Zippin's Theorem

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April 1 2004

Definition 1 *A basis $\{x_j\}$ for a Banach space X is said to be equivalent to a basis $\{y_j\}$ for a Banach space Y if there exists constants C_1, $C_2 \geq 0$ such that for any m and for any real sequence of scalars $\{a_j\}_{j=1}^m$,*

$$\frac{1}{C_1}\left\|\sum_{j=1}^m a_j y_j\right\| \leq \left\|\sum_{j=1}^m a_j x_j\right\| \leq C_2 \left\|\sum_{j=1}^m a_j y_j\right\|.$$

That is, the convergence of $\sum_{j=1}^\infty a_j x_j$ is equivalent to the convergence of $\sum_{j=1}^\infty a_j y_j$.

Definition 2 *The sequence $\{z_j\}_{j=1}^\infty$ is called a block basis with respect to the basis $\{x_j\}_{j=1}^\infty$ if for every j, $z_j = \sum_{i=p_j+1}^{p_{j+1}} a_i x_i$, where $\{p_j\}_{j=1}^\infty$ is an increasing sequence of non negative integers.*

Definition 3 *A basis $\{x_j\}$ is called unconditional if there exists constant $C > 0$ such that for every sequence of scalars $\{a_j\}$, and for every sequence $\{\theta_j\}$, where $\theta_j = \pm 1$, $\|\sum \theta_j a_j x_j\| \leq C\|\sum a_j x_j\|$.*

Definition 4 *A normalized basis $\{x_j\}$ is said to satisfy condition (A) for any two disjoint finite sets U and V of positive integers and $|s| \leq |t|$, then for every real $\{a_j\}$, $j \in U \cup V$,*

$$\left\|\sum_{j\in U} a_j x_j + s\sum_{j\in V} a_j x_j\right\| \leq \left\|\sum_{j\in U} a_j x_j + t\sum_{j\in V} a_j x_j\right\|.$$

Theorem 5 *Let $\{x_j\}$ be an unconditional basis for Banach space X. Then $\{x_j\}$ satisfies condition (A).*

Proof: See [1] pp. 67, 68, 73.

□

Lemma 6 *Let $\{x_j, f_j\}$ denote the biorthogonal sequence of the basis $\{x_j\}$. If $\{x_j\}$ satisfies condition (A), then $\|f_j\| = 1$ for every j.*

Proof: $\|f_j\| \geq f_j(x_j) = 1$. On the other hand,

$$\|f_j\| = \sup_{\|\sum_{i=1}^\infty a_i x_i\| \leq 1} \left| f_j\left(\sum_{i=1}^\infty a_i x_i\right)\right| = \sup_{\|\sum_{i=1}^\infty a_i x_i\| \leq 1} |a_j| \leq 1,$$

since by condition (A), $|a_j| = \|a_j x_j\| \leq \|\sum_{i=1}^\infty a_i x_i\| \leq 1$.

□

Lemma 7 *If $\{x_j\}$ is a basis in a Banach space which satisfies condition (A) and if $|s_j| \leq |t_j|$ for $1 \leq j \leq n$, then $\|\sum_{j=1}^n s_j x_j\| \leq \|\sum_{j=1}^n t_j x_j\|$.*

Proof: Apply condition (A) n times.

□

Lemma 8 *Let $\{x_j\}$ be a normalized basis in a Banach space X which satisfies condition (A). If for some $M \geq 1$, $\| \sum_{j=1}^{n} x_j \| \leq M$ for every n, then $\{x_j\}$ is equivalent to the unit vector basis of c_o.*

Proof: By Lemma 7,

$$\max_{1 \leq j \leq n} |a_j| \leq \left\| \sum_{j=1}^{n} a_j x_j \right\| \leq \left(\max_{1 \leq j \leq n} |a_j| \right) \cdot \left\| \sum_{j=1}^{n} x_j \right\| \leq M \cdot \max_{1 \leq j \leq n} |a_j|.$$

Hence, $\sum_{j=1}^{\infty} a_j x_j$ converges if and only if $a_j \to 0$.

□

Lemma 9 *Let the basis $\{x_j\}$ be equivalent to all of its normalized block bases. Then $\{x_j\}$ is an unconditional basis for X.*

Proof: Notice that for any sequence $\{\theta_j\}$, where $\theta_j = \pm 1$, $\{\theta_j x_j\}$ is a normalized block basis of $\{x_j\}$. An application of the Banach-Steinhaus theorem completes the proof.

□

Theorem 10 *Let X be a Banach space with normalized basis $\{x_j\}$. $\{x_j\}$ is equivalent to every normalized block basis of $\{x_n\}$ if and only if X is isomorphic to c_0, or to l_p for some $1 \leq p < \infty$.*

Proof: The *if* part is clear, since the unit vector bases in c_0 and l_p are equivalent to each of their normalized block bases.

Assume that the normalized basis $\{x_j\}$ is equivalent to all normalized block bases. By Lemma 9 $\{x_j\}$ is unconditional. By Theorem 5, $\{x_j\}$ satisfies condition (A).

Next, using a uniform boundedness argument similar to the one used for the proof of Lemma 9, the following is proved: there exists a constant M such that for every normalized block basis $\{u_j\}$ of $\{x_j\}$, the operator T_u which exhibits the equivalence of the basic sequence (i.e. $T_u x_j = u_j$ for all j) satisfies $\|T_u\|, \ \|T_u^{-1}\| \leq M$ or equivalently,

$$M^{-1} \left\| \sum_{j=1}^{\infty} a_j x_j \right\| \leq \left\| T_u \left(\sum_{j=1}^{\infty} a_j x_j \right) \right\| = \left\| \sum_{j=1}^{\infty} a_j u_j \right\| \leq M \left\| \sum_{j=1}^{\infty} a_j x_j \right\|. \tag{1}$$

for all choices of scalars $\{a_j\}$ such that $\sum_{j=1}^{\infty} a_j x_j$ converges.

We will give the details of the argument. Indeed, let I denote the set of all normalized block basis with respect to $\{x_j\}$. For each $\{z_j\} \in I$, we denote by T_z the operator defined by $T_z(\sum_{j=1}^{\infty} a_j x_j) = \sum_{j=1}^{\infty} a_j z_j$. We will show that the set $\{\|T_z\| : z \in I\}$ is finitely bounded. Assume the contrary. By the theorem of Banach and Steinhaus, there is an $x = \sum_{j=1}^{\infty} b_j x_j \in X$ with $\|x\| = 1$ such that $\sup_{z \in I}\{\|T_z x\|\} = \sup_{z \in I}\{\| \sum_{j=1}^{\infty} b_j z_j \|\} = \infty$. Set $n_0 = 0$. Then we can find a $z^{(1)} = \{z_j^{(1)}\} \in I$ such that $\| \sum_{j=n_0+1}^{\infty} b_j z_j^{(1)} \| \geq 2$, and we can pick an $n_1 > n_0$ large enough so that

$$\left\| \sum_{j=n_0+1}^{n_1} b_j z_j^{(1)} \right\| \geq 1.$$

Let $z_j = z_j^{(1)}$ for $n_0+1 \le j \le n_1$. $\| \sum_{j=n_0+1}^{n_1} b_j z_j \| \le \sum_{j=n_0+1}^{n_1} |b_j| = A_1$. Notice that for each $z^{(k)} \in I$,

$$z_j^{(k)} = \sum_{i=p_j^{(k)}+1}^{p_{j+1}^{(k)}} a_i x_i,$$

where each $\{p_i^{(k)}\}$ is a sequence of positive increasing integers. We can find a $z^{(2)} \in I$ such that $\| \sum_{j=n_0+1}^{\infty} b_j z_j^{(2)} \| \ge A_1 + 2$ with $p_{n_1+1}^{(1)} < p_{n_1+1}^{(2)} + 1$, and also such that for some $n_2 > n_1$,

$$\left\| \sum_{j=n_0+1}^{n_2} b_j z_j^{(2)} \right\| \ge A_1 + 1.$$

Let $z_j = z_j^{(2)}$ for $n_1+1 \le j \le n_2$. Then we have

$$\left\| \sum_{j=n_1+1}^{n_2} b_j z_j \right\| \ge \left\| \sum_{j=n_0+1}^{n_2} b_j z_j^{(2)} \right\| - \left\| \sum_{j=n_0+1}^{n_1} b_j z_j^{(2)} \right\| \ge A_1 + 1 - \left\| \sum_{j=n_0+1}^{n_1} b_j z_j^{(2)} \right\| \ge 1.$$

Continuing in this fashion we have $\{z^{(i)}\} \subset I$ and $\{n_i\}$ a sequence of increasing positive integers such that

$$z_j = \begin{cases} z_j^{(1)} & \text{for } n_0+1 \le j \le n_1 \\ z_j^{(2)} & \text{for } n_1+1 \le j \le n_2 \\ z_j^{(3)} & \text{for } n_2+1 \le j \le n_3 \\ & \vdots \end{cases}$$

$$\left\| \sum_{j=n_i+1}^{n_{i+1}} b_j z_j \right\| \ge 1 \quad \text{for all } i \ge 0 \tag{2}$$

$$p_{n_i+1}^{(i)} < p_{n_i+1}^{(i+1)} + 1 \quad \text{for all } i \ge 1. \tag{3}$$

By (3) the sequence $\{z_j\}$ is a normalized block basis with respect to $\{x_j\}$. By (2) $\sum_{j=1}^{\infty} \sum_{i=n_j+1}^{n_{j+1}} b_i z_i$ does not converge. Since $\{x_j\}$ is an unconditional basis, $\sum_{j=1}^{\infty} \sum_{i=n_j+1}^{n_{j+1}} b_i x_i$ does converge, contradicting the equivalence of $\{x_j\}$ to $\{z_j\}$. Thus $\{\|T_z\| : z \in I\}$ is finitely bounded.

The finite boundedness of the set $\{\|T_z^{-1}\| : z \in I\}$ is shown is a similar fashion. This establishes (1).

Taking $u_j = x_{m_j}$ and

$$a_j = \begin{cases} 1 & \text{if } 1 \le j \le n \\ 0 & \text{if } j > n \end{cases}$$

in (1) for some increasing sequence $\{m_j\}$ of positive integers, we get

$$M^{-1} \left\| \sum_{j=1}^{n} x_j \right\| \le \left\| \sum_{j=1}^{n} x_{m_j} \right\| \le M \left\| \sum_{j=1}^{n} x_j \right\|, \; n = 1, 2, \dots \tag{4}$$

Now, we construct blocks $u_j,\ j = 1, 2, \ldots$ in the following way: we fix integers n and k and we take

$$\begin{aligned}
u_1 &= \frac{x_1 + \ldots + x_{n^{k-1}}}{\|x_1 + \ldots + x_{n^{k-1}}\|},\\
u_2 &= \frac{x_{n^{k-1}+1} + \ldots + x_{2n^{k-1}}}{\|x_{n^{k-1}+1} + \ldots + x_{2n^{k-1}}\|}\\
&\vdots
\end{aligned}$$

We will use the following notation hereafter:

$$\lambda(n) = \left\|\sum_{j=1}^{n} x_j\right\|,$$

By applying (1) for these blocks $\{u_j\}$ and suitably chosen $\{a_j\}$, we have that

$$M^{-2}\lambda(n)\lambda(n^{k-1}) \leq \lambda(n^k) \leq M^2\lambda(n)\lambda(n^{k-1}),\ n, k = 1, 2, \ldots$$

It follows easily by induction on k that

$$M^{-2k}\lambda(n)^k \leq \lambda(n^k) \leq M^{2k}\lambda(n)^k. \tag{5}$$

For any natural N, n and k let $h = h(N, n, k)$ be the non-negative integer for which $N^h \leq n^k < N^{h+1}$. Thus, $h \log N \leq k \log n < (h + 1) \log N$.

It follows from (5) and after some easy (but not obvious) calculations that

$$\left|\frac{\log\lambda(n)}{\log n} - \frac{\log\lambda(N)}{\log N}\right| \leq 2\log M\left(\frac{1}{\log N} + \frac{1}{\log n}\right). \tag{6}$$

Since $1 \leq \lambda(n) \leq n$, we get that the sequence $\{\frac{\log\lambda(n)}{\log n}\}$ converges to a limit c where $0 \leq c \leq 1$.

Passing to a limit in (6) as $N \to \infty$, we get

$$M^{-2}n^c \leq \lambda(n) \leq M^2n^c,\ n = 1, 2, \ldots.$$

If $c = 0$ we have $\lambda(n) \leq M^2$, and thus by Lemma 8, $\{x_n\}$ is equivalent to the unit vector basis of c_0.

If $0 < c \leq 1$, set $c = 1/p$ and we have

$$M^{-2}n^{1/p} \leq \left\|\sum_{j=1}^{n} x_j\right\| \leq M^2n^{1/p}. \tag{7}$$

To prove equivalence between $\{x_n\}$ and the unit vector basis of l_p, we let $r_j,\ j = 1, 2, \ldots, J$ be any positive rational numbers and assume that $r_j = k_j/k$ where k_j and k are positive integers. It follows from (7) that

$$\left\|\sum_{j=1}^{J} r_j^{1/p}x_j\right\| = k^{-1/p}\left\|\sum_{j=1}^{J} k_j^{1/p}x_j\right\| \geq M^{-2}k^{-1/p}\left\|\sum_{j=1}^{J}\left\|\textstyle\sum_{i=1}^{k_j}x_i\right\|x_j\right\|.$$

Using again (1) with the normalized block basis

$$\begin{aligned}
u_1 &= \frac{x_1 + \ldots + x_{k_1}}{\|x_1 + \ldots + x_{k_1}\|}, & a_1 &= \|x_1 + \ldots + x_{k_1}\|, \\
u_2 &= \frac{x_{k_1+1} + \ldots + x_{k_1+k_2}}{\|x_{k_1+1} + \ldots + x_{k_1+k_2}\|}, & a_2 &= \|x_{k_1+1} + \ldots + x_{k_1+k_2}\| \\
&\vdots
\end{aligned}$$

we get that

$$\begin{aligned}
\left\| \sum_{j=1}^{J} \left\| \textstyle\sum_{i=1}^{k_j} x_i \right\| x_j \right\| &\geq M^{-1} \left\| \sum_{j=1}^{J} a_j x_j \right\| \\
&\geq M^{-2} \left\| \sum_{j=1}^{J} a_j u_j \right\| = M^{-2} \left\| \sum_{i=1}^{\sum_{j=1}^{J} k_j} x_i \right\| \\
&\geq M^{-4} \left(\sum_{j=1}^{J} k_j \right)^{1/p} = M^{-4} k^{1/p} \left(\sum_{j=1}^{J} r_j \right)^{1/p},
\end{aligned}$$

where the first inequality follows from (4), the second inequality follows from (1) for a finite sequence $\{a_j\}$, the first equality follows from the definition of $a_j u_j$, and the third inequality follows from (7). Thus

$$\left\| \sum_{j=1}^{J} r_j^{1/p} x_j \right\| \geq M^{-6} \left(\sum_{j=1}^{J} r_j \right)^{1/p}.$$

It follows easily from condition (A) that for any real sequence $\{a_j\}$,

$$\left\| \sum_{j=1}^{J} a_j x_j \right\| \geq M^{-6} \left(\sum_{j=1}^{J} |a_j|^p \right)^{1/p}.$$

Similar arguments show that

$$\left\| \sum_{j=1}^{J} a_j x_j \right\| \leq M^{6} \left(\sum_{j=1}^{J} |a_j|^p \right)^{1/p},$$

and thus $\{x_j\}$ is equivalent to the unit vector basis of l_p.

In both the c_0 and the l_p case, the equivalence of $\{x_j\}$ to the unit vectors of each space induces an isomorphism from X onto c_0, l_p respectively. This is a result if the closed graph theorem. This completes the proof.

$\square$

Bibliography

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[3] M. Zippin, *On Perfectly Homogeneous Bases in Banach Spaces*, Israel J. Math. **4** (1966), 255–272.