Composite Self-concordant Minimization

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Abstract

We propose an algorithmic framework for convex minimization problems of composite functions with two terms: a self-concordant part and a possibly nonsmooth regularization part. Our method is a new proximal Newton algorithm with local quadratic convergence rate. As a specific problem instance, we consider sparse precision matrix estimation problems in graph learning. Via a careful dual formulation and a novel analytic step-size selection, we instantiate an algorithm within our framework for graph learning that avoids *Cholesky decompositions and matrix inversions*, making it attractive for parallel and distributed implementations.

Composite minimization

$$\text{(P)} \qquad \min_{\mathbf{x}\in\mathbb{R}^n} \{F(\mathbf{x}) := f(\mathbf{x}) + g(\mathbf{x})\}$$

f: convex and smooth — **g**: convex and possibly nonsmooth

Motivation

Problem (P) *covers many practical problems*:

- ***Unconstrained basic LASSO/Basic pursuit denoising problems***
- ***Graphical model selections*** / latent variable graphical model selection
- **Poisson imaging reconstruction** / Heteroscedastic LASSO
- Low-rank approximation
- Clustering

Problem (P) *has specific separable structure*

- ***splitting methods***
- ***augmented Lagrangian methods***

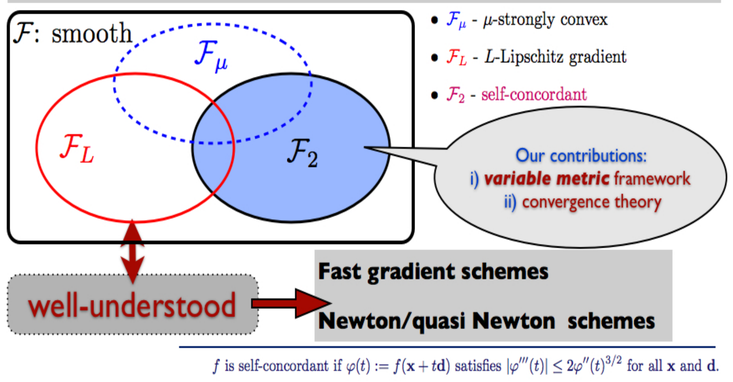
g: ℓ_1-norm, nuclear norm or indicator functions

Composite self-concordant minimization

$$\text{(P)} \qquad \min_{\mathbf{x}\in\mathbb{R}^n} \{F(\mathbf{x}) := f(\mathbf{x}) + g(\mathbf{x})\}$$

f: convex and self-concordant — **g**: convex and possibly nonsmooth

Classes of smooth functions (f)



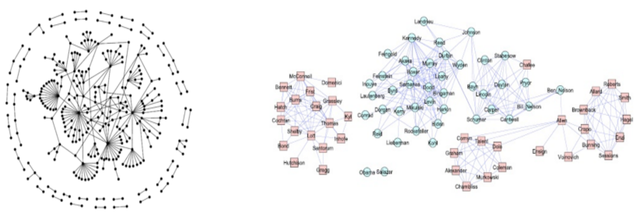
f is self-concordant if $\varphi(t) := f(\mathbf{x}+t\mathbf{d})$ satisfies $|\varphi'''(t)| \le 2\varphi''(t)^{3/2}$ for all $\mathbf{x}$ and $\mathbf{d}$.

Example: Log-determinant for LMIs

Optimization problem

$$\min_{\Theta\succ 0}\left\{\underbrace{-\log\det(\Theta)+\operatorname{trace}(\Sigma\Theta)}_{f(\mathbf{x})} + \underbrace{\rho\|\operatorname{vec}(\Theta)\|_1}_{g(\mathbf{x})}\right\}$$

- ***Application***: Graphical model selection



Log-barrier for linear/quadratic inequalities

- Poisson imaging reconstruction

$$x^* \in \operatorname{argmin}\left\{\underbrace{\sum_{i=1}^m a_i^T x - \sum_{i=1}^m y_i \log(a_i^T x + b_i)}_{f(x)} + g(x)\right\}$$

- Basic pursuit denoising problem (BPDP): Barrier formulation

$$\mathbf{x}_t^* = \operatorname*{argmin}_{\mathbf{x}}\left\{\underbrace{-t\log\left(\sigma^2 - \|\mathbf{A}\mathbf{x}-\mathbf{y}\|_2^2\right)}_{=:f(\mathbf{x})} + g(\mathbf{x})\right\}$$

- LASSO problem with unknown variance

$$\mathbf{x}^* \equiv (\phi^*,\gamma^*) = \operatorname*{argmin}_{\phi,\gamma}\left\{\underbrace{-\log(\gamma)+\frac{1}{2n}\|\gamma\mathbf{y}-\mathbf{X}\phi\|_2^2}_{=:f(\mathbf{x})} + \underbrace{\lambda\|\phi\|_1}_{=:g(\mathbf{x})}\right\}$$

Prior state-of-the-art

- Proximal point scheme with variable metric [Bonnans, 1993]

Proximal point scheme with variable metric (PPA(H))

Given x^0, generate a sequence $\{x^k\}_{k\ge 0}$ such that

$$x^{k+1} = \operatorname{prox}_{H_k}\left(x^k - H_k^{-1}\nabla f(x^k)\right)$$

where H_k is symmetric positive definite

- Special cases
  - If $H_k := \gamma_k I$ then we obtain a proximal-gradient scheme [Nesterov2007, Beck2009]
  - If $H_k \approx \nabla^2 f(x^k)$ then we obtain a proximal-quasi Newton scheme [Becker2012]
  - If $H_k = \nabla^2 f(x^k)$ then we obtain a proximal-Newton scheme [Lee2012]
  - If $\operatorname{prox}_{H_k} \equiv \mathrm{Id}$ then we obtain a classical variable metric descent scheme
- Connection to splitting
  - PPA(H) is a special case of splitting forward-backward methods

Self-concordance: A mathematical tool

- **Local norm** — f **is a self-concordant function**

	Primal norm	Dual norm
Local norm	$\|\mathbf{u}\|_{\mathbf{x}} = [\mathbf{u}^T\nabla^2 f(\mathbf{x})\mathbf{u}]^{1/2}$	$\|\mathbf{u}\|_{\mathbf{x}}^* = [\mathbf{u}^{T-1}\nabla^2 f(\mathbf{x})\mathbf{u}]^{1/2}$
Utility functions	$\omega(t) := t - \ln(1+t)$	$\omega_*(t) := -t - \ln(1-t)$

- **Main properties**

Lower approximation	$f(\mathbf{y}) \ge f(\mathbf{x}) + \nabla f(\mathbf{x})^T(\mathbf{y}-\mathbf{x}) + \omega(\|\mathbf{y}-\mathbf{x}\|_{\mathbf{x}})$	$\mathbf{x},\mathbf{y}\in\operatorname{dom}(f)$
Upper approximation	$f(\mathbf{y}) \le f(\mathbf{x}) + \nabla f(\mathbf{x})^T(\mathbf{y}-\mathbf{x}) + \omega^*(\|\mathbf{y}-\mathbf{x}\|_{\mathbf{x}})$	$\|\mathbf{y}-\mathbf{x}\|_{\mathbf{x}} < 1$
Hessian approximation	$(1-\|\mathbf{y}-\mathbf{x}\|_{\mathbf{x}})^2\nabla^2 f(\mathbf{x}) \preceq \nabla^2 f(\mathbf{y}) \preceq (1-\|\mathbf{y}-\mathbf{x}\|_{\mathbf{x}})^{-2}\nabla^2 f(\mathbf{x})$	$\|\mathbf{y}-\mathbf{x}\|_{\mathbf{x}} < 1$

Local

- **Compare to the main properties of $\mathcal{F}_{\mu,L}$**

Lower approximation	$f(\mathbf{y}) \ge f(\mathbf{x}) + \nabla f(\mathbf{x})^T(\mathbf{y}-\mathbf{x}) + \frac{\mu}{2}\|\mathbf{y}-\mathbf{x}\|_2^2$	$\mathbf{x},\mathbf{y}\in\operatorname{dom}(f)$
Upper approximation	$f(\mathbf{y}) \le f(\mathbf{x}) + \nabla f(\mathbf{x})^T(\mathbf{y}-\mathbf{x}) + \frac{L}{2}\|\mathbf{y}-\mathbf{x}\|_2^2$	$\mathbf{x},\mathbf{y}\in\operatorname{dom}(f)$
Hessian approximation	$\mu\mathbb{I} \preceq \nabla^2 f(\mathbf{x}) \preceq L\mathbb{I}$	$\mathbf{x},\mathbf{y}\in\operatorname{dom}(f)$

Our approach

- Proximal Newton scheme

Given $\mathbf{x}^0$, generate a sequence $\{\mathbf{x}^k\}_{k\ge 0}$ such that

$$\mathbf{x}^{k+1} := \mathbf{x}^k + \alpha_k \mathbf{d}_{\mathbf{H}_k}^k$$

where $\alpha_k \in (0,1]$ is step-size, $\mathbf{d}_{\mathbf{H}_k}^k$ is a search direction

- How to calculate the Proximal Newton direction?

$$\mathbf{d}_{\mathbf{H}_k}^k := \arg\min_{\mathbf{d}}\left\{f(\mathbf{x}^k) + \nabla f(\mathbf{x}^k)^T\mathbf{d} + \frac{1}{2}\mathbf{d}^T\mathbf{H}_k\mathbf{d} + g(\mathbf{x}^k+\mathbf{d})\right\}, \quad \mathbf{H}_k = \nabla^2 f(\mathbf{x}^k)$$

- Highlights

- **Globalization**: a new strategy for finding step-size explicitly, motivate "*forward-looking*" line-search strategy
- **Search direction**: efficient (strongly convex program)
- **Local convergence**: quadratic convergence *without boundedness of the Hessian* analytic quadratic convergence region

How do we compute the step-size?

- Upper approximation of f

$$f(\mathbf{x}^{k+1}) \le f(\mathbf{x}^k) + \nabla f(\mathbf{x}^k)^T(\mathbf{x}^{k+1}-\mathbf{x}^k) + \omega^*(\|\mathbf{x}^{k+1}-\mathbf{x}^k\|_{\mathbf{x}^k}), \quad \|\mathbf{x}^{k+1}-\mathbf{x}^k\|_{\mathbf{x}^k} < 1$$

- Convexity of g and optimality condition of the subproblem

$$g(\mathbf{x}^{k+1}) - g(\mathbf{x}^k) \le -\alpha_k\nabla f(\mathbf{x}^k)^T\mathbf{d}_{\mathbf{H}_k}^k - \alpha_k\|\mathbf{d}_{\mathbf{H}_k}^k\|_{\mathbf{H}_k}^2.$$

$$\phi(\mathbf{x}^{k+1}) \le \phi(\mathbf{x}^k) - \alpha_k\|\mathbf{d}_{\mathbf{H}_k}^k\|_{\mathbf{H}_k}^2 + \omega_*\left(\alpha_k\|\mathbf{d}_{\mathbf{H}_k}^k\|_{\mathbf{x}^k}\right)$$

- When $\mathbf{H}_k \equiv \nabla^2 f(\mathbf{x}^k)$, $\lambda_k := \|\mathbf{d}_{\mathbf{H}_k}^k\|_{\mathbf{x}^k}$

$$\phi(\mathbf{x}^{k+1}) \le \phi(\mathbf{x}^k) - \underbrace{\left[\alpha_k\lambda_k - \omega_*\left(\alpha_k\lambda_k\right)\right]}_{\psi(\alpha_k)}$$

maximize $\psi(\alpha_k)$ to get optimal α_k^*

$$\alpha_k^* = \frac{1}{\lambda_k+1} \in (0,1]$$

- Line-search on $[\alpha_k^*, 1]$ to accelerate the global convergence

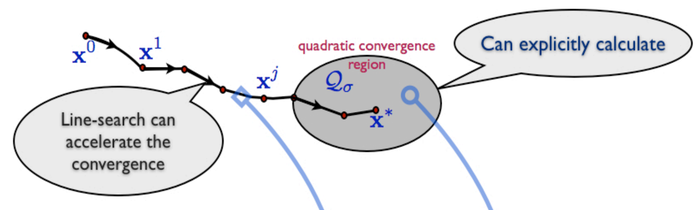
Analytic complexity

Proximal Newton decrement: $\lambda_k := \|\mathbf{d}^k\|_{\mathbf{x}^k}$

Quadratic convergence region: Let $\sigma := (5-\sqrt{17})/4 \approx 0.219224$

$$\mathcal{Q}_\sigma := \{\mathbf{x}^k \mid \lambda_k \le \sigma\}$$

Illustration the convergence behavior



- Worst-case complexity

$$\#\text{iterations} = \left\lfloor\frac{F(\mathbf{x}^0)-F(\mathbf{x}^*)}{0.021}\right\rfloor + O\left(\ln\ln\left(\frac{c}{\varepsilon}\right)\right)$$

Application to graphical model selection

- The underlying convex optimization problem

$$\min_{\Theta\succ 0}\left\{\underbrace{-\log\det(\Theta)+\operatorname{trace}(\Sigma\Theta)}_{f(\mathbf{x})} + \underbrace{\rho\|\operatorname{vec}(\Theta)\|_1}_{g(\mathbf{x})}\right\}$$

- Gradient and Hessian (large-scale, special structure)
  - Gradient of f: $\nabla f(\mathbf{x}) = \mathbf{vec}(\Sigma - \Theta^{-1})$.
  - Hessian of f: $\nabla^2 f(\mathbf{x}) = \Theta^{-1}\otimes\Theta^{-1}$
- Dual approach for solving subproblem (SP)

Unconstrained LASSO problems — No Cholesky decomposition and matrix inversion

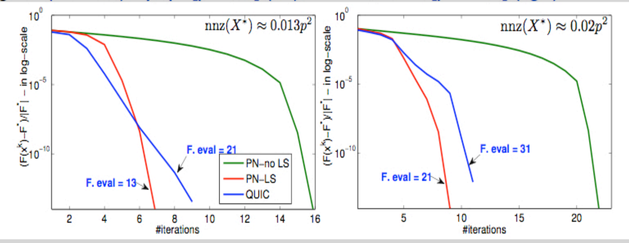
Primal subproblem	Dual subproblem (SPGL)
$\min_{\Delta}\left\{\frac{1}{2}\operatorname{trace}((\Theta_i^{-1}\Delta)^2) + \operatorname{trace}(\mathbf{R}_i\Delta) + \rho\|\operatorname{vec}(\Delta)\|_1\right\}$	$\min_{\|\operatorname{vec}(\mathbf{U})\|_\infty\le 1}\left\{\frac{1}{2}\operatorname{trace}((\Theta_i\mathbf{U})^2) + \operatorname{trace}(\mathbf{Q}_i\mathbf{U})\right\}$
$\mathbf{R}_i := \Sigma - 2\Theta_i^{-1}$	$\mathbf{Q}_i := \rho^{-1}[\Theta_i\Sigma\Theta_i - 2\Theta_i]$

- How to compute proximal Newton decrement $\lambda_i := \|\mathbf{d}^i\|_{\mathbf{x}^i}$?

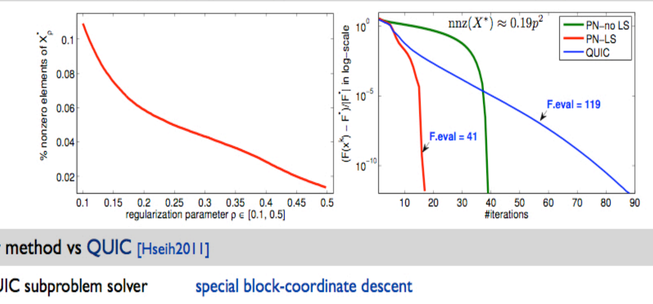
$$\lambda_i := [p - 2\operatorname{trace}(\mathbf{W}_i) + \operatorname{trace}(\mathbf{W}_i^2)]^{1/2}, \quad \mathbf{W}_i = \Theta_i(\Sigma - \rho\mathbf{U}^*)$$

Numerical experiments on graph learning

Convergence (rho = 0.5): Lymph [p = 587] (left), Leukemia [p = 1255] (right)



Sparsity w.r.t. rho for Lymph (left), Convergence for Lymph with rho = 0.05 (right)

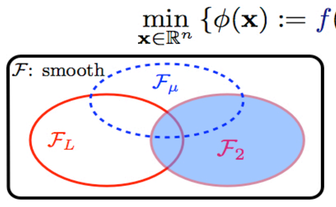


Our method vs QUIC [Hseih2011]

- QUIC subproblem solver: special block-coordinate descent
- Our subproblem solver: general proximal algorithms

Conclusions

$$\min_{\mathbf{x}\in\mathbb{R}^n}\{\phi(\mathbf{x}) := f(\mathbf{x}) + g(\mathbf{x})\}$$



- $\mathcal{F}_\mu$ - μ-strongly convex
- $\mathcal{F}_L$ - L-Lipschitz gradient
- $\mathcal{F}_2$ - self-concordant

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- **Search direction**: efficient (strongly convex program)
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- Practical contributions
  - software package has quasi-Newton, pure-Newton, gradient and fast gradient algorithms
  - leverage fast proximal solvers for g(x) (structured norms etc.)
  - robust to subproblem solver accuracy

Software

- A open-source software package called **SCOPT** is available at http://lions.epfl.ch/software
- **SCOPT** stands for **S**elf-**C**oncordant **OPT**imization