Actions of Semitopological Semigrou
Hausdorff Spaces

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Notation and Definitions

A *semigroup* is a nonempty set G together with an associative b
tion, which we usually call multiplication, from $G \times G$ into G.

A semigroup G together with a Hausdorff topology for which r
in G is continuous from the left is called a *right topological sem*

A semigroup G together with a Hausdorff topology for which r
in G is separately [resp. jointly] continuous is called a *semitopo*
topological] *semigroup*. When the topology of G is locally co
called a *locally compact semitopological* [resp. *topological*] *sem*

A group G equipped with a Hausdorff topology is called a *topolo*
both multiplication and inversion are continuous. When the topo
a group is locally compact, we say G is a *locally compact group*

G shall always denote a semitopological semigroup, unless othe

X shall always denote a Hausdorff topological space, and we w

X has the discrete topology.

Let X be any (nonempty) topological space.

$$\ell^\infty(X) = \text{ all bounded complex-valued functions}$$

$$CB(X) = \text{ all bounded, continuous complex-valued fu}$$

$$C_0(X) = \{f \in CB(X) : \text{for every } \epsilon > 0\text{, there exists a}$$
$$\text{subset } K \subset X \text{ (depending on}$$
$$\text{such that } |f(x)| < \epsilon \text{ for every}$$

$$C_c(X) = \{f \in CB(X) : f \text{ has compact suppor}$$

$\ell^\infty(X)$ is a commutative unital C^*-algebra with resp
supremum norm

$$\|f\|_\infty = \sup_{x \in X} |f(x)|,$$

with pointwise addition and multiplication, and involu

$$f^*(x) = \overline{f(x)}.$$

$C_c(X) \subseteq C_0(X) \subseteq CB(X)$, with equality for compact

Generally we always consider $CB(X)$ to have the supre
topology, noting that $CB(X_d) = \ell^\infty(X)$.

For any $f \in \ell^\infty(G)$ and $a \in G$, we denote the left translations of f by a by

$$\ell_a f(b) = f(ab),$$

and

$$r_a f(b) = f(ba),$$

for all $b \in G$. We denote the left and right orbits of f

$$O_\ell f = \{\ell_a f : a \in G\},$$

and

$$O_r f = \{r_a f : a \in G\}.$$

A linear subspace $Y \subseteq \ell^\infty(G)$ is *left* [resp. *right*] *i*
$O_\ell f \subseteq Y$ [resp. $O_r f \subseteq Y$] for all $f \in Y$. Y is *invariant*
and right invariant.

Let Y be a left [resp. right] invariant norm closed su
$\ell^\infty(G)$, and let Y^* denote the dual space of Y. Y is
right] *introverted* if the function

$$n_\ell f(g) = n(\ell_g f) \qquad \text{[resp. } n_r f(g) = n(r_g f)]$$

is in Y for each $n \in Y^*$. Y is *introverted* if it is left
introverted.

Let X be an arbitrary set, let Y be a norm closed su
$\ell^\infty(X)$ containing the constant functions. Denote t
function in Y by 1_Y. A *mean* on Y is any $m \in Y^*$
$m(1_Y) = 1 = \|m\|$. We denote all means on Y by $M($

Let Y be a left [resp. right] invariant norm closed su
$\ell^\infty(G)$ containing the constant functions. A mean $m \in$
[resp *right*] *invariant mean* (LIM [resp. RIM]) if $m(\ell_g$
[resp. $m(r_g f) = m(f)$] for all $g \in G$ and $f \in Y$.
the set of all such means by $LIM(Y)$ [resp. $RIM(Y$
invariant mean (IM) if m is both a LIM and a RIM

G is said be to *left* [resp. *right*] *amenable* if there ex
[resp. RIM] on $\ell^\infty(G)$. G is *amenable* if there exists
$\ell^\infty(G)$. A topological group G is said to be *amenable*
if there exists a LIM or a RIM for $\ell^\infty(G_d)$.

Let Y be a left introverted subspace of $\ell^\infty(G)$. The A
uct is a map from $Y^* \times Y^*$ to Y^* defined by:

$$(m, n) \mapsto m \bigcirc n,$$

where

$$m \bigcirc n(f) = \langle m, n_\ell f \rangle$$

for all m, $n \in Y^*$ and $f \in Y$.

When Y is a right introverted subspace of $\ell^\infty(G)$, t
Arens product is defined for Y^* by

$$(m, n) \mapsto m \,\Box\, n,$$

where

$$m \,\Box\, n(f) = \langle m, n_r f \rangle.$$

Both products are associative, distributive and weak*-w
tinuous in the first variable. Furthermore, Y^* is a Bana
under either product.

If Y is a left [resp. right] introverted subspace of ℓ^{∞}
Y^* is a right topological semigroup under the first [res
Arens product.

Let Y be left G introverted. The *topological center*
respect to the first Arens product is

$Z_t(Y) = \{m \in Y^*:$ the map $n \mapsto m \bigcirc n$ is w*-w* cont

Let Y be introverted. The *topological center* of Y^* is
be

$$Z_t(Y) = \{m \in Y^* : m \bigcirc n = m \,\square\, n \text{ for all } n \in Y$$

Y is called *Arens regular* when $Z_t(Y) = Y^*$.

Analysis on G-invariant subspaces
Definitions

Definition 1. *A jointly continuous action of* G *on a* *Hausdorff space* X *is a jointly continuous map* (a, x) $G \times X$ *into* X*, such that* $(ab)x = a(bx)$ *for all* a, $b \in G$ *and such that the map* $x \mapsto ax$*, from* X *into* X*, is con* *each* $a \in G$*.*

Example Multiplication in a topological semigroup viewed as a jointly continuous action of S on itself.

Let G have a jointly continuous action on X.

Definition 2. *We denote the left translation of*
$f \in \ell^\infty(X)$ *by an element* $a \in G$ *by*

$$\lambda_a f(x) = f(ax)$$

for all $x \in X$. *We define an analogue to the right*
(defined in (1)): for $f \in \ell^\infty(X)$ *and* $x \in X$, *define the*

$$\rho_x f(a) = f(ax),$$

for all $a \in G$.

Definition 3. *We define the left and right orbits of*
with respect to G *by*

$$O_\lambda f = \{\lambda_a f : a \in G\}, \qquad O_\rho f = \{\rho_x f : x \in X$$

Definition 4. *Consider a subspace of $CB(X)$ that*
purely in terms of the action of G on X and the
of G and X (i.e. defined independently of the algebrai
of X, if any). Hereafter, we will always write such
as $Y(G,X)$. We define the incident space of $Y(G$
$Y(G,G)$, the subspace of $CB(G)$ defined by the same
of $Y(G,X)$, with X replaced by G. For convenience
$Y(G)$ for $Y(G,G)$.

Definition 5. *A linear subspace $Y \subseteq \ell^{\infty}(X)$ is left G*
$O_{\lambda}f \subseteq Y$ for all $f \in Y$. A subspace $Y(G,X)$ of $CB(X)$
invariant if $O_{\rho}f \subseteq Y(G)$ for all $f \in Y(G,X)$. $Y(G,X)$
if it is left G and right X invariant.

Definition 6. *Let* $Y(G,X)$ *be a norm closed, left* $($
subspace of $CB(X)$*, and let* $n \in Y(G,X)^*$*.* $Y(G,X$
introverted if the function

$$n_\lambda f(g) = n(\lambda_g f)$$

is in $Y(G)$ *for all* $f \in Y(G,X)$*. When* $Y(G,X)$ *is a no*
right X *invariant subspace and* $m \in Y(G)^*$*,* $Y(G,X)$
introverted if the function

$$m_\rho f(x) = m(\rho_x f)$$

is in $Y(G,X)$ *for all* $f \in Y(G,X)$*.* $Y(G,X)$ *is introve*
left G *and right* X *introverted.*

Definition 7 (Greenleaf). *Let Y be a left G-invari closed subspace of $\ell^\infty(X)$ containing the constant fu mean $m \in Y^*$ is said to be a left G-invariant mean $m(\lambda_g f) = m(f)$ for all $g \in G$ and $f \in Y$. We denote th such means by $GLIM(Y)$. X is said to be left amena exists a GLIM on $\ell^\infty(X)$.*

Definition 8. *Let $Y_1(G,X)$ be a left introverted su* $CB(X)$*. For $m \in Y_1(G)^*$, $n \in Y_1(G,X)^*$ and $f \in Y_1($ left Arens action of $Y_1(G)^*$ on $Y_1(G,X)^*$ is defined by*

$$(m,n) \mapsto m \odot n,$$

where

$$m \odot n(f) = \langle m, n_\lambda f \rangle.$$

Let $Y_2(G,X)$ be a right introverted subspace of $CB($ $Y_2(G)^*$, $n \in Y_2(G,X)^*$ and $f \in Y_2(G,X)$. We similarly right Arens action of $Y_2(G)^*$ on $Y_2(G,X)^*$ by

$$n \boxdot m(f) = \langle n, m_\rho f \rangle.$$

Proposition 9. *Let $Y(G,X)$ be a left introverted s* $CB(X)$. *The left Arens action of $Y(G)^*$ on $Y(G,X$ *following properties:*

(i) If $n \in GLIM(Y(G,X))$, then for every $m \in$ $m \odot n = n$.

(ii) $Y(G,X)^*$ is a left Banach-$Y(G)^*$ module.

(iii) For any $g \in G$ and $x \in X$, $\delta_g \odot \delta_x = \delta_{gx}$.

Definition 10. *Let $Y(G,X)$ be a left introverted s* $CB(X)$. *We define the topological center of $Y(G,X)^*$* *set*

$Z_Y = \{m \in Y(G)^* : n \mapsto m \odot n$ *is w*-w* continuous on*

G-invariant function spaces

Definition 11. *A function $f \in CB(X)$ is almost perioc relatively compact in the norm topology of $CB(X)$ (ec if $O_\rho f$ is relatively compact in the norm topology of C denote the space of all such functions $AP(G, X)$.*

Definition 12. *A function $f \in CB(X)$ is weakly al odic if $O_\lambda f$ is relatively compact in the weak topology (equivalently, if $O_\rho f$ is relatively compact in the weak t $CB(G)$). We denote the space of all such functions W*

Definition 13. *A function $f \in CB(X)$ is called left continuous if the map $a \mapsto \lambda_a f$ from G into $CB(X)$ is c We denote the set of all such functions by $LUC(G, X$*

Theorem 14. $AP(G,X)$ *is the largest involution clo*
verted C^**-subalgebra of* $CB(X)$ *containing the cons*
tions such that $m \odot n = n \boxdot m$*, and such that* (m,n)
jointly continuous on bounded subsets.

Theorem 15. $WAP(G,X)$ *is the largest involution clo*
ant, left introverted C^**-subalgebra of* $CB(X)$ *containi*
stant functions such that the subalgebra is introverte
$n \boxdot m$*, and such that* $(m,n) \mapsto m \odot n$ *is separately co*

Theorem 16. $LUC(G,X)$ *is an invariant left intro*
subalgebra of $CB(X)$ *containing the constant functio*

- $AP(G,X) \subset WAP(G,X)$

- $AP(G,X) \subset LUC(G,X)$

- If G is a locally compact group, then $WAP(G,X) \subset$

- $f \in WAP(G,X) \Leftrightarrow \lim_i \lim_j f(g_i x_j) = \lim_j \lim_i f(g_i$
quences $\{g_i\} \subset G$ and $\{x_j\} \subset X$, whenever all limit

Arens action of the Banach algebra $LUC(G)^*$ on L

Using ideas of Lau and Wong, we prove that the measu
$\mathcal{M}(G)$ is a subset of Z_{LUC}.

Let Z denote the set of all $m \in LUC(G)^*$ such that th
$m_\rho f$ is in $LUC(G,X)$ for all $f \in LUC(G,X)$, with $m \odot$
for all $n \in LUC(G,X)^*$.

Lemma 17. *Let $m \in LUC(G)^*$. The following are eq*

(i) $m \in Z$.

(ii) $m \in Z_{LUC}$.

(iii) The map $n \mapsto m \odot n$ is weak-weak* continuou bounded subsets of $LUC(G,X)^*$.*

For any locally compact space X, let τ_X denote the lo
vex topology on $\mathcal{M}(X)$ determined by the family of
$\{p_f : f \in LUC(G,X)\}$, where $p_f(\mu) = |\int f\, d\mu|$, $\mu \in \mathcal{M}($

Lemma 18. *Let G be a locally compact semitopolog
group with jointly continuous action on a locally com
dorff space X.*

*(i) For every $\mu \in \mathcal{M}(G)$, the map $n \mapsto \mu \odot n$ is we
continuous on norm bounded subsets of $LUC(G,$*

*(ii) For every $n \in LUC(G,X)^*$, the map $\mu \mapsto \mu \odot n$ is
continuous.*

*(iii) For $\mu \in \mathcal{M}(G)$, $\nu \in \mathcal{M}(X)$, $\mu \odot \nu(f) = \langle \mu *$
$f \in C_0(X)$.*

*(iv) $n \in GLIM(LUC(G,X))$ if and only if $\mu \odot n =$
$\mu \in \mathcal{M}_0(G) = \{\mu \in \mathcal{M}(G) : \mu \geq 0, \|\mu\| = 1\}$.*

Main Result

Theorem 19. *Let G be a locally compact semitopolo group with jointly continuous action on a locally com dorff space X. If $\mu \in \mathcal{M}(G)$, then $\mu \in Z_{LUC}$.*

Proof. By Lemma 18 (i), the map $\mu \mapsto m \odot \mu$ is weak*-v tinuous on norm bounded subsets of $LUC(G, X)^*$. By L $\mu \in Z_{LUC}$.

Remarks

In *Continuity of Arens multiplication on the dual space* *uniformly continuous functions on locally compact g* *topological semigroups* (Math. Proc. Cambridge Ph **99** (1986), 273–283), Lau proved that if G is eithe compact group or a cancellative discrete semigroup a then $Z_{LUC} = \mathcal{M}(G)$.

Recently, Neufang, in his paper *On a unified appro* *topological center problem for certain Banach algeb* *in abstract harmonic analysis* (Arch. Math. (Basel) 8 no. 2, 164–171), used a different technique to prove result when G is a locally compact group.

G-minimal sets and G-invariant measures for βX

Let G, X be discrete. $(\kappa, \beta X)$ = the Stone-Čech com
tion of X. We identify $\kappa(x) \in \beta X$ with $\delta_x \in \ell^\infty(X)^*$.

The left action of G on X extends to an action of G

$$(a, n) \mapsto \kappa(a) \odot n.$$

Notation:
For fixed $n \in \beta X$, $\kappa(G) \odot n = \{\kappa(g) \odot n : g \in G\}$.
For fixed $g \in G$ and any $U \subset \beta X$, $\kappa(g) \odot U = \{\kappa(g) \odot$
If $K \subset G$ and $U \subset \beta X$,

$$\{\kappa(K)\}^{-1} \odot U = \{n \in \beta X : \kappa(k) \odot n \in U \text{ for some } k$$

If $A \subset G$ and K is as above,

$$\{K\}^{-1} A = \{g \in G : kg \in A \text{ for some } k \in K\}.$$

We have an isometric $*$-isomorphism T from $C_c(\beta X)$

onto $\ell^\infty(X)$, $\tilde{f} \mapsto f$, where

$$f(x) = \tilde{f}(\kappa(x)),$$

for $\tilde{f} \in C_c(\beta X)$ and $x \in X$. We identify $CB(\beta X)^*$ wi

in the usual way:

$$\langle T^* n, \tilde{f} \rangle = \int \tilde{f} \, d(T^* n).$$

Proposition 20. *$n \in GLIM(\ell^\infty(X))$ if and only if $T^* n$*

bility measure on βX such that $(T^* n)(\{\kappa(g)\}^{-1} \odot U)$ =

for all $g \in G$ and Borel sets $U \subset \beta X$.

Definition 21. *Let $n \in \ell^\infty(X)^*$. $T^* n$ is called G-*

$n \in GLIM(\ell^\infty(X))$.

Definition 22. *$n \in \beta X$ is called left almost G-periodic* *neighbourhood U of n there exists a subset $A \subset G$ such* *is a finite subset $K \subset G$ with $G = \{K\}^{-1}A$ and $\kappa(A) \odot$* *denote the set of all almost G-periodic elements in βX*

Definition 23. *A nonempty subset U of βX is called* *if $\kappa(g) \odot U \subset U$ for all $g \in G$. U is called G-minim* *closed and minimal with respect to this property. We* *elements of βX which belong to a G-minimal set by*

We denote by $K^{G,X}$ the elements in βX which are in t of some G-invariant measure.

Proposition 24. *Let* $n \in GLIM(\ell^\infty(X))$. *Then* supp
G*-invariant set.*

Proposition 25. *Let* $n \in M(\ell^\infty(X))$*, and let* U *b*
subset of βX. *Then* supp $(T^*n) \subset U$ *if and only if* $n \in$

Main Results

Theorem 26. $A^{G,X} = B^{G,X}$.

Corollary 27. *When* G *acts amenably on* X, $A^{G,X} \subset$

We now find conditions on X and G that imply $K^{G,X}\

We need the following definition:

Definition 28. *For $A \subset X$, let*

$$d(A) = \sup\{m(\chi_A) : m \in GLIM(\ell^\infty(X))\}.$$

A is called a C-subset for the pair (G,X) if $d(A)$
$d(\{K\}^{-1}A) < 1$ for all finite $K \subset G$.

Theorem 29. *If G is left amenable and (G,X) has a C*
then $\overline{\kappa(A)} \cap \overline{A^{G,X}} = \varnothing$ and $\overline{\kappa(A)} \cap K^{G,X} \neq \varnothing$. Thus A^{G}

Proposition 30. *Let G be left amenable. Suppose th*
(G,X) has no C-subsets. Then $K^{G,X} \subset \overline{A^{G,X}}$.

Fixed Point Properties

Let G be a locally compact group, let H be a closed
of G. The coset space $G/H = \{xH : x \in G\}$. The s
admits a quasi-invariant measure λ.

$L^\infty(G/H) =$ ess. bdd. λ-measurable $\mathbb{C}$-valued functio

with norm

$$\begin{aligned} \|f\| &= \text{ess.}\sup_{x\in G} |f(x)| \\ &= \inf\{\alpha > 0 : \{g \in G : |f(g)| > \alpha\} \text{ is locally nu} \end{aligned}$$

An affine transformation from a vector space V to itse
T such that

$$T(\alpha x + (1-\alpha)y) = \alpha T(x) + (1-\alpha)T(y)$$

for all $x, y \in V$ and scalars α.

Existence of a G-invariant measure on coset spa

We prove, using Day's fixed point theorem in *Fixed-p*
rems for compact convex sets (Illinois J. Math. **5** (19
589), that G/H admits a regular Borel measure μ or
space G/H such that $\mu(gE) = \mu(E)$ for all $g \in G$ an
sets E of G/H. The proof uses an idea of Izzo in th
the existence of the Haar measure on locally compa
groups using the Markov-Kakutani fixed point theor
in *On certain actions of semi-groups on L-spaces* (St
29 (1967), 63–77).

Theorem 31 (Day's Fixed Point Theorem). *Let K pact, convex subset of a locally convex Hausdorff vector space. Let S be a semigroup of affine continu formations of K into itself. If S is amenable as a disc group, then there exists $k \in K$ such that $Tk = k$ for a*

Lemma 32. *Let G be a topological group and let U metric neighbourhood of the identity in G. There ex such that for each $g \in G$, the set gUU contains at element of V, and such that the set gU contains at element of V.*

Lemma 33. *Let X be a vector space. If K is a we subset of X^* such that for each $x \in X$ the set $\{\phi(x)$ bounded, then K is compact.*

Main Result

Theorem 34. *Let* G *be a locally compact group which*
as a discrete group, and let H *be a closed subgroup o*
G/H *admits a* G*-invariant measure.*

Proof. For each $a \in G$, define $T_a : C_c(G/H)^* \to C_c(G/$

$$\langle T_a, \phi \rangle(f) = \phi(\lambda_a f)$$

Each T_a is continuous and affine, and $S = \{T_a : a$
representation of G.

Fix a symmetric neighbourhood U of the identity in G s
is compact. Let K be all positive linear functionals $\phi \in$
which satisfy:

> $\phi(f) \leq 1$ for all nonnegative $f \in C_c(G/H)$ that ar
> above by 1 and supported in $(xU)H = \{(xu)H :$
> some $x \in G$, and
>
> $\phi(f) \geq 1$ for all nonnegative $f \in C_c(G/H)$ that are
> on $(xUU)H = \{(xuv)H : u, v \in U\}$ for some $x \in G$.

K is weak* closed and convex. For each $f \in C_c(G/F$
$\{\phi(f) : \phi \in K\}$ is bounded. By Lemma 33, K is weak*
Take V as in Lemma 32. The functional

$$\psi : f \mapsto \sum_{v \in V} f(vH).$$

is in K.

It follows, from the definition of K, that T_a maps K
By Day's fixed point theorem, $S = \{T_a : a \in G\}$ has
fixed point in K.

A Fixed Point Property for the pair $(G : H)$

Eymard defined the following fixed point property fo
$(G : H)$ in *Moyennes invariantes et représentation*
(Lecture Notes in Mathematics, vol. 300, Springer-Ver
1972).

Definition 35. *The pair $(G : H)$ is said to have the*
property (FPP) if every jointly continuous affine acti
a compact convex subset K of a locally convex topolo
X which has a fixed point for H also has a fixed poi

Furthermore, Eymard proved that there exists a GLIM
if and only if $(G : H)$ has the FPP.

In *A remark on groups with the fixed point prope*
Amer. Math. Soc. **23** (1972), no. 2, 623–624), S
sidered the following weaker condition of the action o

Definition 36. *A weakly measurable affine action o*
compact convex subset K *of a locally convex topolog*
X *is a representation of* G *by continuous affine maps o*
such that for each $\phi \in X^*$ *and* $x \in K$*, the map* $g \mapsto \langle$
measurable.

We define a second fixed point property for the pair o

Definition 37. *The pair* $(G : H)$ *is said to have th*
every weakly measurable affine action of G *on a comp*
subset K *of a locally convex topological space* X *w*
fixed point for H *also has a fixed point for* G*.*

Main Result

Theorem 38. *For G a locally compact group with a c*
group H, the following are equivalent:

(i) There exists a GLIM on $L^{\infty}(G/H)$.

(ii) $(G:H)$ has the FPP2.

(iii) $(G:H)$ has the FPP.

(iv) There exists a GLIM on $LUC(G,G/H)$.