Fourier Analysis on Coset Spaces

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1 Introduction

G a locally compact group, H a closed subgroup of G. We define and study natural analogs of the Fourier and Fourier-Stieltjes algebras for G/H, and show that when H is compact, $A(G/H)$ can be used to study the nature of G/H in a manner similar to the group case.

2 Defining $A(G/H)$ and $B(G/H)$

In this section, definitions for $A(G/H)$ and $B(G/H)$ are given, and it is shown that these definitions are useful analogs of the Fourier and Fourier-Stieljes algebras for groups.

Notation
Let H be a closed subgroup of G. $q : G \to G/H$ denotes the canonical quotient map. $\tilde{x}$ denotes the left coset xH. We have an isomorphism between $C(G/H)$ and $C(G : H) = \{f \in C(G) : f(xh) = f(x)\ \forall x \in G,\ h \in H\}$ via the map $\tilde{f} \mapsto f$, where $f = \tilde{f} \circ q$. We denote the equivalence class of all continuous unitary representations of G by Σ_G. For $\pi \in \Sigma_G$, we let A_π denote the closed linear span of the coefficient functions of π, and we denote the weak-* closure of A_π by B_π. For ρ, the left regular representation of G on $L_2(G)$, A_ρ is usually denoted $A(G)$.

Definition 1

$$B(G : H) = \{u \in B(G) : u(xh) = u(x)\ \forall\ x \in G,\ h \in H\},$$

$$A(G : H) = \{u \in B(G : H) : q(supp\ u)\ compact\ in\ G/H\}^{-\|\cdot\|_{B(G)}}.$$

Proposition 2 *(i) $B(G:H)$,$A(G:H)$ are closed subalgebras of $B(G)$. Moreover, $A(G:H)$ is a closed ideal in $B(G:H)$.*
(ii) $B(G:H)$ is unital.
(iii) $A(G:H) \cap A(G) \neq \{0\}$ iff H is compact.
(iv) $A(G:H) = B(G:H)$ iff G/H is compact.

It is known that $A(G:H)$ is isometrically isomorphic to $A(G/H)$ when H is compact and normal. The compactness of H is not necessary:

Proposition 3 *Let H be a closed normal subgroup of G. Then $B(G:H)$ and $A(G:H)$ are isometrically isomorphic to $B(G/H)$ and $A(G/H)$ respectively.*

It is shown below that for compact K, $B(G:K)$ and $A(G:K)$ are complemented subspaces of $B(G)$, $A(G)$ respectively.

Theorem 4 *Let H be a closed subgroup of G. Then there exists a projection $P : B(G) \to B(G:H)$ with $\|P\| \leq 1$.*

In general, P does not map $A(G)$ onto $A(G:H)$. However, for compact subgroup K, we have

$$P(f)(x) = P_K(f)(x) := \int_K f(xk)dk.$$

Thus we have the following corollary:

Corollary 5 *Let K be a compact subgroup of G. Then P_K is a continuous projection of $B(G)$ onto $B(G:K)$. The restriction of P_K to $A(G)$ is a projection of $A(G)$ onto $A(G:K)$.*

The analog of the Fourier Algebra for the coset space G/H is usually considered to be the space A_{π_H}, where π_H is the quasi-regular representation of G determined by H. This definition has two major problems.

Problem I. A_{π_H} is in general not an algebra. For example, when K is a compact subgroup, A_{π_H} is an algebra iff $A_{\pi_H} = A_{\pi_{K_1}}$, where $K_1 = \cap_{x\in G} xKx^{-1}$. Since K_1 is normal, $A_{\pi_{K_1}} = A(G/K_1)$ (Arsac, 1976). From these results it follows that for K compact, $A(G:K) \neq A_{\pi_H}$ unless K is normal.

A little more can be said:

Proposition 6 *Let K be a compact subgroup of G. Then $A(G:K) = A_\pi$ for some $\pi \in \Sigma_G$ iff K is normal.*

Problem II. It is possible to have two distinct closed subgroups H_1, H_2, and yet $A_{\pi_{H_1}} = A_{\pi_{H_2}}$, even when these subgroups are compact (Arsac, 1976). However, we do not have this problem for the space $A(G:K)$ for a compact subgroup K:

Proposition 7 *Let K_1, K_2 be compact subgroups of G. Then $A(G : K_1) = A(G : K_2)$ iff $K_1 = K_2$. If G is a [SIN]-group and H_1, H_2 are closed subgroups of G with $H_1 \neq H_2$, then $A(G : H_1) \neq A(G : H_2)$.*

Proof: We give here the proof for the first case only. Clearly, $K_1 = K_2$ implies that $A(G : K_1) = A(G : K_2)$. Assume that $x_0 \in K_1$ and $x_0 \notin K_2$. Then there is an open $\tilde{U} \subset G/K_2$ with $\tilde{e} \in \tilde{U}$ and $\tilde{x}_0 \notin \tilde{U}$. Let $U = q_{K_2}^{-1}(\tilde{U})$. U is an open neighbourhood of K_2 not containing x_0. We can find $u \in A(G)$ such that $u(x) = 1$ for $x \in K_2$ and $u(x) = 0$ if $x \notin U$. Let $u_1 = P_{K_2}u$. $u_1(xk_2) = \int_{K_2} u(xk_2k)\, dk = \int_{K_2} u(xk)\, dk = u_1(x)$ for $x \in G$, $k_2 \in K_2$. That is, $u_1 \in A(G : K_2)$. $u_1(e) = 1$ while $u_1(x_0) = 0$, and so $u_1 \notin A(G : K_1)$.

$\square$

Corollary 8 *Let K_1, K_2 be compact subgroups of G. Then $B(G : K_1) = B(G : K_2)$ iff $K_1 = K_2$. If G is a [SIN]-group, then $B(G : H_1) = B(G : H_2)$ for H_1, H_2 closed subgroups of G, iff $H_1 = H_2$.*

Proof: in either case, u_1 as constructed above is in $B(G : K_2)$ $[B(G : H_2)]$, but not in $B(G : K_1)$ $[B(G : H_1)]$.

$\square$

In light of the above problems for A_{π_H}, $A(G : H)$ and $B(G : H)$ are more useful analogs for G/H of the Fourier and Fourier-Stieltjes algebras.

Definition 9 *We define $A(G/H)$, the Fourier algebra of the coset space G/H, to be the subalgebra of $C(G/H)$ identified with $A(G : H)$.*

Definition 10 *We define $B(G/H)$, the Fourier-Stieltjes algebra of the coset space G/H, to be the subalgebra of $C(G/H)$ identified with $B(G : H)$.*

When H is a compact subgroup, $A(G/H)$ and $B(G/H)$ have many of the same properties of $A(G)$, $B(G)$.

We have the definitions for the almost periodic and weakly almost periodic functions on a coset space (Skantharajah, 1985):

Definition 11 *$AP(G/H)$ is the set of all $f \in C(G/H)$ such that the set $\{{}_xf : x \in G\}$ is relatively compact in the norm topology of $C(G/H)$.*

Definition 12 *$WAP(G/H)$ is the set of all $f \in C(G/H)$ such that the set $\{{}_xf : x \in G\}$ is relatively compact in the weak topology of $C(G/H)$.*

Proposition 13 *Let H be a closed subgroup of G. Then,*

(i) $B(G/H) \subseteq WAP(G/H)$, and $B(G/H) \cap AP(G/H)$ is the space identified with $B(G : H) \cap AP(G)$.

(ii) $B(G/H) \cap AP(G/H)$ is a complemented subalgebra of $B(G/H)$ with the Radon-Nikodym property.

Proof: (i) follows from Skantharajah (1985). (ii): $B(G/H) \cap AP(G/H)$ is an algebra. $B(G) \cap AP(G)$ has the RNP (Lahoue, 1973) and is complemented in $B(G)$. $B(G) \cap AP(G)$ is of the form A_π where π is the left regular representation of the almost periodic compactification of G. For the projection, take $P = P_\pi \circ P_H$, where P_π is the projection determined by π.

□

Recall that $A(G)$ is sup-norm dense in $C_0(G)$. For a compact subgroup K, $A(G/K)$ is also sup-norm dense in $C_0(G/K)$. However, when H is not compact, there may exist $f \in C_0(G/H)$ such that $f \notin WAP(G/H)$ (Chou). Thus $A(G/H)$ may not separate the points of G/H. The proof of Proposition 7 gives the following result.

Theorem 14 *Let G be a [SIN]-group with a closed subgroup H. Then $A(G/H)$ separates points in G/H.*

This next proposition extends a result of Herz (1973).

Proposition 15 *Let K be a compact subgroup of G. Let H be a closed subgroup of G such that $K \subseteq H$. Then every $\tilde{u} \in A(H/K)$ extends to a function $\tilde{u}_1 \in A(G/K)$ with $\|\tilde{u}\|_{A(H/K)} = \|\tilde{u}_1\|_{A(G/K)}$.*

Proof: Let $u \in A(H : K)$ be the function identified with $\tilde{u}$. By Herz's result, u extends to some $v \in A(G)$ of equal norm. Let $u_1 = P_K(v)$. Since $\|P_K\| \leq 1$, $\tilde{u}_1$ is the desired extension.

□

We can extend a $u \in B(H)$ to some $v \in B(G)$ when G is a [SIN]-group or if H is normal (Cowling, Rodway, 1979). Modifying the argument above produces the following proposition:

Proposition 16 *Let H be a closed subgroup of G. Assume that either G is a [SIN]-group or that H is normal. Let H_1 be another closed subgroup containing H. Then every $\tilde{u} \in B(H_1/H)$ extends to a $\tilde{u}_1 \in B(G/H)$ with the same norm.*

3 Structure of $A(G/K)$ for compact subgroup K

Notation and Definitions
Let $\mathcal{A}$ be a semisimple commutative Banach algebra. $\Delta(\mathcal{A})$ denotes the maximal ideal space of $\mathcal{A}$. Given any closed set A of $\Delta(\mathcal{A})$, we define the following ideals:
$I(A) = \{u \in \mathcal{A} : u(x) = 0 \; \forall \; x \in A\}$
$j(A) = \{u \in I(A) : \text{supp } u \text{ is compact}\}$
$J(A) =$ the norm closure of $j(A)$ in $I(A)$.
A is said to be of spectral synthesis if $I(A) = J(A)$. A is said to be of weak spectral synthesis if for each $u \in I(A)$, there exists a positive integer n such that $u^n \in J(A)$. We say that (weak) spectral synthesis fails if there exists a closed subset A of $\Delta(\mathcal{A})$ that is not a set of (weak) spectral synthesis.

Definition 17 *A multiplier of $\mathcal{A}$ is a linear operator T on $\mathcal{A}$ for which $T(uv) = uT(v)$. We denote the set of all such maps by $\mathcal{M}(A)$, a Banach space with the operator norm.*

Definition 18 *Let $\mathcal{X}$ be a Banach $\mathcal{A}$-bimodule. A derivation of $\mathcal{A}$ on $\mathcal{X}$ is a linear map $D : \mathcal{A} \to \mathcal{X}$ such that $D(uv) = uD(v) + D(u)v$ for every u, $v \in \mathcal{A}$.*

It is known that $\Delta(A(G)) = G$. It is also know that when G is compact, and when K is any closed subgroup, that $\Delta(A(G/K)) = G/K$. It is shown below that we need not assume G is compact.

Theorem 19 *$A(G/K)$ is a regular commutative Banach algebra with $\Delta A(G/K) = G/K$.*

Proof outline: Let $\tilde{x}_0 \in G/K$. $\delta_{\tilde{x}_0}(\tilde{u}) = \tilde{u}(\tilde{x}_0)$ is continuous multiplicative linear functional on $A(G/K)$.

Suppose $\tilde{\Phi} \in \Delta(A(G/K))$. We can identify $\tilde{\Phi}$ with $\Phi \in A(G:K)^*$. $A(G:K)$ is complemented in $A(G)$ by Corollary 5. Thus there exists $\Gamma \in VN(G)$ with $P_K^*(\Phi) = \Gamma$ and $\Gamma|_{A(G:K)} = \Phi$. $\Phi \neq 0$, so $\Gamma \neq 0$. It can be shown that supp$\Gamma = x_0K$ for some $x_0 \in G$. x_0K is a set of spectral synthesis for $A(G)$ (Forrest, 1992), therefor Γ is the weak-* norm limit of $\Psi = \sum_{i=1}^n a_i L_{x_i}$, where $x_i \in x_0K$. But $L_{x_i}|_{A(G:H)} = \Phi = \delta_{\tilde{x}_0}$.

It can be shown that the map $\tilde{x}_0 \mapsto \delta_{\tilde{x}_0}$ is a homeomorphism of G/K onto $\Delta(A(G/K))$.

□

We now examine some structural properties of $A(G/K)$.

Theorem 20 *Let G be a locally compact group with compact subgroup K. The following are equivalent:*

(i) G is amenable

(ii) G/K is an amenable coset space

(iii) $A(G/K)$ has a bounded approximate identity consisting of functions with compact support in G/K

(iv) $A(G/K)$ weakly factorizes

Corollary 21 *Let G be an amenable locally compact group with a compact subgroup K. Then $\mathcal{M}(A(G/K)) = B(G/K)$ and the usual norms agree.*

Forrest showed that for amenable G, every derivation from $A(G)$ into a Banach $A(G)$-bimodule is continuous. He extends the result to coset spaces in Theorem 26 below.

Proposition 22 *Let K be a compact subgroup of G. Let $\tilde{E} \subset G/K$ be a set for which (weak) spectral synthesis fails in $A(G/K)$. Then (weak) spectral synthesis fails for $q^{-1}(\tilde{E})$ in $A(G/K)$. In particular, if (weak) spectral synthesis fails for $A(G/K)$, then (weak) spectral synthesis fails for $A(G)$.*

Proof: We give the proof for the case of spectral synthesis. Assume spectral synthesis fails in $A(G/K)$ for $\tilde{E} \subset G/K$. Then there is $\tilde{v} \in I_{G/K}(\tilde{E})$ such that $\tilde{v} \notin J_{G/K}(\tilde{E})$. Let $v = \tilde{v} \circ q$. $v \in I_G(A)$ for $A = q^{-1}(\tilde{E})$. Suppose now that spectral synthesis holds in $A(G)$ for A. Then $v \in J_G(A)$, and there is a net $\{v_n\} \subset j_G(A)$ such that $\|v - v_n\|_{A(G)} \to 0$ as $n \to \infty$. Then $\|P(v-v_n)\|_{A(G)} = \|v-P(v_n)\|_{A(G)} \to 0$. $P(v_n) = \tilde{v}_n \circ q$ for some $\tilde{v} \in A(G/K)$, and $\lim_n \|\tilde{v} - \tilde{v}_n\|_{A(G/K)} = \lim_n \|v - P(v_n)\|_{A(G)} = 0$. Also, $\operatorname{supp} P(v_n) \subset (\operatorname{supp} v_n)K$. It follows that $\tilde{v}_n \subset q(\operatorname{supp} v_n)$, and that $\tilde{v}_n \in j_{G/K}(\tilde{E})$, a contradiction since $\tilde{v} \notin J_{G/K}(\tilde{E})$.

□

Corollary 23 *Let G be a locally compact group with a compact subgroup K. Then each singleton $\{x\} \subset G/K$ is a set of spectral synthesis for $A(G/K)$. Furthermore, if G is amenable, then every finite subset of G/K is a set of spectral synthesis.*

Proof: The first statement follows from Lemma 22 and from the fact that K and every coset of K is a set of spectral synthesis for $A(G)$. If G is amenable, Forrest (1990) showed that any set of the form $A = \cup_{k=1}^{n} x_k K$ is a set of spectral synthesis for $A(G)$. Hence, every finite set in G/K is also a set of spectral synthesis.

□

Proposition 24 *Let G be amenable with compact subgroup K. Let $\{x_1, \dots, x_n\}$ be a finite subset of G/K. Then $I = I_{G/K}\{x_1, \dots, x_n\}$ has a bounded approximate identity $\{u_\alpha\}$ in $A(G/K) \cap C_c(G/K)$.*

Theorem 25 *Let K be a compact subgroup of G. The following are equivalent:*

(i) G is amenable

(ii) If I is a cofinite ideal of $A(G/K)$, then $I = I(\{x_1, \dots, x_n\})$ where $n =$ codim (I)

(iii) Every cofinite ideal in $A(G/K)$ has a bounded approximate identity

(iv) Each homomorphism of $A(G/K)$ with finite dimensional range is continuous

Theorem 26 *Let K be a compact subgroup of G. The following are equivalent:*

(i) G is amenable

(ii) Every derivation from $A(G/K)$ into a Banach $A(G/K)$-bimodule is continuous.

Proof: $A(G/K)$ is a Silov algebra. Corollary 23 gives us that each closed primary ideal in $A(G/K)$ has codimension 1. Proposition 24 gives us that each maximal ideal has a bounded approximate identity. By a result of Bade and Curtis (1994) we get that each derivation from $A(G/K$ into a Banach $A(G/K)$-bimodule is continuous.

On the other hand, if G is not amenable, $A(G/K)$ does not weakly factorize by Theorem 20. $A(G/K)^2$ is not closed in $A(G/K)$ since since it is dense in $A(G/K)$. Let ϕ be some discontinuous linear functional on $A(G/K)$ with $\phi(u) = 0$ for every $u \in A(G/K)^2$. Let X be a 1-dimensional space, and let $u \cdot x = x \cdot u = 0$ for every $u \in A(G/K)$. Then the derivation $D : A(G/K) \to X$ defined by $D(u) = \phi(u)(x)$ is also discontinuous (Bade, Curtis, 1994).

□

4 Weak amenability of $A(G)$

Definition 27 *A commutative Banach algebra $\mathcal{A}$ is weakly amenable if every continuous derivation from $\mathcal{A}$ into a commutative Banach $\mathcal{A}$-bimodule is identically zero.*

$A(G)$ is weakly amenable if G is discrete (Forrest, 1988). If G is the rotation group on $\mathbb{R}^3$, then $A(G)$ is not weakly amenable (Johnson, 1994). When this paper was published, very little was known about the class of groups G for which $A(G)$ is amenable. It is here shown that this class contains all totally disconnected groups.

Theorem 28 *Let H be an open subgroup of G. Then $A(G/H)$ is weakly amenable.*

Proof: Let $D : A(G/H) \to \mathcal{X}$ be a continuous derivation into a commutative Banach $A(G/H)$-bimodule. Let $\tilde{u}$ be an idempotent in $A(G/H)$. Then $D(\tilde{u}) = D(\tilde{u}^n) = nD(\tilde{u})$ for $n \geq 2$. Thus $D(\tilde{u}) = 0$. H is open, so the linear span of the idempotents is dense in $A(G/H)$. Hence D is identically zero.

$\square$

Lemma 29 *Let G be totally disconnected. Let $u \in A(G)$ and $\epsilon > 0$. Then there exists an open compact subgroup K and a $v \in A(G : K)$ such that $\|u - v\|_{A(G)} < \epsilon$.*

Proof: The map $x \mapsto_x u$, from G into $A(G)$ is continuous. Therefore there exists an open neighbourhood V of e such that $x \in V$ implies $\|u -_x u\|_{A(G)} < \epsilon$. Let K be an open compact subgroup contained in V. Let $v = P_K(u) = \int_K \, _k u \, dk$. Then

$$\|u - v\|_{A(G)} = \left\| \int_K (u -_k u) \, dk \right\|_{A(G)} \leq \int_K \|u -_k u\|_{A(G)} \, dk \leq \epsilon.$$

$\square$

Theorem 30 *Let G be disconnected. Then $A(G)$ is weakly amenable.*

Proof: Let $D : A(G) \to \mathcal{X}$ be a continuous derivation into a commutative Banach $A(G)$-bimodule. Let K be a compact open subgroup of G. The restriction of D to $A(G : K)$ determines a derivation of $A(G/K)$. By Theorem 28, D is zero on each $A(G : K)$. By Lemma 29 each $u \in A(G)$ can be approximated within ϵ by some $v \in A(G : K)$ for some open compact subgroup K. Thus $D = 0$.

$\square$

This shows that for locally compact totally disconnected group G, the span of the idempotents in $A(G)$ is dense. Claim: this characterizes totally disconnected groups. Indeed: the idempotents in $A(G)$ are characteristic functions of open compact subsets in the coset ring of G. Let $\mathcal{K}$ be the intersection of such open compact subgroups. If G is not totally disconnected, $\mathcal{K} \neq \{e\}$. At the same time, the idempotents in $A(G)$ are constant of $\mathcal{K}$, and if follows that their span cannot be dense in $A(G)$.

Proposition 31 *Let G_1, G_2 be such that $A(G_i)$ is weakly amenable for $i = 1, 2$. Then $A(G_1 \times G_2)$ is also weakly amenable.*

Proof: The projective tensor product $A(G_1) \otimes A(G_2)$ is weakly amenable. The map $u \otimes v \to w$, where $w(g_1, g_2) = u(g_1)v(g_2)$ extends to a continuous homomorphism from $A(G_1) \otimes A(G_2)$ onto a dense subalgebra of $A(G_1 \times G_2)$. It follows that $A(G_1 \times G_2)$ is also weakly amenable.

$\square$

Corollary 32 *Let $G = G_1 \times G_2$ where G_1 is Abelian and G_2 is totally disconnected. Then $A(G)$ is weakly amenable.*

Proof: G_1 Abelian implies $A(G_1)$ is amenable, thus weakly amenable. Apply Theorem 30 and Proposition 31.

$\square$