Chapter 1

Examples

1.1 Group Theory

Let G be a group.

Definition 1.1.1. Let $a \in G$. The *cyclic group (of G) generated by a*, which we denote by $\langle a \rangle$, is the set of all powers of a.

Definition 1.1.2. If a, $b \in G$, the *commutator* of a and b, denoted $[a, b]$, is

$$aba^{-1}b^{-1}.$$

The *commutator subgroup* of G, denoted $G^{(1)}$, is the subgroup of G generated by all the commutators. $G^{(1)}$ is a normal subgroup of G ($G^{(1)} \lhd G$).

Definition 1.1.3. If H and K are both subgroups of G, then

$$[H, K] = \langle [h, k] : h \in H,\ k \in K \rangle.$$

Notice that $[G, G] = G^{(1)}$, and, more generally, $G^{(n+1)} = [G^{(n)}, G^{(n)}]$. The series $G = G^{(0)} \geq G^{(1)} \geq G^{(2)} \geq \cdots$ is called the *derived series* of G.

Definition 1.1.4. The *characteristic subgroups* $\gamma_i(G)$ of G are defined inductively:

$$\gamma_1(G) = G, \qquad \gamma_{i+1}(G) = [\gamma_i(G), G].$$

Definition 1.1.5. G is *nilpotent* if $\gamma_i(G) = \{e\}$ for some positive integer i. The least such i is called the *class* of the nilpotent group.

Definition 1.1.6. G is *solvable* if there exists a finite series of subgroups

$$\{e\} = G_n \lhd G_{n-1} \lhd \cdots \lhd G_1 \lhd G_0 = G$$

such that each factor group G_i/G_{i+1} is abelian ($i = 0, \ldots, n-1$). In this case, such a series is called a *solvable series*.

Lemma 1.1.7. *If $\{e\} = G_n \lhd G_{n-1} \lhd \cdots \lhd G_1 \lhd G_0 = G$ is a solvable series, then $G^{(i)} \leq G_i$ for all i.*

Theorem 1.1.8. *G is solvable if and only if $G^{(n)} = \{e\}$ for some positive n.*

Proof. " $\Rightarrow$ " follows from the above lemma.

Conversely, if $G^{(n)} = \{e\}$, it is a fact that the derived series is a normal series. ■

Proposition 1.1.9. *Every nilpotent group is solvable.*

Proof. By induction, $G^{(i)} \leq \gamma_i(G)$. Indeed, $G^{(1)} = [G, G] = [\gamma_1(G), G] = \gamma_2(G)$. Assume now that $G^{(i-1)} \leq \gamma_{i-1}(G)$. Then,

$$G^{(i)} = \langle [g, h] : g, h \in G^{(i-1)} \rangle$$
$$\gamma_i(G) = \langle [x, y] : x \in \gamma_{i-1}(G), y \in G \rangle.$$

An element of $G^{(i)}$ looks like $(ghg^{-1}h^{-1})^k$. But $h, h^{-1} \in G$. By assumption, $g, g^{-1} \in \gamma_{i-1}(G)$. Thus $(ghg^{-1}h^{-1})^k \in \gamma_i(G)$. Next, if $\gamma_{c+1}(G) = \{e\}$, then $G^{(c+1)} = \{e\}$ Thus G is solvable, with derived length of $c + 1$. ■

Example 1.1.10. The affine group is solvable.

Proof. First calculate $G^{(1)} = [G, G]$.

$$\begin{bmatrix} a & b \\ 0 & 1 \end{bmatrix} \begin{bmatrix} c & d \\ 0 & 1 \end{bmatrix} \left(\begin{bmatrix} c & d \\ 0 & 1 \end{bmatrix} \begin{bmatrix} a & b \\ 0 & 1 \end{bmatrix} \right)^{-1} = \begin{bmatrix} 1 & d(a-1) + b(1-c) \\ 0 & 1 \end{bmatrix}.$$

Thus

$$G^{(1)} = \left\langle \begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix} : x \in \mathbb{R} \right\rangle.$$

Next calculate $G^{(2)} = [G^{(1)}, G^{(1)}]$. Notice that for any $x, y \in \mathbb{R}$,

$$\begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & y \\ 0 & 1 \end{bmatrix} \left(\begin{bmatrix} 1 & y \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix} \right)^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Thus $G^{(2)} = \{e\}$. ■

Example 1.1.11. The Heisenberg group is nilpotent.

Proof. $\gamma_1(G) = G$. Calculate $\gamma_2(G) = [\gamma_1(G), G] = [G, G] = G^{(1)}$: Notice that

$$\begin{bmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & d & e \\ 0 & 1 & f \\ 0 & 0 & 1 \end{bmatrix} \left(\begin{bmatrix} 1 & d & e \\ 0 & 1 & f \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{bmatrix} \right)^{-1} = \begin{bmatrix} 1 & 0 & af - cd \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Thus

$$\gamma_2(G) = \left\langle \begin{bmatrix} 1 & 0 & x \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} : x \in \mathbb{R} \right\rangle.$$

Calculate $\gamma_3(G) = [\gamma_2(G), G]$:

$$\begin{bmatrix} 1 & 0 & x \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{bmatrix} \left(\begin{bmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & x \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right)^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Thus $\gamma_3(G) = \{e\}$. ■

1.2 Amenability

Definition 1.2.1. Let G be a l.c. group. A *mean* on $L^\infty(G)$ is a functional $m \in L^\infty(G)^*$ such that $m(1) = \|m\| = 1$.

Theorem 1.2.2. *(i) $m \in L^\infty(G)^*$ is a mean iff $m(1) = 1$ and $m(f) \geq 0$ whenever $f \geq 0$.*

(ii) If m is a mean, then

$$\inf_{x\in G} f(x) \leq m(f) \leq \sup_{x\in G} f(x)$$

for all $\mathbb{R}$-valued $f \in L^\infty(G)$.

Proof. (i) " $\Rightarrow$ " $m(1) = 1$ by definition. Let $f \geq 0$ with $\|f\| \leq 1$ (wlog). Then $\|1 - f\| = \sup_{x\in G} |1 - f(x)| \leq 1$. Thus

$$m(1) - m(f) = m(1 - f) \leq |m(1 - f)| = \|m\|\|1 - f\| \leq 1,$$

and so $m(f) \geq 0$.

" $\Leftarrow$ " Consider (nonzero) real-valued f. $\frac{f}{\|f\|} \leq 1$, and so $1 - \frac{f}{\|f\|} \geq 0$. By assumption, $m\left(1 - \frac{f}{\|f\|}\right) \geq 0$, so

$$\|f\|m(1) \geq m(f).$$

For an arbitrary $\mathbb{C}$-valued f, choose $c \in \mathbb{C}$ such that $|c|$ and $|m(f)| = cm(f)$. Let $g = \Re(cf)$ and $h = \Im(cf)$. Then

$$\begin{aligned} |m(f)| = m(cf) = m(g) + im(h) = m(g) \quad &\text{(since } m(h) \text{ is } \mathbb{R}\text{-valued)} \\ \leq \|g\|m(1) \leq \|cf\|m(1) = \|f\|m(1). \end{aligned}$$

Thus $\|m\| = m(1) = 1$.

(ii) The functions

$$f_1(x) = \sup f - f(x), \qquad f_2(x) = f(x) - \inf f$$

are positive. ■

Theorem 1.2.3. *Let G be a group. If G_d has a LIM, it has a RIM (and an IM)*

Proof. For all $f \in \ell^\infty(G_d)$, define $f^*(x) = f(x^{-1})$. Notice that for $a, g \in G$,

$$(r_a f)^*(g) = r_a f(g^{-1}) = f(g^{-1}a) = f^*(a^{-1}g) = \ell_{a^{-1}}(f^*)(g).$$

If $m \in LIM(\ell^\infty(G))$, then define $n(f) = m(f^*)$. We have that

$$n(r_a f) = m((r_a f)^*) = m(\ell_{a^{-1}}(f)^*) = m(f^*) = n(f).$$

Thus $n \in RIM(\ell^\infty(G_d))$. ∎

Theorem 1.2.4. *Abelian groups are amenable.*

Proof. Let G be an abelian group. Let $\{f_1, \ldots, f_n\} \subset \ell^\infty(G_d)$, and let $\{g_1, \ldots, g_n\} \subset G$. Let

$$h = \sum_{k=1}^{n} (f_k - \ell_{g_k} f_k).$$

Let $\epsilon > 0$, and suppose, for a contradiction, that

$$\sup_{x \in G} h(x) = -\epsilon. \tag{1}$$

Let $p \in \mathbb{Z}^+$, and let $\Phi = \{f : \{1, \ldots, n\} \to \{1, \ldots, p\}\}$. Clearly $|\Phi| = p^n$. Define another function $\tau : \Phi \to G$ by

$$\tau(\phi) = \prod_{k=1}^{n} g_k^{\phi(k)}.$$

Fix k. Then

$$\sum_{\phi \in \Phi} (f_k(\tau(\phi)) - f_k(g_k \tau(\phi))) = \sum_{\phi \in \Phi} \left(f_k(g_1^{\phi(k)} \cdots g_n^{\phi(k)}) - f_k(g_1^{\phi(k)} \cdots g_k^{\phi(k)+1} \cdots g_n^{\phi(k)}) \right).$$

All terms above cancel, except for those $f_k(\tau(\phi))$ such that $\phi(k) = 1$ and those $f_k(g_k \tau(\phi))$ such that $\phi(k) = p$. This is because the range of ϕ is in $\{1, \ldots, p\}$. There are p^{n-1} of each of these elements, for a total of $2p^{n-1}$. Thus

$$\sum_{\phi \in \Phi} (f_k(\tau(\phi)) - f_k(g_k \tau(\phi))) \geq -2p^{n-1} \|f_k\|.$$

By (1),

$$
\begin{aligned}
-\epsilon p^n \geq \sum_{\phi\in\Phi} h(\tau(\phi)) &= \sum_{\phi\in\Phi}\sum_{k=1}^{n}(f_k(\tau(\phi)) - f_k(g_k\tau(\phi))) \\
&= \sum_{k=1}^{n}\sum_{\phi\in\Phi}(f_k(\tau(\phi)) - f_k(g_k\tau(\phi))) \\
&\leq -\sum_{k=1}^{n} 2p^{n-1}\max\{\|f_k\|\} \\
&= -\,2np^{n-1}\max\{\|f_k\|\}.
\end{aligned}
$$

Thus we get that

$$\epsilon p \leq 2n\max\{\|f_k\|\},$$

and taking the limit $p \to \infty$ we arrive at a contradiction: $0 < \epsilon \leq 0$. ∎

A second proof used the Markov-Kakutani fixed point theorem:

Theorem 1.2.5 (Markov-Kakutani Fixed Point Theorem)**.** *Let K be a compact, convex subset of a locally convex linear topological space X. Let S be a commutative family of continuous affine maps from K to K. Then there exists a point $k \in K$ such that $T(k) = k$ for all $T \in S$.*

Proof. (of the amenability of an Abelian group): The set K of all means is a w*-compact, convex subset of $L^\infty(G)^*$. Define a family of maps from K to K by

$$\langle T_a(m), f\rangle = \langle m, \ell_a f\rangle,$$

for $a \in G$, $f \in L^\infty(G)$. The family $\{T_a : a \in G\}$ is commuting since G is Abelian. Thus there is an $m \in K$ such that $m(f) = m(\ell_a f)$ for all $a \in G$ and $f \in L^\infty(G)$. ∎

Theorem 1.2.6. *If $H \lhd G$ and if H and G/H are amenable, then G is amenable.*

Proof. Let $m \in LIM(\ell^\infty(H))$ and $n \in LIM(\ell^\infty(G/H))$. For $f \in \ell^\infty(G)$ and $g \in G$, let $\hat{f}(g) = m(\ell_g f|_H)$. We claim that the function $\hat{f}$ is constant on cosets. i.e., suppose $xH = yH$. Then, $x = yh_0$ for some $h_0 \in H$. For $h \in H$,

$$\ell_x f(h) = f(xh) = f(yh_0h) = \ell_{h_0}(\ell_y f)(h).$$

Thus

$$\hat{f}(x) = m(\ell_x f|_H) = m(\ell_{h_0}(\ell_y f)|_H) = m(\ell_y f|_H) = \hat{f}(y).$$

Since $\hat{f}$ is constant on cosets, we may define $\tilde{f} \in \ell^\infty(G/H)$ via

$$\tilde{f}(xH) = \hat{f}(x).$$

The we define a mean $\phi(f) = n(\tilde{f})$. To see that ϕ is left invariant, for $a, g \in G$,

$$\widetilde{\ell_a f}(xH) = \hat{f}(ax) = \tilde{f}(axH) = \ell_{aH}\tilde{f}(xH),$$

and thus,

$$\phi(\ell_a f) = n(\widetilde{\ell_a f}) = n(\ell_{aH}\tilde{f}) = n(\tilde{f}) = \phi(f).$$

■

Proposition 1.2.7. *A solvable group is amenable.*

Proof. This follows from repeated applications of Theorems 1.2.4 and 1.2.6. ■

Example 1.2.8. The free group on two generators is not amenable.

Proof. A reduced word x contains no subwords of the form aa^{-1}, $a^{-1}a$, bb^{-1} or $b^{-1}b$. Divide G into the disjoint sets $\{H_i : i \in \mathbb{Z}\}$ where

$$x \in H_i \Leftrightarrow x = a^i b^{n_1} a^{n_2} \cdots, \text{ as a reduced word},$$

where $\{n_j\}$ is any sequence in $\mathbb{Z}$ with $n_1 \neq 0$, unless $x = a^i$. Clearly $G = \cup_{i \in \mathbb{Z}} H_i$.

Notice that $\ell_a(H_i) = H_{i+1}$. Also, $\ell_a \chi_{H_i} = \chi_{aH_i} = \chi_{H_{i-1}}$. For any mean m and a fixed $k \in \mathbb{Z}$,

$$1 = m(\chi_G) = \sum_{i<k} m(\chi_{H_i}) + \sum_{i \geq k} m(\chi_{H_i}).$$

If m is left invariant, then

$$1 = m(\ell_a \chi_G) = \sum_{i<k} m(\chi_{H_{i-1}}) + \sum_{i \geq k} m(\chi_{H_i})$$
$$= \sum_{j \leq k} m(\chi_{H_j}) + \sum_{i \geq k} m(\chi_{H_i}),$$

where we make the substitution $j = i - 1$. Thus $m(\chi_{H_k}) = 0$. Since k was arbitrary, it follows that $m(\chi_{H_i}) = 0$ for all $i \in \mathbb{Z}$.

Next notice that $\ell_b(H_i) \subsetneq H_0$. Assuming still that m is a LIM,

$$\sum_{i \neq 0} m(\chi_{H_i}) = m(\chi_{\cup_{i \neq 0} H_i}) = m(\ell_b \chi_{\cup_{i \neq 0} H_i}) = m(\chi_{\cup_{i \neq 0} bH_i}) \leq m(\chi_{H_0}).$$

But since

$$1 = m(\chi_G) = m(\chi_{H_0}) + \sum_{i \neq 0} m(\chi_{H_i}),$$

we have that

$$m(\chi_{H_0}) \geq \frac{1}{2},$$

a contradiction. ■

1.3 Functions on groups and semigroups

Example 1.3.1. Let $G = (\mathbb{R}, +)$, the addivitive semigroup of reals. The function

$$f(x) = \sum_{k=1}^{N} \xi_k e^{i a_k x}$$

is almost periodic.

Proof. For this G, almost periodicity is equivalent to the following definition.

Definition 1.3.2. $f \in CB(\mathbb{R})$ is almost periodic iff for any $\epsilon > 0$ there exists a positive constant d_ϵ such that every interval of length d_ϵ contains a number t with the property that $\|\ell_t f - f\| < \epsilon$

The function on $x \mapsto e^{ix}$ has period 2π. Let $\epsilon > 0$. Take $d_\epsilon = 2\pi$. Clearly this function is in $AP(G)$. Since $AP(G)$ is an algebra, the result follows. ■

Example 1.3.3. Let $G = (\mathbb{R}, +)$, the addivitive semigroup of reals. The function

$$f(x) = \frac{x}{1+|x|}$$

is in $LUC(G)$ but not it $WAP(G)$.

Proof. To see that $f \in LUC(G)$, let $y_\alpha \to y \in G$. Then

$$\begin{aligned}
\|\ell_{y_\alpha} f - \ell_y f\| =& \sup_{x\in G} \left| \frac{y_\alpha + x}{1+|y_\alpha + x|} - \frac{y+x}{1+|y+x|} \right| \\
=& \sup_{x\in G} \left| \frac{y_\alpha + y_\alpha|y+x| + x + x|y+x| - y - y|y_\alpha + x| - x - x|y_\alpha + x|}{1 + |y+x| + |y_\alpha + x| + |y_\alpha + x||y+x|} \right|.
\end{aligned}$$

Consider the numerator:

$$\begin{aligned}
& |y_\alpha - y + y_\alpha|y+x| - y|y_\alpha + x| + x - x + x|y+x| - x|y_\alpha + x|| \\
\leq& |y_\alpha - y| + |y_\alpha|y+x| - y|y_\alpha + x|| + |x(|y+x| - |y_\alpha + x|)| \\
\leq& |y_\alpha - y| + |y_\alpha|y+x| - y|y+x| + y|y+x| - y|y_\alpha + x|| + |x|||y+x| - |y_\alpha + x|| \\
\leq& |y_\alpha - y| + |y+x||y_\alpha - y| + |y(|y+x| - |y_\alpha + x|)| + |x||y - y_\alpha| \\
\leq& |y_\alpha - y| + |y+x||y_\alpha - y| + |y||y - y_\alpha| + |x||y - y_\alpha| \to 0.
\end{aligned}$$

Thus $f \in LUC(G)$. ∎

Chapter 2

Other Theorems

2.1 Day's Fixed Point Theorem

Theorem 2.1.1 (Day's Fixed Point Theorem)**.** *Let K be a convex, compact subset of a locally convex linear topological space X. Let S be a semigroup (under composition) of continuous affine maps from K to K. If S is amenable as discrete, then there exists $k \in K$ such that $T(k) = k$ for all $T \in S$.*

Proof. (Outline) Let $y \in K$. Define linear map $F : X^* \to \ell^\infty(S)$ by

$$\langle F\phi, T\rangle = \langle \phi, T(y)\rangle.$$

F is linear. ϕ is bounded on the compact set K, thus $F(\phi) \in \ell^\infty(S)$. Then we have the adjoint map, $F^\# : \ell^\infty(S)^* \to X^{*\#}$,

$$\langle F^\# \mu, \phi\rangle = \langle \mu, F\phi\rangle.$$

Let $K' = Q(K)$, where $Q : X \to X^{*\#}$, $Q(x)(\phi) = \phi(x)$. It can be shown that $F^\# \mu \in K' \subset X^{*\#}$ for all means $\mu \in \ell^\infty(S)^*$. But then $Q^{-1}(F^\# \mu) \in K$. i.e. the map $Q^{-1}F^\#$ maps all means into K.

Notice that if μ is left invariant, then

$$\langle \ell_T^* \mu, f\rangle = \langle \mu, \ell_T f\rangle = \langle \mu, f\rangle$$

for all $f \in \ell^\infty(S)$ and $T \in S$. It can be shown that for any mean μ and $T \in S$ that

$$Q^{-1}F^{\#}(\ell_T^*\mu) = T(Q^{-1}F^{\#}(\mu)).$$

Thus, for left invariant μ,

$$T(Q^{-1}F^{\#}(\mu)) = Q^{-1}F^{\#}(\ell_T^*\mu) = Q^{-1}F^{\#}(\mu).$$

i.e., the point $Q^{-1}F^{\#}(\mu)$ is a fixed point in K. ■

Chapter 3

More definitions

3.1 What is $\beta\mathbb{N}$?

Definition 3.1.1. A topological space X is *regular* if for every closed set $C \subset X$ and point $x \notin C$, there exists two open disjoint sets U and V such that $C \subset U$ and $x \in V$. X is *completely regular* if there exists a continuous function $f : X \to [0,1]$ such that $f(x) = 0$ and $f(C) = 1$.

Example 3.1.2. A discrete space is completely regular.

Definition 3.1.3. Let X be a topological space. The *Stone-Čech compactification* of X is a compact space, βX, with an embedding $\kappa : X \to \beta X$ such that $\kappa(X)$ is dense in βX, and such that every continuous function f from X to any compact Hausdorff space K extends to a continuous $\tilde{f} : \beta X \to K$. βX is unique up to homeomorphism (bijection, continuous and inversely continuous).

Example 3.1.4. Let $X = \mathbb{N}$. The space of all multiplicative means for $\mathbb{N}$, $MM(\ell^\infty(\mathbb{N}))$ is weak* compact. The mapping $x \mapsto \delta_x$ maps $\mathbb{N}$ into $MM(\ell^\infty(\mathbb{N}))$ such that $\{\delta_x : x \in \mathbb{N}\}$ is weak* dense in $MM(\ell^\infty(\mathbb{N}))$. This is just the character space of the commutative C^*-algebra $\ell^\infty(\mathbb{N})$.

3.2 Quasi Invariant Measures

Definition 3.2.1. Let λ be a Radon measure on the homogenous space G/H. Define

$$\lambda_x(E) = \lambda(xE)$$

for $x \in G$. λ is *quasi-invariant* if there exists a continuous function $f : G \times (G/H) \to (0, \infty)$ such that

$$d\lambda_x(p) = f(x,p)d\lambda(p)$$

for all $x \in G$ and $p \in G/H$

Proposition 3.2.2. *For any lcg G and closed subgroup H, $(G : H)$ admits a function*

$$\rho(x\xi) = \frac{\Delta_H(\xi)}{\Delta_G(\xi)}\rho(x).$$

Theorem 3.2.3. *Given any function ρ as above for $(G : H)$, there is a quasi-invariant measure λ on G/H such that*

$$\frac{d\lambda_x}{d\lambda}(yH) = \frac{\rho(xy)}{\rho(y)}.$$

($x, y \in G$).