# MPPDC Derivation

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**Indices**

| | |
|---|---|
| $i$ | Node index |
| $n$ | Reference node index |
| $k$ | Edge index |
| $g$ | Generator index |
| $t$ | Trading interval index |
| $p$ | The first element (from node) for AC edge $k$ |
| $q$ | The last element (to node) for AC edge $k$ |

**Sets**

| | |
|---|---|
| $I$ | Nodes in the network |
| $N$ | Set of reference nodes |
| $K$ | AC network edges |
| $M$ | HVDC network edges |
| $T$ | Trading intervals |
| $G$ | Generators |
| $G_i$ | Generators connected to node $i$ |
| $K_i$ | AC edges connected to node $i$ |
| $\Theta$ | Upper-level program primal variables |
| $\Lambda$ | Lower-level program primal variables |
| $\Xi$ | Lower-level program dual variables |

**Primal variables for upper-level program**

| | |
|---|---|
| $\tau$ | Permit price [\$/tCO$_2$] |
| $\phi$ | EmiSubions intensity baseline [tCO$_2$/MWh] |

**Primal variables for lower-level program**

| | |
|---|---|
| $p_{g,t}$ | Power output of generator $g$ during interval $t$ [MW] |
| $\theta_{i,t}$ | Voltage angle (with respect to reference node) at node $i$ for trading interval $t$ [rad] |
| $h_{m,t}$ | Power flow over HVDC link $m$ during period $t$ [MW] |

**Dual variables**

| | |
|---|---|
| $\alpha_{g,t}$ | Minimum generation constraint dual variable |
| $\beta_{g,t}$ | Maximum generation constraint dual variable |
| $\gamma_{pq,t}$ | Lower voltage angle difference constraint dual variable for branch $k$ connecting nodes $i$ and $j$ |
| $\delta_{pq,t}$ | Upper voltage angle difference constraint dual variable for branch $k$ connecting nodes $i$ and $j$ |
| $\zeta_{i,t}$ | Reference node voltage angle constraint dual variable for $i \in N$ |
| $\lambda_{i,t}$ | Power balance constraint dual variable at node $i$ during period $t$ |
| $\kappa_{pq,t}$ | Minimum power flow constraint dual variable for flow between nodes $i$ and $j$ during period $t$ |
| $\eta_{pq,t}$ | Maximum power flow constraint dual variable for flow between nodes $i$ and $j$ during period $t$ |
| $\omega_{m,t}$ | Power flow lower bound dual variable for HVDC link $m$ |
| $\psi_{m,t}$ | Power flow upper bound dual variable for HVDC link $m$ |

**Parameters**

| | |
|---|---|
| $L_t$ | Duration of trading interval $t$ [hrs] |
| $D_{i,t}$ | Power demand at node $i$ during period $t$ [MW] |
| $Y_{i,t}$ | Fixed power injection at node $i$ during period $t$ [MW] |
| $Z$ | Target electricity price [\$/MWh] |
| $E_g$ | Emissions intensity of generator $g$ [tCO$_2$/MWh] |
| $R$ | Scheme revenue constraint [\$] |
| $X$ | Emissions intensity constraint [tCO$_2$/MWh] |
| $\overline{P}_g$ | Maximum output for generator $g$ [MW] |
| $\underline{P}_g$ | Minimum output for generator $g$ [MW] |
| $\overline{\theta}$ | Maximum voltage angle difference between buses $i$ and $j$ [rad] |
| $\underline{\theta}$ | Minimum voltage angle difference between buses $i$ and $j$ [rad] |

| | |
|---|---|
| $B_{pq}$ | Susceptance between nodes $i$ and $j$ [pu] |
| $\underline{H}_m$ | Lower bound for power flow from node $i$ to node $j$ for HVDC link $m$ [MW] |
| $\overline{H}_m$ | Upper bound for power flow from node $i$ to node $j$ for HVDC link $m$ [MW] |
| $U_{in}$ | Matrix describing network reference nodes |
| $C_{mi}$ | HVDC incidence matrix |
| **Mapping functions** | |
| $f(g)$ | Maps generator $g$ to the node at which it is located |

Table 1: Model notation

# 1 Introduction

This document describes the reformulation procedure used to convert the policy maker's bi-level program into a single level program. This is accomplished by converting the lower-level optimisation problem into a system of equations. The reformulation strategy involves constructing the dual of the primal problem, and then combining constraints of the primal and dual programs, and adding a strong duality constraint. The resulting system of equations is known as a Mathematical Program with Primal and Dual Constraints (MPPDC).

## 2 Bi-level formulation

The policy maker's bi-level optimisation program is as follows:

$$\min_{\Theta,\Lambda,\Xi} \left\| \frac{\sum\limits_{t\in T}\sum\limits_{i\in I} \lambda_{i,t} L_t D_{i,t}}{\sum\limits_{t\in T}\sum\limits_{i\in I} L_t D_{i,t}} - Z \right\|_1$$

s.t.

$$\sum_{t\in T}\sum_{g\in G}(E_g-\phi)\tau p_{g,t}L_t \geq R$$

$$\frac{\sum\limits_{t\in T}\sum\limits_{g\in G} E_g p_{g,t} L_t}{\sum\limits_{t\in T}\sum\limits_{g\in G} D_{i,t} L_t} \leq X$$

$$\left.\begin{aligned}
&\min_{\Lambda_t} \sum_{g\in G} \left[A_g + (E_g-\phi)\tau\right] p_{g,t} \\
&\text{s.t.} \\
&\underline{P}_g - p_{g,t} \leq 0 \quad (\alpha_{g,t}) \quad \forall g\in G \\
&p_{g,t} - \overline{P}_g \leq 0 \quad (\beta_{g,t}) \quad \forall g\in G \\
&\underline{\theta} - \theta_{p,t} + \theta_{q,t} \leq 0 \quad (\gamma_{pq,t}) \quad \forall k\in K \\
&\theta_{p,t} - \theta_{q,t} - \overline{\theta} \leq 0 \quad (\delta_{pq,t}) \quad \forall k\in K \\
&\theta_i = 0 \quad (\zeta_{i,t}) \quad \forall i\in N \\
&S\sum_{j\in I} B_{ij}(\theta_{i,t}-\theta_{j,t}) + \sum_{m\in M} C_{mi}h_{m,t} + D_{i,t} - Y_{i,t} - \sum_{g\in G_i} p_{g,t} = 0 \quad (\lambda_{i,t}) \quad \forall i\in I \\
&\underline{F}_k - SB_{pq}(\theta_{p,t}-\theta_{q,t}) \leq 0 \quad (\kappa_{pq,t}) \quad \forall k\in K \\
&SB_{pq}(\theta_{p,t}-\theta_{q,t}) - \overline{F}_k \leq 0 \quad (\eta_{pq,t}) \quad \forall k\in K \\
&\underline{H}_m - h_{m,t} \leq 0 \quad (\omega_{m,t}) \quad \forall m\in M \\
&h_{m,t} - \overline{H}_m \leq 0 \quad (\psi_{m,t}) \quad \forall m\in M
\end{aligned}\right\} \quad \forall t\in T$$

## 3 Lagrangian for lower level problems

The Lagrangian for each lower-level optimisation problem is as follows:

$$
\begin{aligned}
\mathcal{L}(\Lambda,\Xi) = & \sum_{g\in G} \left[A_g + (E_g - \phi)\tau\right] p_{g,t} + \alpha_{g,t}(\underline{P}_g - p_{g,t}) + \beta_{g,t}(p_{g,t} - \overline{P}_g) \\
& + \sum_{k\in K} \gamma_{pq,t}(\underline{\theta} - \theta_{p,t} + \theta_{q,t}) + \delta_{pq,t}(\theta_{p,t} - \theta_{q,t} - \overline{\theta}) + \kappa_{pq,t}\left[\underline{F}_k - SB_{pq}(\theta_{p,t} - \theta_{q,t})\right] + \eta_{pq,t}\left[SB_{pq}(\theta_{p,t} - \theta_{q,t}) - \overline{F}_k\right] \\
& + \sum_{m\in M} \omega_{m,t}\left[\underline{H}_m - h_{m,t}\right] + \psi_{m,t}\left[h_{m,t} - \overline{H}_m\right] \\
& + \sum_{i\in I} \lambda_{i,t}\left[S\sum_{j\in I} B_{ij}(\theta_{i,t} - \theta_{j,t}) + \sum_{m\in M} C_{mi}h_{m,t} + D_{i,t} - Y_{i,t} - \sum_{g\in G_i} p_{g,t}\right] \\
& + \sum_{i\in N} \zeta_{i,t}\theta_{i,t}
\end{aligned}
\tag{1}
$$

### 3.1 Re-writing terms

Some terms in the Lagrangian must be re-written, in particular voltage angles indexed by $j$ and $q$. Using a four node network, we verify that the original and re-formulated terms match.

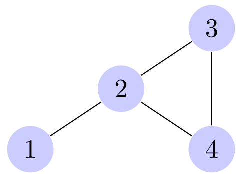


Figure 1: 4 node network

The set of edges is given by: $K = \{(1,2),(2,3),(2,4),(3,4)\}$

#### 3.1.1 Re-writing: $\sum_{k\in K} \gamma_{pq,t}\left[\underline{\theta} - \theta_{p,t} + \theta_{q,t}\right] = \sum_{i\in I} \theta_{i,t} \sum_{k\in K_i} \gamma_{pq,t}\left[\text{sign}(i-p) + \text{sign}(i-q)\right] + \underline{\theta}\sum_{k\in K} \gamma_{pq,t}$

LHS (original) expression:

$$
\begin{aligned}
& \sum_{k\in K} \gamma_{pq,t}\left[\underline{\theta} - \theta_{p,t} + \theta_{j,t}\right] \\
& = \gamma_{12,t}\left[\underline{\theta} - \theta_{1,t} + \theta_{2,t}\right] + \gamma_{23,t}\left[\underline{\theta} - \theta_{2,t} + \theta_{3,t}\right] + \gamma_{24,t}\left[\underline{\theta} - \theta_{2,t} + \theta_{4,t}\right] + \gamma_{34,t}\left[\underline{\theta} - \theta_{3,t} + \theta_{4,t}\right] \\
& = \gamma_{12,t}\underline{\theta} - \gamma_{12,t}\theta_{1,t} + \gamma_{12,t}\theta_{2,t} + \gamma_{23,t}\underline{\theta} - \gamma_{23,t}\theta_{2,t} + \gamma_{23,t}\theta_{3,t} + \gamma_{24,t}\underline{\theta} - \gamma_{24,t}\theta_{2,t} + \gamma_{24,t}\theta_{4,t} + \gamma_{34,t}\underline{\theta} - \gamma_{34,t}\theta_{3,t} + \gamma_{34,t}\theta_{4,t} \\
& = \theta_{1,t}\left[-\gamma_{12,t}\right] + \theta_{2,t}\left[\gamma_{12,t} - \gamma_{23,t} - \gamma_{24,t}\right] + \theta_{3,t}\left[\gamma_{23,t} - \gamma_{34,t}\right] + \theta_{4,t}\left[\gamma_{24,t} + \gamma_{34,t}\right] + \underline{\theta}\left[\gamma_{12,t} + \gamma_{23,t} + \gamma_{24,t} + \gamma_{34,t}\right]
\end{aligned}
\tag{2}
$$

RHS (reformulated) expression:

$$
\begin{aligned}
& \sum_{i\in I} \theta_{i,t} \sum_{k\in K_i} \gamma_{pq,t}\left[\text{sign}(i-p) + \text{sign}(i-q)\right] + \underline{\theta}\sum_{k\in K} \gamma_{pq,t} \\
& = \theta_{1,t}\left[\gamma_{12,t}\left[\text{sign}(1-1) + \text{sign}(1-2)\right]\right] \\
& + \theta_{2,t}\left[\gamma_{12,t}\left[\text{sign}(2-1) + \text{sign}(2-2)\right] + \gamma_{23,t}\left[\text{sign}(2-2) + \text{sign}(2-3)\right] + \gamma_{24,t}\left[\text{sign}(2-2) + \text{sign}(2-4)\right]\right] \\
& + \theta_{3,t}\left[\gamma_{23,t}\left[\text{sign}(3-2) + \text{sign}(3-3)\right] + \gamma_{34,t}\left[\text{sign}(3-3) + \text{sign}(3-4)\right]\right] \\
& + \theta_{4,t}\left[\gamma_{24,t}\left[\text{sign}(4-2) + \text{sign}(4-4)\right] + \gamma_{34,t}\left[\text{sign}(4-3) + \text{sign}(4-4)\right]\right] \\
& + \underline{\theta}\left[\gamma_{12,t} + \gamma_{23,t} + \gamma_{24,t} + \gamma_{34,t}\right] \\
& = \theta_{1,t}\left[-\gamma_{12,t}\right] + \theta_{2,t}\left[\gamma_{12,t} - \gamma_{23,t} - \gamma_{24,t}\right] + \theta_{3,t}\left[\gamma_{23,t} - \gamma_{34,t}\right] + \theta_{4,t}\left[\gamma_{24,t} - \gamma_{34,t}\right] + \underline{\theta}\left[\gamma_{12,t} + \gamma_{23,t} + \gamma_{24,t} + \gamma_{34,t}\right]
\end{aligned}
\tag{3}
$$

Note that the expanded expressions for (2) and (3) match.

### 3.1.2 Re-writing: $\sum_{k\in K} \delta_{pq,t}\left[\theta_{p,t}-\theta_{q,t}-\overline{\theta}\right] = \sum_{i\in I}\theta_{i,t}\sum_{k\in K_i} -\delta_{pq,t}\left[\text{sign}(i-p)+\text{sign}(i-q)\right] - \overline{\theta}\sum_{k\in K}\delta_{pq,t}$

LHS (original) expression:

$$
\begin{aligned}
&\sum_{k\in K} \delta_{pq,t}\left[\theta_{p,t}-\theta_{q,t}-\overline{\theta}\right] \\
&= \delta_{12,t}\left[\theta_{1,t}-\theta_{2,t}-\overline{\theta}\right] + \delta_{23,t}\left[\theta_{2,t}-\theta_{3,t}-\overline{\theta}\right] + \delta_{24,t}\left[\theta_{2,t}-\theta_{4,t}-\overline{\theta}\right] + \delta_{34,t}\left[\theta_{3,t}-\theta_{4,t}-\overline{\theta}\right] \\
&= \delta_{12,t}\theta_{1,t} - \delta_{12,t}\theta_{2,t} - \delta_{12,t}\overline{\theta} + \delta_{23,t}\theta_{2,t} - \delta_{23,t}\theta_{3,t} - \delta_{23,t}\overline{\theta} + \delta_{24,t}\theta_{2,t} - \delta_{24,t}\theta_{4,t} - \delta_{24,t}\overline{\theta} + \delta_{34,t}\theta_{3,t} - \delta_{34,t}\theta_{4,t} - \delta_{34,t}\overline{\theta} \\
&= \theta_{1,t}\delta_{12,t} + \theta_{2,t}\left[-\delta_{12,t}+\delta_{23,t}+\delta_{24,t}\right] + \theta_{3,t}\left[-\delta_{23,t}+\delta_{34,t}\right] + \theta_{4,t}\left[-\delta_{24,t}-\delta_{34,t}\right] - \overline{\theta}\left[\delta_{12,t}+\delta_{23,t}+\delta_{24,t}+\delta_{34,t}\right]
\end{aligned}
\tag{4}
$$

RHS (reformulated) expression

$$
\begin{aligned}
&\sum_{i\in I}\theta_{i,t}\sum_{k\in K_i} -\delta_{pq,t}\left[\text{sign}(i-p)+\text{sign}(i-q)\right] - \overline{\theta}\sum_{k\in K}\delta_{pq,t} \\
&= \theta_{1,t}\left[-\delta_{12,t}\left[\text{sign}(1-1)+\text{sign}(1-2)\right]\right] \\
&+ \theta_{2,t}\left[-\delta_{12,t}\left[\text{sign}(2-1)+\text{sign}(2-2)\right] - \delta_{23,t}\left[\text{sign}(2-2)+\text{sign}(2-3)\right] - \delta_{24,t}\left[\text{sign}(2-2)+\text{sign}(2-4)\right]\right] \\
&+ \theta_{3,t}\left[-\delta_{23,t}\left[\text{sign}(3-2)+\text{sign}(3-3)\right] + -\delta_{34,t}\left[\text{sign}(3-3)+\text{sign}(3-4)\right]\right] \\
&+ \theta_{4,t}\left[-\delta_{24,t}\left[\text{sign}(4-2)+\text{sign}(4-4)\right] + -\delta_{34,t}\left[\text{sign}(4-3)+\text{sign}(4-4)\right]\right] \\
&- \overline{\theta}\left[\delta_{12,t}+\delta_{23,t}+\delta_{24,t}+\delta_{34,t}\right] \\
&= \theta_{1,t}\delta_{12,t} + \theta_{2,t}\left[-\delta_{12,t}+\delta_{23,t}+\delta_{24,t}\right] + \theta_{3,t}\left[-\delta_{23,t}+\delta_{34,t}\right] + \theta_{4,t}\left[-\delta_{24,t}-\delta_{34,t}\right] - \overline{\theta}\left[\delta_{12,t}+\delta_{23,t}+\delta_{24,t}+\delta_{34,t}\right]
\end{aligned}
\tag{5}
$$

Note that the expanded expressions for (4) and (5) match.

### 3.1.3 Re-writing: $\sum_{k\in K} \kappa_{pq,t}\left[\underline{F}_k - SB_{pq}(\theta_{p,t}-\theta_{q,t})\right] = \sum_{i\in I}\theta_{i,t}\sum_{k\in K_i}\kappa_{pq,t}SB_{pq}\left[\text{sign}(i-p)+\text{sign}(i-q)\right] + \sum_{k\in K}\kappa_{pq,t}\underline{F}_k$

LHS (original) expression:

$$
\begin{aligned}
&\sum_{k\in K} \kappa_{pq,t}\left[\underline{F}_k - SB_{pq}(\theta_{p,t}-\theta_{q,t})\right] \\
&= \kappa_{12,t}\left[\underline{F}_k - SB_{12}(\theta_{1,t}-\theta_{2,t})\right] + \kappa_{23,t}\left[\underline{F}_k - SB_{23}(\theta_{2,t}-\theta_{3,t})\right] \\
&\quad + \kappa_{24,t}\left[\underline{F}_k - SB_{24}(\theta_{2,t}-\theta_{4,t})\right] + \kappa_{34,t}\left[\underline{F}_k - SB_{34}(\theta_{3,t}-\theta_{4,t})\right] \\
&= \kappa_{12,t}\underline{F}_k - \kappa_{12,t}SB_{12}\theta_{1,t} - \kappa_{12,t}SB_{12}\theta_{2,t} + \kappa_{23,t}\underline{F}_k - \kappa_{23,t}SB_{23}\theta_{2,t} - \kappa_{23,t}SB_{23}\theta_{3,t} \\
&\quad + \kappa_{24,t}\underline{F}_k - \kappa_{24,t}SB_{24}\theta_{2,t} - \kappa_{24,t}SB_{24}\theta_{4,t} + \kappa_{34,t}\underline{F}_k - \kappa_{34,t}SB_{34}\theta_{3,t} - \kappa_{34,t}SB_{34}\theta_{4,t} \\
&= \theta_{1,t}\left[-\kappa_{12,t}SB_{12}\right] + \theta_{2,t}\left[\kappa_{12,t}SB_{12} - \kappa_{23,t}SB_{23} - \kappa_{24,t}SB_{24}\right] + \theta_{3,t}\left[\kappa_{23,t}SB_{23} - \kappa_{34,t}SB_{34}\right] \\
&\quad + \theta_{4,t}\left[\kappa_{24,t}SB_{24} + \kappa_{34,t}SB_{34}\right] + \kappa_{12,t}\underline{F}_{12} + \kappa_{23,t}\underline{F}_{23} + \kappa_{24,t}\underline{F}_{24} + \kappa_{34,t}\underline{F}_{34}
\end{aligned}
\tag{6}
$$

RHS (reformulated) expression:

$$
\begin{aligned}
&\sum_{i\in I}\theta_{i,t}\sum_{k\in K_i}\kappa_{pq,t}SB_{pq}\left[\text{sign}(i-p)+\text{sign}(i-q)\right] + \sum_{k\in K}\kappa_{pq,t}\underline{F}_k \\
&= \theta_{1,t}\left[\kappa_{12,t}SB_{12}\left[\text{sign}(1-1)+\text{sign}(1-2)\right]\right] \\
&\quad + \theta_{2,t}\left[\kappa_{12,t}SB_{12}\left[\text{sign}(2-1)+\text{sign}(2-2)\right] + \kappa_{23,t}SB_{23}\left[\text{sign}(2-2)+\text{sign}(2-3)\right] + \kappa_{24,t}SB_{24}\left[\text{sign}(2-2)+\text{sign}(2-4)\right]\right] \\
&\quad + \theta_{3,t}\left[\kappa_{23,t}SB_{23}\left[\text{sign}(3-2)+\text{sign}(3-3)\right] + \kappa_{34,t}SB_{34}\left[\text{sign}(3-3)+\text{sign}(3-4)\right]\right] \\
&\quad + \theta_{4,t}\left[\kappa_{24,t}SB_{24}\left[\text{sign}(4-2)+\text{sign}(4-4)\right] + \kappa_{34,t}SB_{34}\left[\text{sign}(4-3)+\text{sign}(4-4)\right]\right] \\
&= \theta_{1,t}\left[-\kappa_{12,t}SB_{12}\right] + \theta_{2,t}\left[\kappa_{12,t}SB_{12} - \kappa_{23,t}SB_{23} - \kappa_{24,t}SB_{24}\right] + \theta_{3,t}\left[\kappa_{23,t}SB_{23} - \kappa_{34,t}SB_{34}\right] \\
&\quad + \theta_{4,t}\left[\kappa_{24,t}SB_{24} + \kappa_{34,t}SB_{34}\right] + \kappa_{12,t}\underline{F}_{12} + \kappa_{23,t}\underline{F}_{23} + \kappa_{24,t}\underline{F}_{24} + \kappa_{34,t}\underline{F}_{34}
\end{aligned}
\tag{7}
$$

Note that the expanded expressions for (6) and (7) match.

#### 3.1.4 Re-writing: $\sum_{k\in K} \eta_{pq,t}\left[SB_{pq}(\theta_{p,t}-\theta_{q,t})-\overline{F}_k\right] = \sum_{i\in I}\theta_{i,t}\sum_{k\in K_i} -\eta_{pq,t}SB_{pq}\left[\text{sign}(i-p)+\text{sign}(i-q)\right] - \sum_{k\in K}\eta_{pq,t}\overline{F}_k$

LHS (original) expression:

$$
\begin{aligned}
&\sum_{k\in K} \eta_{pq,t}\left[SB_{pq}(\theta_{p,t}-\theta_{q,t})-\overline{F}_k\right] \\
&= \eta_{12,t}\left[SB_{12}(\theta_{1,t}-\theta_{2,t})-\overline{F}_k\right] + \eta_{23,t}\left[SB_{23}(\theta_{2,t}-\theta_{3,t})-\overline{F}_k\right] \\
&\quad + \eta_{24,t}\left[SB_{24}(\theta_{2,t}-\theta_{4,t})-\overline{F}_k\right] + \eta_{34,t}\left[SB_{34}(\theta_{3,t}-\theta_{4,t})-\overline{F}_k\right] \\
&= \eta_{12,t}SB_{12}\theta_{1,t} - \eta_{12,t}SB_{12}\theta_{2,t} - \eta_{12,t}\overline{F}_k + \eta_{23,t}SB_{23}\theta_{2,t} - \eta_{23,t}SB_{23}\theta_{3,t} - \eta_{23,t}\overline{F}_k \\
&\quad + \eta_{24,t}SB_{24}\theta_{2,t} - \eta_{24,t}SB_{24}\theta_{4,t} - \eta_{24,t}\overline{F}_k + \eta_{34,t}SB_{34}\theta_{3,t} - \eta_{34,t}SB_{34}\theta_{4,t} - \eta_{34,t}\overline{F}_k \\
&= \theta_{1,t}\left[\eta_{12,t}SB_{12}\right] + \theta_{2,t}\left[-\eta_{12,t}SB_{12}+\eta_{23,t}SB_{23}+\eta_{24,t}SB_{24}\right] + \theta_{3,t}\left[-\eta_{23,t}SB_{23}+\eta_{34,t}SB_{34}\right] \\
&\quad + \theta_{4,t}\left[-\eta_{24,t}SB_{24}-\eta_{34,t}SB_{34}\right] - \eta_{12,t}\overline{F}_{12} - \eta_{23,t}\overline{F}_{23} - \eta_{24,t}\overline{F}_{24} - \eta_{34,t}\overline{F}_{34}
\end{aligned}
\tag{8}
$$

RHS (reformulated) expression:

$$
\begin{aligned}
&\sum_{i\in I}\theta_{i,t}\sum_{k\in K_i} -\eta_{pq,t}SB_{pq}\left[\text{sign}(i-p)+\text{sign}(i-q)\right] - \sum_{k\in K}\eta_{pq,t}\overline{F}_k \\
&= \theta_{1,t}\left[-\eta_{12,t}SB_{12}\left[\text{sign}(1-1)+\text{sign}(1-2)\right]\right] \\
&\quad + \theta_{2,t}\left[-\eta_{12,t}SB_{12}\left[\text{sign}(2-1)+\text{sign}(2-2)\right] - \eta_{23,t}SB_{23}\left[\text{sign}(2-2)+\text{sign}(2-3)\right] - \eta_{24,t}SB_{24}\left[\text{sign}(2-2)+\text{sign}(2-4)\right]\right] \\
&\quad + \theta_{3,t}\left[-\eta_{23,t}SB_{23}\left[\text{sign}(3-2)+\text{sign}(3-3)\right] - \eta_{34,t}SB_{34}\left[\text{sign}(3-3)+\text{sign}(3-4)\right]\right] \\
&\quad + \theta_{4,t}\left[\eta_{24,t}SB_{24}\left[\text{sign}(4-2)+\text{sign}(4-4)\right] - \eta_{34,t}SB_{34}\left[\text{sign}(4-3)+\text{sign}(4-4)\right]\right] \\
&= \theta_{1,t}\left[\eta_{12,t}SB_{12}\right] + \theta_{2,t}\left[-\eta_{12,t}SB_{12}+\eta_{23,t}SB_{23}+\eta_{24,t}SB_{24}\right] + \theta_{3,t}\left[-\eta_{23,t}SB_{23}+\eta_{34,t}SB_{34}\right] \\
&\quad + \theta_{4,t}\left[-\eta_{24,t}SB_{24}-\eta_{34,t}SB_{34}\right] - \eta_{12,t}\overline{F}_{12} - \eta_{23,t}\overline{F}_{23} - \eta_{24,t}\overline{F}_{24} - \eta_{34,t}\overline{F}_{34}
\end{aligned}
\tag{9}
$$

Note that the expanded expressions for (8) and (9) match.

#### 3.1.5 Re-writing: $\sum_{i\in I}\lambda_{i,t}\sum_{j\in I}B_{ij}(\theta_{i,t}-\theta_{j,t}) = \sum_{i\in I}\theta_{i,t}\sum_{j\in I}\left[\lambda_{i,t}B_{ij}-\lambda_{j,t}B_{ji}\right]$

$$
\begin{aligned}
&\sum_{i\in I}\lambda_{i,t}\sum_{j\in I}B_{ij}(\theta_{i,t}-\theta_{j,t}) \\
&= \lambda_{1,t}\left[B_{12}\theta_{1,t}-B_{12}\theta_{2,t}+B_{23}\theta_{2,t}-B_{23}\theta_{3,t}+B_{24}\theta_{2,t}-B_{24}\theta_{4,t}+B_{34}\theta_{3,t}-B_{34}\theta_{4,t}\right] \\
&\quad + \lambda_{2,t}\left[B_{12}\theta_{1,t}-B_{12}\theta_{2,t}+B_{23}\theta_{2,t}-B_{23}\theta_{3,t}+B_{24}\theta_{2,t}-B_{24}\theta_{4,t}+B_{34}\theta_{3,t}-B_{34}\theta_{4,t}\right] \\
&\quad + \lambda_{3,t}\left[B_{12}\theta_{1,t}-B_{12}\theta_{2,t}+B_{23}\theta_{2,t}-B_{23}\theta_{3,t}+B_{24}\theta_{2,t}-B_{24}\theta_{4,t}+B_{34}\theta_{3,t}-B_{34}\theta_{4,t}\right] \\
&\quad + \lambda_{4,t}\left[B_{12}\theta_{1,t}-B_{12}\theta_{2,t}+B_{23}\theta_{2,t}-B_{23}\theta_{3,t}+B_{24}\theta_{2,t}-B_{24}\theta_{4,t}+B_{34}\theta_{3,t}-B_{34}\theta_{4,t}\right] \\
&= \theta_{1,t}\left[\lambda_{1,t}B_{11}-\lambda_{1,t}B_{11}+\lambda_{1,t}B_{12}-\lambda_{2,t}B_{21}+\lambda_{1,t}B_{13}-\lambda_{3,t}B_{31}+\lambda_{1,t}B_{14}-\lambda_{4,t}B_{41}\right] \\
&\quad + \theta_{2,t}\left[\lambda_{2,t}B_{21}-\lambda_{1,t}B_{12}+\lambda_{2,t}B_{22}-\lambda_{2,t}B_{22}+\lambda_{2,t}B_{23}-\lambda_{3,t}B_{32}+\lambda_{2,t}B_{24}-\lambda_{4,t}B_{42}\right] \\
&\quad + \theta_{3,t}\left[\lambda_{3,t}B_{31}-\lambda_{1,t}B_{13}+\lambda_{3,t}B_{32}-\lambda_{2,t}B_{23}+\lambda_{3,t}B_{33}-\lambda_{3,t}B_{33}+\lambda_{3,t}B_{34}-\lambda_{4,t}B_{43}\right] \\
&\quad + \theta_{4,t}\left[\lambda_{4,t}B_{41}-\lambda_{1,t}B_{14}+\lambda_{4,t}B_{42}-\lambda_{2,t}B_{24}+\lambda_{4,t}B_{43}-\lambda_{3,t}B_{34}+\lambda_{4,t}B_{44}-\lambda_{4,t}B_{44}\right] \\
&= \sum_{i\in I}\theta_{i,t}\sum_{j\in I}\left[\lambda_{i,t}B_{ij}-\lambda_{j,t}B_{ji}\right]
\end{aligned}
\tag{10}
$$

#### 3.1.6 Re-writing: $\sum_{i\in N}\theta_{i,t}\zeta_{i,t} = \sum_{i\in I}U_{in}\theta_{i,t}\zeta_{i,t}$

Let $U^{I\times N}$ be matrix with entries defined by the following rule:

$$
U_{in} = \begin{cases} 1 \text{ if node } i \text{ is zone } n\text{'s reference node} \\ 0 \text{ otherwise} \end{cases}
\tag{11}
$$

For example, consider the case where nodes 1 and 2 belong to zone 1, and nodes 3 and 4 belong to zone 2. Let node 1 be zone 1's reference node, and node 3 be zone 2's reference node i.e. $N = \{1, 3\}$. $U^{I \times N}$ is given by (12).

$$U_{in} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \tag{12}$$

The original expression for terms in the Lagrangian that fix reference node angles is give by (13).

$$\sum_{i \in N} \theta_{i,t}\zeta_{i,t} = \theta_{1,t}\zeta_{1,t} + \theta_{3,t}\zeta_{3,t} \tag{13}$$

A reformulated expression is given by (14).

$$\begin{aligned} &\sum_{i \in I} \theta_{i,t} \sum_{n \in N} U_{in}\zeta_{i,t} \\ &= \theta_{1,t}\left[U_{11}\zeta_{1,t} + U_{12}\zeta_{1,t}\right] + \theta_{2,t}\left[U_{21}\zeta_{2,t} + U_{22}\zeta_{2,t}\right] + \theta_{3,t}\left[U_{31}\zeta_{3,t} + U_{32}\zeta_{3,t}\right] + \theta_{4,t}\left[U_{41}\zeta_{4,t} + U_{42}\zeta_{4,t}\right] \\ &= \theta_{1,t}\zeta_{1,t} + \theta_{3,t}\zeta_{3,t} \end{aligned} \tag{14}$$

Note that the expanded expressions for (13) and (14) match.

# 4 Lagrangian with re-formulated terms

The Lagrangian is now re-written using the reformulated terms.

$$\begin{aligned} \mathcal{L}(\Lambda, \Xi) = &\sum_{k \in K} \left[\underline{\theta}\gamma_{pq,t} - \overline{\theta}\delta_{pq,t} + \underline{F}_k\kappa_{pq,t} - \overline{F}_k\eta_{pq,t}\right] + \sum_{g \in G} \left[\underline{P}_g\alpha_{g,t} - \overline{P}_g\beta_{g,t}\right] + \sum_{i \in I} \left[D_{i,t} - Y_{i,t}\right]\lambda_{i,t} + \sum_{m \in M} \left[\underline{H}_m\omega_{m,t} - \overline{H}_m\psi_{m,t}\right] \\ &+ \sum_{g \in G} p_{g,t}\left[A_g + (E_g - \phi)\tau - \alpha_{g,t} + \beta_{g,t} - \lambda_{f(g)}\right] \\ &+ \sum_{i \in I} \theta_{i,t}\left[\sum_{k \in K_i} \left(\gamma_{pq,t} - \delta_{pq,t} + SB_{pq}\left[\kappa_{pq,t} - \eta_{pq,t}\right]\right)\left[\text{sign}(i-p) + \text{sign}(i-q)\right] + S\sum_{j \in I}\left[\lambda_{i,t}B_{ij} - \lambda_{j,t}B_{ji}\right] + \sum_{n \in N} U_{in}\zeta_{i,t}\right] \\ &+ \sum_{m \in M} h_{m,t}\left[\sum_{i \in I} \lambda_{i,t}C_{mi} + \psi_{m,t} - \omega_{m,t}\right] \end{aligned} \tag{15}$$

## 4.1 Lagrange-dual function

$$g(\Xi) = \inf_{\Lambda} \mathcal{L}(\Lambda, \Xi) \tag{16}$$

Objective is to maximize $g(\Xi)$.

## 4.2 Dual problem

$$\max_{\Xi} \sum_{k \in K} \left[\underline{\theta}\gamma_{pq,t} - \overline{\theta}\delta_{pq,t} + \underline{F}_k\kappa_{pq,t} - \overline{F}_k\eta_{pq,t}\right] + \sum_{g \in G} \left[\underline{P}_g\alpha_{g,t} - \overline{P}_g\beta_{g,t}\right] + \sum_{i \in I} \left[D_{i,t} - Y_{i,t}\right]\lambda_{i,t} + \sum_{m \in M} \left[\underline{H}_m\omega_{m,t} - \overline{H}_m\psi_{m,t}\right] \tag{17}$$

s.t.

$$A_g + (E_g - \phi)\tau - \alpha_{g,t} + \beta_{g,t} - \lambda_{f(g)} = 0 \quad \forall g \in G \tag{18}$$

$$\sum_{k \in K_i} \left(\gamma_{pq,t} - \delta_{pq,t} + SB_{pq}\left[\kappa_{pq,t} - \eta_{pq,t}\right]\right)\left[\text{sign}(i-p) + \text{sign}(i-q)\right] + S\sum_{j \in I}\left[\lambda_{i,t}B_{ij} - \lambda_{j,t}B_{ji}\right] + \sum_{n \in N} U_{in}\zeta_{i,t} = 0 \quad \forall i \in I \tag{19}$$

$$\sum_{i\in I}\lambda_{i,t}C_{mi}+\psi_{m,t}-\omega_{m,t}=0 \quad \forall m\in M \tag{20}$$

# 5 Reformulated program - MPPDC

Constraints from the primal and dual formulations are combined, and strong duality constraints are added by equating the objectives of these formulations. This yields the following single-level MPPDC:

$$\min_{\Theta,\Lambda,\Xi}\left\|\frac{\sum\limits_{t\in T}\sum\limits_{i\in I}\lambda_{i,t}L_tD_{i,t}}{\sum\limits_{t\in T}\sum\limits_{i\in I}L_tD_{i,t}}-Z\right\|_1 \tag{21}$$

s.t.
Upper-level primal constraints

$$\sum_{t\in T}\sum_{g\in G}(E_g-\phi)\tau p_{g,t}L_t\geq R \tag{22}$$

$$\frac{\sum\limits_{t\in T}\sum\limits_{g\in G}E_gp_{g,t}L_t}{\sum\limits_{t\in T}\sum\limits_{g\in G}D_{i,t}L_t}\leq X \tag{23}$$

Lower-level primal constraints

$$\underline{P}_g-p_{g,t}\leq 0 \quad (\alpha_{g,t}) \quad \forall g\in G \quad \forall t\in T \tag{24}$$

$$p_{g,t}-\overline{P}_g\leq 0 \quad (\beta_{g,t}) \quad \forall g\in G \quad \forall t\in T \tag{25}$$

$$\underline{\theta}-\theta_{p,t}+\theta_{q,t}\leq 0 \quad (\gamma_{pq,t}) \quad \forall k\in K \quad t\in T \tag{26}$$

$$\theta_{p,t}-\theta_{q,t}-\overline{\theta}\leq 0 \quad (\delta_{pq,t}) \quad \forall k\in K \quad \forall t\in T \tag{27}$$

$$\theta_i=0 \quad (\zeta_{i,t}) \quad \forall i\in N \quad \forall t\in T \tag{28}$$

$$S\sum_{j\in I}B_{ij}(\theta_{i,t}-\theta_{j,t})+\sum_{m\in M}C_{mi}h_{m,t}+D_{i,t}-Y_{i,t}-\sum_{g\in G_i}p_{g,t}=0 \quad (\lambda_{i,t}) \quad \forall i\in I \quad \forall t\in T \tag{29}$$

$$\underline{F}_k-SB_{pq}(\theta_{p,t}-\theta_{q,t})\leq 0 \quad (\kappa_{pq,t}) \quad \forall k\in K \quad \forall t\in T \tag{30}$$

$$SB_{pq}(\theta_{p,t}-\theta_{q,t})-\overline{F}_k\leq 0 \quad (\eta_{pq,t}) \quad \forall k\in K \quad \forall t\in T \tag{31}$$

$$\underline{H}_m-h_{m,t}\leq 0 \quad (\omega_{m,t}) \quad \forall m\in M \quad \forall t\in T \tag{32}$$

$$h_{m,t}-\overline{H}_m\leq 0 \quad (\psi_{m,t}) \quad \forall m\in M \quad \forall t\in T \tag{33}$$

Lower-level dual constraints

$$A_g+(E_g-\phi)\tau-\alpha_{g,t}+\beta_{g,t}-\lambda_{f(g)}=0 \quad \forall g\in G \quad \forall t\in T \tag{34}$$

$$\begin{aligned}&\sum_{k\in K_i}(\gamma_{pq,t}-\delta_{pq,t}+SB_{pq}\left[\kappa_{pq,t}-\eta_{pq,t}\right])\left[\operatorname{sign}(i-p)+\operatorname{sign}(i-q)\right]\\&\qquad+S\sum_{j\in I}\left[\lambda_{i,t}B_{ij}-\lambda_{j,t}B_{ji}\right]+\sum_{n\in N}U_{in}\zeta_{i,t}=0 \quad \forall i\in I \quad \forall t\in T\end{aligned} \tag{35}$$

$$\sum_{i \in I} \lambda_{i,t} C_{mi} + \psi_{m,t} - \omega_{m,t} = 0 \quad \forall m \in M \quad \forall t \in T \tag{36}$$

Strong duality constraints

$$\begin{aligned}
&\sum_{g \in G} \left[A_g + (E_g - \phi)\tau\right] p_{g,t} \\
&\quad = \sum_{k \in K} \left[\underline{\theta}\gamma_{pq,t} - \overline{\theta}\delta_{pq,t} + \underline{F}_k \kappa_{pq,t} - \overline{F}_k \eta_{pq,t}\right] + \sum_{g \in G} \left[\underline{P}_g \alpha_{g,t} - \overline{P}_g \beta_{g,t}\right] + \sum_{i \in I} \left[D_{i,t} - Y_{i,t}\right] \lambda_{i,t} \\
&\quad + \sum_{m \in M} \left[\underline{H}_m \omega_{m,t} - \overline{H}_m \psi_{m,t}\right] \quad \forall t \in T
\end{aligned} \tag{37}$$