Independent Study Final Report
Automatic Music Generation

Yuheng Ma

Mentor: Sayan Mukherjee, Anna Yanchenko

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1 Introduction

Automatic music generation, which means a process of using some formal process to make music with minimal human intervention [8], has centuries of history. An early example is Dice Music by Wolfgang Amadeus Mozart [5] in 18th century, a game involved assembling a number of small musical fragments and combining them randomly, piecing together a new piece. A notably wide range of method are implemented in algorithmic composition with the help of computers. A well-known early work on algorithmic composition is that Illiac Suite has utilized the classical rules for counterpoint in the generation of the first and second movements (Hiller & Isaacson, 1958) [7]. Since 2016, Google's Magenta [1] has been exploring the role of neural network and deep learning in the process of creating art and music. Hidden Markov Model(HMM) is a statistical model that has been utilized in many aspect of music analysis and generation. Farbood and Schoner (2001) [6] were the first to apply HMM to composing dealing with fixed melody patterns. This report is a part of future work of Anna Yanchenko's [11], which explored application of variations of HMM and time varying autoregression models in algorithmic composition of piano pieces in Romantic era, focusing especially on melody.

The main goal is to apply a variation of HMM called first order dilated convolution to compose piano pieces and evaluate the generated pieces' quality with respect to different target. We also adopt multiple input way to average influence of single piece and modify a style in general. Some result gives positive response to intuition of the model, while some remains to be further explored.

2 Data and Processing

To learn from input pieces and return generated pieces, the algorithm will need digital representation of music, which carries information about pitch, velocity, clock signal etc. A common format is the Musical Instrument Digital Interface (MIDI) [2]. However, if we directly use MIDI and regard all possible status at each time signature as state space, there will be too many possibilities. Thus underfitting may happen and also it takes more time to run. Also, it's not reasonable to combine notes and velocities since there is no direct relevance between. To be compatible with HMM, MIDI is transferred into a sequence of note pitches (either one note or chords) at discrete beat represented by one of finite many positive integer. This allows the modeling of notes. For single dilated convolution, we simply take exactly same timing information from original piece. For multiple sequence version, timing information are randomly taken from original pieces. As for velocities, interpolation method is applied. These together form a generated piece.

In order to adopt multiple input method, several pieces in similar composing style are required. The input pieces for this report are 20 Romantic piano pieces, listed in Appendix A. These are all originally MIDI files, transformed to CSV file [3] and then processed to sequences of integer notes [4].

3 Model

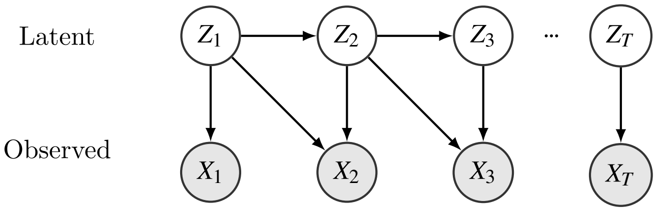
The new model and its multiple input version are derived from standard first order HMM.

First Order HMM Denote length of sequence as T, observation sequence and latent sequence as $\{X_1, \cdots, X_T\} := X_{1:T}$, $\{Z_1, \cdots, Z_T\} := Z_{1:T}$. Then the likelihood for the regular HMM is

$$P(X_{1:T}, Z_{1:T}|A, b, \pi) = P(Z_1|\pi)P(X_1|Z_1, b)\prod_{t=2}^{T} P(Z_t|Z_{t-1}, A)P(X_t|Z_t, b)$$

where T represents transition probability, b represents emmision probability and π represents initial distribution. The expectation maximum algorithm, which is to optimize the parameters, is Baum-Welch algorithm. [9]

Dilated Convolution Some interpretable variations can be made basing features of music. Besides an approximate distribution of notes and chords, we also want the model to grab some long-term structures, such as motifs. In regular HMM, the previous latent states influence next observation through next latent states which is by determining latent distribution and then emission distribution. We can allow a direct influence between last hidden states and observation to improve effect of a short past. The graphical model is as follows.



The model is called **Dilated Convolution** (**first order**). We are expecting a better performance of model in grabbing long-term pattern. Likelihood function for this model is

$$P(X_{1:T}, Z_{1:T}|A, b, \pi) = P(Z_1|\pi)P(X_1|Z_1, b)\prod_{t=2}^{T} P(Z_t|Z_{t-1}, A)P(X_t|Z_{t-1}, Z_t, b)$$

The expectation maximum algorithm for this model is in Appendix B.

Multiple Input In order to have sufficient data to make a reliable estimate of parameters, we may consider learning from multiple observation sequence. The joint likelihood is

$$P(X^1_{1:T}, \cdots, X^K_{1:T}, Z_{1:T}) = \prod_{k=1}^{K} P(X^k_{1:T}, Z_{1:T})$$

The expectation maximum algorithm for this model is in Appendix C.

We consider in total 4 models. First we generate by dilated convolution with 5 hidden states and 10 hidden states respectively. Then we apply multiple input version. All these model are compared with homologous first order model.

4 Result

4.1 Evaluation Metrics

The metrics we used in the following section are to evaluate the performance of generated pieces by criterion of Romantic style in the sense of originality, musicality and temporal structure. The originality tells about the relevance between generated and original pieces, as well as complexity of generated pieces. Empirical entropy, edit distance and mutual information are considered. Musicality metrics include counts of notes, dissonant harmonic intervals and melodic intervals, all normalized by length of the piece. We also use percentage of intervals that are perfect consonances, imperfect consonances and dissonances. For temporal structure, PACF and ACF are metrics representing decay correlation of the system.

After learning and optimizing the parameters by Baum-Welch, 1000 new pieces are generated. Each metric of the algorithm (except mutual information and edit distance) takes the optimized value regrading RMSE as loss function. For mutual information and edit distance, we take their average over all pieces.

4.2 Dilated Convolution Single Input

For the single input version, we take ”Pachelbel” as an example and set number of hidden states 5 as default. As previously mentioned, the intuition of dilated convolution model is to grab a longer-term structure. As a result, ACF/PACF, which can reflect whether sequences have a high degree of global structure. The expected ACF/PACF plot of dilated convolution have higher lags than first order plot.

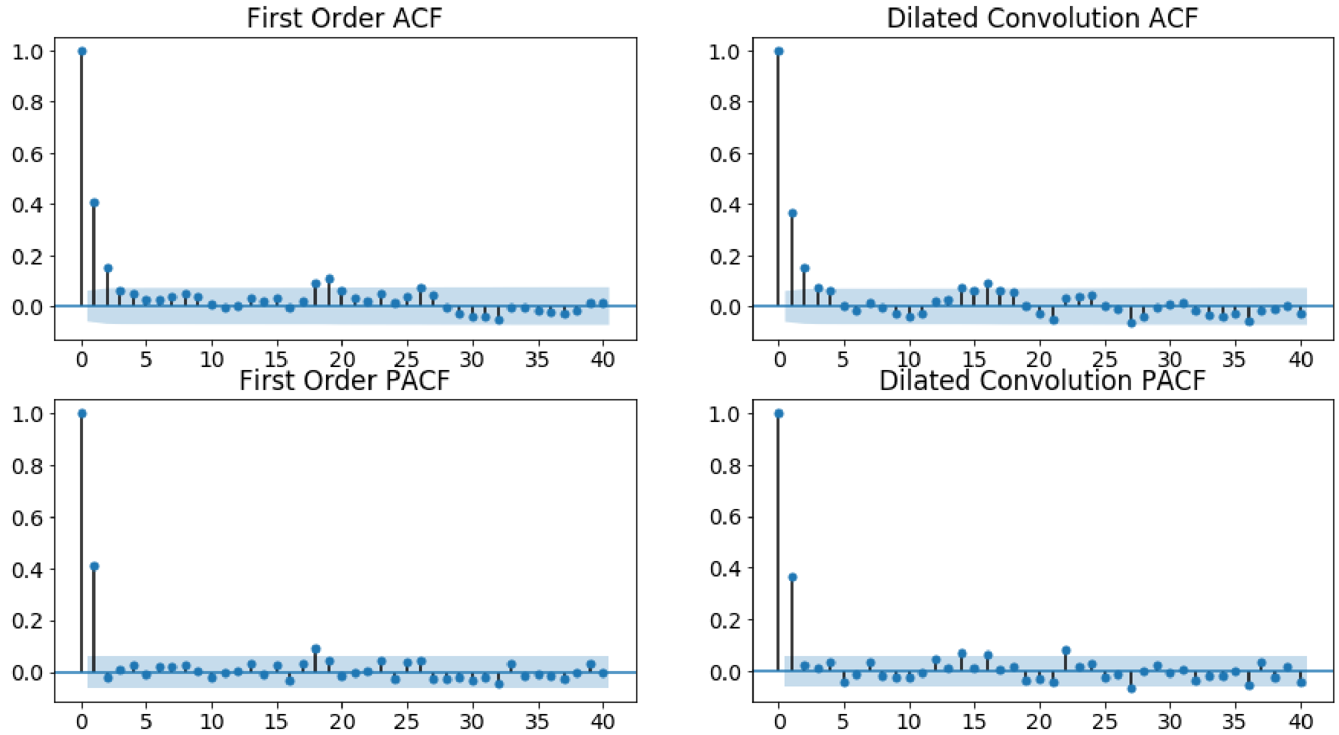


Figure 1: ACF/PACF plot for first order and dilated convolution training on Pachelbel.

From Figure 1, dilated convolutions model shows some persistence in ACF/PACF, but it’s not sufficient to draw the conclusion that this modification of first order model leads to better global structure. The other calculated metrics are listed in Table 1.

	Entropy	Mutual Info	Edit	H Intervals	M Intervals	Percent	Note Count
First-Order Pachelbel	0.04029	0.33813	0.87228	68.16021	94.17695	0.16198	0.00829
Dilated-Convolution Pachelbel	0.02361	0.33659	0.87317	68.90035	95.73763	0.16175	0.00614

Table 1: Top metrics trained from Pachelbel. Metrics are successively empirical entropy, mutual information, edit distance, count of harmonic intervals normalized by piece length, count of harmonic intervals normalized by piece length, percentage of intervals and count of notes.

From the table, we can see that dilated convolution model generally didn’t outperform first order model or even worse. What relatively significant are empirical entropy and count of harmonic intervals. This is reasonable since dilated convolution tend to generate a system with less freedom. Meanwhile, the result in Table 2 illustrates that comparison result depend not only on algorithms but also on original training pieces, thus it is more proper to compare multiple input performance. Also, a promising future work is about relationship between properties of original pieces and metric performance, which eventually answer the question that what kind of Romantic era musics are best materials to learn from.

	Entropy	Mutual Info	Edit	H Intervals	M Intervals	Percent	Note Count
First-Order Ode-to-Joy	0.03283	0.16695	0.71798	22.15956	36.73144	0.13401	0.00943
Dilated-Convolution Ode-to-Joy	0.02930	0.16639	0.69042	22.14071	36.90521	0.13417	0.00876

Table 2: Top metrics trained from Ode to joy.

Interpretation explain caught The hidden states have specific meanings representing the underlying dynamics of the piece. Decompose the note into single pitches and look at the emission matrix (the matrix of dilated convolution is three dimension with entries $b_{ijk} = p(P_t = k|Z_{t-1} = i, Z_t = j)$ where P_t is some pitch, so we fix the second index j). For instance, if a hidden state only has high emission distribution of (C) and (G), then the state catch the perfect fifth. As is shown in Figure 2, fixing $Z_{t-1} = 0$, the hidden states 1 catches D3, 2 catches G chord, 3 catches Gmaj 7 and 4 catches A2.

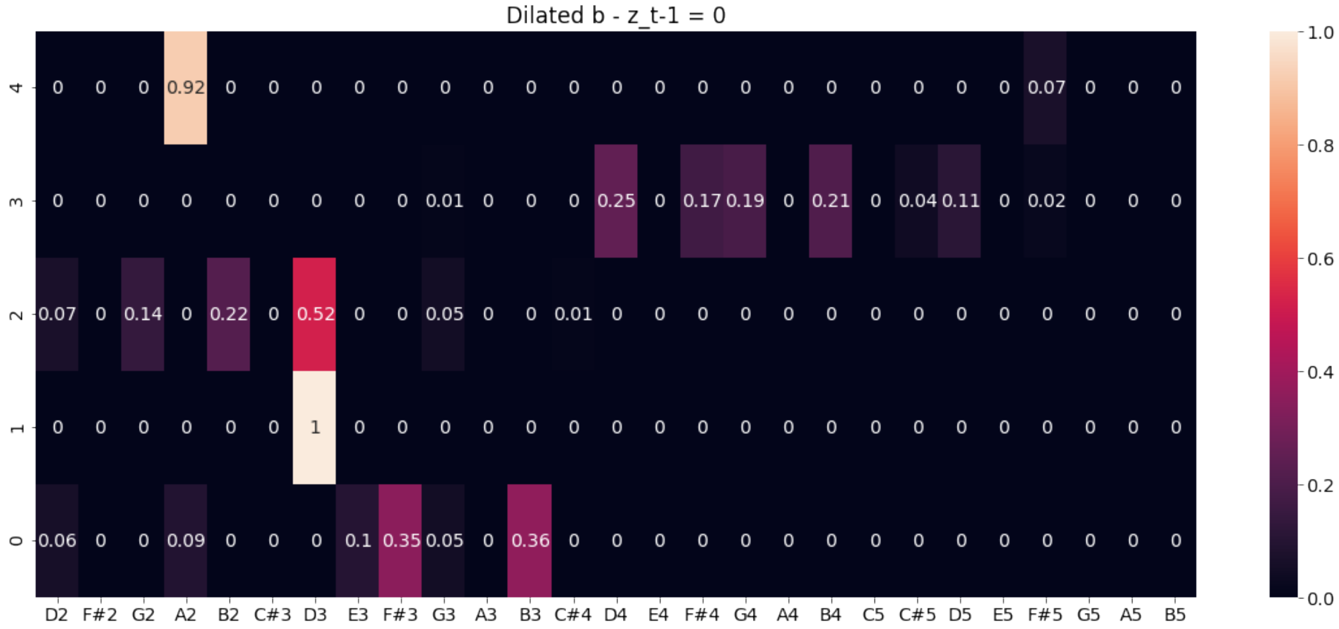


Figure 2: Emission distribution of pitches fixing Z_{t-1}=0

4.3 Dilated Convolution Multiple Input

Difference in Metric Calculation When calculating metrics, edit distance and mutual information need that two pieces being compared have the same length. Different timing information were randomly assigned to generated pieces. Thus, edit distance and mutual information of each generated piece is calculated with the corresponding original piece. Due to the large scale of data, we calculate KL-divergence between distribution of metrics on original pieces and on all generated pieces for other metrics.

Label Switching Baum-welch for multiple input takes weighted average of each set of parameters trained from one original pieces. However, directly doing so might ruin the subtle structure that we already have before averaging. Because of randomness of initial value, even if the hidden states caught the same thing, transition matrix might look completely different. For example, if five hidden states catch successively perfect five, sixth chord, seventh chord, ninth chord and eleventh chord, it's unreasonable to average this transition matrix with another one which catch sixth chord, seventh chord, ninth chord, eleventh chord and perfect five.

Resolution lies in find the permutation of hidden states. One way is by fixing one transition matrix and permute others that each of them has the minimum distance to ruler matrix. We can also reorder the hidden states that the first has the highest probability in latent sequence, and the last has the lowest. Then averaging all parameters seems reasonable.

The calculated metrics are listed in Table 3.

	Entropy	Harmonic Intervals	Melodic Intervals	Percent	ACF	PACF
HMM-5	11.0815	3.0508	4.0215	8.054	4.9671	2.5001
HMM-10	11.0815	3.0172	3.9405	7.8203	5.1167	2.4965
Dilated-5	10.9881	4.5698	7.5747	9.4979	5.1253	2.511
Dilated-10	10.9882	4.8365	7.6679	9.8692	5.1895	2.5347

Table 3: KL-divergence of between distribution of metrics of multiple input pieces and generated pieces

Noting that KL-divergence is distance between distribution, the first order HMM generally outperformed dilated convolution in there metrics. But speaking of number of hidden states, it seems that 10 hidden states didn't do as well as 5 in dilated model. This remains to be explored.

	First-Order-5	First-Order-10	Dilated-Convolution-5	Dilated-Convolution-10
Mutual Information	0.2637	0.2629	0.7162	0.6966
Edit Distance	0.915	0.916	0.9633	0.9641

Table 4: Mutual information and edit distance comparison of the four model

As we can see from Table 4, dilated model carries much more information about original pieces, while edit distance is also high. Combining Figure 3, dilated model tends to take information from input but generate pieces with more complexity. This is reasonable because 20 pieces are put in. It's easy for the model to conserve their information, but using 20 times large information to recompose will lead to a mess.

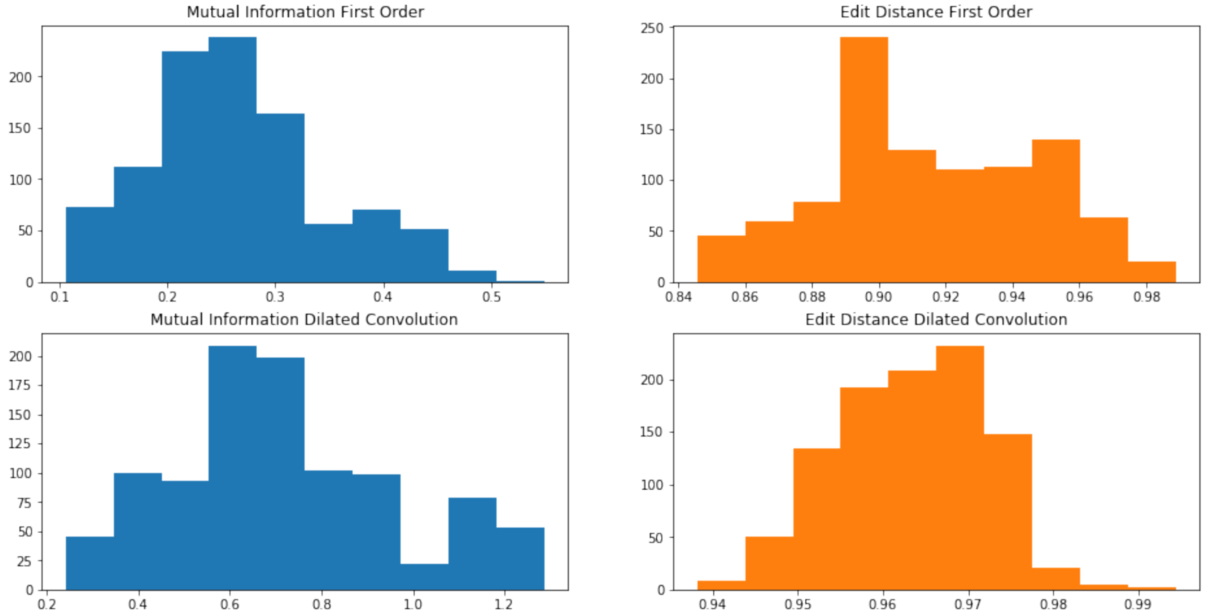


Figure 3: Distribution of mutual information and edit distance. First and second line are respectively first order model and dilated convolution model

5 Conclusion and Future Work

We applied dilated convolution model on pieces from Romantic era. The model generally satisfied the propose of finding long term structure with higher ACF/PACF, but it's not significant. The generated pieces appear melodic progressions sometimes, but still remain in short-term pattern. Based on the results, some future work directions are considered.

- The emission probability can be extended to depend on more latent states, which might help to find significantly better performance in global structure or melodic progression.

- The way to resolve label switching issue can be reconsidered to be more interpretable. Further more, even if we adopted the right way, averaging over all pieces is still unreasonable. For instance, consider averaging two set of parameters such that one from five hidden states catching successively perfect five, sixth chord, seventh chord, ninth chord and eleventh chord, one from five hidden states catching A2, B2, D3, A3 and D4. These requires close meaning of hidden states, which leads to the third direction.
- We tend to find what kind of pieces best to learn from in the sense of musicality (count of melodic intervals) and complexity (empirical entropy). Also, we want to know what qualities makes it reasonable to put some pieces together for multiple input in the sense of musicality (unique notes and melodic intervals).

References

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[8] John A. Maurer. A brief history of algorithmic composition. https://ccrma.stanford.edu/~blackrse/algorithm.html, March 1999.

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[10] Anna K. Yanchenko. Music theory. https://aky4wn.github.io/Classical-Music-Composition-Using-State-Space-Models/.

[11] Anna K. Yanchenko and Sayan Mukherjee. Classical music composition using state space models, 2017.

Appendices

A Musical Resource

Name	Composer	Name	Composer
Ode to Joy	Beethoven	Once in Royal David's City	
Carol of the Bells		Pachelbel's Canon	
Deutschlandlied		Shall we Gather at the River	
God Rest you Merry Gentlemen		Song without Words - 06	Mendelssohn
Greensleeves		Swing Low, Sweet Chariot	
Hark! The Herald Angels Sing	Mendelssohn	Beautiful Blue Danube, Theme	Strauss, Jr.
I Vow to Thee, My Country	Holst	Third Mode Melody	Tallis
In the Bleak Midwinter	Holst	Twinkle, Twinkle, Little Star	
Pictures at an Exhibition - Gnomus	Mussorgsky	We Three Kings	
Old 100th		When Jonny Comes Marching Home	

Table 5: List of input pieces and composer (if available)

For reference in musical theory, please check [10].

B Baum-welch for Dilated Model

B.1 Expectation

Generally we need to know $P(X_{1:T})$ for future computation. From property of HMM we know

$$P(X_{1:T},Z_{1:T}) = P(Z_1)P(X_1|Z_1)\prod_{t=2}^{T}P(Z_t|Z_{t-1})P(X_t|Z_{t-1},Z_t)$$

Thus we repeat same operation,

$$\begin{aligned}
P(X_{1:T}) &= \sum_{Z_{1:T}} P(X_{1:T},Z_{1:T}) \\
&= \sum_{Z_{1:T}} \underbrace{P(Z_1)P(X_1|Z_1)}_{S_1(Z_1)}\prod_{t=2}^{T}P(Z_t|Z_{t-1})P(X_t|Z_{t-1},Z_t) \\
&= \sum_{Z_{2:T}} \underbrace{\left(\sum_{Z_1} S_1 P(Z_2|Z_1)P(X_2|Z_1,Z_2)\right)}_{S_2(Z_2)}\prod_{i=3}^{T}P(Z_t|Z_{t-1})P(X_t|Z_{t-1},Z_t) \\
&\dots \\
&= \sum_{Z_{j+1:T}} \underbrace{\left(\sum_{Z_j} S_j P(Z_{j+1}|Z_j)P(X_{j+1}|Z_j,Z_{j+1})\right)}_{S_{j+1}(Z_{j+1})}\prod_{t=j+2}^{T}P(Z_t|Z_{t-1})P(X_t|Z_{t-1},Z_t)
\end{aligned}$$

By same kind of trick, we also have

$$\begin{aligned}
P(X_{1:T}) &= \sum_{Z_{1:T}} P(X_{1:T}, Z_{1:T}) \\
&= \sum_{Z_{1:T-1}} \underbrace{\sum_{Z_T} P(Z_T|Z_{T-1})P(X_T|Z_{T-1},Z_T)}_{R_{T-1}(Z_{T-1})} P(Z_1)P(X_1|Z_1) \prod_{t=2}^{T-1} P(Z_t|Z_{t-1})P(X_t|Z_{t-1},Z_t) \\
&= \sum_{Z_{1:T-1}} \underbrace{\Big(\sum_{Z_{T-1}} R_{T-1}P(Z_{T-1}|Z_{T-2})P(X_{T-1}|Z_{T-2},Z_{T-1})\Big)}_{R_{T-2}(Z_{T-2})} P(Z_1)P(X_1|Z_1) \prod_{i=2}^{T-2} P(Z_t|Z_{t-1})P(X_t|Z_{t-1},Z_t) \\
&\cdots \\
&= \sum_{Z_{1:j}} \underbrace{\Big(\sum_{Z_j} R_j P(Z_j|Z_{j-1})P(X_j|Z_{j-1},Z_j)\Big)}_{R_{j-1}(Z_{j-1})} P(Z_1)P(X_1|Z_1) \prod_{t=2}^{j-1} P(Z_t|Z_{t-1})P(X_t|Z_{t-1},Z_t)
\end{aligned}$$

Thus we decomposed $P(X_{1:T})$ as function of $P(Z_{j+1}|Z_j)$, $P(X_{j+1}|Z_j,Z_{j+1})$(or $P(X_1|Z_1)$) and $P(Z_1)$ in two ways. Then we set the parameter vector θ as

- $\pi = (\pi_1, \cdots, \pi_m)$, where $\pi_i = P(Z_1 = i)$
- $\phi_0 = \Big(b_{in}\Big)_{m\times k}$, where $b_{in} = P(X_1 = n|Z_1 = i)$
- $\phi = \big(b_{ijn}\big)_{m\times m\times k}$, where $b_{ijn} = P(X_t = n|Z_{t-1} = i, Z_t = j)$
- $T = \Big(t_{ij}\Big)_{m\times m}$, where $t_{ij} = P(Z_{t+1} = j|Z_t = i)$
- $\theta = (\pi, \phi_0, \phi, T)$

And the two notations, S and R, have specific meanings and we introduce α and β here. (If having priors θ, we may condition on it).

$$\alpha_t(Z_t) := S_t(Z_t) = P(X_1, \cdots, X_t, Z_t|\theta)$$
$$\beta_t(Z_t) := R_t(Z_t) = P(X_T, \cdots, X_{t+1}|Z_t, \theta)$$

They are easy to see by checking the first term and induction relations:

$$\alpha_t(Z_t) = \sum_{Z_{t-1}} \alpha_{t-1}P(Z_t|Z_{t-1})P(X_t|Z_{t-1},Z_t)$$
$$\beta_t(Z_t) = \sum_{Z_{t+1}} \beta_{t+1}P(Z_{t+1}|Z_t)P(X_{t+1}|Z_t,Z_{t+1})$$

We also tend to find ξ and γ that

$$\xi_t(Z_t, Z_{t+1}) = P(Z_t, Z_{t+1}|X_{1:T}, \theta)$$
$$\gamma_t(Z_t) = P(Z_t|X_{1:T}, \theta)$$

Clearly, γ can be acquired by summing over Z_{t+1} of ξ and

$$\begin{aligned}
\xi_t(Z_t, Z_{t+1}) =& P(Z_t, Z_{t+1}|X_{1:T}, \theta) \\
=& \frac{P(Z_t, Z_{t+1}, X_{1:T}|\theta)}{P(X_{1:T}|\theta)} \\
=& \frac{P(X_{1:t}, Z_t|\theta)P(Z_{t+1}|Z_t, \theta)P(X_{t+2:T}|Z_{t+1}, \theta)P(X_{t+1}|Z_t, Z_{t+1})}{P(X_{1:T}|\theta)} \\
=& \frac{\alpha_t(Z_t)T_{Z_t,Z_{t+1}}\beta_{t+1}(Z_{t+1})\phi(Z_t, Z_{t+1}, X_{t+1})}{\sum_{Z_t,Z_{t+1}} \alpha_t(Z_t)T_{Z_t,Z_{t+1}}\beta_{t+1}(Z_{t+1})\phi(Z_t, Z_{t+1}, X_{t+1})}
\end{aligned}$$

B.2 Maximization

Then we will use these terms for Baum-Welch algorithm. Recall that

$$Q(\theta,\theta_k) = \mathbb{E}_{\theta_k}\left(\log p(X,Z|\theta)|X\right)$$

thus followed by

$$\begin{aligned}
\log P(X,Z|\theta) =& \log p(Z_1|\theta) + \sum_{t=1}^{T-1} \log p(Z_{t+1}|Z_t,\theta) + \log p(X_1|Z_1,\theta) + \sum_{t=1}^{T-1} \log p(X_{t+1}|Z_{t+1},Z_t,\theta) \\
=& \sum_{i=1}^{m} 1(Z_1 = i)\log \pi_i + \sum_{t=1}^{T-1}\sum_{i=1}^{m}\sum_{j=1}^{m} 1(Z_t = i, Z_{t+1} = j)\log T_{ij} \\
&+ \sum_{i=1}^{m} 1(Z_1 = i)\log \phi_0(i,X_1) + \sum_{t=1}^{T-1}\sum_{i=1}^{m}\sum_{j=1}^{m} 1(Z_t = i, Z_{t+1} = j)\log \phi_{ij}(X_{t+1})
\end{aligned}$$

Expectation of indicator is just probability of condition in it, thus

$$\begin{aligned}
Q(\theta,\theta_k) =& \sum_{i=1}^{m} P_{\theta_k}(Z_1 = i|X)\log \pi_i + \sum_{t=1}^{T-1}\sum_{i=1}^{m}\sum_{j=1}^{m} P_{\theta_k}(Z_t = i, Z_{t+1} = j|X)\log T_{ij} \\
&+ \sum_{i=1}^{m} P_{\theta_k}(Z_1 = i|X)\log \phi_0(i,X_1) + \sum_{t=1}^{T-1}\sum_{i=1}^{m}\sum_{j=1}^{m} P_{\theta_k}(Z_t = i, Z_{t+1} = j|X)\log \phi_{ij}(X_{t+1})
\end{aligned}$$

And by out definition, this is

$$Q(\theta,\theta_k) = \sum_{i=1}^{m} \gamma_{1i}\log \pi_i\phi_0(i,X_1) + \sum_{t=1}^{T-1}\sum_{i,j=1}^{m} \xi_{tij}\log T_{ij} + \sum_{t=1}^{T-1}\sum_{i,j=1}^{m} \xi_{tij}\log \phi_{ij}(X_{t+1})$$

By method of Lagrangian multipliers, we have

$$\begin{aligned}
\hat{\pi}_i =& \frac{\gamma_{1i}}{\sum_{j=1}^{m}\gamma_{1j}} = \gamma_{1i} \\
\hat{T_{ij}} =& \frac{\sum_{t=1}^{T-1}\xi_{tij}}{\sum_{t=1}^{T-1}\sum_{j=1}^{m}\xi_{tij}} = \frac{\sum_{t=1}^{T-1}\xi_{tij}}{\sum_{t=1}^{T-1}\gamma_{ti}} \\
\hat{\phi_{0in}} =& \begin{cases} 1 & n = X_1 \\ 0 & otherwise \end{cases} \\
\hat{\phi_{ijn}} =& \frac{\sum_{X_{t+1}=n}\xi_{ijt}}{\sum_{t=1}^{T-1}\xi_{ijt}}
\end{aligned}$$

Notice that optimization of ϕ_0 is rather simple, we can solve this by using multiple input.

C Baum-welch for Multiple Input

When we have multiple observations, we tend to average them in some sense to get better approximation. We consider different way of optimizing following

$$\begin{aligned}
\pi_i =& P(Z_1 = i) \\
b_{0in} =& P(X_1 = n|Z_1 = i) \\
b_{ijn} =& P(X_t = n|Z_{t-1} = i, Z_t = j) \\
t_{ij} =& P(Z_{t+1} = j|Z_t = i)
\end{aligned}$$

For multiple input, we have

$$\begin{aligned}
P(X|\theta) =& \prod_{k=1}^{K} P\left(X^k|\theta\right) \\
=& \prod_{k=1}^{K} P_k
\end{aligned}$$

Since observations are independent, we have following result and by maximizing every single P_k, we get our multiple input approximation.

$$\begin{aligned}
t_{ij} &= P\left(Z_{t+1} = j | Z_t = i, (X^1, \cdots, X^K)\right) \\
&= \frac{P\left(Z_{t+1} = j, Z_t = i | (X^1, \cdots, X^K)\right)}{P(Z_t = i | (X^1, \cdots, X^K))} \\
&= \frac{\sum_j P\left(Z_{t+1} = j, Z_t = i | X^j) P(X_j | (X^1, \cdots, X^K)\right)}{\sum_j P(Z_t = i | X_j) P(X^j | (X^1, \cdots, X^K))} \\
&= \frac{\frac{1}{K} \sum_j P(Z_{t+1} = j, Z_t = i | X^j)}{\frac{1}{K} \sum_j P(Z_t = i | X^j)} \\
&= \frac{\sum_j P(Z_{t+1} = j, Z_t = i, X^j) / P(X^j)}{\sum_j P(Z_t = i, X^j) / P(X^j)}
\end{aligned}$$

Thus

$$\bar{t}_{ij} = \frac{\sum_{k=1}^{K} \frac{1}{P_k} \sum_{t=1}^{T_k - 1} \alpha_t^k(i) t_{ij} b_{ij} \left(Z_{t+1}^{(k)}\right) \beta_j^k(Z_{t+1})}{\sum_{k=1}^{K} \frac{1}{P_k} \sum_{t=1}^{T_k - 1} \alpha_t^k(i) \beta_t^k(i)}$$

and by same way

$$\begin{aligned}
\overline{b_{ijn}} &= \frac{\sum_{k=1}^{K} \frac{1}{P_k} \sum_{X_{t+1} = n} \alpha_t^k(i) \beta_{t+1}^k(j)}{\sum_{k=1}^{K} \frac{1}{P_k} \sum_{t=1}^{T_k - 1} \alpha_t^k(i) \beta_{t+1}^k(j)} \\
b_{0in} &= \frac{\sum_{X^k = n} 1}{K} \\
\pi_i &= \frac{\sum_k \pi_i^k}{K}
\end{aligned}$$