} SB_{12} [\text{sign}(2 - 1) + \text{sign}(2 - 2)] - \eta_{23,t} SB_{23} [\text{sign}(2 - 2) + \text{sign}(2 - 3)] - \eta_{24,t} SB_{24} [\text{sign}(2 - 2) + \text{sign}(2 - 4)]] \\
&\quad + \theta_{3,t} [-\eta_{23,t} SB_{23} [\text{sign}(3 - 2) + \text{sign}(3 - 3)] - \eta_{34,t} SB_{34} [\text{sign}(3 - 3) + \text{sign}(3 - 4)]] \\
&\quad + \theta_{4,t} [\eta_{24,t} SB_{24} [\text{sign}(4 - 2) + \text{sign}(4 - 4)] - \eta_{34,t} SB_{34} [\text{sign}(4 - 3) + \text{sign}(4 - 4)]] \\
&= \theta_{1,t} [\eta_{12,t} SB_{12}] + \theta_{2,t} [-\eta_{12,t} SB_{12} + \eta_{23,t} SB_{23} + \eta_{24,t} SB_{24}] + \theta_{3,t} [-\eta_{23,t} SB_{23} + \eta_{34,t} SB_{34}] \\
&\quad + \theta_{4,t} [-\eta_{24,t} SB_{24} - \eta_{34,t} SB_{34}] - \eta_{12,t} \bar{F}_{12} - \eta_{23,t} \bar{F}_{23} - \eta_{24,t} \bar{F}_{24} - \eta_{34,t} \bar{F}_{34}
\end{aligned} \tag{9}$$

Note that the expanded expressions for (8) and (9) match.

3.1.5 Re-writing: $\sum_{i \in \Omega^I} \lambda_{i,t} \sum_{j \in \Omega^I} B_{ij}(\theta_{i,t} - \theta_{j,t}) = \sum_{i \in \Omega^I} \theta_{i,t} \sum_{j \in \Omega^I} [\lambda_{i,t} B_{ij} - \lambda_{j,t} B_{ji}]$

$$\begin{aligned}
& \sum_{i \in \Omega^I} \lambda_{i,t} \sum_{j \in \Omega^I} B_{ij}(\theta_{i,t} - \theta_{j,t}) \\
&= \lambda_{1,t} [B_{12} \theta_{1,t} - B_{12} \theta_{2,t} + B_{23} \theta_{2,t} - B_{23} \theta_{3,t} + B_{24} \theta_{2,t} - B_{24} \theta_{4,t} + B_{34} \theta_{3,t} - B_{34} \theta_{4,t}] \\
&\quad + \lambda_{2,t} [B_{12} \theta_{1,t} - B_{12} \theta_{2,t} + B_{23} \theta_{2,t} - B_{23} \theta_{3,t} + B_{24} \theta_{2,t} - B_{24} \theta_{4,t} + B_{34} \theta_{3,t} - B_{34} \theta_{4,t}] \\
&\quad + \lambda_{3,t} [B_{12} \theta_{1,t} - B_{12} \theta_{2,t} + B_{23} \theta_{2,t} - B_{23} \theta_{3,t} + B_{24} \theta_{2,t} - B_{24} \theta_{4,t} + B_{34} \theta_{3,t} - B_{34} \theta_{4,t}] \\
&\quad + \lambda_{4,t} [B_{12} \theta_{1,t} - B_{12} \theta_{2,t} + B_{23} \theta_{2,t} - B_{23} \theta_{3,t} + B_{24} \theta_{2,t} - B_{24} \theta_{4,t} + B_{34} \theta_{3,t} - B_{34} \theta_{4,t}] \\
&= \theta_{1,t} [\lambda_{1,t} B_{11} - \lambda_{1,t} B_{11} + \lambda_{1,t} B_{12} - \lambda_{2,t} B_{21} + \lambda_{1,t} B_{13} - \lambda_{3,t} B_{31} + \lambda_{1,t} B_{14} - \lambda_{4,t} B_{41}] \\
&\quad + \theta_{2,t} [\lambda_{2,t} B_{21} - \lambda_{1,t} B_{12} + \lambda_{2,t} B_{22} - \lambda_{2,t} B_{22} + \lambda_{2,t} B_{23} - \lambda_{3,t} B_{32} + \lambda_{2,t} B_{24} - \lambda_{4,t} B_{42}] \\
&\quad + \theta_{3,t} [\lambda_{3,t} B_{31} - \lambda_{1,t} B_{13} + \lambda_{3,t} B_{32} - \lambda_{2,t} B_{23} + \lambda_{3,t} B_{33} - \lambda_{3,t} B_{33} + \lambda_{3,t} B_{34} - \lambda_{4,t} B_{43}] \\
&\quad + \theta_{4,t} [\lambda_{4,t} B_{41} - \lambda_{1,t} B_{14} + \lambda_{4,t} B_{42} - \lambda_{2,t} B_{24} + \lambda_{4,t} B_{43} - \lambda_{3,t} B_{34} + \lambda_{4,t} B_{44} - \lambda_{4,t} B_{44}] \\
&= \sum_{i \in \Omega^I} \theta_{i,t} \sum_{j \in \Omega^I} [\lambda_{i,t} B_{ij} - \lambda_{j,t} B_{ji}]
\end{aligned} \tag{10}$$

3.1.6 Re-writing: $\sum_{i \in \Omega^N} \theta_{i,t} \zeta_{i,t} = \sum_{i \in \Omega^I} U_{in} \theta_{i,t} \zeta_{i,t}$

Let $U^{\Omega^I \times \Omega^N}$ be matrix with entries defined by the following rule:

$$U_{in} = \begin{cases} 1 & \text{if node } i \text{ is zone } n\text{'s reference node} \\ 0 & \text{otherwise} \end{cases} \tag{11}$$

For example, consider the case where nodes 1 and 2 belong to zone 1, and nodes 3 and 4 belong to zone 2. Let node 1 be zone 1's reference node, and node 3 be zone 2's reference node i.e. $N = \{1, 3\}$. $U^{\Omega^I \times \Omega^N}$ is given by (12).

$$U_{in} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \quad (12)$$

The original expression for terms in the Lagrangian that fix reference node angles is give by (13).

$$\sum_{i \in \Omega^N} \theta_{i,t} \zeta_{i,t} = \theta_{1,t} \zeta_{1,t} + \theta_{3,t} \zeta_{3,t} \quad (13)$$

A reformulated expression is given by (14).

$$\begin{aligned} & \sum_{i \in \Omega^I} \theta_{i,t} \sum_{n \in \Omega^N} U_{in} \zeta_{i,t} \\ &= \theta_{1,t} [U_{11} \zeta_{1,t} + U_{12} \zeta_{1,t}] + \theta_{2,t} [U_{21} \zeta_{2,t} + U_{22} \zeta_{2,t}] + \theta_{3,t} [U_{31} \zeta_{3,t} + U_{32} \zeta_{3,t}] + \theta_{4,t} [U_{41} \zeta_{4,t} + U_{42} \zeta_{4,t}] \\ &= \theta_{1,t} \zeta_{1,t} + \theta_{3,t} \zeta_{3,t} \end{aligned} \quad (14)$$

Note that the expanded expressions for (13) and (14) match.

4 Lagrangian with re-formulated terms

The Lagrangian is now re-written using the reformulated terms.

$$\begin{aligned} \mathcal{L}(\Omega^\Lambda, \Omega^\Xi) &= \sum_{k \in \Omega^K} [\theta \gamma_{pq,t} - \bar{\theta} \delta_{pq,t} + \underline{F}_k \kappa_{pq,t} - \bar{F}_k \eta_{pq,t}] + \sum_{g \in \Omega^G} [\underline{P}_g \alpha_{g,t} - \bar{P}_g \beta_{g,t}] + \sum_{i \in \Omega^I} [D_{i,t} - Y_{i,t}] \lambda_{i,t} + \sum_{m \in \Omega^M} [\underline{H}_m \omega_{m,t} - \bar{H}_m \psi_{m,t}] \\ &+ \sum_{g \in \Omega^G} p_{g,t} [A_g + (E_g - \phi) \tau - \alpha_{g,t} + \beta_{g,t} - \lambda_{f(g)}] \\ &+ \sum_{i \in \Omega^I} \theta_{i,t} \left[\sum_{k \in \Omega_i^K} (\gamma_{pq,t} - \delta_{pq,t} + S B_{pq} [\kappa_{pq,t} - \eta_{pq,t}]) [\text{sign}(i - p) + \text{sign}(i - q)] + S \sum_{j \in \Omega^I} [\lambda_{i,t} B_{ij} - \lambda_{j,t} B_{ji}] + \sum_{n \in \Omega^N} U_{in} \zeta_{i,t} \right] \\ &+ \sum_{m \in \Omega^M} h_{m,t} \left[\sum_{i \in \Omega^I} \lambda_{i,t} C_{mi} + \psi_{m,t} - \omega_{m,t} \right] \end{aligned} \quad (15)$$

4.1 Lagrange-dual function

$$g(\Omega^\Xi) = \inf_{\Omega^\Lambda} \mathcal{L}(\Omega^\Lambda, \Omega^\Xi) \quad (16)$$

Objective is to maximize $g(\Omega^\Xi)$.

4.2 Dual problem

$$\max_{\Omega^\Xi} \sum_{k \in \Omega^K} [\theta \gamma_{pq,t} - \bar{\theta} \delta_{pq,t} + \underline{F}_k \kappa_{pq,t} - \bar{F}_k \eta_{pq,t}] + \sum_{g \in \Omega^G} [\underline{P}_g \alpha_{g,t} - \bar{P}_g \beta_{g,t}] + \sum_{i \in \Omega^I} [D_{i,t} - Y_{i,t}] \lambda_{i,t} + \sum_{m \in \Omega^M} [\underline{H}_m \omega_{m,t} - \bar{H}_m \psi_{m,t}] \quad (17)$$

s.t.

$$A_g + (E_g - \phi) \tau - \alpha_{g,t} + \beta_{g,t} - \lambda_{f(g)} = 0 \quad \forall g \in \Omega^G \quad (18)$$

$$\sum_{k \in \Omega_i^K} (\gamma_{pq,t} - \delta_{pq,t} + S B_{pq} [\kappa_{pq,t} - \eta_{pq,t}]) [\text{sign}(i - p) + \text{sign}(i - q)] + S \sum_{j \in \Omega^I} [\lambda_{i,t} B_{ij} - \lambda_{j,t} B_{ji}] + \sum_{n \in \Omega^N} U_{in} \zeta_{i,t} = 0 \quad \forall i \in \Omega^I \quad (19)$$

$$\sum_{i \in \Omega^I} \lambda_{i,t} C_{mi} + \psi_{m,t} - \omega_{m,t} = 0 \quad \forall m \in \Omega^M \quad (20)$$

$$\alpha_{g,t} \geq 0 \quad g \in \Omega^G \quad t \in \Omega^T \quad (21)$$

$$\beta_{g,t} \geq 0 \quad g \in \Omega^G \quad t \in \Omega^T \quad (22)$$

$$\gamma_{pq,t} \geq 0 \quad k \in \Omega^K \quad t \in \Omega^T \quad (23)$$

$$\delta_{pq,t} \geq 0 \quad k \in \Omega^K \quad t \in \Omega^T \quad (24)$$

$$\kappa_{pq,t} \geq 0 \quad k \in \Omega^K \quad t \in \Omega^T \quad (25)$$

$$\eta_{pq,t} \geq 0 \quad k \in \Omega^K \quad t \in \Omega^T \quad (26)$$

$$\omega_{m,t} \geq 0 \quad m \in \Omega^M \quad t \in \Omega^T \quad (27)$$

$$\psi_{m,t} \geq 0 \quad m \in \Omega^M \quad t \in \Omega^T \quad (28)$$

5 Reformulated program - MPPDC

Constraints from the primal and dual formulations are combined, and strong duality constraints are added by equating the objectives of these formulations. This yields the following single-level MPPDC:

$$\min_{\Omega^\Theta, \Omega^\Lambda, \Omega^\Xi} \left\| \frac{\sum_{t \in \Omega^T} \sum_{i \in \Omega^I} \lambda_{i,t} L_t D_{i,t}}{\sum_{t \in \Omega^T} \sum_{i \in \Omega^I} L_t D_{i,t}} - Z \right\|_1 \quad (29)$$

s.t.

Upper-level primal constraints

$$\sum_{t \in \Omega^T} \sum_{g \in \Omega^G} (E_g - \phi) \tau p_{g,t} L_t \geq R \quad (30)$$

$$\frac{\sum_{t \in \Omega^T} \sum_{g \in \Omega^G} E_g p_{g,t} L_t}{\sum_{t \in \Omega^T} \sum_{g \in \Omega^G} D_{i,t} L_t} \leq X \quad (31)$$

Lower-level primal constraints

$$\underline{P}_g - p_{g,t} \leq 0 \quad (\alpha_{g,t}) \quad \forall g \in \Omega^G \quad \forall t \in \Omega^T \quad (32)$$

$$p_{g,t} - \bar{P}_g \leq 0 \quad (\beta_{g,t}) \quad \forall g \in \Omega^G \quad \forall t \in \Omega^T \quad (33)$$

$$\underline{\theta} - \theta_{p,t} + \theta_{q,t} \leq 0 \quad (\gamma_{pq,t}) \quad \forall k \in \Omega^K \quad t \in \Omega^T \quad (34)$$

$$\theta_{p,t} - \theta_{q,t} - \bar{\theta} \leq 0 \quad (\delta_{pq,t}) \quad \forall k \in \Omega^K \quad \forall t \in \Omega^T \quad (35)$$

$$\theta_i = 0 \quad (\zeta_{i,t}) \quad \forall i \in \Omega^N \quad \forall t \in \Omega^T \quad (36)$$

$$S \sum_{j \in \Omega^I} B_{ij} (\theta_{i,t} - \theta_{j,t}) + \sum_{m \in \Omega^M} C_{mi} h_{m,t} + D_{i,t} - Y_{i,t} - \sum_{g \in \Omega_i^G} p_{g,t} = 0 \quad (\lambda_{i,t}) \quad \forall i \in \Omega^I \quad \forall t \in \Omega^T \quad (37)$$

$$\underline{F}_k - SB_{pq}(\theta_{p,t} - \theta_{q,t}) \leq 0 \quad (\kappa_{pq,t}) \quad \forall k \in \Omega^K \quad \forall t \in \Omega^T \quad (38)$$

$$SB_{pq}(\theta_{p,t} - \theta_{q,t}) - \overline{F}_k \leq 0 \quad (\eta_{pq,t}) \quad \forall k \in \Omega^K \quad \forall t \in \Omega^T \quad (39)$$

$$\underline{H}_m - h_{m,t} \leq 0 \quad (\omega_{m,t}) \quad \forall m \in \Omega^M \quad \forall t \in \Omega^T \quad (40)$$

$$h_{m,t} - \overline{H}_m \leq 0 \quad (\psi_{m,t}) \quad \forall m \in \Omega^M \quad \forall t \in \Omega^T \quad (41)$$

Lower-level dual constraints

$$A_g + (E_g - \phi)\tau - \alpha_{g,t} + \beta_{g,t} - \lambda_{f(g)} = 0 \quad \forall g \in \Omega^G \quad \forall t \in \Omega^T \quad (42)$$

$$\begin{aligned} \sum_{k \in \Omega_i^K} (\gamma_{pq,t} - \delta_{pq,t} + SB_{pq}[\kappa_{pq,t} - \eta_{pq,t}]) [\text{sign}(i - p) + \text{sign}(i - q)] \\ + S \sum_{j \in \Omega^I} [\lambda_{i,t} B_{ij} - \lambda_{j,t} B_{ji}] + \sum_{n \in \Omega^N} U_{in} \zeta_{i,t} = 0 \quad \forall i \in \Omega^I \quad \forall t \in \Omega^T \end{aligned} \quad (43)$$

$$\sum_{i \in \Omega^I} \lambda_{i,t} C_{mi} + \psi_{m,t} - \omega_{m,t} = 0 \quad \forall m \in \Omega^M \quad \forall t \in \Omega^T \quad (44)$$

$$\alpha_{g,t} \geq 0 \quad g \in \Omega^G \quad t \in \Omega^T \quad (45)$$

$$\beta_{g,t} \geq 0 \quad g \in \Omega^G \quad t \in \Omega^T \quad (46)$$

$$\gamma_{pq,t} \geq 0 \quad k \in \Omega^K \quad t \in \Omega^T \quad (47)$$

$$\delta_{pq,t} \geq 0 \quad k \in \Omega^K \quad t \in \Omega^T \quad (48)$$

$$\kappa_{pq,t} \geq 0 \quad k \in \Omega^K \quad t \in \Omega^T \quad (49)$$

$$\eta_{pq,t} \geq 0 \quad k \in \Omega^K \quad t \in \Omega^T \quad (50)$$

$$\omega_{m,t} \geq 0 \quad m \in \Omega^M \quad t \in \Omega^T \quad (51)$$

$$\psi_{m,t} \geq 0 \quad m \in \Omega^M \quad t \in \Omega^T \quad (52)$$

Strong duality constraints

$$\begin{aligned} \sum_{g \in \Omega^G} [A_g + (E_g - \phi)\tau] p_{g,t} \\ = \sum_{k \in \Omega^K} [\underline{\theta} \gamma_{pq,t} - \overline{\theta} \delta_{pq,t} + \underline{F}_k \kappa_{pq,t} - \overline{F}_k \eta_{pq,t}] + \sum_{g \in \Omega^G} [\underline{P}_g \alpha_{g,t} - \overline{P}_g \beta_{g,t}] + \sum_{i \in \Omega^I} [D_{i,t} - Y_{i,t}] \lambda_{i,t} \\ + \sum_{m \in \Omega^M} [\underline{H}_m \omega_{m,t} - \overline{H}_m \psi_{m,t}] \quad \forall t \in \Omega^T \end{aligned} \quad (53)$$