Randomized Low-Memory Singular Value Projection

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Motivation: Quantum Tomography

- **Quantum computers:** much faster computation than conventionally possible.

Problem	Complexity	(Quantum) Complexity	Algorithm
Perform discrete Fourier transform in N-dimensions	$\mathcal{O}(N \log N)$	$\mathcal{O}\left((\log N)^2\right)$	quantum Fourier algorithm
Factor integer N into prime factors	$\mathcal{O}\left(\exp(1.9(\log N)^{1/3}(\log\log N)^{2/3})\right)$	$\mathcal{O}\left((\log N)^3\right)$	Shor's algorithm

- **State-of-the-art:** far away from desiderata...
- **Current status:** measure quantum systems for further understanding:

Quantum Tomography

- **Definition of QT:** verifying a quantum state of q-bits
  - Mathematical representation via density matrix X:

$$X = \sum_{i=1}^{r} c_i |\psi_i\rangle\langle\psi_i|$$

  - $X \succeq 0 \in \mathbb{R}^d$, $d = 2^q$-dimensional complex space, $\text{rank}(X) = r \ll d$, $\text{trace}(X) = 1$.
- **Challenge:** X is huge even for moderate number of qubits q... $\longrightarrow$ Solution: subsample X!

$$u = \mathcal{A}(X) + z \quad \text{where} \quad \mathcal{A} \in \mathbb{S}_+^{d\times d} \to \mathbb{C}^m,\ m \ll d^2 \text{ is a Rank-RIP linear operator.}$$

- R-RIP: $(1-\delta_k) \leq \|\mathcal{A}(X)\|_2 / \|X\|_F \leq (1+\delta_k),\ \forall X \text{ s.t. } \text{rank}(X) \leq k$
- **Contributions**:
  1. **Fast** quantum tomography through **approximate matrix decomposition schemes** - e.g, $q = 16$.
  2. **Low-memory implementation** with working space $\propto$ degrees of freedom.
  3. Provable recovery guarantees.

Optimization criteria

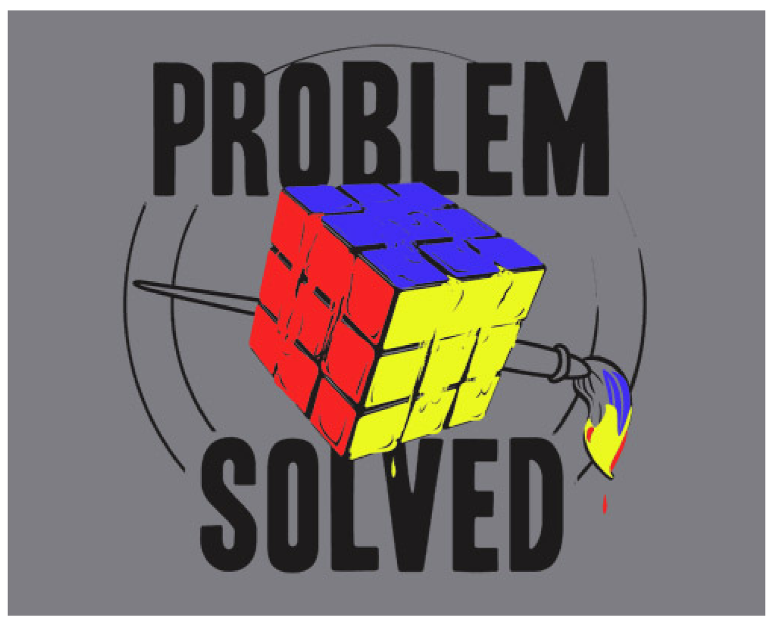
- Affine rank minimization (ARM) problem:

$$\min_{X\in\mathbb{R}^{m\times n}} f(X) \quad \text{s.t.} \quad \text{rank}(X) \leq r, \qquad \textbf{Non-convex!}$$

where $f(X) := \frac{1}{2}\|u - \mathcal{A}X\|_2^2$.

- Convexify $\text{rank}(\cdot) \longrightarrow$ Nuclear norm metric:

$$\min_{X\in\mathbb{R}^{m\times n}} f(X) \quad \text{s.t.} \quad \|X\|_* \leq \lambda, \qquad \min_{X\in\mathbb{R}^{m\times n}} \|X\|_* \quad \text{s.t.} \quad f(X) \leq \varepsilon,$$



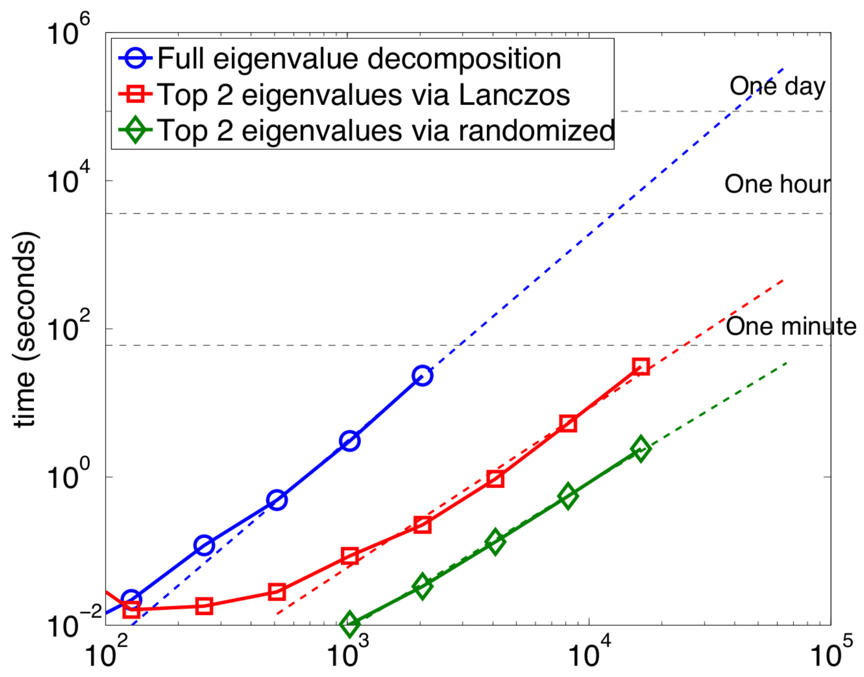
Not exactly...

Why $\|\cdot\|_*$-metric is not applicable to large-scale QT?

➡ **computation/storage:**

Examples		nnz(A)	storage	range(A*)
	netflix	O(rd)	~0.9GB	sparse
	QT	O(rd²)	~400GB	dense

- Computational overheads:
  - (i) Calculation of $\nabla f(X) := -\mathcal{A}^*(u - \mathcal{A}(X))$ is expensive.
  - (ii) Nuclear norm-based algorithms often require at least one *full* eigenvalue / singular value decomposition.
- Overall: the curse of dimensionality rules out $\|\cdot\|_*$-based schemes



- True constraints result in fixed $\|\cdot\|_*$ metric:

$$\left.\begin{array}{l} \text{trace}(X) = 1 \\ \text{rank}(X) = 1 \\ X \succeq 0 \end{array}\right\} \Longrightarrow \|X\|_* = 1$$

- Nuclear norm minimization:
  - (i) Not always meaningful results
  - (ii) Noise overfitting
  - (iii) Requires heuristic tuning

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- Low-rank + trace recovery via projected gradient descent:

$$X_{i+1} \in \mathcal{P}_{\mathcal{C}}\left(\mathcal{P}_r^{\epsilon}(X_i - \frac{\mu}{2}\nabla f(X_i))\right)$$

where $\mathcal{C} = \{X \succeq 0,\ \text{trace}(X) = 1\}$ or $\{X \succeq 0\}$.

- Provably: $\mathcal{P}_{r\cap\mathcal{C}} = \mathcal{P}_{\mathcal{C}} \circ \mathcal{P}_r$.
- **Efficient eig. decomposition using randomized schemes.**
- For QT: Kronecker form of Pauli operator results in low-memory distributed $\mathcal{A}^*$ operations.

THEOREM. Pick an accuracy $\epsilon < 1/12$ and define $\ell = r + \rho$. Let c be an integer such that $\ell = (c-1)r$. Let $\mu = \frac{1}{2(1+\delta_{cr})}$. The projected gradient descent scheme has the following iteration invariant

$$\mathbb{E}f(X_{i+1}) \leq \theta f(X_i) + \tau\|z\|^2,$$

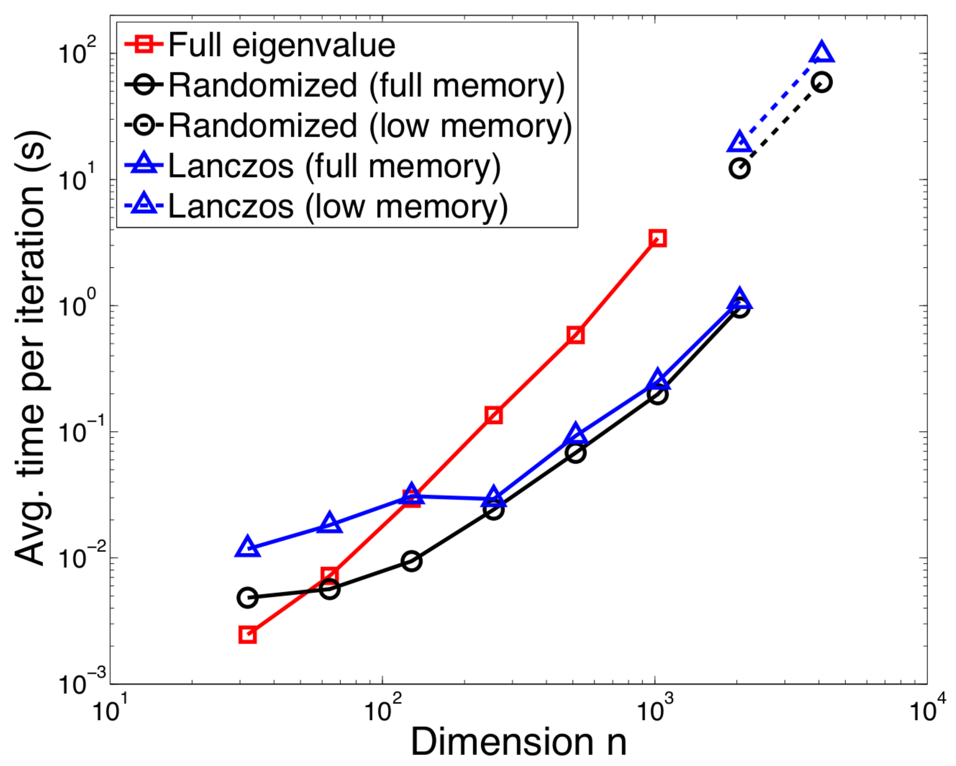
in expectation, where

$$\theta \leq 12 \cdot \frac{1+\delta_{2r}}{1-\delta_{cr}} \cdot \left(\epsilon + (1+\epsilon)\frac{3\delta_{cr}}{1-\delta_{2r}}\right) \quad \text{and} \quad \tau \leq \frac{1+\delta_{2r}}{1-\delta_{cr}} \cdot \left(12\cdot(1+\epsilon)\left(1 + \frac{2\delta_{cr}}{1-\delta_{2r}}\right) + 8\right).$$

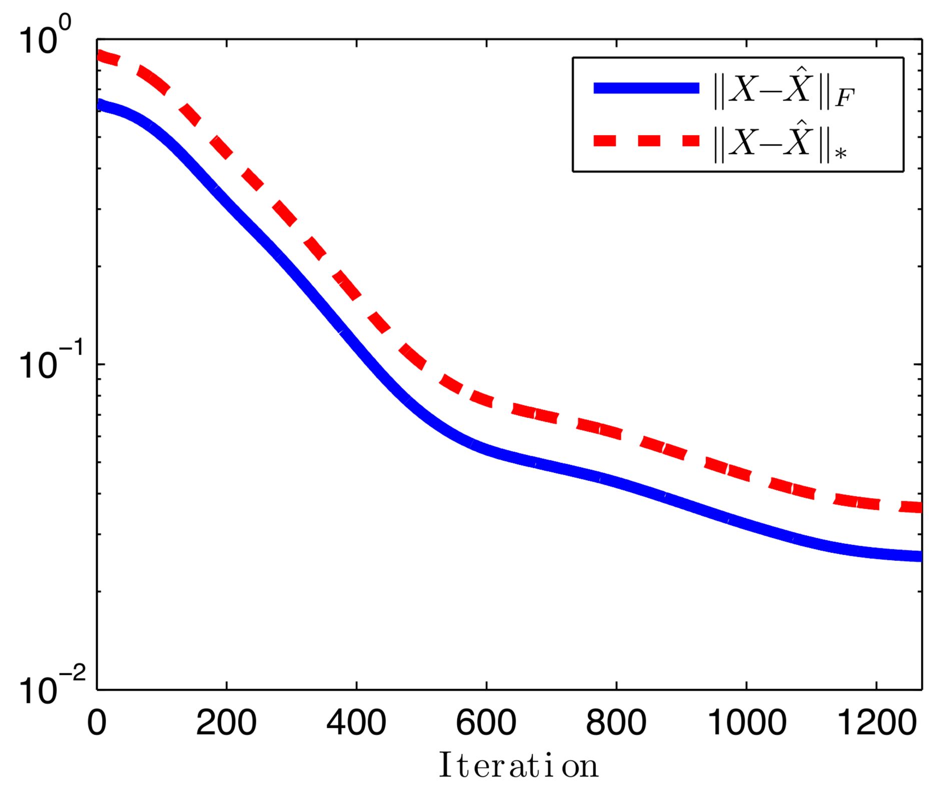
The expectation is taken with respect to Gaussian random designs.

QT experimental results

- Scalability:



- 16-qubit Convergence ($2^{16} \times 2^{16}$):



- Comparison with splitting schemes:

Qubits	Dimension	Time per iteration		Time to 10^{-1} error	
		SVP	Splitting	SVP	Splitting
8	256	0.012 s	0.006 s	0.64 s	5.25 s
9	512	0.045 s	0.028 s	2.90 s	47.4 s
10	1024	0.225 s	0.156 s	17.1 s	516.3 s