The L^p-Conjecture and Young's Inequality

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The L^p-Conjecture: Let G be a locally compact group. If L^p is closed under convulution for some $p \in (1, \infty)$ (i.e. $f * g \in L^p(G)$ for all $f, g \in L^p(G)$), then G is compact.

Young's Inequality for Convolution: Let G be a locally compact unimodular group. Let p, q be real numbers such that $1 < p, q < \infty$, and $1/p + 1/q > 1$, and let r be defined by $1/r = 1/p + 1/q - 1$. Then

1. $L^p(G) * L^q(G) \subset L^r(G)$, and

2. for $f \in L^p(G)$ and $g \in L^q(G)$, we have

$$\|f * g\|_r \leq \|f\|_p \|g\|_q$$

History of the L^p-Conjecture

1. G abelian, Zelazko (1961)
2. G arbitrary with $p > 2$, Zelazko (1963) and Rajagopalan (1966)
3. G discrete with $p \geq 2$, G totally diconnected with $p = 2$, or G either nilpotent or a semi-direct product of two LCA groups, by Rajagopalan (1963, 1966, 1967)
4. G solvable and $p > 1$, Rajagopalan and Zelazko (1965)
5. G arbitrary with $p = 2$ by Rickert (1968)
6. G amenable with $p > 1$ by Greenleaf (1969)
7. G arbitrary, with $p \in (1, \infty)$ by Saeki (1990)

Notation and Definitions

For $p \in [1, \infty]$, we let $p' = p/(p-1)$ for $p > 1$ and $p' = 1$ when $p = \infty$.

For λ, a left Haar measure on G, we write $|A|$ for $\lambda(A)$ whenever A is a Haar measurable subset of G.

For any function f on G, we define $f^{\vee}$ by $f^{\vee}(x) := f(x^{-1})$.

A function f is called symmetric if $f^{\vee} = f$.

We also define $L^p_s := \{f \in L^p : f^{\vee} = f\}$.

Lemma 1.1. Let A be a compact subset of the general locally compact group G. Then we have

$$|A|^2|A^{m+n}| \leq |A^4| \cdot |A^m| \cdot |A^n| \text{ for } m, n \geq 1.$$

Lemma 1.2. Let $p, q, r \in [1, \infty]$ be such that $1/p + 1/q - 1/r \neq 1$. Suppose that $L_s^p * L_s^q \subset L^r$. Then G is unimodular, $L^p * L^q \subset L^r$, and there exists constant $0 < c_o < \infty$ such that

$$\|f * g\|_r \leq c_o \|f\|_p \cdot \|g\|_q \text{ for } f \in L^p, g \in L^q.$$

Lemma 1.3. Let p, q, r and c_o be as in Lemma 1.2. Then

$$(|A| \cdot |B|)^{1/p' + 1/q'} \leq c_o^2 |AB|^{2/r'}$$

for all compact $A, B \subset G$.

Theorem 1. Suppose that there exists $p \in (1, \infty)$ such that $f * g \in L^p(G)$ for all symmetric $f, g \in L^p(G)$. Then G is compact.

On Young's Inequality

Let $p, q, r \in [0, \infty]$. If $1/r = 1/p + 1/q - 1$, we have

$$\|f * g\|_r \leq \|f\|_p \max(\|g\|_q, \|g^\vee\|_q) \text{ for } f \in L^p_+, g \in L^q_+$$

by Young's Inequality (see Theorem (20.18) of Hewitt and Ross, *Abstract harmonic analysis*, Vol. I). It is well known that for $s \geq r \geq 1$, $L^r \subset L^s$ when G is discrete, and $L^s \subset L^r$ when G is compact. Together, the above statements give us that

1. $\frac{1}{r} \leq \frac{1}{p} + \frac{1}{q} - 1$ implies that $L^p * L^q \subset L^r$ for G discrete, and

2. $\frac{1}{r} \geq \frac{1}{p} + \frac{1}{q} - 1$ implies that $L^p * L^q \subset L^r$ for G compact.

Questions

1. If $\frac{1}{r} < \frac{1}{p} + \frac{1}{q} - 1$ and $L^p * L^q \subset L^r$, is G discrete?

2. If $\frac{1}{r} > \frac{1}{p} + \frac{1}{q} - 1$ and $L^p * L^q \subset L^r$, is G compact?

Answers

Both are true for abelian groups, Quek and Yap (1983).
The answer to the second question is "no", for if $G = SL(2, \mathbf{R})$ and $1 \leq p < 2$, then $L^p * L^2 \subset L^2$, Kunze and Stein (1960).

Let $\|\cdot\|_u$ denote the uniform norm, $\|f\|_u = \sup\{|f(x)| : x \in G\}$. Define a complete norm $\|\cdot\|_{p,u}$ on $L^p_s \cap C_0^+(G)$ by

$$\|f\|_{p,u} = \max\{\|f\|_p, \|f\|_u\}.$$

Lemma 2.1. Suppose that $p, q, r \in [1.\infty]$, $p > 1$ and G satisfies $(L^p \cap C_0) * (L^q \cap C_0) \subset L^r$. Then G is unimodular, and there exists $0 < c_1 < \infty$ such that

$$\|f * g\|_r \leq c_1 \|f\|_{p,u} \|g\|_{q,u} \text{ for } f \in L^p \cap C_0, g \in L^q \cap C_0.$$

If, in addition, G is compact, then $r \geq \max\{p, q\}$.

Lemma 2.2. Let G, p, q, r be as in Lemma 2.1. Then we have

$$(|A| \cdot |B|)^{1/p' + 1/q'} \leq c_1^2 |AB|^{2/r'}$$

for all compact $A, B \subset G$ with $|A|, |B| \leq 1$.

Theorem 2. Suppose that the noncompact group G is such that given any $\epsilon > 0$, there exists a compact $A \subset G$, with large enough $|A|$, such that

$$\liminf_{n\to\infty} n^{-1} \log\log |A^{2^n}| < \epsilon.$$

Let $1 < p < \infty$. Then there exists $f \in L^p_s \cap C_0^+(G)$ such that

$$f * L^q_s \not\subset L^r$$

for all $r, q \in [1, \infty]$ satifying $\frac{1}{r} > \frac{1}{p} + \frac{1}{q} - 1$.

Corollary 2.3. Let $p, q, r \in [1, \infty]$ and $p > 1$. Suppose that G is an infinite LCA group and that $L^p * L^q \subset L^r$. Then,

1. If G is discrete, then $1/r \leq 1/p + 1/q - 1$.
2. If G is compact, then $1/r \geq 1/p + 1/q - 1$.
3. If G is neither discrete nor compact, then $1/r = 1/p + 1/q - 1$.