Polynomial-Complexity Computation of the *M*-phase Vector that Maximizes a Rank-Deficient Quadratic Form

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Outline

- Problem Presentation
- Previous Work and Applications
- Problem Reformulation
- Proposed Algorithm
- Results
- Conclusion and Further Work

Quadratic Form Maximization over the M-ary field

- Optimization problem:

$$\mathbf{s}_{\text{opt}} \triangleq \arg\max_{\mathbf{s}\in\mathcal{A}_M^N} \mathbf{s}^{\mathcal{H}}\mathbf{Q}\mathbf{s}$$

  where

  - $\mathbf{Q} \in \mathbb{C}^{N\times N}$: positive (semi)definite matrix.
  - $\mathbf{s} \in \mathcal{A}_M^N$: a M-PSK N-tuple vector argument.
  - $\mathcal{A}_M = \left\{ e^{\frac{j2\pi m}{M}} \mid m = 0, 1, \ldots, M-1 \right\}$: M-phase alphabet.
  - $M \in \{2^k \mid k = 1, 2 \ldots\}$.

- Rank of observation matrix $\mathbf{Q}$: $D \leq N$.

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- Rank of observation matrix $\mathbf{Q}$: $D \leq N$.
- Full rank case $\rightarrow$ known $\mathcal{NP}$-hard problem with an obvious approach: *Exhaustive search*.
- Complexity: $\mathcal{O}(M^N)$.
- We focus on: rank-deficient case.

Previous Work

- Optimal Approaches - Computational Geometry (CG) Algorithms:
  - Assume rank-deficiency of **Q**.
  - Reverse Search Method [D. Avis et. al.:1996] - only for BPSK and QPSK vector argument.
  - Incremental Algorithm [H.Edelsbrunner et. al.: 1986] - applied for arbitrary M.
  - A fast algorithm for multiple-symbol differential detection of MPSK [K.M. Mackenthun Jr.: 1994].
- Suboptimal Approaches - Relaxation Algorithms:
  - No assumption of rank-deficiency of **Q**.
  - SemiDefinite Relaxation (SDR) method [S. Zhang and Y. Huang: 2006] - approximation algorithm with randomization (most of the times).
- Our work is based on:
  - Quadratic Form Maximization over the binary field $\{\pm 1\}$ [G. Karystinos and A. Liavas: 2010].

Applications

- A common optimization problem in digital communications.
- Examples:
  - ML block noncoherent MPSK detection in SIMO systems [D.S. Papailiopoulos and G.N. Karystinos: 2008].
  - MLSD noncoherent M-PSK detection [I. Motedayen et. al.: 2007, V. Pauli et. al.: 2008].
  - Multiuser detection in M-PSK CDMA systems:
    - [G. Manglani and A.K. Chaturvedi: 2006, W.-K. Ma et. al.: 2002] - SDR over binary field.
    - [P. Tan and L.K. Rasmussen: 2001] - SDR over BPSK and QPSK.
    - [W.-K. Ma et. al.: 2004] - SDR over M-ary PSK CDMA.
  - Blind ML detection of orthogonal space-time block codes (OSTBC) [W.-K. Ma et.al.: 2006]
  - QAM and PSK codebooks for limited MIMO beamforming [D.J. Ryan et. al.: 2009].
  - etc.

Quadratic Form Maximization over the M-ary field

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- We will show that if rank($\mathbf{Q}$) = D, $D < N$ and D is independent of N, then:

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Quadratic Form Maximization over the M-ary field

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    - Exhaustive search is not necessary.
    - $\mathbf{s}_{\text{opt}}$ belongs into a polynomial-size set of candidates.
    - $\mathbf{s}_{\text{opt}}$ can be obtained in polynomial time $\mathcal{O}\left(\left(\frac{MN}{2}\right)^{2D}\right)$.

Problem Reformulation

- Let $\mathbf{V} \triangleq \begin{bmatrix} \mathbf{v}_1 \ \mathbf{v}_2 \ \ldots \mathbf{v}_D \end{bmatrix} \in \mathbb{C}^{N \times D}$ such that:

$$\mathbf{Q} = \mathbf{V}\mathbf{V}^{\mathcal{H}}.$$

- Rank($\mathbf{V}$) = D.

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$$\mathbf{Q} = \mathbf{V}\mathbf{V}^{\mathcal{H}}.$$

- Rank($\mathbf{V}$) = D.
- Optimal solution:

$$\begin{aligned} \mathbf{s}_{\text{opt}} &= \arg\max_{\mathbf{s}\in\mathcal{A}_M^N} \mathbf{s}^{\mathcal{H}}\mathbf{Q}\mathbf{s} \\ &= \arg\max_{\mathbf{s}\in\mathcal{A}_M^N} \mathbf{s}^{\mathcal{H}}\mathbf{V}\mathbf{V}^{\mathcal{H}}\mathbf{s} \\ &= \arg\max_{\mathbf{s}\in\mathcal{A}_M^N} \|\mathbf{V}^{\mathcal{H}}\mathbf{s}\|^2 = \arg\max_{\mathbf{s}\in\mathcal{A}_M^N} \|\mathbf{V}^{\mathcal{H}}\mathbf{s}\|. \end{aligned}$$

Problem Reformulation

- Let $\phi_{i:j} \triangleq [\phi_i, \phi_{i+1}, \ldots, \phi_j]^T$. We introduce:
  (*i*) $2D - 1$ auxiliary hyperspherical coordinates:

$$\phi_{1:2D-1} \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right]^{2D-2} \times (-\pi, \pi].$$

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  - (ii) the hyperspherical real vector:

$$\tilde{\mathbf{c}}(\boldsymbol{\phi}_{1:2D-1}) \triangleq \begin{bmatrix} \sin\phi_1 \\ \cos\phi_1 \sin\phi_2 \\ \cos\phi_1 \cos\phi_2 \sin\phi_3 \\ \vdots \\ \left[\prod_{i=1}^{2D-2} \cos\phi_i\right] \sin\phi_{2D-1} \\ \left[\prod_{i=1}^{2D-2} \cos\phi_i\right] \cos\phi_{2D-1} \end{bmatrix}_{2D\times 1}$$

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  (*iii*) the $D \times 1$ hyperspherical complex vector:

$$\mathbf{c}(\phi_{1:2D-1}) \triangleq \tilde{\mathbf{c}}_{2:2:2D}(\phi_{1:2D-1}) + j\tilde{\mathbf{c}}_{1:2:2D-1}(\phi_{1:2D-1})$$

$$= \begin{bmatrix} \cos\phi_1 \sin\phi_2 + j\sin\phi_1 \\ \cos\phi_1 \cos\phi_2 \cos\phi_3 \sin\phi_4 + j\cos\phi_1 \cos\phi_2 \sin\phi_3 \\ \vdots \\ \left[\prod_{i=1}^{2D-2}\cos\phi_i\right]\cos\phi_{2D-1} + j\left[\prod_{i=1}^{2D-2}\cos\phi_i\right]\sin\phi_{2D-1} \end{bmatrix} = \begin{bmatrix} c_1(\phi_{1:2}) \\ c_2(\phi_{1:4}) \\ \vdots \\ c_D(\phi_{1:2D-1}) \end{bmatrix}_{D\times 1}$$

Problem Reformulation

- Using the following inequalities $\forall \mathbf{a} \in \mathbb{C}^D, \forall \hat{\theta} \in (-\pi, \pi]$ and $\forall \boldsymbol{\phi}_{2D-1} \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right]^{2D-2} \times (-\pi, \pi]$:

$$\Re\left\{\mathbf{a}^{\mathcal{H}}\mathbf{c}(\phi_{1:2D-1})e^{-j\hat{\theta}}\right\} \overset{(2)}{\leq} \left|\mathbf{a}^{\mathcal{H}}\mathbf{c}(\phi_{1:2D-1})\right| \overset{(1)}{\leq} \|\mathbf{a}\| \underbrace{\|\mathbf{c}(\phi_{1:2D-1})\|}_{=1} = \|\mathbf{a}\|.$$

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$$\Re\left\{\mathbf{a}^{\mathcal{H}}\mathbf{c}(\phi_{1:2D-1})e^{-j\hat{\theta}}\right\} \overset{(2)}{\leq} \left|\mathbf{a}^{\mathcal{H}}\mathbf{c}(\phi_{1:2D-1})\right| \overset{(1)}{\leq} \|\mathbf{a}\| \underbrace{\|\mathbf{c}(\phi_{1:2D-1})\|}_{=1} = \|\mathbf{a}\|.$$

- The maximization problem becomes (substituting $\mathbf{a} = \mathbf{V}^{\mathcal{H}}\mathbf{s}$ and applying some computations):

$$\begin{aligned}\mathbf{s}_{\text{opt}} = \arg\max_{\mathbf{s}\in\mathcal{A}_M^N} \|\mathbf{V}^{\mathcal{H}}\mathbf{s}\| &\overset{(1)}{=} \arg\max_{\mathbf{s}\in\mathcal{A}_M^N} \max_{\phi_{1:2D-1}\in(-\frac{\pi}{2},\frac{\pi}{2}]^{2D-2}\times(-\pi,\pi]} \left|\mathbf{s}^{\mathcal{H}}\mathbf{V}\mathbf{c}(\phi_{1:2D-1})\right| \\ &\overset{(2)}{=} \arg\max_{\mathbf{s}\in\mathcal{A}_M^N} \max_{\phi_{1:2D-1}\in(-\frac{\pi}{2},\frac{\pi}{2}]^{2D-2}\times(-\pi,\pi]} \Re\left\{\mathbf{s}^{\mathcal{H}}\mathbf{V}\mathbf{c}(\phi_{1:2D-1})\right\}.\end{aligned}$$

Problem Reformulation

- Maximization over the whole range of $\phi_{1:2D-1}$ generates M-ary equivalent duplicates of candidate vectors $\mathbf{s}$.

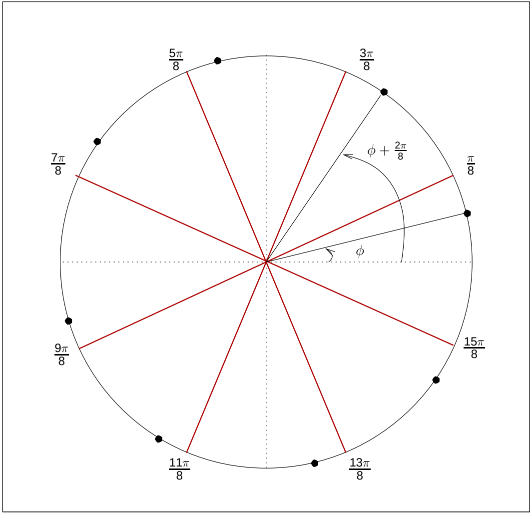
- We observed and proved that:

  - We can divide the space $(-\frac{\pi}{2}, \frac{\pi}{2}]^{2D-2} \times (-\pi, \pi]$ into M equivalent subspaces.
  - The division is performed by separating the range of values of ϕ_{2D-1} into M consecutive parts.
  - E.g. $\phi_{2D-1} \in \left\{ \left(-\frac{\pi}{M}, \frac{\pi}{M} \right], \left(\frac{\pi}{M}, \frac{3\pi}{M} \right], \dots, \left(\frac{(2M-3)\pi}{M}, \frac{(2M-1)\pi}{M} \right] \right\}$.

- Without loss of generality, we choose $\phi_{2D-1} \in (-\frac{\pi}{M}, \frac{\pi}{M}]$.

Problem Reformulation

- Illustrative example: Rank($\mathbf{Q}$) $= D = 1$, arbitrary N, $M = 8$.



Problem Reformulation

- The optimization problem becomes:

$$\mathbf{s}_{\mathrm{opt}} = \arg\max_{\mathbf{s}\in\mathcal{A}_M^N} \max_{\phi_{1:2D-1}\in(-\frac{\pi}{2},\frac{\pi}{2}]^{2D-2}\times(-\frac{\pi}{M},\frac{\pi}{M}]} \Re\left\{\mathbf{s}^{\mathcal{H}}\mathbf{V}\mathbf{c}(\phi_{1:2D-1})\right\}.$$

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- Interchanging maximizations:

$$\max_{\phi_{1:2D-1}\in\Phi^{2D-2}\times(-\frac{\pi}{M},\frac{\pi}{M}]} \sum_{n=1}^{N} \max_{s_n\in\mathcal{A}_M} \Re\left\{s_n^*\mathbf{V}_{n,1:D}\mathbf{c}(\phi_{1:2D-1})\right\}, \quad \Phi \triangleq \left(-\frac{\pi}{2},\frac{\pi}{2}\right].$$

Decision Functions and Candidate Set $\mathcal{S}(\mathbf{V}_{N\times D})$

- Element-by-element decision rule for fixed set of $\phi_{1:2D-1}$:

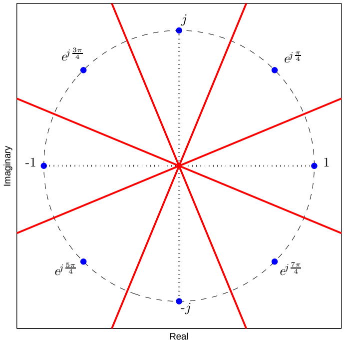
$$s(\mathbf{v}^T; \phi_{1:2D-1}) \triangleq \arg\max_{s\in\mathcal{A}_M} \Re\{s^* \mathbf{v}^T \mathbf{c}(\phi_{1:2D-1})\}.$$

- Sequence decision rule for fixed set of $\phi_{1:2D-1}$:

$$\begin{aligned}\mathbf{s}(\mathbf{V}_{N\times D}; \phi_{1:2D-1}) &\triangleq \begin{bmatrix} s(\mathbf{V}_{1,1:D}; \phi_{1:2D-1}) \\ s(\mathbf{V}_{2,1:D}; \phi_{1:2D-1}) \\ \vdots \\ s(\mathbf{V}_{N,1:D}; \phi_{1:2D-1}) \end{bmatrix} \\ &= \arg\max_{\mathbf{s}\in\mathcal{A}_M^N} \Re\{\mathbf{s}^{\mathcal{H}} \mathbf{V}\mathbf{c}(\phi_{1:2D-1})\}.\end{aligned}$$

- Computing $\mathbf{s}(\mathbf{V}_{N\times D}; \phi_{1:2D-1})$ for $\forall \phi_{1:2D-1} \in \Phi^{2D-2} \times (-\frac{\pi}{M}, \frac{\pi}{M}]$:

$$\mathcal{S}(\mathbf{V}_{N\times D}) \triangleq \bigcup_{\phi_{1:2D-1}\in\Phi^{2D-2}\times(-\frac{\pi}{M},\frac{\pi}{M}]} \left\{\mathbf{s}(\mathbf{V}_{N\times D}; \phi_{1:2D-1})\right\} \subseteq \mathcal{A}_M^N.$$

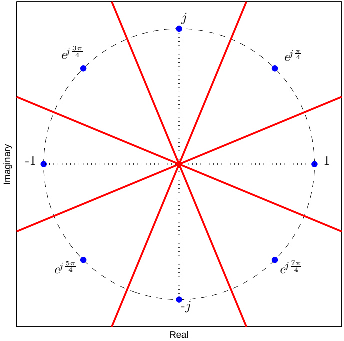


- We rewrite the previous expression:

$$\max_{\phi_{1:2D-1}\in\Phi^{2D-2}\times(-\frac{\pi}{M},\frac{\pi}{M}]} \sum_{n=1}^{N} \max_{s_n\in\mathcal{A}_M} \Re\left\{s_n^* \mathbf{V}_{n,1:D}\mathbf{c}(\phi_{1:2D-1})\right\}, \quad \Phi \triangleq \left(-\frac{\pi}{2},\frac{\pi}{2}\right].$$

- As $\phi_{1:2D-1}$ vary, the decision in favor of s_n is maintained as long as a decision boundary is not crossed.

Hypersurfaces and Cardinality of $\mathcal{S}(\mathbf{V}_{N\times D})$



- The $\frac{M}{2}$ decision boundaries for the determination of s_n are given by

$$\mathbf{V}_{n,1:D}\mathbf{c}(\phi_{1:2D-1}) = Ae^{j\pi\frac{2k+1}{M}}, \;\; A \in \mathbb{R}, \;\; k = 0, 1, \ldots, \frac{M}{2} - 1,$$

or equivalently

$$\Im\left\{e^{-j\pi\frac{2k+1}{M}}\mathbf{V}_{n,1:D}\mathbf{c}(\phi_{1:2D-1})\right\} = 0, \;\; k = 0, 1, \ldots, \frac{M}{2} - 1.$$

- In matrix form:

$$\Im\left(\underbrace{\begin{bmatrix} e^{-j\frac{\pi}{M}}\mathbf{V}_{N\times D} \\ \vdots \\ e^{-j\pi\frac{M-1}{M}}\mathbf{V}_{N\times D} \end{bmatrix}}_{\hat{\mathbf{V}}_{\frac{MN}{2}\times D}} \mathbf{c}(\phi_{1:2D-1})\right) = \mathbf{0}_{\frac{MN}{2}\times 1} \Rightarrow \dots$$

$$\tilde{\mathbf{V}}_{l,1:2D}\tilde{\mathbf{c}}(\phi_{1:2D-1}) = 0, \quad l = 1, 2, \dots, \frac{MN}{2}$$

where

$$\tilde{\mathbf{V}}_{:,1:2:2D-1} = \Re(\hat{\mathbf{V}}) \quad \text{and} \quad \tilde{\mathbf{V}}_{:,2:2:2D} = \Im(\hat{\mathbf{V}}).$$

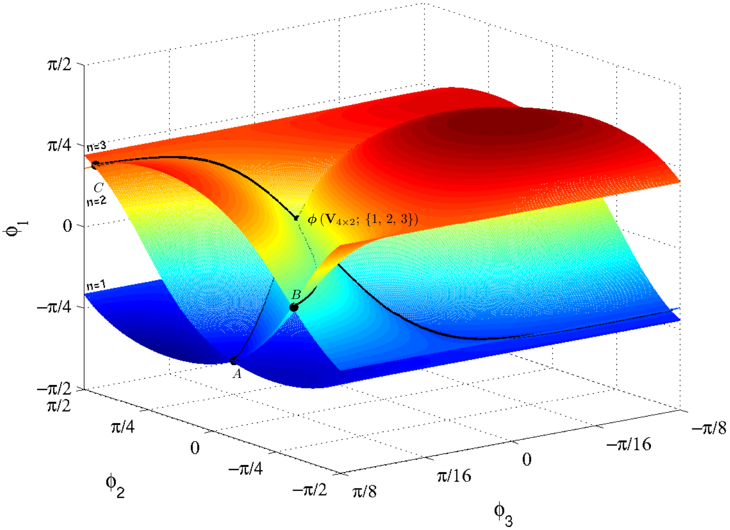
Hypersurfaces and Cardinality of $\mathcal{S}(\mathbf{V}_{N\times D})$

- From the construction of $\tilde{\mathbf{V}}_{\frac{MN}{2}\times 2D}$, we observe:
  - each row of $\mathbf{V}$ is rotated by each of the $\frac{M}{2}$ exponentials $e^{-j\pi\frac{2k+1}{M}}$.
  - We can define N different groups $\boldsymbol{\mathcal{G}}^{(n)}, n \in \{1, 2, \ldots, N\}$:

$$\boldsymbol{\mathcal{G}}^{(n)} \triangleq \begin{bmatrix} \tilde{\mathbf{V}}_{n,1:2D} \\ \tilde{\mathbf{V}}_{(n+N),1:2D} \\ \vdots \\ \tilde{\mathbf{V}}_{\left(n+(\frac{M}{2}-1)N\right),1:2D} \end{bmatrix}_{\frac{M}{2}\times 2D}$$

- The set of $\frac{MN}{2}$ hypersurfaces $\boldsymbol{\mathcal{H}} = \left\{\mathcal{H}(\tilde{\mathbf{V}}_{1,1:2D}), \mathcal{H}(\tilde{\mathbf{V}}_{2,1:2D}), \ldots, \mathcal{H}(\tilde{\mathbf{V}}_{\frac{MN}{2},1:2D})\right\}$:
  - Separate the $\Phi^{2D-2} \times (-\frac{\pi}{M}, \frac{\pi}{M}]$ space into distinct regions.
  - Each of which is associated with a different M-PSK candidate vector $\mathbf{s}$.

- Illustrative example: $D = 2, N = 4$ and $M = 8$.



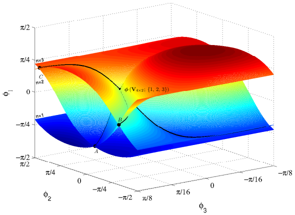
- Let $\boldsymbol{\mathcal{H}} = \{\mathcal{H}(\tilde{\mathbf{V}}_{1,1:2D}), \mathcal{H}(\tilde{\mathbf{V}}_{2,1:2D}), \ldots, \mathcal{H}(\tilde{\mathbf{V}}_{\frac{MN}{2},1:2D})\}$ the set of $\frac{MN}{2}$ hypersurfaces.

Proposition 2a

Each subset of $\boldsymbol{\mathcal{H}}$ that consists of $2D-1$ hypersurfaces has either a single or uncountably many intersections in $\Phi^{2D-2} \times (-\frac{\pi}{M}, \frac{\pi}{M}]$.

Proposition 2b

Each combination of $2D-1$ hypersurfaces from the set $\boldsymbol{\mathcal{H}}$ has a unique intersection point that constitutes a vertex of a cell if and only if no more than two hypersurfaces originate from the same row of the matrix **V**.



- Let $\mathcal{I}_{2D-1} \triangleq \{i_1, i_2, \ldots, i_{2D-1}\} \subset \{1, 2, \ldots, \frac{MN}{2}\}$ be the indices of $2D-1$ hypersurfaces $\mathcal{H}(\tilde{\mathbf{V}}_{i_1,1:2D}), \mathcal{H}(\tilde{\mathbf{V}}_{i_2,1:2D}), \ldots, \mathcal{H}(\tilde{\mathbf{V}}_{i_{2D-1},1:2D})$. We detect the following cases:
  - (a) At most two surfaces originate from the same row of **V**.
  - (b) At least three surfaces originate from the same row of **V**.

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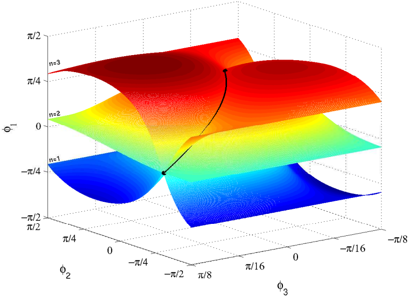
  - (a) At most two surfaces originate from the same row of $\mathbf{V}$.
  - (b) At least three surfaces originate from the same row of $\mathbf{V}$.

- Combinations of the form (b) do not have a unique intersection point.

  - Illustrative example: $D = 2, N = 4$ and $M = 8$.
  - $2D-1$ hypersurfaces originating from the same row of $\mathbf{V}$.



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  - (a) At most two surfaces originate from the same row of **V**.
  - (b) At least three surfaces originate from the same row of **V**.

Corollary

All $\frac{M}{2}$ hypersurfaces originating from the same row of **V** intersect to a common axis.

- The dimensionality of the common axis depends on the rank D of the observation matrix **V** and equals to $2(D-1)$.
  - E.g. common one-dimensional line if $D = 2$, common four-dimensional hyperplane if $D = 3$, etc.

Hypersurfaces and Cardinality of $\mathcal{S}(\mathbf{V}_{N\times D})$

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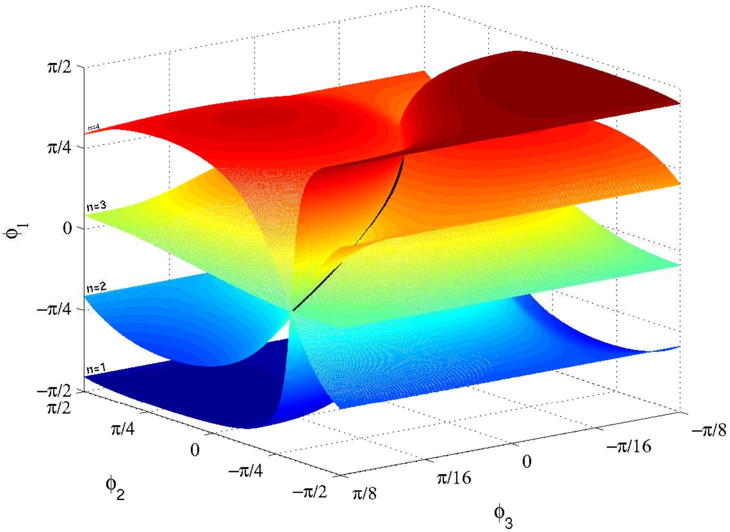

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  - (a) At most two surfaces originate from the same row of **V**.
  - (b) At least three surfaces originate from the same row of **V**.

- Combinations of the form (a) have a unique intersection point $\phi(\mathbf{V}_{N\times D}; \mathcal{I}_{2D-1}) \in \Phi^{2D-2} \times (-\frac{\pi}{M}, \frac{\pi}{M}]$ such that:
  - "Leads" $\mathcal{Q}$ cells, say $C_q(\mathbf{V}_{N\times D}; \mathcal{I}_{2D-1})$, where $q = 1, 2, \ldots, \mathcal{Q}$ and $\mathcal{Q} \in \left\{ (\frac{M}{2}-1)^0, (\frac{M}{2}-1)^1, \ldots, (\frac{M}{2}-1)^{D-1} \right\}$.

Hypersurfaces and Cardinality of $\mathcal{S}(\mathbf{V}_{N\times D})$

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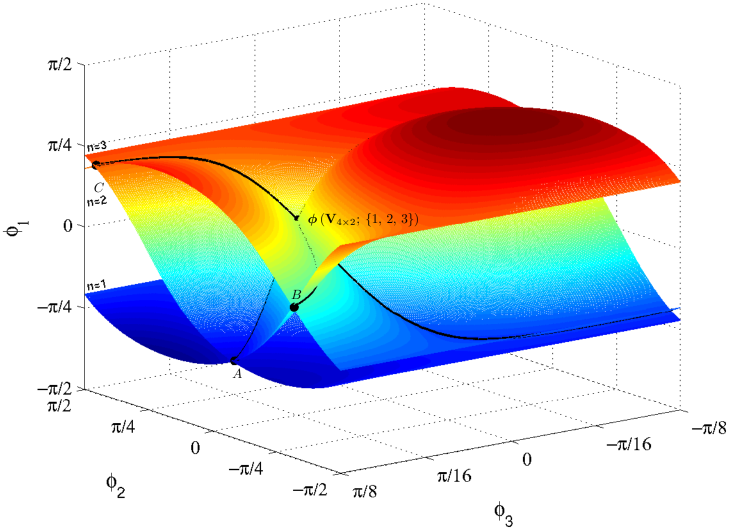
  - (b) At least three surfaces originate from the same row of $\mathbf{V}$.

- Combinations of the form (a) have a unique intersection point $\phi(\mathbf{V}_{N\times D}; \mathcal{I}_{2D-1}) \in \Phi^{2D-2} \times (-\frac{\pi}{M}, \frac{\pi}{M}]$ such that:
  - "Leads" $\mathcal{Q}$ cells, say $C_q(\mathbf{V}_{N\times D}; \mathcal{I}_{2D-1})$, where $q = 1, 2, \ldots, \mathcal{Q}$ and $\mathcal{Q} \in \left\{(\frac{M}{2}-1)^0, (\frac{M}{2}-1)^1, \ldots, (\frac{M}{2}-1)^{D-1}\right\}$.
  - Each cell is associated with a distinct M-phase candidate vector $\mathbf{s}_q(\mathbf{V}_{N\times D}; \mathcal{I}_{2D-1})$ in the sense that:
    - (i) $\mathbf{s}_q(\mathbf{V}_{N\times D}; \phi_{1:2D-1}) = \mathbf{s}_q(\mathbf{V}_{N\times D}; \mathcal{I}_{2D-1})$ for all $\phi_{1:2D-1} \in C_q(\mathbf{V}_{N\times D}; \mathcal{I}_{2D-1})$.
    - (ii) $\phi(\mathbf{V}_{N\times D}; \mathcal{I}_{2D-1})$ is a single point of $C_q(\mathbf{V}_{N\times D}; \mathcal{I}_{2D-1})$ where ϕ_{2D-1} is minimized.

Hypersurfaces and Cardinality of $\mathcal{S}(\mathbf{V}_{N\times D})$

- Illustrative example: $D = 2, N = 4$ and $M = 8$.

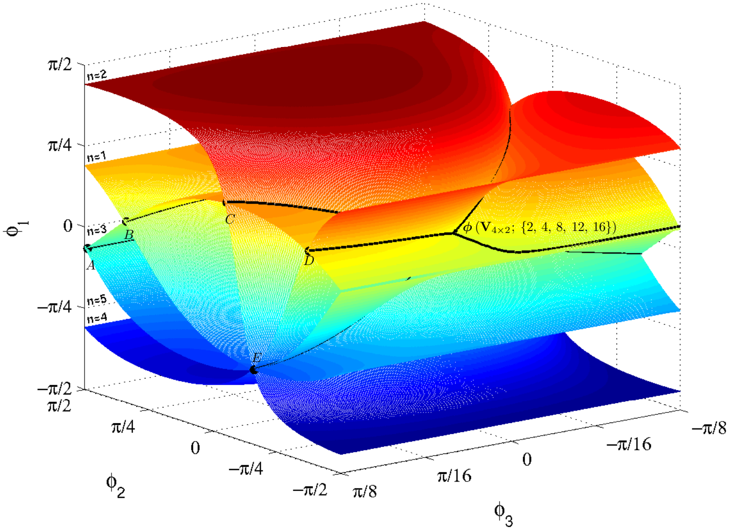


Hypersurfaces and Cardinality of $\mathcal{S}(\mathbf{V}_{N\times D})$

- What's the value of $\mathcal{Q}$? Depends on the number of participating pairs of hypersurfaces that originate from the same row of matrix $\mathbf{V}$.

- E.g.
  - No pairs (i.e. all hypersurfaces originate from different rows of $\mathbf{V}$) $\rightarrow \phi(\mathbf{V}_{N\times D}; \mathcal{I}_{2D-1})$ "leads" only one cell,
  - One pair $\rightarrow \phi(\mathbf{V}_{N\times D}; \mathcal{I}_{2D-1})$ "leads" $(\frac{M}{2}-1)$ cells,
  - $D-1$ pairs $\rightarrow \phi(\mathbf{V}_{N\times D}; \mathcal{I}_{2D-1})$ "leads" $(\frac{M}{2}-1)^{D-1}$ cells,

- Illustrative example: $D = 2, N = 4$ and $M = 8$.



- Each cell is associated with a distinct M-PSK candidate vector.
- We can collect all these vectors into:

$$\mathcal{J}(\mathbf{V}_{N\times D}) \triangleq \bigcup_{\substack{\mathcal{I}_{2D-1}\subset\{1,2,\ldots,\frac{MN}{2}\},\\ \phi(\mathbf{V}_{N\times D};\mathcal{I}_{2D-1})\in\Phi^{2D-2}\times(-\frac{\pi}{M},\frac{\pi}{M}]}} \left\{\mathbf{s}(\mathbf{V}_{N\times D};\mathcal{I}_{2D-1})\right\} \subseteq \mathcal{A}_M^N$$

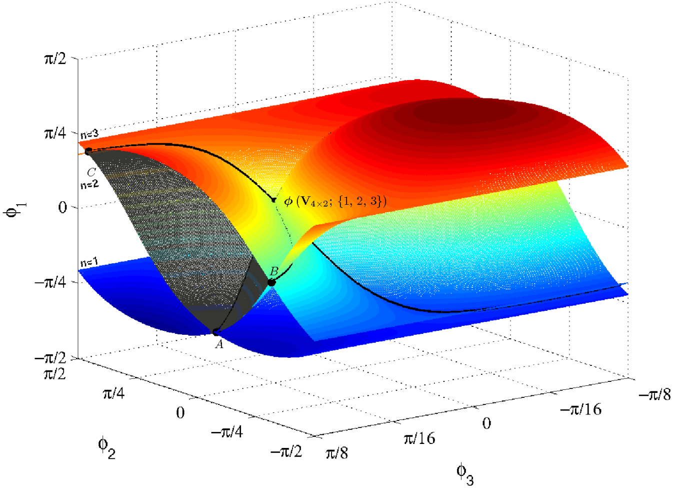
with cardinality

$$|\mathcal{J}(\mathbf{V}_{N\times D})| = \sum_{i=0}^{D-1}\binom{N}{i}\binom{N-i}{2(D-i)-1}\frac{M}{2}^{2(D-i)-2}\left(\frac{M}{2}-1\right)^{i}.$$

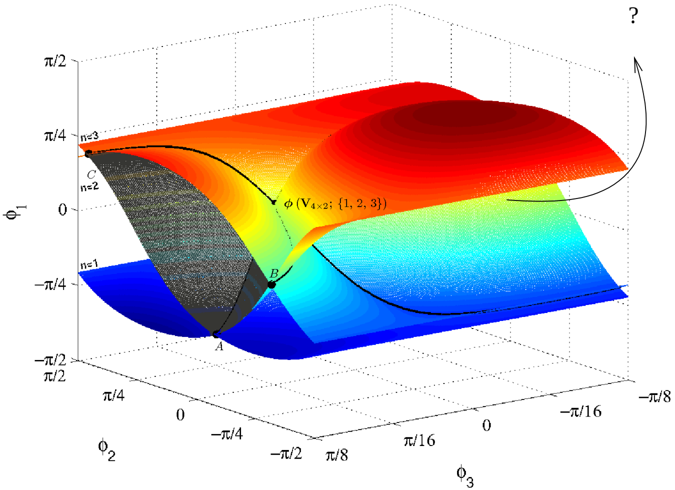
- Are there any regions missing?

Hypersurfaces and Cardinality of $\mathcal{S}(\mathbf{V}_{N\times D})$

- Illustrative example: $D = 2, N = 4$ and $M = 8$.

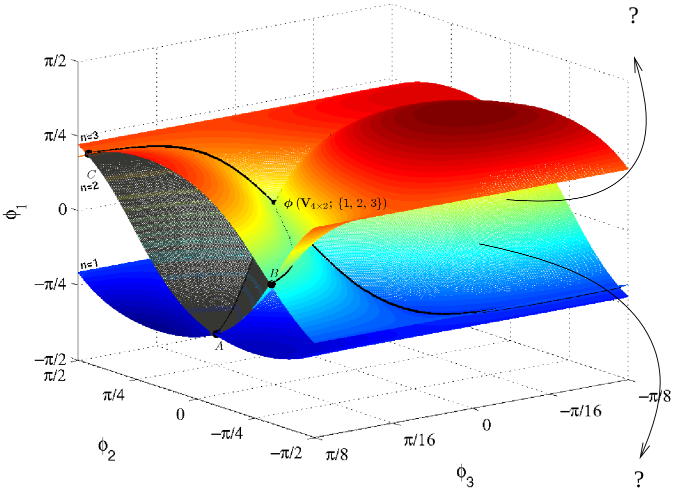


- Illustrative example: $D = 2, N = 4$ and $M = 8$.

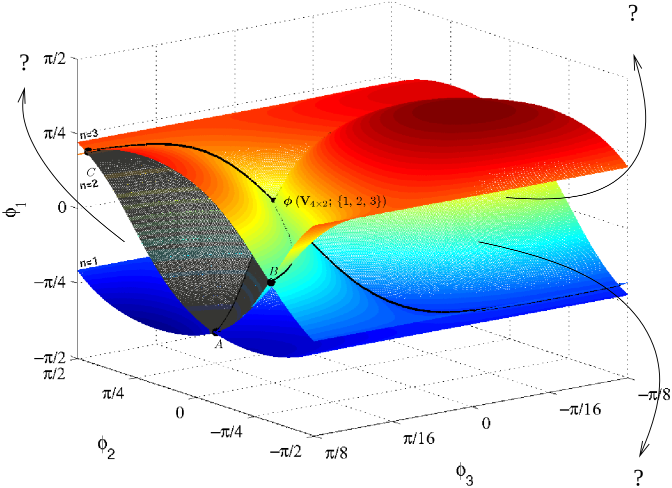


Hypersurfaces and Cardinality of $\mathcal{S}(\mathbf{V}_{N \times D})$

- Illustrative example: $D = 2, N = 4$ and $M = 8$.



- Illustrative example: $D = 2, N = 4$ and $M = 8$.



Proposition

For any $\phi_{1:2D-1} \in \Phi^{2D-2} \times (-\frac{\pi}{M}, \frac{\pi}{M}]$ the following hold true:

(i) $\mathbf{s}(\mathbf{V}_{N\times D}; \phi_{1:2D-2}, -\frac{\pi}{M}) = e^{j\frac{2\pi}{M}} \mathbf{s}(\mathbf{V}_{N\times D}; \phi'_{1:2D-2}, \frac{\pi}{M})$ for some $\phi'_{1:2D-2} \in \Phi^{2D-2}$,

(ii) $\mathbf{s}(\mathbf{V}_{N\times D}; \phi_{1:2D-3}, \frac{\pi}{2}, \phi_{2D-1}) = \mathbf{s}(\mathbf{V}_{N\times(D-1)}; \phi_{1:2D-3})$,

(iii) $\mathbf{s}(\mathbf{V}_{N\times D}; \phi_{1:2D-3}, -\frac{\pi}{2}, \phi_{2D-1}) = -\mathbf{s}(\mathbf{V}_{N\times D}; -\phi_{1:2D-3}, \frac{\pi}{2}, \phi'_{2D-1}), \forall \phi'_{2D-1} \in (-\frac{\pi}{M}, \frac{\pi}{M}]$,

(iv) $\mathbf{s}(\mathbf{V}_{N\times D}; \phi_{1:2D-3}, \pm\frac{\pi}{2}, \phi_{2D-1}) = \mathbf{s}(\mathbf{V}_{N\times D}; \phi_{1:2D-3}, \pm\frac{\pi}{2}, \phi'_{2D-1}), \forall \phi'_{2D-1} \in (-\frac{\pi}{M}, \frac{\pi}{M}]$.

- Hence, $\mathcal{S}(\mathbf{V}_{N\times D}) = \mathcal{J}(\mathbf{V}_{N\times D}) \cup \mathcal{S}(\mathbf{V}_{N\times(D-1)})$ and thus:

$$\mathcal{S}(\mathbf{V}_{N\times D}) = \mathcal{J}(\mathbf{V}_{N\times D}) \cup \mathcal{J}(\mathbf{V}_{N\times(D-1)}) \cup \cdots \cup \mathcal{J}(\mathbf{V}_{N\times 1}) = \bigcup_{d=0}^{D-1} \mathcal{J}(\mathbf{V}_{N\times(D-d)}).$$

- Cardinality of $\mathcal{S}(\mathbf{V}_{N\times D})$:

$$
\begin{aligned}
|\mathcal{S}(\mathbf{V}_{N\times D})| &= |\mathcal{J}(\mathbf{V}_{N\times D})| + |\mathcal{J}(\mathbf{V}_{N\times (D-1)})| + \cdots + |\mathcal{J}(\mathbf{V}_{N\times 1})| \\
&= \sum_{d=1}^{D}\sum_{i=0}^{d-1} \binom{N}{i} \binom{N-i}{2(d-i)-1} \left(\frac{M}{2}\right)^{2(d-i)-2} \left(\frac{M}{2}-1\right)^{i} \\
&= \mathcal{O}\left(\left(\frac{MN}{2}\right)^{2D-1}\right).
\end{aligned}
$$

- If $\mathbf{Q}$ is full rank,

Remark

If $D = N$, then the computation of $\mathbf{s}_{\text{opt}}$ is $\mathcal{NP}$-hard and can be implemented by applying exhaustive search or the proposed algorithm among all elements of $\mathcal{A}_M^{N-1}$ since $|\mathcal{A}_M^{N-1}| = |\mathcal{S}(\mathbf{V}_{N\times D})| = M^{N-1}$.

Summary of Theoretic Developments

- We have utilized $2D-1$ auxiliary hyperspherical coordinates.
- Partitioned the hypercube $\Phi^{2D-2} \times (-\frac{\pi}{M}, \frac{\pi}{M}]$ into a finite number of cells that are associated with distinct M-phase vectors.
- $|\mathcal{S}(\mathbf{V}_{N\times D})| = \mathcal{O}\left(\left(\frac{MN}{2}\right)^{2D-1}\right) \subseteq \mathcal{A}_M^N$ and proved that $\mathbf{s}_{\text{opt}} \in \mathcal{S}(\mathbf{V}_{N\times D})$.
- Complexity to build $\mathcal{S}(\mathbf{V}_{N\times D})$: $\mathcal{O}\left(\left(\frac{MN}{2}\right)^{2D}\right)$.
- But how?

Algorithm

- For each one of $|\mathcal{J}(\mathbf{V}_{N\times D})|$ intersections, find the coordinates $\phi_1, \ldots, \phi_{2D-1}$ and the corresponding complex hyperspherical vector $\mathbf{c}(\phi_{1:2D-1})$ solving the system of hypersurfaces:

$$\tilde{\mathbf{V}}_{\mathcal{I}_{2d-1},1:2d}\tilde{\mathbf{c}}(\phi_{1:2d-1}) = \mathbf{0}_{(2d-1)\times 1}.$$

- Build candidate vectors using:

$$\mathbf{s}(\mathbf{V}_{N\times D};\phi_{1:2D-1}) \triangleq \begin{bmatrix} s(\mathbf{V}_{1,1:D};\phi_{1:2D-1}) \\ s(\mathbf{V}_{2,1:D};\phi_{1:2D-1}) \\ \vdots \\ s(\mathbf{V}_{N,1:D};\phi_{1:2D-1}) \end{bmatrix}$$

Note: vector elements associated with $2D-1$ intersecting hypersurfaces lead to ambiguous decisions.

- Use disambiguation rules for the ambiguous decisions.

Disambiguation rules

- Let $\mathcal{N}_{\mathcal{I}_{2d-1}}$ the set of indices of rows from $\mathbf{V}$ participating in the intersection $\mathcal{I}_{2d-1}$.

(*i*) For any $n \in \{1, 2, \ldots, N\}$ and $n \notin \mathcal{N}_{\mathcal{I}_{2d-1}}$:

$$s_{q,n}(\mathbf{V}_{N\times d}; \mathcal{I}_{2d-1}) = s(\mathbf{V}_{n,1:d}; \phi(\mathbf{V}_{N\times d}; \mathcal{I}_{2d-1})), \ \forall q \in \{1, 2, \ldots, \mathcal{Q}\}$$

(*ii*) For any $n \in \mathcal{N}_{\mathcal{I}_{2d-1}}$ such that there is only one hypersurface related with the n-th row of $\mathbf{V}$:

$$s_{q,n}(\mathbf{V}_{N\times d}; \mathcal{I}_{2d-1}) = s(\mathbf{V}_{n,1:d}; \phi(\mathbf{V}_{N\times d}; \mathcal{I}_{2d-1} - \{i_k\})), \ \forall q \in \{1, 2, \ldots, \mathcal{Q}\}$$

where $\Im\{\mathbf{V}_{n,d}\} = 0$.

(*iii*) For any $n \in \mathcal{N}_{\mathcal{I}_{2d-1}}$ such that there is a pair of hypersurfaces say $\mathcal{H}(\tilde{\mathbf{V}}_{i_k,1:2d})$, $\mathcal{H}(\tilde{\mathbf{V}}_{i_m,1:2d})$ related with the n-th row of $\mathbf{V}$:

$$\begin{aligned} s_{q,n}(\mathbf{V}_{N\times d}; \mathcal{I}_{2d-1}) = & s\Big(\mathbf{V}_{n,1:d}; \phi(\mathbf{V}_{N\times d}; \mathcal{I}_{2d-1} - \{i_k, i_m\}) + \{i_k'\}\Big) \bigcap \\ & s\Big(\mathbf{V}_{n,1:d}; \phi(\mathbf{V}_{N\times d}; \mathcal{I}_{2d-1} - \{i_k, i_m\}) + \{i_m'\}\Big), \end{aligned}$$

where $\Im\{\mathbf{V}_{n,d}\} = 0$ for $\forall q \in \{1, 2, \ldots, \mathcal{Q}\}$.

Algorithm

- For each one of $|\mathcal{J}(\mathbf{V}_{N\times D})|$ intersections, find the coordinates $\phi_1,\ldots,\phi_{2D-1}$ and the corresponding complex hyperspherical vector $\mathbf{c}(\phi_{1:2D-1})$ solving the system of hypersurfaces:

$$\tilde{\mathbf{V}}_{\mathcal{I}_{2d-1},1:2d}\tilde{\mathbf{c}}(\phi_{1:2d-1}) = \mathbf{0}_{(2d-1)\times 1}.$$

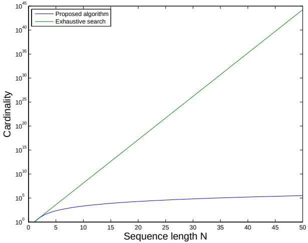
- Build candidate vectors using:

$$\mathbf{s}(\mathbf{V}_{N\times D};\phi_{1:2D-1}) \triangleq \begin{bmatrix} s(\mathbf{V}_{1,1:D};\phi_{1:2D-1}) \\ s(\mathbf{V}_{2,1:D};\phi_{1:2D-1}) \\ \vdots \\ s(\mathbf{V}_{N,1:D};\phi_{1:2D-1}) \end{bmatrix}$$

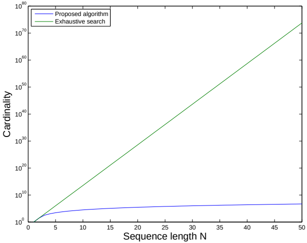
Note: vector elements associated with $2D-1$ intersecting hypersurfaces lead to ambiguous decisions.

- Use disambiguation rules for the ambiguous symbols.
- Set $D \leftarrow D-1$ (drop the rightmost column of complex matrix $\mathbf{V}$) and repeat steps.

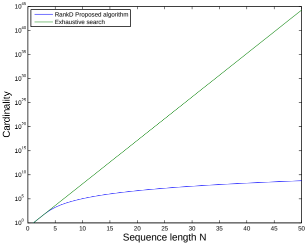
Complexity Study



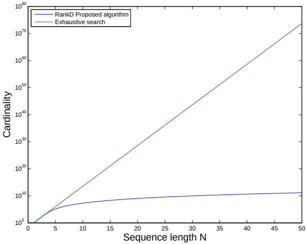
(a) $D = 2, M = 8$



(b) $D = 2, M = 32$



(c) $D = 3, M = 8$



(d) $D = 3, M = 32$

Figure: Proposed algorithm cardinality (blue line) Vs. exhaustive search cardinality (green line).

Conclusions and Further Developments

Conclusions:

- Maximization of a rank-deficient form with M-phase argument proved to be polynomially solvable.
- Implementable algorithm even for large sequence lengths.
- Time and memory efficient, fully parallelizable and rank-scalable algorithm.