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ACTIONS OF SEMITOPOLOGICAL SEMIGROUPS ON LOCALLY COMPACT HAUSDORFF SPACES

by

Alan Glenn Kydd

A thesis submitted to the Faculty of Graduate Studies and Research
in partial fulfillment of the requirements for the degree of
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in

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The undersigned certify that they have read, and recommend to the Faculty of Graduate Studies and Research for acceptance, a thesis entitled **Actions of semitopological semigroups on locally compact Hausdorff spaces** submitted by **Alan Glenn Kydd** in partial fulfillment of the requirements for the degree of **Master of Science** in **Mathematics**.

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ABSTRACT

Let a semitopological semigroup G have a jointly continuous left action on a topological Hausdorff space X. In this setting, several subspaces of $CB(G)$ have analogues in $CB(X)$. We define an action of the duals of these subspaces on the duals of their analogues. This action is an analogue to the first Arens product. We consider the topological center of this action of $LUC(G)^*$ on $LUC(G,X)^*$. As an application, we examine the relationship between almost G-periodic points, G-minimal sets, and elements of βX that are in the support of a G-invariant measure for βX.

We use Day's fixed point theorem to show the existence of an invariant measure on coset spaces of certain groups. When H is a closed subgroup of a topological group G, we define a new fixed point property for the pair $(G : H)$ that is equivalent to Eymard's stronger fixed point property for $(G : H)$.

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Chapter 1

Introduction

A semigroup G together with a Hausdorff topology for which multiplication in G is separately [resp. jointly] continuous is called a semitopological [resp. topological] semigroup. A jointly continuous action of G on a topological Hausdorff space X is a jointly continuous map $(a, x) \mapsto ax$ from $G \times X$ into X, such that $(ab)x = a(bx)$ for all a, $b \in G$ and $x \in X$, and such that the map $x \mapsto ax$, from X into X, is continuous for each $a \in G$.

Chapter 2 contains basic definitions and results that are used throughout this thesis. The results are stated without proof. We also list some well known examples that demonstrate the ideas contained within this chapter.

It is obvious that multiplication for a topological semigroup G can be viewed as a jointly continuous action of G onto itself. It is with this idea in mind that we develop our ideas in the third chapter. Under the assumption that a semitopological semigroup G has a jointly continuous left action on a topological Hausdorff space X, we modify the existing definitions of invariance and introversion for subspaces of $CB(G)$ to obtain similar definitions for subspaces of $CB(X)$. These modified definitions allow us to define analogues to the first and second Arens products.

In Section 3.3 we define and examine the properties of several G-invariant function spaces in $CB(X)$ which are analogues of common invariant function spaces in $CB(G)$. In Section 3.4 we examine, in detail, the action of the dual space of the left uniformly continuous functions on G on the dual space of the left G uniformly continuous functions on X.

The last section of Chapter 3 is an application of the Arens action. βG, the Stone-Čech compactification of G, has an Arens action on βX, the Stone-Čech compactification of X. We examine the relationship between the almost G-periodic points of βX, the G-minimal subsets of βX, and the elements of βX that are in the support of a G-invariant measure on βX.

The last chapter of this thesis investigates fixed point properties of the action of a locally compact group G on a locally convex topological space X. We use Day's fixed point theorem to show the existence of an invariant measure on coset spaces for certain groups. We define a fixed point property for the pair $(G : H)$, where H is a closed subgroup of G, such that the action of G on X need not be jointly continuous. We prove that this fixed point property is in fact equivalent to a stronger fixed point property for $(G : H)$ due to Eymard, who requires that the action of G on X be jointly continuous.

Chapter 2

Preliminaries

2.1 Definitions and Notation

A semigroup G together with a Hausdorff topology for which multiplication in G is continuous from the left is called a *right topological semigroup*. If multiplication in G is separately [resp. jointly] continuous, G is called a *semitopological* [resp. *topological*] *semigroup*. When the topology of G is locally compact, G is called a *locally compact semitopological* [resp. *topological*] *semigroup*. A group G equipped with a Hausdorff topology is called a *topological group* if both multiplication and inversion are continuous. When the topology of such a group is locally compact, we say G is a *locally compact group*. It is well known that a locally compact group G admits a unique left Haar measure; that is, there exists a unique (up to a constant) regular Borel measure ν on G such that $\nu(gE) = \nu(E)$, for every Borel set $E \subset G$ and every $g \in G$.

For an arbitrary topological space X, we denote by X_d the space X taken with its discrete topology. When G is a semigroup [resp. group], we call G_d the *discrete semigroup* [resp. *group*].

Throughout this chapter, G will denote a semitopological semigroup, unless otherwise stated.

For any nonempty set X, we denote by $\ell^\infty(X)$ the space of all bounded complex-valued functions on X. Then $\ell^\infty(X)$ is a commutative unital C^*-

algebra with respect to the supremum norm $\|f\|_\infty = \sup_{x\in X} |f(x)|$, where the multiplication and addition are defined pointwise and the involution is defined by pointwise complex conjugation, $f^*(x) = \overline{f(x)}$.

For a topological space X, there are several important subspaces of $\ell^\infty(X)$ that we will consider. We denote by $CB(X)$ the space of all bounded, continuous complex-valued functions on X. $C_0(X)$ denotes all $f \in CB(X)$ such that for every $\epsilon > 0$, there exists a compact subset $K \subset X$ (depending on f and ϵ) such that $|f(x)| < \epsilon$ for every $x \in X\backslash K$. $C_c(X)$ denotes all $f \in CB(X)$ having compact support. We have the natural inclusion relation $C_c(X) \subseteq C_0(X) \subseteq CB(X)$, where equality holds when X is compact. With the operations and topology inherited from $\ell^\infty(X)$, both $CB(X)$ and $C_0(X)$ are C^*-subalgebras of $\ell^\infty(X)$. Throughout this thesis, we will always consider the space $CB(X)$ to have the supremum norm topology, unless otherwise stated, noting that $CB(X_d) = \ell^\infty(X)$.

For any function $f \in \ell^\infty(G)$ and $a \in G$, we denote the left and right translations of f by a by

$$\ell_a f(b) = f(ab), \qquad r_a f(b) = f(ba) \tag{2.1}$$

for all $b \in G$. We may also view ℓ_a and r_a as operators from $\ell^\infty(G)$ into $\ell^\infty(G)$. We denote the left and right orbits of f by

$$O_\ell f = \{\ell_a f : a \in G\}, \qquad O_r f = \{r_a f : a \in G\}.$$

A linear subspace $Y \subseteq \ell^\infty(G)$ is called *left* [resp. *right*] *invariant* if $O_\ell f \subseteq Y$ [resp. $O_r f \subseteq Y$] for all $f \in Y$. Y is said to be *invariant* if it is both left and right invariant.

Let Y be a left [resp. right] invariant norm closed subspace of $\ell^\infty(G)$, and let Y^* denote the dual space of Y. Y is called *left* [resp. *right*] *introverted* if the function

$$n_\ell f(g) = n(\ell_g f) \qquad [\text{resp. } n_r f(g) = n(r_g f)] \tag{2.2}$$

is in Y for each $n \in Y^*$. Y is called *introverted* if it is both left and right introverted.

For any nonempty subset A of a vector space V, we denote the convex hull of A and the convex circled hull of A by conv A and cconv A, respectively.

Finally, we say that a semitopological semigroup G has a *left action* on a topological space X when there is a map $(a, x) \mapsto ax$ from $G \times X$ into X such that

$$(ab)x = a(bx)$$

for all $a, b \in G$ and $x \in X$, with the map

$$x \mapsto ax$$

continuous for each fixed $a \in G$. Such an action is called *jointly continuous* if the map $(a, x) \mapsto ax$ is jointly continuous.

2.2 Subspaces of $CB(G)$

We give below a brief overview of common subspaces of the C^*-algebra $CB(G)$ that are encountered in analysis. Proofs of the stated results can be found in Chapter 4 of the useful book on analysis on semigroups by Berglund *et al.* [2].

A function $f \in CB(G)$ is called *almost periodic* if the left orbit $O_\ell f$ is relatively compact in $CB(G)$. We denote the set of all such functions by $AP(G)$. $AP(G)$ is an introverted C^*-subalgebra of $CB(X)$ containing the constant functions.

A function $f \in CB(G)$ is called *weakly almost periodic* if the left orbit $O_\ell f$ is relatively weakly compact in $CB(G)$. We denote the set of all such functions by $WAP(G)$. $WAP(G)$ is an introverted C^*-subalgebra of $CB(G)$ containing the constant functions (see [11, Theorems 4.2 and 12.1]).

A function $f \in CB(G)$ is called *left uniformly continuous* when the map $a \mapsto \ell_a f$ from G into $CB(G)$ is continuous. We denote the set of all such functions by $LUC(G)$. $LUC(G)$ is an invariant, left introverted C^*-subalgebra of $CB(G)$ containing the constant functions (see [26, pp. 64, 68, 72]).

It is well known that equivalently, a function $f \in CB(G)$ is in $AP(G)$ [resp. $WAP(G)$] if the right orbit $O_r f$ is relatively [weakly] compact in $CB(G)$.

Furthermore, Grothendieck proved that $f \in WAP(G)$ is equivalent to the equality $\lim_i \lim_j f(a_i b_j) = \lim_j \lim_i f(a_i b_j)$, whenever the limits exist, where $\{a_i\}$ and $\{b_j\}$ are sequences in G [17].

Since the norm topology of $CB(G)$ is stronger than the weak topology, we know that $AP(G) \subseteq WAP(G)$. It can also be shown that $AP(G) \subseteq LUC(G)$. In general, there is no inclusion relationship between $WAP(G)$ and $LUC(G)$ (see [2, p.165] for an example). However, when G is a locally compact topological group, $WAP(G) \subseteq LUC(G)$, $C_0(G) \cap AP(G) = \{0\}$ and $C_0(G) \subseteq WAP(G)$. Consequently, $WAP(G) = AP(G)$ if and only if G is a compact topological group. Thus, in the case that G is a compact topological group, $C_0(G) = AP(G) = WAP(G) = LUC(G) = CB(G)$.

2.3 Means and Amenability

We state the definitions for a mean and invariant mean on subspaces of $\ell^\infty(G)$. We state some well known results concerning means, and we state the definitions of amenability for semigroups and for discrete groups.

Let X be an arbitrary set, and let Y be a norm closed subspace of $\ell^\infty(X)$ containing the constant functions. Denote the characteristic function of a subset $A \subseteq X$ by χ_A. A *mean* on Y is any linear functional $m \in Y^*$ such that $m(\chi_X) = 1 = \|m\|$. We denote the set of all means on Y by $M(Y)$.

Remark 2.3.1. (a) The set $M(Y)$ is a convex, weak* compact subset of Ball (Y^*), the unit ball of Y^*. (b) We define the evaluation map $\delta\colon X \to \text{Ball}\,(Y^*)$ by

$$\delta(x) = \delta_x,$$

where

$$\delta_x(f) = f(x)$$

for all $f \in Y$. The set conv $\{\delta_x : x \in X\}$ is weak* dense in $M(Y)$ (see [6, p. 281] and [7, p. 513]), and the set cconv $\{\delta_x : x \in X\}$ is weak* dense in $\overline{\text{Ball}\,(Y^*)}^{w^*}$.

Let Y be a left [resp. right] invariant norm closed subspace of $\ell^\infty(G)$ containing the constant functions. A mean $m \in Y^*$ is said to be a *left* [resp *right*] *invariant mean* (LIM [resp. RIM]) if $m(\ell_g f) = m(f)$ [resp. $m(r_g f) = m(f)$] for all $g \in G$ and $f \in Y$. We denote the set of all such means by $LIM(Y)$ [resp. $RIM(Y)$]. We say m is an *invariant mean* (IM) if m is both a LIM and a RIM

G is said be to *left* [resp. *right*] *amenable* if there exists a LIM [resp. RIM] on $\ell^\infty(G)$. G is *amenable* if there exists an IM on $\ell^\infty(G)$.

Let G be a topological group. It is well known that there exists a LIM for $\ell^\infty(G_d)$. In this case, there is also a RIM for $\ell^\infty(G_d)$. For this reason, a topological group G is said to be *amenable as discrete* if there exists a LIM, RIM, or an IM for $\ell^\infty(G_d)$.

2.4 Arens Product

In this section we recall the Arens product on subspaces of $\ell^\infty(G)$. We refer the reader to [1] for a description of the construction of the Arens multiplication, and to [7, §6] for results on the application of Arens' ideas to semigroup algebras.

Let Y be a left introverted subspace of $\ell^\infty(G)$. The *Arens product* is a map from $Y^* \times Y^*$ to Y^* defined by:

$$(m, n) \mapsto m \bigcirc n,$$

where

$$m \bigcirc n(f) = \langle m, n_\ell f \rangle$$

for all m, $n \in Y^*$ and $f \in Y$. The Arens product is associative, distributive, and weak*-weak* continuous in the first variable. Also,

$$\|m \bigcirc n\| \leq \|m\| \|n\|,$$

showing that the Arens product makes Y^* Banach algebra. $\bigcirc$ is commonly referred to as the *first Arens product* on Y^*. The *topological center* of Y^* with respect to the first Arens product is defined to be

$$Z_t(Y) = \{m \in Y^* : \text{ the map } n \mapsto m \bigcirc n \text{ is w*-w* continuous on } Y^*\}.$$

When Y is a right introverted subspace of $\ell^\infty(G)$, the *second Arens product* is defined for Y^* by

$$(m, n) \mapsto m \,\square\, n,$$

where

$$m \,\square\, n(f) = \langle m, n_r f \rangle.$$

The second Arens product is associative, distributive, and weak*-weak* continuous in the first variable. Furthermore, Y^* is Banach algebra with this product.

If Y is a left [resp. right] introverted subspace of $\ell^\infty(G)$, then Y^* is a right topological semigroup under the first [resp. second] Arens product.

Let Y be introverted. It generally not the case that the first and second Arens products agree. In this case, the *topological center* of Y^* is defined to be

$$Z_t(Y) = \{m \in Y^* : m \bigcirc n = m \,\square\, n \text{ for all } n \in Y^*\}.$$

Y is called *Arens regular* when $Z_t(Y) = Y^*$,

The topological centers of various Banach algebras have been studied by Dales and Lau [5], Lau [23], Lau and Ülger [24], and Neufang [27], to name a few. For a discussion of the Arens products on the second duals of Banach algebras, and for a discussion of the topological center of the Arens products in such a setting, we refer the reader to [24]. The reader will also find many new results and examples regarding the topological center of second duals of various Banach algebras in [5].

2.5 Examples

Example 2.5.1. Let G be a topological group with a closed subgroup H. G has a jointly continuous left action on the coset space G/H defined by the map

$$(a, gH) \mapsto agH.$$

Example 2.5.2. Let $G = (\mathbb{N}, +)$, the commutative additive semigroup of the natural numbers. Let $X = \beta\mathbb{N}$, the Stone-Čech compactification of G. Addition in G extends uniquely to a binary operation $\star$ on X such that $a+b = a \star b$ for every a, $b \in G$, and such that G has a continuous left action on X defined by

$$(a, x) \mapsto a \star x,$$

for $a \in G$ and $x \in X$.

Example 2.5.3 (Bohr [3])**.** Let $G = (\mathbb{R}, +)$, the additive semigroup of the real numbers, with the usual topology. Every continuous periodic function on G is almost periodic. It follows that every trigonometric polynomial

$$f(x) = \sum_{n=1}^{N} \xi_n e^{i\alpha_n x}, \qquad x, \alpha_n \in \mathbb{R},\ \xi_n \in \mathbb{C}$$

is almost periodic.

Example 2.5.4. Let $G = (\mathbb{R}, +)$. The function

$$x \mapsto \frac{x}{1 + |x|}$$

is a function in $LUC(G)$, but not in $WAP(G)$ [2, 4.4.19].

Example 2.5.5. The free group on two generators is not amenable [18, 17.16].

Example 2.5.6 (Day)**.** As a discrete group, the special orthogonal group $SO(3, \mathbb{R})$ contains the free group on two generators as a closed subgroup. Day proved that if a discrete group G is amenable, then every closed subgroup H of G is also amenable. By Example 2.5.5 and the previous statement, $SO(3, \mathbb{R})$ is not amenable as a discrete group.

Example 2.5.7. The "$ax+b$" group, or affine group of $\mathbb{R}$, consists of matrices of the form

$$\begin{bmatrix} a & b \\ 0 & 1 \end{bmatrix}, \qquad a \in \mathbb{R}^+,\, b \in \mathbb{R}$$

with the topology inherited from the half plane in $\mathbb{R}^2$, and with the group operations of matrix inversion and matrix multiplication. It is well known that the "$ax+b$" group is a solvable, hence amenable, locally compact group.

Example 2.5.8. The Heisenberg group over the real numbers consists of matrices of the form

$$\begin{bmatrix} 1 & x & y \\ 0 & 1 & z \\ 0 & 0 & 1 \end{bmatrix}, \qquad x, y, z \in \mathbb{R}$$

with the topology inherited from $\mathbb{R}^3$, and with the group operations of matrix inversion and matrix multiplication. The Heisenberg group is a nilpotent locally compact group. Hence it is solvable, and therefore amenable.

Example 2.5.9. Let $\mathcal{H}$ be a Hilbert space, and denote the space of bounded linear operators from $\mathcal{H}$ to $\mathcal{H}$ by $\mathcal{B}(\mathcal{H})$. Ball$\,(\mathcal{B}(\mathcal{H}))$ with the weak operator topology (WOT) is a compact, right topological semigroup, where the semigroup operation is the composition of operators.

Let S, $T \in$ Ball$\,(\mathcal{B}(\mathcal{H}))$. Clearly $\|ST\| \leq \|S\|\|T\| \leq 1$. To see that multiplication from the left is WOT continuous, take a net $\{S_\alpha\} \subset$ Ball$\,(\mathcal{B}(\mathcal{H}))$ such that $S_\alpha \to S$ in the WOT. Then

$$\langle (S_\alpha T)x, y\rangle = \langle S_\alpha(Tx), y\rangle \to \langle (ST)x, y\rangle$$

for all x, $y \in \mathcal{H}$. The WOT and weak* topologies agree on norm bounded sets, hence Ball$\,(\mathcal{B}(\mathcal{H}))$ is WOT compact.

Chapter 3

Main results

3.1 Introduction

Let a semitopological semigroup G have a left action on a Hausdorff space X. We begin this chapter by defining translations, invariance, introversion and amenability of subspaces of $\ell^\infty(X)$ with respect to the left action of G on X. These definitions allow us to define actions that are analogues to the first and second Arens products, and that reduce to the first and second Arens products when $X = G$. In Section 3.3 we introduce a number of G-invariant subspaces of $CB(X)$, We discuss some properties of each subspace; in particular, we examine the conditions of invariance and introvertibility. The focus of Section 3.4 is the space $LUC(G, X)$, where X is a locally compact space. We consider the action of the Banach algebra $LUC(G)^*$ on $LUC(G, X)^*$. Using ideas of Lau [23] and Wong [31], we prove that when G and X are locally compact, the measure algebra $\mathcal{M}(G)$ is a subset of the topological center of the action of $LUC(G)^*$ on $LUC(G, X)^*$. In Section 3.5 we show that for discrete G and X, the set of almost G-periodic points of βX is exactly the set of elements in βX which belong to some G-minimal set. We also show that under certain conditions, there exist non almost G-periodic points in βX which are in the support of some G-invariant measure of βX.

3.2 Invariance, introversion, and the Arens action

Let a semitopological semigroup G have a left action on a nonempty set X. In the case that X has its own semigroup structure (that is. an associative binary operation, $X \times X \to X$), it is convenient to distinguish between translations of a function f on X by elements in G and translations of f by elements in X. With this in mind, we denote the left translation of a function $f \in \ell^\infty(X)$ by an element $a \in G$ by

$$\lambda_a f(x) = f(ax)$$

for all $x \in X$. We can regard λ_a as an operator from $\ell^\infty(X)$ into $\ell^\infty(X)$. We also define an analogue to the right translation defined in (2.1) as follows: for $f \in \ell^\infty(X)$ and $x \in X$, define the function

$$\rho_x f(a) = f(ax),$$

for all $a \in G$, noting that $\rho_x f$ is a function of G. Both functions are well defined. We can regard ρ_x as an operator from $\ell^\infty(X)$ into $\ell^\infty(G)$. We define the left and right orbits of $f \in \ell^\infty(X)$ with respect to G by

$$O_\lambda f = \{\lambda_a f : a \in G\}, \qquad O_\rho f = \{\rho_x f : x \in X\},$$

noting that $O_\rho f$ is a subset of $\ell^\infty(G)$.

Let G have a jointly continuous left action on a topological Hausdorff space X.

Definition 3.2.1. Consider a subspace of $CB(X)$ that is defined purely in terms of the action of G on X and the topologies of G and X (i.e. defined independently of the algebraic structure of X, if any). Hereafter, we will always write such a subspace as $Y(G, X)$. We define the *incident space of* $Y(G, X)$ to be $Y(G, G)$, the subspace of $CB(G)$ defined by the same conditions of $Y(G, X)$, with X replaced by G. For convenience, we write $Y(G)$ for $Y(G, G)$.

The notion of the incident space allows us describe an analogue of right invariance within the framework of the left action from G.

A linear subspace $Y \subseteq \ell^\infty(X)$ is called *left G invariant* if $O_\lambda f \subseteq Y$ for all $f \in Y$. A subspace $Y(G,X)$ of $CB(X)$ is called *right X invariant* if $O_\rho f \subseteq Y(G)$ for all $f \in Y(G,X)$. $Y(G,X)$ is *invariant* if it is both left G and right X invariant.

We define functions similar to those in (2.2), and use them to define analogues of left and right introversion. Let $Y(G,X)$ be a norm closed, left G invariant subspace of $CB(X)$, and let $n \in Y(G,X)^*$, the dual space of $Y(G,X)$. $Y(G,X)$ is called *left G introverted* if the function

$$n_\lambda f(g) = n(\lambda_g f)$$

is in $Y(G)$ for all $f \in Y$. When $Y(G,X)$ is a norm closed, right X invariant subspace and $m \in Y(G)^*$, $Y(G,X)$ is called *right X introverted* if the function

$$m_\rho f(x) = m(\rho_x f)$$

is in $Y(G,X)$ for all $f \in Y(G,X)$. $Y(G,X)$ is *introverted* if it is both left G and right X introverted.

We have the following useful lemma concerning the above functions $n_\lambda f$ and $m_\rho f$.

Lemma 3.2.2. *Let $Y(G,X)$ be a norm closed invariant linear subspace of $CB(X)$ containing the constant functions. For every $f \in Y(G,X)$,*

(i) $\{n_\lambda f : n \in Y(G,X)^*,\ \|n\| \leq 1\} = \overline{\text{cconv}\,\{\rho_x f : x \in X\}}^p$

(ii) $\{m_\rho f : m \in Y(G)^*,\ \|m\| \leq 1\} = \overline{\text{cconv}\,\{\lambda_g f : g \in G\}}^p$

where the closures are taken in the pointwise topologies of the respective function spaces.

Proof. (i) Let $f \in Y(G,X)$, $n \in \text{Ball}\,(Y(G,X)^*)$. By Remark 2.3.1(b), we may take a net $\{n_\alpha\} \subset Y(G,X)^*$ of linear combinations of point evaluations

$\sum_{i=1}^{k} \xi_i \delta_{x_i}$, with $\sum_{i=1}^{k} |\xi_i| \leq 1$, such that $n_\alpha \to n$ weak*. For $g \in G$, write $n_\lambda f(g)$ as

$$\lim_\alpha \left(\sum_{i=1}^{k} \xi_i \delta_{x_i} \right)_\alpha (\lambda_g f) = \lim_\alpha \left(\sum_{i=1}^{k} \xi_i \rho_{x_i} \right)_\alpha f(g).$$

The proof of (ii) is similar. ■

Remark 3.2.3. (a) We will say that $Y(G, X)$ is left or right invariant [resp. introverted], with the understanding that we refer to left G or right X invariance [resp. introversion]. (b) When X has its own associative binary operation and $X \neq G$, do not confuse our definition of right X invariance with the usual definition of right invariance for a subspace of $\ell^\infty(X)$.

The following definition of a G invariant mean on subspaces of $\ell^\infty(X)$ is due to Greenleaf, who first studied such means when G is a locally compact group (see [15]).

Definition 3.2.4. Let Y be a left G-invariant, norm closed subspace of $\ell^\infty(X)$ containing the constant functions. A mean $m \in Y^*$ is said to be a *left G-invariant mean* [GLIM] if $m(\lambda_g f) = m(f)$ for all $g \in G$ and $f \in Y$. We denote the set of all such means by $GLIM(Y)$. X is said to be *left amenable* if there exists a GLIM on $\ell^\infty(X)$.

The new notions of introvertibility allow us to define analogues of the first and second Arens product as introduced by Arens in [1].

Definition 3.2.5. Let $Y_1(G, X)$ be a left introverted subspace of $CB(X)$. For $m \in Y_1(G)^*$, $n \in Y_1(G, X)^*$ and $f \in Y_1(G, X)$, the *left Arens action* of $Y_1(G)^*$ on $Y_1(G, X)^*$ is defined by the map

$$(m, n) \mapsto m \odot n,$$

where

$$m \odot n(f) = \langle m, n_\lambda f \rangle.$$

Let $Y_2(G, X)$ be a right introverted subspace of $CB(X)$, let $m \in Y_2(G)^*$, $n \in Y_2(G, X)^*$ and $f \in Y_2(G, X)$. We similarly define the *right Arens action* of $Y_2(G)^*$ on $Y_2(G,X)^*$ by

$$n \boxdot m(f) = \langle n, m_\rho f \rangle.$$

It is easy to see that the left and right Arens actions are each weak*-weak* continuous in the first variable.

Proposition 3.2.6. *Let $Y(G,X)$ be a left introverted subspace of $CB(X)$. The left Arens action of $Y(G)^*$ on $Y(G,X)^*$ has the following properties:*

(i) If $Y(G,X)$ has a GLIM n, then for every $m \in M(Y(G))$, $m \odot n = n$.

(ii) $Y(G,X)^*$ is a left Banach-$Y(G)^*$ module.

(iii) For any $g \in G$ and $x \in X$, $\delta_g \odot \delta_x = \delta_{gx}$.

Proof. (i) Let $n \in GLIM(Y(G,X))$. Then $n_\lambda f(g) = n(f)$ for all $g \in G$ and $f \in Y(G,X)$. Hence

$$m \odot n(f) = \langle m, n_\lambda f \rangle = \langle m, n(f) \cdot 1 \rangle = n(f)$$

for all $m \in M(Y(G))$.

(ii) $Y(G)^*$ is a Banach algebra with the first Arens product, $\bigcirc$. For $l, m \in Y(G)^*$, $n \in Y(G,X)^*$, $f \in Y(G,X)$ and g, $h \in G$,

$$\lambda_{gh} f = \lambda_h(\lambda_g f),$$

$$\ell_g(n_\lambda f) = n_\lambda(\lambda_g f),$$

and so

$$\begin{aligned}(m_\ell(n_\lambda f))(g) &= \langle m,\ \ell_g(n_\lambda f) \rangle = \langle m, n_\lambda(\lambda_g f) \rangle = m \odot n(\lambda_g f) \\ &= (m \odot n)_\lambda f(g).\end{aligned} \tag{3.1}$$

Thus

$$\begin{aligned}(l \bigcirc m) \odot n(f) = l \bigcirc m\,(n_\lambda f) = \langle l, m_\ell(n_\lambda f) \rangle &\overset{(3.1)}{=} \langle l, (m \odot n)_\lambda f \rangle \\ &= l \odot (m \odot n)\,(f).\end{aligned}$$

Furthermore,

$$\|m \odot n\| = \sup_{\|f\|=1} |m(n_\lambda f)| \leq \|m\| \sup_{\|f\|=1} \|n_\lambda f\| = \|m\| \sup_{\|f\|=1} \sup_{a \in G} |n(\lambda_a f)|$$
$$\leq \|m\|\|n\|.$$

(iii) For $g \in G$, $x \in X$ and $f \in Y(G, X)$,

$$\delta_g \odot \delta_x(f) = \langle \delta_g, (\delta_x)_\lambda f \rangle = f(gx) = \delta_{gx}(f).$$

■

Definition 3.2.7. Let $Y(G, X)$ be a left introverted subspace of $CB(X)$. We define the *topological center* of $Y(G, X)^*$ to be the set

$Z_Y = \{m \in Y(G)^* :$ the map $n \mapsto m \odot n$ is w*-w* continuous on $Y(G, X)^*\}$.

3.3 Invariant subspaces of $CB(X)$

When a semitopological semigroup G has a jointly continuous left action on a Hausdorff space X, we are able to define the almost periodic, weakly almost periodic, and left uniformly continuous function spaces on X. We examine the properties of these function spaces.

Definition 3.3.1. A function $f \in CB(X)$ is *almost periodic* if $O_\lambda f$ is relatively compact in the norm topology of $CB(X)$. We denote the space of all such functions $AP(G, X)$.

Lemma 3.3.2. *$AP(G, X)$ is an invariant C^*-subalgebra of $CB(X)$.*

Proof. Clearly we have left invariance, since $\{\lambda_g f : g \in G\} = \{\lambda_a(\lambda_g f) : g \in G\}$ for all $f \in CB(X)$ and $a \in G$.

Let $f \in AP(G, X)$ and $x \in X$. To see that $\rho_x f \in AP(G)$, start with sequences $\{g_{n_k}\} \subset \{g_n\} \subset G$ such that $\|\lambda_{g_{n_k}} f - F\| \to 0$ for some $F \in CB(X)$. Then,

$$\|\ell_{g_{n_k}}(\rho_x f) - \rho_x F\| = \sup_{z \in G \cdot x} |\lambda_{g_{n_k}} f(z) - F(z)| \leq \|\lambda_{g_{n_k}} f - F\| \to 0,$$

showing that $AP(G,X)$ is right invariant.

Take $f_1,\ f_2 \in AP(G)$, a sequence $\{g_n\} \subset G$, a subsequence $\{g_{n_k}\}$ of $\{g_n\}$, and $F_1,\ F_2 \in CB(X)$ such that

$$\lim_k \|\lambda_{g_{n_k}} f_i - F_i\| = 0, \qquad i = 1, 2.$$

Then,

$$\|\lambda_{g_{n_k}}(f_1 + f_2) - (F_1 + F_2)\| \le \sum_{i=1,2} \|\lambda_{g_{n_k}} f_i - F_i\| \to 0,$$

and

$$\|\lambda_{g_{n_k}}(f_1 \cdot f_2) - (F_1 \cdot F_2)\| \le \|\lambda_{g_{n_k}} f_1 - F_1\|\|f_2\| + \|\lambda_{g_{n_k}} f_2 - F_2\|\|F_1\| \to 0.$$

Thus, $f_1 + f_2$ and $f_1 \cdot f_2 \in AP(G,X)$. Consequently, $AP(G,X)$ is a subalgebra of $CB(X)$.

To see that $AP(G,X)$ is norm closed, take $\{f_n\} \subset A(G,X)$ such that $\|f_n - f\| \to 0$ for some $f \in CB(X)$. Let $\{g_m\}$ be a sequence in G. For every m and n, the function $\lambda_{g_m} f_n$ is in $AP(G,X)$. Using the diagonal process, we find a subsequence $\{g_{m_k}\}$ of $\{g_m\}$ such that for every i,

$$\lim_k \|\lambda_{g_{m_k}} f_i - F_i\| = 0 \tag{3.2}$$

for some $F_i \in CB(X)$. Now, for $j > i$ and large values of k,

$$\begin{aligned} \|F_i - F_j\| &\le \|F_j - \lambda_{g_{m_k}} f_j\| + \|F_i - \lambda_{g_{m_k}} f_j\| \\ &\le \|F_j - \lambda_{g_{m_k}} f_j\| + \|F_i - \lambda_{g_{m_k}} f_i\| + \|\lambda_{g_{m_k}}(f_i - f_j)\| \\ &\overset{(3.2)}{\le} \|f_i - f_j\|. \end{aligned}$$

Thus there exists $F \in CB(X)$ such that $F_n \to F$ uniformly. We claim that $\|\lambda_{g_{m_k}} f - F\| \to 0$. Indeed, for every i,

$$\|\lambda_{g_{m_k}} f - F\| \le \|\lambda_{g_{m_k}} f_i - F_i\| + \|\lambda_{g_{m_k}} f_i - \lambda_{g_{m_k}} f\| + \|F_i - F\|,$$

and so, for large values of i,

$$\lim_k \|\lambda_{g_{m_k}} f - F\| \le \lim_k \|\lambda_{g_{m_k}} f_i - F_i\| \overset{(3.2)}{=} 0.$$

Thus $f \in AP(G,X)$. ■

Theorem 3.3.3. *Let $Y(G,X)$ be a norm closed, involution closed, translation invariant, linear subspace of $CB(X)$ containing the constant functions. For any $f \in Y(G,X)$, the following are equivalent:*

(i) $f \in AP(G,X)$.

(ii) $\{\rho_x f : x \in X\}$ *is relatively norm compact in* $CB(G)$.

(iii) *The map* $m \mapsto m_\rho f$, $\mathrm{Ball}\,(Y(G)^*) \to \ell^\infty(X)$ *is weak*-norm continuous.*

(iv) *The map* $n \mapsto n_\lambda f$, $\mathrm{Ball}\,(Y(G,X)^*) \to \ell^\infty(G)$ *is weak*-norm continuous.*

(v) *For every* $m \in Y(G,X)^*$ *and* $n \in Y(G)^*$, *the function* $m_\rho f$ *is in* $Y(G,X)$, *the function* $n_\lambda f$ *is in* $Y(G)$, $m \odot n(f) = n \boxdot m(f)$, *and the map* $(m,n) \mapsto m \odot n(f)$ *is weak* continuous on* $\mathrm{Ball}\,(Y(G)^*) \times \mathrm{Ball}\,(Y(G,X)^*)$.

Proof. (i) $\Leftrightarrow$ (iii) Define a map $V_1 : \mathrm{Ball}\,(Y(G)^*) \to \ell^\infty(X)$ by

$$V_1(m) = m_\rho f.$$

V_1 is weak*-pointwise continuous. By Lemma 3.2.2,

$$V_1(\mathrm{Ball}\,(Y(G)^*)) = \overline{\mathrm{cconv}\,\{\lambda_g f : g \in G\}}^{p}.$$

If (i) holds, then by Mazur's Theorem [10, V.2.6], $\overline{\mathrm{cconv}\,\{\lambda_g f : g \in G\}}^{\|\cdot\|}$ is compact. Thus the norm and pointwise topologies coincide on $V_1(\mathrm{Ball}\,(Y(G)^*))$, and V_1 is weak*-norm continuous. If we assume (iii), then $\{m_\rho f : \|m\| \leq 1\}$ is norm compact. But by Lemma 3.2.2,

$$\overline{\{\lambda_g f : g \in G\}}^{\|\cdot\|} \subset \overline{\{m_\rho f : \|m\| \leq 1\}}^{\|\cdot\|},$$

showing that $f \in AP(G,X)$.

Similarly, (ii) $\Leftrightarrow$ (iv). We define $V_2 : \mathrm{Ball}\,(Y(G,X)^*) \to \ell^\infty(G)$ by

$$V_2(n) = n_\lambda f.$$

V_2 is weak*-pointwise continuous. By Lemma 3.2.2,

$$V_2(\mathrm{Ball}\,(Y(G,X)^*)) = \overline{\mathrm{cconv}\,\{\rho_x f : x \in X\}}^{p}.$$

If we assume (ii) then, $\overline{\mathrm{cconv}\,\{\rho_x f : x \in X\}}^{\|\cdot\|}$ is compact, and the norm and pointwise topologies coincide on $V_2(\mathrm{Ball}\,(Y(G,X)^*))$. If we assume (iv), then $\{n_\lambda f : \|n\| \leq 1\}$ is norm compact. Relative norm compactness of $\{\rho_x f : x \in X\}$ follows from noticing that

$$\overline{\{\rho_x f : x \in X\}}^{\|\cdot\|} \subset \overline{\{n_\lambda f : \|n\| \leq 1\}}^{\|\cdot\|}.$$

(v) $\Rightarrow$ (i) Let $f \in Y(G,X)$. Define a map $W_1 : \mathrm{Ball}\,(Y(G)^*) \to CB(\mathrm{Ball}\,(Y(G,X)^*))$ by

$$[W_1(m)](n) = m \odot n(f).$$

W_1 is weak*-norm continuous by [2, Lemma B.3]. Then $\{W_1(\delta_g) : g \in G\}$ is relatively norm compact, as $\{\delta_g : g \in G\} \subset \mathrm{Ball}\,(Y(G)^*)$. Notice that

$$\lambda_g f(x) = \delta_g \odot \delta_x(f)$$

for all $x \in X$, and so

$$\{\lambda_g f : g \in G\} = \{[W_1(\delta_g)] \circ \delta : g \in G\},$$

which is a relatively norm compact set in $CB(X)$.

To prove (v) $\Rightarrow$ (ii), let $f \in Y(G,X)$. We define a map $W_2 : \mathrm{Ball}\,(Y(G)^*) \to CB(\mathrm{Ball}\,(Y(G)^*))$ by

$$[W_2(n)](m) = n \boxdot m(f).$$

The proof continuous in the same manner as the proof of (v) $\Rightarrow$ (i). W_2 is weak*-norm continuous, and the set $\{W_2(\delta_x) : x \in X\}$ is relatively norm compact.

$$\rho_x f(g) = \delta_x \boxdot \delta_g(f)$$

for all $g \in G$, and thus

$$\{\rho_x f : x \in X\} = \{[W_2(\delta_x)] \circ \delta : x \in X\},$$

which is a relatively norm compact set in $CB(G)$.

(ii) $\Rightarrow$ (v) Let $f \in Y(G,X)$. Recall that from Lemma 3.2.2 and the equivalence of (ii) and (iv),

$$\{n_\lambda f : \|n\| \leq 1\} = \overline{\mathrm{cconv}\,\{\rho_x f : x \in X\}}^{\|\cdot\|} \subset Y(G).$$

Assuming (ii), and since $Y(G,X)^* = \overline{\mathrm{span}\,\{\delta_x : x \in X\}}^{w^*}$, $n_\lambda f \in Y(G)$ for all $n \in Y(G,X)^*$. Next, we claim that the map $(m,n) \mapsto m \odot n(f)$ is continuous for all $m \in \mathrm{Ball}\,(Y(G)^*)$, $n \in \mathrm{Ball}\,(Y(G,X)^*)$ and $f \in Y(G,X)$. Indeed, let $\{m_\alpha\} \subset \mathrm{Ball}\,(Y(G)^*)$ and $\{n_\beta\} \subset \mathrm{Ball}\,(Y(G,X)^*)$ such that $m_\alpha \to m$ and $n_\beta \to n$, both weak*. By (iv),

$$|m \odot n(f) - m_\alpha \odot n_\beta(f)| \leq \|m\|\|n_\lambda f - (n_\beta)_\lambda f\| + \|m - m_\alpha\|\|(n_\beta)_\lambda f\| \to 0.$$

To see that $m_\rho f \in Y(G,X)$, notice that the continuity of the map $(m,n) \mapsto m \odot n(f)$ is all we need to show that $\{\lambda_g f : g \in G\}$ is relatively norm compact (see the proof of (v) $\Rightarrow$ (i)), and then by (iv), the map V_1 is weak*-norm continuous. As already seen in the proof of (i) $\Rightarrow$ (iii),

$$\{m_\rho f : \|m\| \leq 1\} = \overline{\mathrm{cconv}\,\{\lambda_g f : g \in G\}}^{\|\cdot\|} \subset Y(G,X).$$

The fact that $Y(G,X)^* = \overline{\mathrm{span}\,\{\delta_x : x \in X\}}^{w^*}$ proves that $m_\rho f \in Y(G,X)$ for all $m \in Y(G,X)^*$. It remains to be seen that the map $(n,m) \mapsto n \boxdot m(f)$ is continuous for all $m \in \mathrm{Ball}\,(Y(G)^*)$, $n \in \mathrm{Ball}\,(Y(G,X)^*)$ and $f \in Y(G,X)$. Indeed, let $\{m_\alpha\} \subset \mathrm{Ball}\,(Y(G)^*)$ and $\{n_\beta\} \subset \mathrm{Ball}\,(Y(G,X)^*)$ such that $m_\alpha \to m$ and $n_\beta \to n$ in the weak* topology. By the weak*-norm continuity of V_1,

$$|n \boxdot m(f) - n_\beta \boxdot m_\alpha(f)| \leq \|n\|\|m_\rho f - (m_\alpha)_\rho f\| + \|n - n_\beta\|\|(m_\alpha)_\rho f\| \to 0.$$

Clearly, $\delta_g \odot \delta_x(f) = \delta_x \boxdot \delta_g(f)$ for every $g \in G$ and every $x \in X$. By the joint continuity of the maps $(m,n) \mapsto m \odot n(f)$ and $(m,n) \mapsto n \boxdot m(f)$, $m \odot n(f) = n \boxdot m(f)$ for all $m \in \mathrm{Ball}\,(Y(G)^*)$ and $n \in \mathrm{Ball}\,(Y(G,X)^*)$, and thus holds for all $m \in Y(G)^*$ and $n \in Y(G,X)^*$.

(i) $\Rightarrow$ (v) Let $f \in AP(G,X)$. By Lemma 3.2.2 and Mazur,

$$\{m_\rho f : \|m\| \leq 1\} = \overline{\mathrm{cconv}\,\{\lambda_g f : g \in G\}}^{\|\cdot\|} \subset Y(G,X).$$

Assuming (i), and since $Y(G)^* = \overline{\text{span}\,\{\delta_g : g \in G\}}^{w^*}$, $m_\rho f \in Y(G,X)$ for all $m \in Y(G)^*$.

Next we show that $n_\lambda f \in Y(G)$ for all $n \in Y(G,X)^*$. Let $\{m_\alpha\} \subset Y(G)^*$ and $\{n_\beta\} \subset Y(G,X)^*$ such that $m_\alpha \to m$ and $n_\beta \to n$ in the weak* topologies of their respective spaces. By the continuity of V_1,

$$|n \boxdot m(f) - n_\beta \boxdot m_\alpha(f)| \leq \|n\| \|m_\rho f - (m_\alpha)_\rho f\| + \|n - n_\beta\| \|(m_\alpha)_\rho f\| \to 0.$$

As in the proof of (v) $\Rightarrow$ (i), $\{\rho_x f : x \in X\}$ is relatively norm compact. Then, applying (iv), Lemma 3.2.2 and Mazur,

$$\{n_\lambda f : \|n\| \leq 1\} = \overline{\text{cconv}\,\{\rho_x f : x \in G\}}^{\|\cdot\|} \subset Y(G).$$

Once again, using $Y(G,X)^* = \overline{\text{span}\,\{\delta_x : x \in X\}}^{w^*}$, we reason that $n_\lambda f \in Y(G)$ for all $n \in Y(G,X)^*$. By the weak*-norm continuity of V_2, the map $(m,n) \mapsto m \odot n(f)$ is continuous on $\text{Ball}\,(Y(G)^*) \times \text{Ball}\,(Y(G,X)^*)$. Finally, the equality $m \odot n(f) = n \boxdot m(f)$ follows as in the proof of (ii) $\Rightarrow$ (v). ■

Corollary 3.3.4. *$AP(G,X)$ is the largest involution closed introverted subspace $Y(G,X)$ of $CB(X)$ containing the constant functions with the following properties:*

(i) $m \odot n = n \boxdot m$ for all $m \in Y(G)^*$, $n \in Y(G,X)^*$.

(ii) The mapping $(m,n) \to m \odot n$ is jointly continuous on norm bounded subsets of $Y(G)^* \times Y(G,X)^*$.

Definition 3.3.5. A function $f \in CB(X)$ is *weakly almost periodic* if $O_\lambda f$ is relatively compact in the weak topology of $CB(X)$. We denote the space of all such functions $WAP(G,X)$.

Lemma 3.3.6. *$WAP(G,X)$ is an invariant C^*-subalgebra of $CB(X)$ containing $AP(G,X)$.*

Proof. The inclusion $AP(G,X) \subset WAP(G,X)$ is trivial, as norm compactness implies weak compactness. Left invariance follows from observing that

$\{\lambda_a(\lambda_g f) : a \in G\} \subset \{\lambda_a f : a \in G\}$ for all $g \in G$. To show right invariance, we need only to show that $\lim_i \lim_j \rho_x f(a_i b_j) = \lim_j \lim_i \rho_x f(a_i b_j)$ for sequences $\{a_i\}$ and $\{b_j\}$ in G. Notice that $\rho_x f(a_i b_x) = \delta_{b_j x}(\lambda_{a_i} f)$. $\lambda_{a_i} f$ has the double limit property, as shown in the next theorem.

Let $f_1,\, f_2 \in WAP(G, X)$. Choose a sequence $\{g_n\} \subset G$ and a subsequence $\{g_{n_k}\} \subset \{g_n\}$ such that for all $\phi \in CB(X)^*$,

$$|\phi(\lambda_{g_{n_k}} f_i) - \phi(F_i)| \to 0, \quad i = 1, 2, \tag{3.3}$$

where $F_1,\, F_2 \in CB(X)$. Clearly,

$$|\phi(\lambda_{g_{n_k}}(f_1+f_2)) - \phi(F_1+F_2)| \leq |\phi(\lambda_{g_{n_k}} f_1) - \phi(F_1)| + |\phi(\lambda_{g_{n_k}} f_2) - \phi(F_2)| \overset{(3.3)}{\to} 0,$$

and so $f_1 + f_2 \in WAP(G, X)$. To see that $f_1 \cdot f_2 \in WAP(G, X)$, we identify $CB(X)$ with the space $C(\Omega)$ of continuous complex-valued functions on the compact spectrum of the unital C^*-algebra $CB(X)$. The weak convergence of $\lambda_{g_{n_k}} f_1$ and $\lambda_{g_{n_k}}$ to F_1 and F_2 (respectively) implies the pointwise convergence of $\lambda_{g_{n_k}}(f_1 \cdot f_2)$ to $F_1 \cdot F_2$. Furthermore, $\|\lambda_{g_{n_k}}(f_1 \cdot f_2)\| \leq \|f_1\| \|f_2\|$. The pointwise convergence and norm boundedness is equivalent to the weak sequential compactness in $C(\Omega)$ (see [11, Theorem 1.3]). As a result, $f_1 \cdot f_2 \in WAP(G, X)$, and so $WAP(G, X)$ is a subalgebra of $CB(X)$.

To see that $WAP(G, X)$ is norm closed, take $\{f_n\} \subset WAP(G, X)$ such that $\|f_n - f\| \to 0$ in $CB(X)$. For a sequence $\{g_m\} \subset G$ use the diagonal process to produce a subsequence $\{g_{m_k}\}$ such that for each i and each $\phi \in CB(X)^*$,

$$|\phi(\lambda_{g_{m_k}} f_i) - \phi(F_i)| \to 0 \tag{3.4}$$

for some $F_i \in CB(X)$. For $j > i$,

$$\begin{aligned} \|F_i - F_j\| &= \sup_{\substack{\|\phi\| \leq 1 \\ \phi \in CB(X)^*}} |\langle \phi, F_j \rangle - \langle \phi, F_i \rangle| \\ &= \sup_{\substack{\|\phi\| \leq 1 \\ \phi \in CB(X)^*}} \lim_k |\langle \phi, \lambda_{g_{m_k}} f_j \rangle - \langle \phi, \lambda_{g_{m_k}} f_i \rangle| \\ &= \|f_j - f_i\|. \end{aligned}$$

This implies that $\{F_n\}$ is a norm Cauchy sequence in $CB(X)$, and as such, F_n is norm convergent to a function $F \in CB(X)$. For every i and $\phi \in CB(X)^*$,

$$\begin{aligned}|\phi(\lambda_{g_{m_k}} f) - \phi(F)| &\leq |\phi(F) - \phi(F_i)| + |\phi(F_i) - \phi(\lambda_{g_{m_k}} f_i)| \\ &\quad + |\phi(\lambda_{g_{m_k}} f_i) - \phi(\lambda_{g_{m_k}} f)| \\ &\leq \|\phi\|\|F - F_i\| + |\phi(F_i) - \phi(\lambda_{g_{m_k}} f_i)| + \|\phi\|\|f_i - f\|.\end{aligned}$$

In particular, for very large i,

$$\lim_k |\phi(\lambda_{g_{m_k}} f) - \phi(F)| \leq |\phi(F_i) - \phi(\lambda_{g_{m_k}} f_i)| \overset{(3.4)}{=} 0,$$

proving that $WAP(G, X)$ is norm closed in $CB(X)$. ■

Theorem 3.3.7. *Let $Y(G, X)$ be a norm closed, involution closed, translation invariant, linear subspace of $CB(X)$ containing the constant functions. For any $f \in Y(G, X)$, the following are equivalent:*

(i) $f \in WAP(G, X)$.

(ii) $\{\rho_x f : x \in X\}$ *is relatively weakly compact.*

(iii) $\lim_i \lim_j f(g_i x_j) = \lim_j \lim_i f(g_i x_j)$ *for sequences* $\{g_i\} \subset G$ *and* $\{x_j\} \subset X$*, whenever all limits exist.*

(iv) for every $m \in Y(G)^*$ *and* $n \in Y(G, X)^*$*, the function* $m_\rho f$ *is in* $Y(G, X)$*, the function* $n_\lambda f$ *is in* $Y(G)$*, and* $m \odot n(f) = n \boxdot m(f)$*.*

Proof. (i) ⇔ (iii) Let $f \in WAP(G, X)$. In [17], Grothendieck showed that the relatively weak compactness of $\{\lambda_g f : g \in G\}$ is equivalent to that set having the double limit property in $\{\delta_x : x \in X\}$. Thus, for sequences $\{g_i\} \subset G$ and $\{x_j\} \subset X$,

$$\lim_i \lim_j \delta_{x_j}(\lambda_{g_i} f) = \lim_j \lim_i \delta_{x_j}(\lambda_{g_i} f),$$

whenever all the limits exist.

(iii) ⇔ (ii) Notice that for sequences $\{g_i\}$ and $\{x_j\}$ as above,

$$f(g_i x_j) = \rho_{x_j} f(g_i) = \delta_{g_i}(\rho_{x_j} f),$$

showing that $\{\rho_x f : x \in X\}$ has the double limit property in a bounded subset of $CB(G)^*$. By the double limit theorem, this is equivalent to $\{\rho_x f : x \in X\}$ being relatively weak compact in $CB(G)$.

(i) $\Rightarrow$ (iv) Let $f \in WAP(G,X)$ and $m \in \mathrm{Ball}\,(Y(G)^*)$. We will only show that $m_\rho f \in Y(G,X)$. The proof that $n_\lambda f \in Y(G)$ for $n \in \mathrm{Ball}\,(Y(G,X)^*)$ is similar, so we safely omit the proof. Define a map $\phi : \mathrm{Ball}\,(Y(G)^*) \to \ell^\infty(X)$ by

$$\phi(m) = m_\rho f.$$

ϕ is weak*-pointwise continuous, and by Lemma 3.2.2

$$\phi(\mathrm{Ball}\,(Y(G)^*)) = \overline{\mathrm{cconv}\,\{\lambda_g f : g \in G\}}^p.$$

Now, consider a bounded function $F \in \overline{\mathrm{cconv}\,\{\lambda_g f : g \in G\}}^p$,

$$F(x) = \lim_\alpha \left(\sum_{i=1}^k \xi_i \lambda_{g_i} \right)_\alpha f(x)$$

where $\sum_{i=1}^k |\xi_i| \leq 1$, for all $x \in X$. Then,

$$\sum_j \zeta_j \delta_{x_j}(F) = \sum_j \zeta_j \delta_{x_j} \left(\lim_\alpha \left(\sum_{i=1}^k \xi_i \lambda_{g_i} \right)_\alpha f \right).$$

Since $Y(G,X)^* = \overline{\mathrm{span}\,\{\delta_x : x \in X\}}^{w^*}$, $F \in \overline{\mathrm{cconv}\,\{\lambda_g f : g \in G\}}^w$. Assuming (i) and applying a theorem [10, V.6.4] of Krein-Šmulian,

$$\phi(\mathrm{Ball}\,(Y(G)^*)) = \overline{\mathrm{cconv}\,\{\lambda_g f : g \in G\}}^w$$

and is weak compact in $Y(G,X)$. Thus $m_\rho f \in Y(G,X)$ for every $m \in \mathrm{Ball}\,(Y(G)^*)$.

Next we show that $m \odot n(f) = n \boxdot m(f)$ for all $m \in Y(G)^*$ and $n \in Y(G,X)^*$. The weak topology is stronger than the pointwise topology, and so the weak and pointwise topologies coincide on $\phi(\mathrm{Ball}\,(Y(G)^*))$. Thus ϕ is weak*-weak continuous from $\mathrm{Ball}\,(Y(G)^*)$ to $\ell^\infty(X)$. We use a special case of Grothendieck's completeness theorem [2, Proposition A.8] to extend the continuity of ϕ to $Y(G)^*$. For any $n \in Y(G,X)^*$, define a linear functional ω_n on $Y(G)^*$ by

$$\omega_n(m) = \langle n, m_\rho f \rangle.$$

ω_n is weak* continuous on $\mathrm{Ball}\,(Y(G)^*)$ by the definition of ϕ. Next, apply Grothendieck's completeness theorem to show that there exists a function $F_n \in Y(G)$ such that for all $m \in Y(G)^*$,

$$\omega_n(m) = m(F_n).$$

Then ω_n is a linear weak* continuous functional on $Y(G)^*$. Since $n \in Y(G,X)^*$ was arbitrary, ϕ is weak*-weak continuous from $Y(G)^*$. We claim that the mapping $n \mapsto F_n$ is weak*-weak continuous and linear. Indeed, linearity follows from the linearity of n. If $\{n_\alpha\} \subset Y(G,X)^*$ and $n_\alpha \to n$ weak*, then for any $m \in Y(G)^*$,

$$m(F_{n_\alpha}) = \omega_{n_\alpha}(m) = \langle n_\alpha, m_\rho f \rangle \to \langle n, m_\rho f \rangle = m(F_n).$$

We also have

$$F_{\delta_x}(g) = \delta_g(F_{\delta_x}) = \omega_{\delta_x}(\delta_g) = \langle \delta_x, (\delta_g)_\rho f \rangle = \delta_x \boxdot \delta_g(f).$$

The linear span of the point evaluations is equal to $Y(G,X)^*$, and thus for all $n \in Y(G,X)^*$,

$$F_n(g) = n \boxdot \delta_g(f).$$

That is, $F_n = n_\lambda f$, and so

$$n \boxdot m(f) = \langle n, m_\rho f \rangle = m(F_n) = \langle m, n_\lambda f \rangle = m \odot n(f).$$

(iv) $\Rightarrow$ (iii) For $f \in Y(G,X)$ and sequences $\{g_i\} \subset G$, $\{x_j\} \subset X$,

$$\begin{aligned}
\lim_i \lim_j f(g_i x_j) &= \lim_i \left(\lim_j \delta_{x_j} \boxdot \delta_{g_i}(f) \right) = \lim_i \delta_x \boxdot \delta_{g_i}(f) = \lim_i \delta_{g_i} \odot \delta_x(f) \\
&= \delta_g \odot \delta_x(f) = \delta_x \boxdot \delta_g(f) = \lim_j \delta_{x_j} \boxdot \delta_g(f) = \lim_j \delta_g \odot \delta_{x_j}(f) \\
&= \lim_j \left(\lim_i \delta_{g_i} \odot \delta_{x_j}(f) \right) = \lim_j \lim_i f(g_i x_j)
\end{aligned}$$

whenever all the limits exist. ■

Corollary 3.3.8. *$WAP(G,X)$ is the largest involution closed, invariant, left introverted subspace of $CB(X)$ containing the constant functions that satisfies the following:*

(i) $Y(G,X)$ is introverted.

(ii) $m \odot n = n \boxdot m$ for all $m \in Y(G)^*$ and $n \in Y(G,X)^*$.

(iii) The mapping $(m,n) \mapsto m \odot n$ is separately continuous.

Definition 3.3.9. A function $f \in CB(X)$ is called *left uniformly continuous* if the map $a \mapsto \lambda_a f$ from G into $CB(X)$ is continuous. We denote the set of all such functions by $LUC(G,X)$.

Lemma 3.3.10. *$LUC(G,X)$ is an invariant, left introverted C^*-subalgebra of $CB(X)$.*

Proof. Let $\{g_\alpha\} \subset G$ such that $g_\alpha \to g$ in G, and let $f \in LUC(G,X)$. Left invariance follows from observing that

$$\|\lambda_{g_\alpha}(\lambda_a f) - \lambda_g(\lambda_a f)\| = \|\lambda_{a g_\alpha} f - \lambda_{ag} f\|$$

for all $a \in G$.

To show right invariance, fix $x \in X$. Then

$$\begin{aligned}
\|\ell_{g_\alpha}(\rho_x f) - \ell_g(\rho_x f)\| &= \sup_{a\in G} |\ell_{g_\alpha}(\rho_x f)(a) - \ell_g(\rho_x f)(a)| \\
&= \sup_{a\in G} |f(g_\alpha a x) - f(gax)| \\
&= \sup_{z\in G\cdot x} |f(g_\alpha z) - f(gz)| \\
&\le \sup_{y\in X} |f(g_\alpha y) - f(gy)| \to 0,
\end{aligned}$$

since $G \cdot x \subseteq X$ and $f \in LUC(G,X)$.

$n_\lambda f$ is in $LUC(G)$ for all $n \in LUC(G,X)^*$, since

$$\|\ell_{g_\alpha}(n_\lambda f) - \ell_g(n_\lambda f)\| = \sup_{a\in G} |n(\lambda_{g_\alpha a} f) - n(\lambda_{ga} f)| \to 0,$$

by the weak* continuity of n, and by the left uniform continuity of f.

To see that $LUC(G,X)$ is a subalgebra of $CB(X)$, let $f_1,\, f_2 \in LUC(G,X)$, and take $\{g_\alpha\}$ as above. Then,

$$\|\lambda_{g_\alpha}(f_1+f_2)-\lambda_g(f_1+f_2)\| \le \|\lambda_{g_\alpha}f_1-\lambda_g f_1\|+\|\lambda_{g_\alpha}f_2-\lambda_g f_2\| \to 0,$$

and

$$\|\lambda_{g_\alpha}(f_1\cdot f_2)-\lambda_g(f_1\cdot f_2)\| \le \|f_1\|\|\lambda_{g_\alpha}f_2-\lambda_g f_2\|+\|f_2\|\|\lambda_{g_\alpha}f_1-\lambda_g f_1\| \to 0$$

by the left uniform continuity of f_1 and f_2.

Finally, to see that $LUC(G,X)$ is norm closed, take $\{g_\alpha\}$ as above and let $\{f_\beta\} \subset LUC(G,X)$ such that f_β converges uniformly to some $F \in CB(X)$. Then

$$\|\lambda_{g_\alpha}F-\lambda_g F\| \le \|\lambda_{g_\alpha}F-\lambda_{g_\alpha}f_\beta\|+\|\lambda_{g_\alpha}f_\beta-\lambda_g f_\beta\|+\|\lambda_g f_\beta-\lambda_g F\| \to 0,$$

by the norm convergence of f_β to F, and by the left uniform continuity of each f_β. Thus $F \in LUC(G,X)$. ∎

We will take a closer look at the Arens action of $LUC(G)^*$ on $LUC(G,X)^*$ in Section 3.4.

Lemma 3.3.11. $AP(G,X) \subset LUC(G,X)$.

Proof. Let $f \in AP(G,X)$. Clearly the map $g \mapsto \lambda_g f$ is pointwise continuous. $O_\lambda f$ is relatively norm compact, so the norm and pointwise topologies agree on this set. ∎

Lemma 3.3.12. *Let G be a locally compact group. Then $WAP(G,X) \subset LUC(G,X)$.*

Proof. Let $f \in WAP(G,X)$. We claim that the map $g \mapsto \lambda_g f$, from G to $WAP(G,X)$ is weakly continuous. Indeed, if the net $\{g_\alpha\} \subset G$ converges to $g \in G$, then $\langle \delta_x, \lambda_{g_\alpha} f\rangle \to \langle \delta_x, \lambda_g f\rangle$. Therefore $\{\lambda_{g_\alpha} f\}$ has at least one weak cluster point, $\lambda_g f$. The relative weak compactness of $O_\lambda f$ implies that $\{\lambda_{g_\alpha} f\}$ converges weakly to $\lambda_g f$. A result of Lau [22, p. 151] (see also [25, Theorem 7] for the proof when $X = G$) shows that the map $g \mapsto \lambda_g f$ is norm continuous. For the sake of completeness we give the details of the argument.

Define an action of G on $M(WAP(G,X))$ by:

$$\langle am, f\rangle = \langle m, \lambda_{a^{-1}} f\rangle,$$

for $a \in G$, $m \in M(WAP(G,X))$ and $f \in WAP(G,X)$. The map $(a,m) \mapsto am$ is continuous when $WAP(G,X)^*$ has the weak* topology. Indeed, we show that the map is separately continuous. Let $\{a_\alpha\} \subset G$ such that $a_\alpha \to a \in G$. Then

$$\langle a_\alpha m, f\rangle = \langle m, \lambda_{a_\alpha^{-1}} f\rangle \to \langle m, \lambda_{a^{-1}} f\rangle = \langle am, f\rangle,$$

by the weak continuity of the map $g \mapsto \lambda_g f$. Let $\{m_\beta\} \subset M(WAP(G,X))$ such that $m_\beta \to m$ weak* in $M(WAP(G,X))$. Then

$$\langle am_\beta, f\rangle = \langle m_\beta, \lambda_{a^{-1}} f\rangle \to \langle am, f\rangle.$$

Ellis' result [12, Theorem 1] implies that the action $(a,m) \mapsto am$ is jointly continuous. Next, suppose that $f \notin LUC(G,X)$. That is, suppose there exists $\epsilon > 0$ and nets $\{a_\alpha\} \subset G$, $\{x_\alpha\} \subset X$ with $a_\alpha \to a \in G$ such that

$$\epsilon \leq |\lambda_{a_\alpha} f(x_\alpha) - \lambda_a f(x_\alpha)|$$

for each α. Choose a subnet $\{x_{\alpha_\gamma}\} \subset \{x_\alpha\}$ such that $\delta_{x_{\alpha_\gamma}} \to m$ weak*. Then,

$$\begin{aligned} 0 < \epsilon &\leq \lim_\gamma |\langle \delta_{x_{\alpha_\gamma}}, \lambda_{a_{\alpha_\gamma}} f\rangle - \langle \delta_{x_{\alpha_\gamma}}, \lambda_a f\rangle| \\ &= \lim_\gamma |\langle a_{\alpha_\gamma}^{-1} \delta_{x_{\alpha_\gamma}}, f\rangle - \langle a^{-1} \delta_{x_{\alpha_\gamma}}, f\rangle| = 0. \end{aligned}$$

As a result of this contradiction, $f \in LUC(G,X)$. ■

Remark 3.3.13. Easy calculations will show that the space $CB(X)$ is invariant.

3.4 Arens action of the Banach algebra $LUC(G)^*$ on $LUC(G,X)^*$

In this chapter we shall consider the topological center Z_{LUC} of the Arens action of $LUC(G)^*$ on $LUC(G,X)^*$. Using ideas of Lau and Wong, we show that if G and X are locally compact, then the measure algebra $\mathcal{M}(G)$ is a subset of Z_{LUC}, and that in general $\mathcal{M}(G) \subsetneq Z_{LUC}$.

Let Z denote the set of all $m \in LUC(G)^*$ such that the function $m_\rho f$ is in $LUC(G,X)$ for all $f \in LUC(G,X)$, with $m \odot n = n \boxdot m$ for all $n \in LUC(G)^*$. We begin by proving the following equivalence:

Lemma 3.4.1. *Let $m \in LUC(G)^*$. The following are equivalent:*

(i) $m \in Z$.

(ii) $m \in Z_{LUC}$.

(iii) The map $n \mapsto m \odot n$ is weak-weak* continuous on norm bounded subsets of $LUC(G,X)^*$.*

Proof. (i) $\Rightarrow$ (ii) Suppose $m \in Z$, and suppose $\{n_\alpha\}$ is a net in $LUC(G,X)^*$ which converges weak* to n. Then, for all $f \in LUC(G,X)$,

$$m \odot n_\alpha(f) = n_\alpha \boxdot m(f) \to n \boxdot m(f) = m \odot n(f)$$

(ii) $\Rightarrow$ (iii) is trivial.

(iii) $\Rightarrow$ (i) Consider $f \in LUC(G,X)$. First we establish that $m_\rho f$ is in $CB(X)$. Let $\{x_\alpha\}$ be a net in X converging to x. The norm bounded net $\{\delta_{x_\alpha}\}$ converges weak* to δ_x in $LUC(G,X)^*$. We then have that

$$m_\rho f(x_\alpha) = m(\rho_{x_\alpha} f) = m \odot \delta_{x_\alpha}(f) \longrightarrow m \odot \delta_x(f) = m_\rho f(x),$$

thus $m_\rho f \in CB(X)$.

Suppose that $m_\rho f \notin LUC(G,X)$. That is, suppose for a net $\{a_\alpha\}$ which converges to a in G,

$$0 < \epsilon \leq \|\lambda_{a_\alpha}(m_\rho f) - \lambda_a(m_\rho f)\|. \tag{3.5}$$

Let $\{\theta_\alpha\} \subset \mathrm{Ball}\,(CB(X)^*)$ such that for every α,

$$\langle \theta_\alpha, \lambda_{a_\alpha}(m_\rho f) - \lambda_a(m_\rho f)\rangle = \|\lambda_{a_\alpha}(m_\rho f) - \lambda_a(m_\rho f)\|. \tag{3.6}$$

We claim that for $\theta \in CB(X)^*$, $f \in LUC(G, X)$, and $a \in G$,

$$\langle \theta,\ \lambda_a(m_\rho f)\rangle = \langle m \odot (\delta_a \odot \theta)\,, f\rangle.$$

This claim holds for the case $\theta = \delta_x$, $x \in X$. Indeed, by Proposition 3.2.6(iii),

$$\begin{aligned}\langle \delta_x,\ \lambda_a(m_\rho f)\rangle &= m_\rho f(ax) = m(\rho_{ax} f) = \langle m, (\delta_{ax})_\lambda f\rangle = m \odot \delta_{ax}(f)\\ &= \langle m \odot (\delta_a \odot \delta_x), f\rangle.\end{aligned}$$

Then, if $\theta \geq 0$, $\|\theta\| = 1$, apply Remark 2.3.1 to obtain a net of convex combinations of point evaluations $\theta_\beta = \sum_{i=1}^k \xi_i \delta_{x_i}$, such that $\theta_\beta \to \theta$ weak* in $CB(G)^*$. By (iii),

$$\begin{aligned}\langle \theta,\ \lambda_a(m_\rho f)\rangle &= \lim_\beta \langle \theta_\beta,\ \lambda_a(m_\rho f)\rangle = \lim_\beta \langle m \odot (\delta_a \odot \theta_\beta)\,, f\rangle\\ &= \langle m \odot (\delta_a \odot \theta)\,, f\rangle,\end{aligned}$$

which proves our claim. From (3.5) and (3.6), we have that

$$\begin{aligned}\epsilon \leq \langle \theta_\alpha, \lambda_{a_\alpha}(m_\rho f) - \lambda_a(m_\rho f)\rangle &= |\langle m \odot (\delta_{a_\alpha} \odot \theta_\alpha) - m \odot (\delta_a \odot \theta_\alpha), f\rangle|\\ \leq |\langle m \odot (\delta_{a_\alpha} \odot \theta_\alpha) - m \odot (\delta_a \odot \theta), f\rangle| & \qquad (3.7)\\ + |\langle m \odot (\delta_a \odot \theta) - m \odot (\delta_a \odot \theta_\alpha), f\rangle|. & \qquad (3.8)\end{aligned}$$

We may assume that θ_α converges weak* to θ in $LUC(G, X)^*$ after passing to a subnet, if necessary. Furthermore, the action of G on the unit ball of $LUC(G, X)^*$ defined by $(a, \theta) \mapsto \delta_a \odot \theta$ is jointly continuous. Indeed, let $\{a_\sigma\} \subset G$ and $\{\theta_\iota\} \subset LUC(G, X)^*$, $\theta_\iota \geq 1$, $\|\theta_\iota\| = 1$ such that $a_\sigma \to a$ in G and $\theta_\iota \to \theta$ weak* in $LUC(G, X)^*$. Then for all $f \in LUC(G, X)$,

$$\delta_{a_\sigma} \odot \theta_\iota(f) = \langle \delta_{a_\sigma}, (\theta_\iota)_\lambda f\rangle = \theta_\iota(\lambda_{a_\sigma} f) \to \theta(\lambda_a f) = \delta_a \odot \theta(f),$$

by the definitions of f and θ_ι. Assuming (iii) and taking α sufficiently large, (3.7) and (3.8) are each less than $\epsilon/2$, which is a contradiction. Therefore $m_\rho f \in LUC(G, X)$.

To show that $n \boxdot m = m \odot n$, we first claim that this is true when $n = \delta_x$, $x \in X$. Indeed, for all $f \in LUC(G, X)$,

$$\begin{aligned}(\delta_x \boxdot m)(f) &= \langle \delta_x, m_\rho f\rangle = m_\rho f(x) = m(\rho_x f) \\ &= \langle m, (\delta_x)_\lambda f\rangle = (m \odot \delta_x)(f).\end{aligned}$$

This also holds for convex combinations of point measures. Applying Remark 2.3.1 by writing any $n \in LUC(G, X)^*$ such that $n \geq 0$ and $\|n\| = 1$ as the weak* limit of a net of such combinations, we obtain that for every $f \in LUC(G, X)$,

$$\begin{aligned}n \boxdot m(f) &= \langle n, m_\rho f\rangle = \lim_\alpha \langle n_\alpha, m_\rho f\rangle \\ &= \lim_\alpha m \odot n_\alpha(f) = m \odot n(f).\end{aligned}$$

Thus, $m \in Z$. ■

When X is locally compact, let τ_X denote the locally convex topology on $\mathcal{M}(X)$ determined by the family of seminorms $\{p_f : f \in LUC(G, X)\}$, where $p_f(\mu) = |\int f\, d\mu|$, $\mu \in \mathcal{M}(X)$. Each $\mu \in \mathcal{M}(X)$ defines an element in $LUC(G, X)^*$ via $\langle \mu, f\rangle = \int f\, d\mu$. Let τ_G denote the locally convex topology above for the case $X = G$.

Lemma 3.4.2. *Let G be a locally compact semitopological semigroup with jointly continuous action on a locally compact Hausdorff space X.*

(i) For every $\mu \in \mathcal{M}(G)$, the map $n \mapsto \mu \odot n$ is weak-weak* continuous on norm bounded subsets of $LUC(G, X)^*$.*

(ii) For every $n \in LUC(G, X)^*$, the map $\mu \mapsto \mu \odot n$ is τ_G-weak continuous.*

(iii) For $\mu \in \mathcal{M}(G)$, $\nu \in \mathcal{M}(X)$, $\mu \odot \nu(f) = \langle \mu * \nu, f\rangle$ for all $f \in C_0(X)$.

(iv) $n \in GLIM(LUC(G, X))$ if and only if $\mu \odot n = n$ for any $\mu \in \mathcal{M}_0(G) = \{\mu \in \mathcal{M}(G) : \mu \geq 0, \|\mu\| = 1\}$.

Proof. (i) Consider a net $\{n_\alpha\} \subset LUC(G,X)^*$ such that $n_\alpha \to n$ weak* in $LUC(G,X)^*$ with $\|n_\alpha\|, \|n\| \leq K$.

$$\begin{aligned} |(n_\alpha)_\lambda f(a) - (n_\alpha)_\lambda f(b)| = |n_\alpha(\lambda_a f) - n_\alpha(\lambda_b f)| &\leq \|n_\alpha\| \|\lambda_a f - \lambda_b f\| \\ &\leq K\|\lambda_a f - \lambda_b f\| \end{aligned}$$

for all a, $b \in G$, which shows that $\{(n_\alpha)_\lambda f\}$ is a equicontinuous family of functions ([21, p. 232]). We have pointwise convergence of $(n_\alpha)_\lambda f$ to $n_\lambda f$, since

$$|(n_\alpha)_\lambda f(g) - n_\lambda f(g)| = |n_\alpha(\lambda_g f) - n(\lambda_g f)| \to 0$$

by the weak* convergence of n_α to n. By [21, Theorem 7.15], $(n_\alpha)_\lambda$ converges uniformly to $n_\lambda f$ on compact subsets of G. Now, for μ with compact support U,

$$\left| \int (n_\alpha)_\lambda f \, d\mu - \int n_\lambda f \, d\mu \right| \leq \int \sup_{g \in U} |(n_\alpha)_\lambda f(g) - n_\lambda f(g)| \, d\mu \to 0.$$

Such measures are norm dense in $\mathcal{M}(G)$, and

$$\|(n_\alpha)_\lambda f\| = \sup_{g \in G} |(n_\alpha)_\lambda f(g)| = \sup_{g \in G} |n_\alpha(\lambda_g f)| \leq \|n_\alpha\| \|f\| \leq K\|f\|.$$

Thus $\mu \odot n_\alpha \to \mu \odot n$ weak* for all $\mu \in \mathcal{M}(G)$.

(ii) Consider $n \in LUC(G,X)^*$. Let $\{\mu_\alpha\} \subset \mathcal{M}(G)$ converge to μ in the τ_G topology. Then for all $f \in LUC(G,X)$,

$$\mu_\alpha \odot n(f) - \mu \odot n(f) = \int n_\lambda f \, d\mu_\alpha - \int n_\lambda f \, d\mu \to 0,$$

since $n_\lambda f \in LUC(G)$. Therefore $\mu_\alpha \odot n \to \mu \odot n$ weak* in $LUC(G,X)^*$.

(iii) Let $\mu \in \mathcal{M}(G)$ and $\nu \in \mathcal{M}(X)$. $\mathcal{M}(G)$ has an action on $\mathcal{M}(X)$ defined by

$$\iint f(gx) \, d\mu(g) d\nu(x) = \int f(gx) \, d(\mu * \nu)(gx),$$

for $f \in C_0(X)$ which makes $\mathcal{M}(X)$ a Banach module over $\mathcal{M}(G)$. From [15,

p. 299], $C_0(X) \subset LUC(G,X)$. Thus, for $f \in C_0(X)$,

$$\begin{aligned}
\mu \odot \nu(f) &= \mu(\nu_\lambda f) = \int \nu_\lambda f(g)\, d\mu(g) = \int \nu(\lambda_g f)\, d\mu(g) \\
&= \iint (\lambda_g f)(x)\, d\nu(x) d\mu(g) = \iint f(gx)\, d\nu(x) d\mu(g) \\
&= \iint f(gx)\, d\mu(g) d\nu(x) = \int f(gx) d(\mu * \nu)(gx) \\
&= \langle \mu * \nu, f \rangle.
\end{aligned}$$

(iv) To see "$\Leftarrow$", notice that $\mathcal{M}_0(G)$ contains the point measures δ_g. Let $\mu = \delta_g$. If $\mu \odot n = n$,

$$n(f) = \delta_g \odot n(f) = \int n_\lambda f\, d\delta_g = n(\lambda_g f)$$

for all $g \in G$, which proves that n is G-invariant.

"$\Rightarrow$" For any $\mu \in \mathcal{M}_0(G)$, take a net $\{\mu_\alpha\}$ of convex combinations of point evaluations such that μ_α converges weak* to μ in $LUC(G)^*$. Then μ_α converges to μ in the τ_G topology in $\mathcal{M}(G)$. By (ii), $\mu_\alpha \odot n \to \mu \odot n$ weak*. By Proposition 3.2.6, $\mu_\alpha \odot n = n$ since n is G-invariant. Thus $\mu \odot n = n$ for all $\mu \in \mathcal{M}_0(G)$. ■

Theorem 3.4.3. *Let G be a locally compact semitopological semigroup with jointly continuous action on a locally compact Hausdorff space X. If $\mu \in \mathcal{M}(G)$, then $\mu \in Z_{LUC}$.*

Proof. By Lemma 3.4.2 (i), the map $\mu \mapsto m \odot \mu$ is weak*-weak* continuous on norm bounded subsets of $LUC(G,X)^*$. By Lemma 3.4.1, $\mu \in Z_{LUC}$. ■

The following result was suggested to me by Lau (personal communication).

Example 3.4.4. In general, $\mathcal{M}(G) \subsetneq Z_{LUC}$. Take G to be an infinite semigroup with the discrete topology and with a product defined by $ab = b$ for all a, $b \in G$. Let $X = G$. Then $LUC(G) = LUC(G,X) = \ell^\infty(G)$, and $\mathcal{M}(G) = \ell^1(G)$. Clearly $Z_{LUC} \subseteq \ell^\infty(G)^*$. To see that $\ell^\infty(G)^* \subseteq Z_{LUC}$, let

$\{n_\alpha\} \subset \ell^\infty(G)^*$ such that $n_\alpha \to n$ weak*. Then,

$$\|(n_\alpha)_\ell f - n_\ell f\| = \sup_{g\in G} |(n_\alpha)_\ell f(g) - n_\ell f(g)| = \sup_{g\in G} |n_\alpha(\ell_g f) - n(\ell_g f)|$$
$$= \sup_{g\in G} |n_\alpha(f) - n(f)| \to 0,$$

and so

$$m \odot n_\alpha(f) = \langle m, (n_\alpha)_\ell f\rangle \to \langle m, n_\ell f\rangle = m \odot n(f).$$

Thus $Z_{LUC} = \ell^\infty(G)^* = \ell^1(G)^{**}$, while $\mathcal{M}(G) = \ell^1(G)$, a non-reflexive space.

Remark 3.4.5. (a) Lau proved that if G is either a locally compact group or a cancellative discrete semigroup and $X = G$, then $Z_{LUC} = \mathcal{M}(G)$ (see [23, Theorem 1]). Recently, Neufang [27] used a different technique to prove the same result when G is a locally compact group. (b) Lemma 3.4.1 and its proof are based on [23, Lemma 2]. Lemma 3.4.2 and its proof are based on [31, Lemma 3.1].

3.5 G-minimal sets and supports of G-invariant measures

Let G be a discrete semigroup with a left action on a discrete space X. In this setting, the spaces $LUC(G)$ and $LUC(G,X)$ are simply $\ell^\infty(G)$ and $\ell^\infty(X)$ respectively. We then consider the action of $\ell^\infty(G)^*$ on $\ell^\infty(X)^*$ as defined by the Arens action.

Let $(\kappa, \beta X)$ be the Stone-Čech compactification of X. For discrete X, βX is homeomorphic to the character space of $\ell^\infty(X)$. In other words, we regard βX as the space of all multiplicative linear functionals ϕ on $\ell^\infty(X)$ such that $\phi(1) = \|\phi\| = 1$, with the weak* topology inherited from $\ell^\infty(X)^*$. More precisely, for $x \in X$ we identify $\kappa(x) \in \beta X$ with the evaluation functional $\delta_x \in \ell^\infty(X)^*$. It is easy to see that for any subset $U \subset X$, the set $\overline{\{\kappa(x) : x \in U\}}^{w^*}$ is weak* closed, and that its complement in open in βX. Thus all closed and open subsets of βX are of the form $\overline{\{\kappa(x) : x \in U\}}^{w^*}$, for $U \subset X$. For convenience, we denote the weak* closure of a set V by $\overline{V}$ hereafter.

The action of G on X extends to an action of βG on βX. For any $a \in G$ and $f_1, f_2 \in \ell^\infty(X)$, $\lambda_a(f_1 \cdot f_2) = (\lambda_a f_1 \cdot \lambda_a f_2)$. Then

$$m \odot n(f_1 \cdot f_2) = \langle m, n_\lambda(f_1 \cdot f_2)\rangle = \langle m, n_\lambda f_1 \cdot n_\lambda f_2\rangle = m \odot n(f_1) \cdot m \odot n(f_2)$$

for any $m \in \beta G$, $n \in \beta X$, which shows that $m \odot n \in \beta X$. Since $\kappa(G) \subset \beta G$, we may also define an action of G on βX by

$$(a, n) \mapsto \kappa(a) \odot n$$

for $a \in G$ and $n \in \beta X$.

Remark 3.5.1. By Lemma 3.4.1 and Theorem 3.4.3, the map

$$n \mapsto \kappa(a) \odot n$$

is weak*-weak* continuous for all $a \in G$ and $n \in \beta X$.

For a fixed $n \in \beta X$ we denote the set of all products

$$\kappa(G) \odot n = \{\kappa(g) \odot n : g \in G\}.$$

For a fixed $g \in G$ and any subset $U \subset \beta X$ we denote the set of all products

$$\kappa(g) \odot U = \{\kappa(g) \odot n : n \in U\}.$$

If $K \subset G$ and $U \subset \beta X$, we use the following notation:

$$\{\kappa(K)\}^{-1} \odot U = \{n \in \beta X : \kappa(k) \odot n \in U \text{ for some } k \in K\}.$$

If $A \subset G$ and K is as above, we denote

$$\{K\}^{-1}A = \{g \in G : kg \in A \text{ for some } k \in K\}.$$

We have an isometric $*$-isomorphism T from $C_c(\beta X) = CB(\beta X)$ onto $\ell^\infty(X)$, $\tilde{f} \mapsto f$, where

$$f(x) = \tilde{f}(\kappa(x)),$$

for $\tilde{f} \in C_c(\beta X)$ and $x \in X$. By the Reisz representation theorem, we identify $CB(\beta X)^*$ with $\mathcal{M}(\beta X)$ in the usual way:

$$\langle T^* n, \tilde{f}\rangle = \int \tilde{f}\, d(T^* n).$$

Proposition 3.5.2. *$n \in GLIM(\ell^\infty(X))$ if and only if T^*n is a probability measure on βX such that $(T^*n)(\{\kappa(g)\}^{-1} \odot U) = (T^*n)(U)$ for all $g \in G$ and Borel sets $U \subset \beta X$.*

Proof. "⇒" Suppose $n \in GLIM(\ell^\infty(X))$. T^*n is a regular Borel measure for βX such that $\|T^*n\| = 1$. Let $\tilde{f} \in CB(\beta X)$, $f \geq 0$. Then

$$(T\lambda_{\kappa(g)}\tilde{f})(x) = \lambda_{\kappa(g)}\tilde{f}(\kappa(x)) = \tilde{f}(\kappa(gx)) = (T\tilde{f})(gx) = \lambda_g f(x),$$

for any $x \in X$ and $g \in G$. By the invariance of n,

$$\int \lambda_{\kappa(g)}\tilde{f}\, d(T^*n) = \langle T^*n, \lambda_{\kappa(g)}\tilde{f}\rangle = \langle n, \lambda_g f\rangle = \langle n, f\rangle = \langle T^*n, \tilde{f}\rangle = \int \tilde{f}\, d(T^*n).$$

Since $\tilde{f} \in CB(\beta X)$ was arbitrary,

$$T^*n(\{\kappa(g)\}^{-1} \odot U) = T^*n(U)$$

for all Borel sets $U \subset \beta X$.

To see "⇐", notice that

$$\chi_{\{\kappa(g)\}^{-1}\odot U} = \lambda_{\kappa(g)}\chi_U.$$

for any $g \in G$ and Borel set $U \subset \beta X$. For any function $f \in \ell^\infty(X)$, we can approximate $\tilde{f} \in CB(\beta X)$ by simple functions. As a result,

$$\langle n, f\rangle = \int \tilde{f}\, d(T^*n) = \int \lambda_{\kappa(g)}\tilde{f}\, d(T^*n) = \langle n, \lambda_g f\rangle.$$

■

Definition 3.5.3. Let $n \in \ell^\infty(X)^*$. T^*n is called *G-invariant* if $n \in GLIM(\ell^\infty(X))$.

Definition 3.5.4. $n \in \beta X$ is called *left almost G-periodic* if for every neighbourhood U of n there exists a subset $A \subset G$ such that there is a finite subset $K \subset G$ with $G = \{K\}^{-1}A$ and $\kappa(A) \odot n \subset U$. We denote the set of all almost G-periodic elements in βX by $A^{G,X}$.

Definition 3.5.5. A nonempty subset U of βX is called *G-invariant* if $\kappa(g) \odot U \subset U$ for all $g \in G$. U is called *G-minimal* if it is closed and minimal with respect to this property. We denote the elements of βX which belong to a G-minimal set by $B^{G,X}$.

We denote by $K^{G,X}$ the elements in βX which are in the support of some G-invariant measure.

Example 3.5.6. Take $n \in \beta X$. Clearly, for all $a \in G$, $\kappa(a) \odot \kappa(G) \odot n \subset \kappa(G) \odot n$. Thus, $\overline{\kappa(G) \odot n}$ is a closed G-invariant set.

Proposition 3.5.7. *Let $n \in GLIM(\ell^\infty(X))$. Then $\operatorname{supp}(T^*n)$ is a G-invariant set.*

Proof. Suppose $m \in \operatorname{supp}(T^*n)$ and let $g \in G$. Let $\overline{\kappa(A)}$ be an open neighbourhood of $\kappa(g) \odot m$, so that $m \in \{\kappa(g)\}^{-1} \odot \overline{\kappa(A)}$. By Proposition 3.5.2,

$$0 < T^*n(\{\kappa(g)\}^{-1} \odot \overline{\kappa(A)}) = T^*n(\overline{\kappa(A)}),$$

showing that $\overline{\kappa(A)} \subset \operatorname{supp}(T^*n)$. Consequently $\kappa(g) \odot m \in \operatorname{supp}(T^*n)$. ■

Let U be a closed subset of βG. In the case where $X = G$, it is well known that for all $n \in M(\ell^\infty(G))$, $\operatorname{supp}(T^*n) \subset U$ if and only if $n \in \overline{\operatorname{conv} U}$ [30]. The same can be said when X is an arbitrary discrete space.

Proposition 3.5.8. *Let $n \in M(\ell^\infty(X))$, and let U be a closed subset of βX. Then $\operatorname{supp}(T^*n) \subset U$ if and only if $n \in \overline{\operatorname{conv} U}$.*

Proof. "$\Rightarrow$" The set U is of the form $\overline{\{\delta_x : x \in V\}}$, where $V \subset X$. Next, write n as the weak* limit of convex combinations of point evaluations to obtain the formula

$$\langle T^*n, \chi_U \rangle = \langle n, \chi_V \rangle = \lim_\alpha \left(\sum_{i=1}^k \xi_i \delta_{x_i} \right)_\alpha (\chi_V), \tag{3.9}$$

But $(\sum_{i=1}^k \xi_i)_\alpha = 1$ for each α. Since $\langle T^*n, \chi_U \rangle = 1$, $\{\delta_{x_i}\}_\alpha \subset U$ for each α.

"$\Leftarrow$" Notice that for $n = w^* \lim_\alpha \left(\sum_{i=1}^k \xi_i \delta_{x_i} \right)_\alpha \in \overline{\operatorname{conv} U}$, equation (3.9) $= 1$. ■

Theorem 3.5.9. $A^{G,X} = B^{G,X}$.

Proof. "$\supset$" Let $U \subset \beta X$ be G-minimal, and suppose $m \in U$. Clearly, $\kappa(g) \odot m \in U$ for all $g \in G$. i.e., $\kappa(G) \odot w \subset U$. Since U is closed, $\overline{\kappa(G) \odot m} \subset U$. However, since U is G-minimal, $U \subset \overline{\kappa(G) \odot m}$. Thus $U = \overline{\kappa(G) \odot m}$. Let V be an open neighbourhood of m. Suppose there is some $v \in U \backslash (\{\kappa(G)\}^{-1} \odot V)$. Also suppose that $\kappa(g) \odot v \in \{\kappa(G)\}^{-1} \odot V$ for some $g \in G$. Then there exists some $h \in G$ such that

$$\kappa(h) \odot \kappa(g) \odot v = \kappa(hg) \odot v \in V,$$

and so $v \in \{\kappa(hg)\}^{-1} \odot V$, contradicting $v \notin \{\kappa(G)\}^{-1} \odot V$. We have $\kappa(g) \odot v \notin \{\kappa(G)\}^{-1} \odot V$, and $\kappa(g) \odot v \in U$ since U is G-minimal. Thus $U \backslash (\{\kappa(G)\}^{-1} \odot V)$ is a closed and G-invariant set. By the G-minimality of U,

$$U \cap (\kappa(G)\}^{-1} \odot V) = \varnothing,$$

i.e. $m \notin \{\kappa(G)\}^{-1} \odot V$ and $(\kappa(G) \odot m) \cap V = \varnothing$. This is impossible since V is open and $m \in U = \overline{\{\kappa(G)\}^{-1} \odot V}$. Therefore $U \backslash (\{\kappa(G)\}^{-1} \odot V) = \varnothing$ and $U \subset \{\kappa(G)\}^{-1} \odot V$. Now, U is closed and thus compact. So there exists some finite $K \subset G$ such that

$$U \subset \{\kappa(K)\}^{-1} \odot V.$$

Let $A = \{g \in G : \kappa(g) \odot m \in V\}$. By the definition of U, for any $g \in G$ there must exist some $k \in K$ such that

$$\kappa(k) \odot (\kappa(g) \odot m) = \kappa(kg) \odot m \in V,$$

giving us that $kg \in A$. Since g was arbitrary, we have that $G = \{K\}^{-1} A$. Clearly, $\kappa(A) \odot m \in V$. Thus $m \in A^{G,X}$.

"$\subset$" Suppose $m \in A^{G,X}$. Let U be a G-minimal subset of $\overline{\kappa(G) \odot m}$ and suppose that $m \notin U$. Then there exists an open neighbourhood V of m such that $\overline{V} \cap U = \varnothing$. Since $m \in A^{G,X}$, there exists $A \subset G$ and finite $K \subset G$ such that $\kappa(A) \odot m \in V$ and $G = \{K\}^{-1} A$. Now, $k \in K$, $g \in G$ and $kg \in A$ imply that

$$\kappa(g) \odot m \in \{\kappa(k)\}^{-1} \odot (\kappa(A) \odot m) \subset \{\kappa(k)\}^{-1} \odot V.$$

Hence

$$\kappa(G) \odot m \subset \{\kappa(K)\}^{-1} \odot V.$$

An easy application of Remark 3.5.1 show that

$$\overline{\{\kappa(K)\}^{-1} \odot V} \subset \{\kappa(K)\}^{-1} \odot \overline{V}.$$

Therefore

$$U \subset \overline{\kappa(G) \odot m} \subset \{\kappa(K)\}^{-1} \odot \overline{V}.$$

Choosing $k \in K$ such that $(\{\kappa(k)\}^{-1} \odot \overline{V}) \cap U \neq \varnothing$, we see that for some $n \in \beta X$,

$$\kappa(k) \odot n \in \overline{V} \cap (\kappa(k) \odot U) \subset \overline{V} \cap U = \varnothing$$

by the G-invariance of U. Since this is impossible, we must have

$$U \subset (\overline{\kappa(G) \odot m}) \cap U \subset (\{\kappa(K)\}^{-1} \odot \overline{V}) \cap U = \varnothing.$$

This contradicts $U \neq \varnothing$. Hence we have that $m \in U$, and we conclude that $m \in B^{G,X}$. ■

Remark 3.5.10. The proof for Theorem 3.5.9 is essentially due to Fairchild, who states the theorem with the additional condition that G be left amenable ([14, p. 85]). The reader will notice that neither of the proofs make use of an invariant mean.

Corollary 3.5.11. *When G acts amenably on X, $A^{G,X} \subset K^{G,X}$.*

Proof. Let $n \in GLIM(\ell^\infty(X))$, and let U be a G-minimal subset of βX. Suppose that $\operatorname{supp}(T^*n) \subset U$. By Proposition 3.5.7, $\operatorname{supp}(T^*n)$ is G-invariant. Since U is G-minimal, $\operatorname{supp}(T^*n) = U$. ■

We now find conditions on X and G that imply $K^{G,X} \backslash A^{G,X} \neq \varnothing$. We make use of the following definition, lemma and theorem.

Definition 3.5.12. For $A \subset X$, let

$$d(A) = \sup\{m(\chi_A) : m \in GLIM(\ell^\infty(X))\}.$$

A is called a *C-subset* for the pair (G, X) if $d(A) > 0$ and $d(\{K\}^{-1}A) < 1$ for all finite $K \subset G$.

Lemma 3.5.13. *Let $A \subset X$, $g \in G$. Then*

$$\{\kappa(g)\}^{-1} \odot \overline{\kappa(A)} = \overline{\{\kappa(g)\}^{-1} \odot \kappa(A)} = \overline{\kappa(\{g\}^{-1}A)}.$$

Proof. First take $\kappa(x) \in \kappa(\{g\}^{-1}A)$. Then $\kappa(g) \odot \kappa(x) = \kappa(gx)$, but $gx \in A$. Thus

$$\overline{\kappa(\{g\}^{-1}A)} \subset \overline{\{\kappa(g)\}^{-1} \odot \kappa(A)}.$$

Applying Remark 3.5.1,

$$\overline{\{\kappa(g)\}^{-1} \odot \kappa(A)} \subset \{\kappa(g)\}^{-1} \odot \overline{\kappa(A)}.$$

We now need only show that $\{\kappa(g)\}^{-1} \odot \overline{\kappa(A)} \subset \overline{\kappa(\{g\}^{-1}A)}$. Take $n \in \{\kappa(g)\}^{-1} \odot \overline{\kappa(A)}$, and let $\kappa(x_\alpha)$ be a net converging to n. By Lemma 3.4.2,

$$\kappa(gx_\alpha) = \kappa(g) \odot \kappa(x_\alpha) \to \kappa(g) \odot n \in \overline{\kappa(A)}.$$

Since $\overline{\kappa(A)}$ is open in βX, there exists some α_0 such that $\alpha > \alpha_0$ implies $\kappa(gx_\alpha) \in \overline{\kappa(A)}$. But $\overline{\kappa(A)} \cap \kappa(X) = \kappa(A)$. Indeed, clearly $\overline{\kappa(A)} \cap \kappa(X) \supset \kappa(A)$. Now take some $m \in \overline{\kappa(A)} \cap \kappa(X)$. Then $m = \kappa(x_0)$ for some $x_0 \in X$, and $m = w^* \lim \kappa(a_\gamma)$ for a net $\{a_\gamma\} \subset A$. In particular,

$$|\langle m, \chi_{x_0}\rangle - \langle \kappa(a_\gamma), \chi_{x_0}\rangle| = |\chi_{x_0}(x_0) - \chi_{x_0}(a_\gamma)| \to 0.$$

i.e. for large values of γ, $x_0 = a_\gamma \in A$, showing that $m \in \kappa(A)$. Hence $\kappa(gx_\alpha) \in \kappa(A)$, and so $x_\alpha \in \{g\}^{-1}A$. Then

$$n = w^* \lim \kappa(x_\alpha) \in \kappa(\{g\}^{-1}A) \subset \overline{\kappa(\{g\}^{-1}A)}.$$

■

Theorem 3.5.14. *If left amenable G acts on X and U is a closed G-invariant subset of βX, then for each $n \in U$ and $m \in LIM(\ell^\infty(G))$, $T^*(m \odot n)$ is a G-invariant measure with* $\operatorname{supp}(T^*(m \odot n)) \subset U$.

Proof. To the see that $T^*(m \odot n)$ is G-invariant, we need only check that $m \odot n$ is G-invariant. Let $f \in \ell^\infty(X)$, $g \in G$. Then,

$$m \odot n(\lambda_g f) = \langle m, n_\lambda(\lambda_g f)\rangle = \langle m,\ \lambda_g(n_\lambda f)\rangle = \langle m, n_\lambda f\rangle = m \odot n(f).$$

Let $\{m_\alpha\}$ be a net of convex combinations of point measures with $m_\alpha \to m$ weak*. By the G-invariance of U, $m_\alpha \odot n \in \text{conv } U$ for each α. But then

$$m \odot n = \lim_\alpha m_\alpha \odot n \in \overline{\text{conv } U}.$$

By Proposition 3.5.8, $\text{supp}\,(T^*(m \odot n)) \subset U$. ■

Theorem 3.5.15. *If G is left amenable and (G, X) has a C-subset A, then $\overline{\kappa(A)} \cap \overline{A^{G,X}} = \varnothing$ and $\overline{\kappa(A)} \cap K^{G,X} \neq \varnothing$. Thus $A^{G,X} \subsetneq K^{G,X}$.*

Proof. First suppose that there exists some $n \in \overline{\kappa(A)} \cap A^{G,X}$. Notice that $\overline{\kappa(A)}$ is an open set, and so, since $n \in A^{G,X}$, there exists a subset $B \subset G$ and a finite $K \subset G$ such that $G = \{K\}^{-1}B$ and $\kappa(B) \odot n \subset \overline{\kappa(A)}$. By Lemma 3.5.13

$$\begin{aligned} \kappa(G) \odot n = \kappa(\{K\}^{-1}B) \odot n &\subset \{\kappa(K)\}^{-1} \odot \kappa(B) \odot n \\ &\subset \{\kappa(K)\}^{-1} \odot \overline{\kappa(A)} = \overline{\kappa(\{K\}^{-1}A)}, \end{aligned}$$

which gives us that $\overline{\kappa(G) \odot n} \subset \overline{\kappa(\{K\}^{-1}A)}$. By Proposition 3.2.6, for any $l \in \overline{\kappa(G) \odot n}$ and any $m \in LIM(\ell^\infty(G))$, $m \odot l \in GLIM(\ell^\infty(X))$. Apply Theorem 3.5.14 to see that

$$\text{supp}\,(T^*(m \odot l)) \subset \overline{\kappa(G) \odot n} \subset \overline{\kappa(\{K\}^{-1}A)}.$$

i.e. $T^*(m \odot l)(\chi_{\overline{\kappa(\{K\}^{-1}A)}}) = 1$, contradicting that A is a C-subset for (G, X). $\overline{\kappa(A)}$ is open, and so

$$\overline{\kappa(A)} \cap A^{G,X} = \overline{\kappa(A)} \cap \overline{A^{G,X}} = \varnothing.$$

Next, notice that for some $\phi \in GLIM(\ell^\infty(X))$,

$$0 < \phi(\chi_A) = \phi(T\chi_{\kappa(A)}) = (T^*\phi)(\chi_{\kappa(A)}).$$

Therefore there exists some $a \in A$ such that $\kappa(a)$ is in the support of $T^*\phi$, a G-invariant measure. We conclude that $\overline{\kappa(A)} \cap K^{G,X} \neq \varnothing$, ■

We next consider the case when (G, X) has no C-subsets.

Proposition 3.5.16. *Let G be left amenable. Suppose that the pair (G,X) has no C-subsets. Then $K^{G,X} \subset \overline{A^{G,X}}$.*

Proof. Take $n \in K^{G,X}$, and let U be an open neighbourhood of n. There exists a subset $A \subset X$ such that $n \in \overline{\kappa(A)} \subset U$. Consequently there exists $\phi \in GLIM(\ell^\infty(X))$ such that $\phi(\chi_A) > 0$. A is not a C-subset for (G,X), and so there exists a finite subset $K \subset G$ such that $d(\{K\}^{-1}A) = 1$. By Remark 3.5.1 and Proposition 3.5.2, there exists $m \in GLIM(\ell^\infty(X))$ such that

$$\operatorname{supp}(T^*m) \subset \overline{\kappa(\{K\}^{-1}A)} \subset \{\kappa(A)\}^{-1} \odot \overline{\kappa(A)}.$$

By Proposition 3.5.7, there exists a G-minimal set $V \subset \{\kappa(K)\}^{-1} \odot \overline{\kappa(A)}$. Choose $l \in V$ and $k \in K$ such that $\kappa(k) \odot l \in \overline{\kappa(A)} \subset U$. Then,

$$\kappa(k) \odot l \in U \cap (\kappa(k) \odot V) = U \cap V$$

by the G-invariance of V. By Theorem 3.5.9,

$$U \cap B^{G,X} = U \cap A^{G.X} \neq \varnothing.$$

Since U was arbitrary, $n \in A^{G,X}$. ■

Remark 3.5.17. Section 3.5 is an adaptation of a paper by Fairchild [14], in which it is shown that the set of almost periodic points of βG is exactly the set of elements which belong to a minimal set, and that if (G,G) contains a C-subset, then there exist non almost periodic points of βG that are in the support of some invariant measure for βG.

Chapter 4

Fixed Point Properties

4.1 Introduction

This chapter is concerned with fixed point properties of locally compact groups on compact convex subsets of locally convex topological spaces. In Section 4.2 we utilize Day's fixed point theorem in a functional analysis proof of the existence of a G-invariant measure on the coset space G/H, where G is amenable as a discrete group and H is a closed subgroup of G. Simon proved that a locally compact group G has the strong fixed point property if and only if it has the fixed point property [28]. In Section 4.3 we prove a similar result in the setting of coset spaces.

4.2 Existence of a G-invariant measure on coset spaces

Let locally compact group G be amenable as discrete, and let H be a closed subgroup of G. We prove, using Day's fixed point theorem [8], that G/H admits a G-invariant Radon measure; that is, a regular Borel measure μ on the coset space G/H such that $\mu(gE) = \mu(E)$ for all $g \in G$ and all Borel sets E of G/H. The proof uses an idea of Izzo in the proof of the existence of the Haar measure on locally compact abelian groups using the Markov-Kakutani fixed point theorem (see [20]).

Day's fixed point theorem is as follows.

Theorem 4.2.1 (Day)**.** *Let K be a compact, convex subset of a locally convex Hausdorff topological vector space. Let S be a semigroup of affine continuous transformations of K into itself. If S is amenable as a discrete semigroup, then there exists a point $k \in K$ such that $Tk = k$ for all $T \in S$.*

For our proof, we also need the following two lemmas, both of which are given in [20]. For the sake of completeness, we provide the details of Izzo's argument.

Lemma 4.2.2. *Let G be a topological group and let U be a symmetric neighbourhood of the identity in G. Then there exists a subset V of G such that for each $g \in G$, the set gUU contains at least one element of V, and such that the set gU contains at most one element of V.*

Proof. Define the family $\mathcal{F}$ of subsets of G by

$$\mathcal{F} = \{T \subset G : p^{-1}q \notin UU \text{ for all } p, q \in T\}.$$

$\mathcal{F}$ has a maximal element V by Zorn's Lemma. Consider $g \in G$. There exists some $v \in V$ such that $g^{-1}v \in UU$, for if this were not the case then $V \cup \{g\} \in \mathcal{F}$, contradicting the maximality of V. Thus gUU contains at least one element of V.

Now suppose that gU contains two distinct elements $u, v \in V$. Then

$$u^{-1}v = u^{-1}gg^{-1}v \in U^{-1}U = UU,$$

contradicting the definition of V. Thus gU contains at most one element of V. ■

Lemma 4.2.3. *Let X be a vector space. Give X^* the weak* topology. If K is a closed subset of X^* such that for each $x \in X$ the set $\{\phi(x) : \phi \in K\}$ is bounded, then K is compact.*

Proof. For each $x \in X$, denote by $b(x)$ the least scalar such that $|\phi(x)| \le b(x)$ for all $\phi \in K$, and let $D(x)$ be the set of all scalars α such that $|\alpha| \le b(x)$. Let $I = \prod_{x \in X} D(x)$, and give it the product topology. Tychonoff's theorem gives us that I is compact with this topology.

The elements of I are functions ψ on X such that $|\psi(x)| \le b(x)$ for all $x \in X$. Thus $K \subset X^* \cap I$, and as such it has both the weak* and the product topology of X^* and I respectively. In fact, these topologies coincide on K. Indeed, fix a $\phi_0 \in K$, and choose $\{x_i\}_{i=1}^n \subset X$ and $\epsilon > 0$. We define the following sets,

$$\begin{aligned} W_1 &= \{\phi \in X^* : |\phi(x_i) - \phi_0(x_i)| < \epsilon,\ i = 1, \ldots, n\} \\ W_2 &= \{\psi \in I : |\psi(x_i) - \phi_0(x_i)| < \epsilon,\ i = 1, \ldots, n\}. \end{aligned}$$

Letting $\{x_i\}_{i=1}^n$ and ϵ range over all possible values, we have that the sets W_1 form a local base of the weak* star topology of X^* at ϕ_0, and also that the sets W_2 form a local base for the product topology of I at ϕ_0. Now, since $K \subset X^* \cap I$, we have $W_1 \cap K = W_2 \cap K$. Thus the two topologies coincide on K.

Next we claim that K is closed in the product topology of I. To prove this claim, first consider some $\psi_0 \in \overline{K}^I$. Suppose $x, y \in X$, $\alpha, \beta \in \mathbb{C}$, and let $z = \alpha x + \beta y$ and $\epsilon > 0$. The set

$$V_{z,\epsilon} = \{\psi \in I : |(\psi - \psi_0)(x)|, |(\psi - \psi_0)(y)|, |(\psi - \psi_0)(z)| < \epsilon\}$$

is a product topology neighbourhood of ψ_0, and so there exists $\psi_{z,\epsilon} \in K \cap V_{z,\epsilon}$. We have that ψ_0 is in X^*. Indeed, for any $x, y \in X$, $\alpha, \beta \in \mathbb{C}$, any $\epsilon > 0$, and setting $z = \alpha x + \beta y$,

$$\begin{aligned} &\psi_0(\alpha x + \beta y) - \alpha\psi_0(x) - \beta\psi_0(y) = \\ &\qquad (\psi_0 - \psi_{z,\epsilon})(\alpha x + \beta y) + \alpha(\psi_{z,\epsilon} - \psi_0)(x) + \beta(\psi_{z,\epsilon} - \psi_0)(y), \end{aligned}$$

and thus

$$|\psi_0(\alpha x + \beta y) - \alpha\psi_0(x) - \beta\psi_0(y)| < (1 + |\alpha| + |\beta|)\epsilon.$$

Finally, for $x \in X$ and $\epsilon > 0$, a similar argument shows that there exists a $\psi_{x,\epsilon} \in K$ such that

$$|\psi_0(x) - \psi_{x,\epsilon}(x)| < \epsilon.$$

Since K is weak* closed, $\psi_0 \in K$. K is then compact in the product topology. But the topologies coincide on K, and thus K is weak* compact in X^*. ∎

Theorem 4.2.4. *Let G be a locally compact group which is amenable as a discrete group, and let H be a closed subgroup of G. Then G/H admits a G-invariant measure.*

Proof. Let G be a locally compact group which is amenable as discrete. Let H be a closed subgroup of G. We proceed by showing that there is a nonzero positive linear functional on $C_c(G/H)$ which is invariant under the translations of $f \in C_c(G/H)$ by elements in G.

For each $a \in G$ we define $T_a : C_c(G/H)^* \to C_c(G/H)^*$ by

$$\langle T_a, \phi \rangle(f) = \phi(\lambda_a f), \qquad \phi \in C_c(G/H)^*,\ f \in C_c(G/H).$$

Each T_a is continuous and affine, and $S = \{T_a : a \in G\}$ is a representation of G.

Fix a symmetric neighbourhood U of the identity in G such that the closure of U is compact. Let K be all positive linear functionals $\phi \in C_c(G/H)^*$ which satisfy:

1. $\phi(f) \leq 1$ for all nonnegative $f \in C_c(G/H)$ that are bounded above by 1 and supported in $(xU)H = \{(xu)H : u \in U\}$ for some $x \in G$, and

2. $\phi(f) \geq 1$ for all nonnegative $f \in C_c(G/H)$ that are equal to 1 on $(xUU)H = \{(xuv)H : u, v \in U\}$ for some $x \in G$.

Clearly K is weak* closed and convex in $C_c(G/H)$. Also notice that by a partition of unity argument, every nonnegative function in $C_c(G/H)$ can be expressed as a finite sum of nonnegative functions each with support contained in $(xU)H$ for some $x \in G/H$. By condition 1 of the definition of K, it follows that for each $f \in C_c(G/H)$, the set $\{\phi(f) : \phi \in K\}$ is bounded. By Lemma 4.2.3, K is weak* compact.

To see that K is nonempty, take V as in Lemma 4.2.2 and consider the functional

$$\psi : f \mapsto \sum_{v \in V} f(vH).$$

We show that $\psi \in K$. Indeed, take f_1 as in condition 1 of the definition of K. The support of f_1 is contained in $(aU)H$ for some $a \in G$. aU contains at most one element of V, and thus

$$\psi(f_1) = \sum_{v \in V} f_1(vH) \leq 1.$$

Taking f_2 as in condition 2 of the definition of K, we similarly notice that

$$\psi(f_2) = \sum_{v \in V} f_2(vH) \geq 1.$$

Thus $\psi \in K$.

It remains to show that each T_a maps K onto itself. This follows from the definition of K; for if $\phi \in K$ and f is as in condition 1 of the definition of K, $\|\lambda_a f\| \leq 1$ and the support of $\lambda_a f$ is contained in $(a^{-1}gU)H$ for some $g \in G$. Thus $\langle T_a, \phi \rangle(f) = \phi(\lambda_a f) \leq 1$. Similarly, if f is as in condition 2 of the definition of K, $\langle T_a, \phi \rangle(f) \geq 1$.

By Day's fixed point theorem, $S = \{T_a : a \in G\}$ has a common fixed point in K. K contains positive linear functionals on $C_c(G/H)$. The Riesz representation theorem states that each positive linear functional on $C_c(G/H)$ determines a regular Borel measure on G/H. We have shown that there exists a regular Borel measure for G/H which is G invariant. ■

Remark 4.2.5. (a) Lemma 4.2.3 is essentially Alaoglu's theorem [10, V.4.2]. (b) The proof presented in Theorem 4.2.4 is new even when $H = \{e\}$.

4.3 A Fixed Point Property for the pair $(G : H)$

Let G be a locally compact group with a closed subgroup H. The homogeneous space G/H possesses a quasi-invariant measure. We denote the space of all essentially bounded complex-valued functions on G/H by $L^\infty(G/H)$.

Eymard defined the following fixed point property for the pair $(G : H)$ (see [13, p. 11]).

Definition 4.3.1 (Eymard)**.** The pair $(G : H)$ is said to have the *fixed point property* (FPP) if every jointly continuous affine action of G on a compact convex subset K of a locally convex topological space X which has a fixed point for H also has a fixed point for G.

Furthermore, Eymard proved that there exists a GLIM on $L^\infty(G/H)$ if and only if $(G : H)$ has the FPP.

In [28], Simon considered the following weaker condition of the action of G on K.

Definition 4.3.2 (Simon)**.** A *weakly measurable affine action* of G on a compact convex subset K of a locally convex topological space X is a representation of G by continuous affine maps $\alpha_g : K \to K$ such that for each $\phi \in X^*$ and $x \in K$, the map $g \mapsto \langle \phi, \alpha_g(x) \rangle$ is measurable.

We define a second fixed point property for the pair $(G : H)$, and we show that $(G : H)$ possesses this fixed point property if and only if it has the FPP.

Definition 4.3.3. The pair $(G : H)$ is said to have the FPP2 if every weakly measurable affine action of G on a compact convex subset K of a locally convex topological space X which has a fixed point for H also has a fixed point for G.

Theorem 4.3.4. *For G a locally compact group with a closed subgroup H, the following are equivalent:*

(i) There exists a GLIM on $L^\infty(G/H)$.

(ii) $(G : H)$ has the FPP2.

(iii) $(G : H)$ has the FPP.

(iv) There exists a GLIM on $LUC(G, G/H)$.

Proof. (ii) $\Rightarrow$ (iii) is trivial, since every jointly continuous affine action is also a weakly measurable affine action.

(iii) $\Leftrightarrow$ (iv) follows immediately from the fixed point theorem of Eymard [13, p. 12], and a theorem of Greenleaf [15, Theorem 3.3].

That (iv) $\Leftrightarrow$ (i) is due to Greenleaf [15, Theorem 3.3].

(i) $\Rightarrow$ (ii) Consider a weakly measurable action of G on K such that

$$hx_o = \alpha_h(x_o) = x_o$$

for all $h \in H$. Let $\mathcal{A}(K)$ be the Banach space of all continuous affine maps from K to K. For every $\phi \in \mathcal{A}(K)$, we define the function f_ϕ by

$$f_\phi(gH) = \langle \phi, gx_o \rangle.$$

ϕ is bounded on K. Thus $f_\phi \in L^\infty(G/H)$ since the action is weakly measurable. Let m be the left invariant mean on $L^\infty(G/H)$.

Next, we define a function T on $\mathcal{A}(K)$ by

$$\langle T, \phi \rangle = \langle m, f_\phi \rangle.$$

Notice that T is linear. Indeed, let α, $\beta \in \mathbb{C}$, and let ϕ, $\psi \in \mathcal{A}(K)$. Then,

$$\begin{aligned} f_{\alpha\phi+\beta\psi}(gH) &= \langle \alpha\phi + \beta\psi, gx_o \rangle = \alpha\langle \phi, gx_o \rangle + \beta\langle \psi, gx_o \rangle \\ &= (\alpha f_\phi + \beta f_\psi)(gH). \end{aligned}$$

This gives us that

$$\langle T, \alpha\phi + \beta\psi \rangle = \langle m, f_{\alpha\phi+\beta\psi} \rangle = \langle m, \alpha f_\phi + \beta f_\psi \rangle = \alpha\langle T, \phi \rangle + \beta\langle T, \psi \rangle.$$

Moreover, from the definition of T we find that $\inf_{k\in K}\phi(k) \leq \langle T,\phi\rangle \leq \sup_{k\in K}\phi(k)$ for any real valued ϕ. Thus $T|_{X^*} \in X^{**}$.

Let Q be the canonical embedding of X into X^{**} via $\langle Q(x),\phi\rangle = \phi(x)$. We proceed by first showing that for some $k_o \in X$, $T|_{X^*} = Q(k_o)$. Then we use the existence of a left invariant mean m to show that k_o is a fixed point for G.

To show that $T|_{X^*} = Q(k_o)$ for some $k_o \in X$, we must show that T is weak* continuous. The Mackey topology on X^* is the finest topology such that X^* has X^{**} as its dual space. Thus, it suffices to show that the convergence of ϕ_γ to ϕ in the Mackey topology implies convergence of $\langle T,\phi_\gamma\rangle$ to $\langle T,\phi\rangle$. ϕ_γ converges to ϕ in the Mackey topology if ϕ_γ converges uniformly to ϕ on convex weakly compact subsets of X^{**}. But $Q(K)$ is such a set. In particular, we may assume that

$$\langle \phi_\gamma, gx_o\rangle = f_{\phi_\gamma}(gH) \to f_\phi(gH) = \langle \phi, gx_o\rangle$$

uniformly, i.e., f_{ϕ_γ} is norm convergent to f_ϕ in $L^\infty(G/H)$. Now m is a mean on $L^\infty(G/H)$, and so T is indeed weak* continuous.

For any $\phi \in X^*$ and any $a, g \in G$, notice that

$$f_{\lambda_a\phi}(gH) = (\lambda_a\phi)(gx_o) = \lambda_a(\phi(gx_o)) = \lambda_a(f_\phi)(gH). \tag{4.1}$$

Then, for all $g \in G$,

$$\begin{aligned}\phi(gk_o) &= (\lambda_g\phi)(k_o) = \langle T,\lambda_g\phi\rangle = \langle m, f_{\lambda_g\phi}\rangle \overset{(4.1)}{=} \langle m, \lambda_g(f_\phi)\rangle \\ &= \langle m, f_\phi\rangle = \langle T,\phi\rangle = \phi(k_o),\end{aligned}$$

by the invariance of m. Hence $gk_o = k_o$ for all $g \in G$. ∎

Bibliography

[1] R. Arens, *The adjoint of a bilinear operation*, Proc. Amer. Math. Soc. **2** (1951), 839–848.

[2] J. F. Berglund, H. D. Junghenn, and P. Milnes, *Analysis on semigroups: function spaces, compactifications, representations*, Wiley-Interscience, New York, 1989.

[3] H. A. Bohr, *Almost periodic functions*, Chelsea, New York, 1947.

[4] R. B. Burckel, *Weakly almost periodic functions on semigroups*, Gordon and Breach, New York, 1970.

[5] H. G. Dales and A. T.-M. Lau, *The second duals of Beurling algebras*, Mem. Amer. Math. Soc. (to appear).

[6] M. M. Day, *Means for the bounded functions and ergodicity of the bounded representations*, Trans. Amer. Math. Soc. **69** (1950), 276–291.

[7] ———, *Amenable semigroups*, Illinois J. Math. **1** (1957), 509–544.

[8] ———, *Fixed-point theorems for compact convex sets*, Illinois J. Math. **5** (1961), 585–589.

[9] ———, *Correction to my paper "Fixed-point theorems for compact convex sets"*, Illinois J. Math. **8** (1964), 713.

[10] N. Dunford and J. T. Schwartz, *Linear operators Part I: General theory*, Interscience, New York, 1957.

[11] W. F. Eberlein, *Abstract ergodic theorems and weak almost periodic functions*, Trans. Amer. Math. Soc. **67** (1949), 217–240.

[12] R. Ellis, *Locally compact transformation groups*, Duke Math. J. **24** (1957), 119–125.

[13] P. Eymard, *Moyennes invariantes et représentations unitaries*, Lecture Notes in Mathematics, vol. 300, Springer-Verlag, Berlin, 1972.

[14] L. Fairchild, *Extreme invariant means without minimal support*, Trans. Amer. Math. Soc. **172** (1972), 83–93.

[15] F. P. Greenleaf, *Amenable actions of locally compact groups*, J. Funct. Anal. **4** (1969), 295–315.

[16] ______, *Invariant means on topological groups and their applications*, van Nostrand, New York, 1969.

[17] A. Grothendieck, *Critères de compacité dans les espaces fonctionnels generaux*, Amer. J. Math. **74** (1952), 168–186.

[18] E. Hewitt and K. A. Ross, *Abstract harmonic analysis*, second ed., vol. 1, Springer-Verlag, Berlin, 1979.

[19] N. Hindman and D. Strauss, *Algebra in the Stone-Čech compactification : theory and applications*, de Gruyter Expositions in Mathematics, vol. 27, Walter de Gruyter, Berlin, 1998.

[20] A. J. Izzo, *A functional analysis proof of the existence of Haar measure on locally compact abelian groups*, Proc. Amer. Math. Soc. **115** (1992), no. 2, 581–583.

[21] J. L. Kelley, *General topology*, van Nostrand, New York, 1955.

[22] A. T.-M. Lau, *Actions of topological semigroups, invariant means, and fixed points*, Studia Math. **43** (1972), 139–156.

[23] ______, *Continuity of Arens multiplication on the dual space of bounded uniformly continuous functions on locally compact groups and topological semigroups*, Math. Proc. Cambridge Philos. Soc. **99** (1986), 273–283.

[24] A. T.-M. Lau and A. Ülger, *Topological centers of certain dual algebras*, Trans. Amer. Math. Soc. **348** (1996), 1191–1212.

[25] T. Mitchell, *Topological semigroups and fixed points*, Illinois J. Math. **14** (1970), 630–641.

[26] I. Namioka, *On certain actions of semi-groups on L-spaces*, Studia Math. **29** (1967), 63–77.

[27] M. Neufang, *On a unified approach to the topological center problem for certain Banach algebras arising in abstract harmonic analysis*, Arch. Math. (Basel) **82** (2004), no. 2, 164–171.

[28] B. Simon, *A remark on groups with the fixed point property*, Proc. Amer. Math. Soc. **23** (1972), no. 2, 623–624.

[29] M. Skantharajah, *Amenable actions of locally compact groups on coset spaces*, Master's thesis, University of Alberta, 1985.

[30] C. Wilde and K. Witz, *Invariant means and the Stone-Čech compactification*, Pacific J. Math. **21** (1967), 577–586.

[31] J. C. S. Wong, *Invariant means of locally compact semigroups*, Proc. Amer. Math. Soc. **31** (1972), no. 1, 39–45.