

COMP 414/514: Optimization – Algorithms, Complexity and Approximations

Lecture 3

Overview

- In the last lecture, we:
 - Introduced some notions on smooth optimization
 - Introduced gradient descent and what we can say about its convergence rate
- In this lecture, we will:
 - Discuss briefly **smooth continuous optimization**
 - Introduce the important class of **convex optimization**
 - Discuss about **convergence rates** and some **lower bounds** on such rates

“What does convexity bring onto the table?”

Convex functions

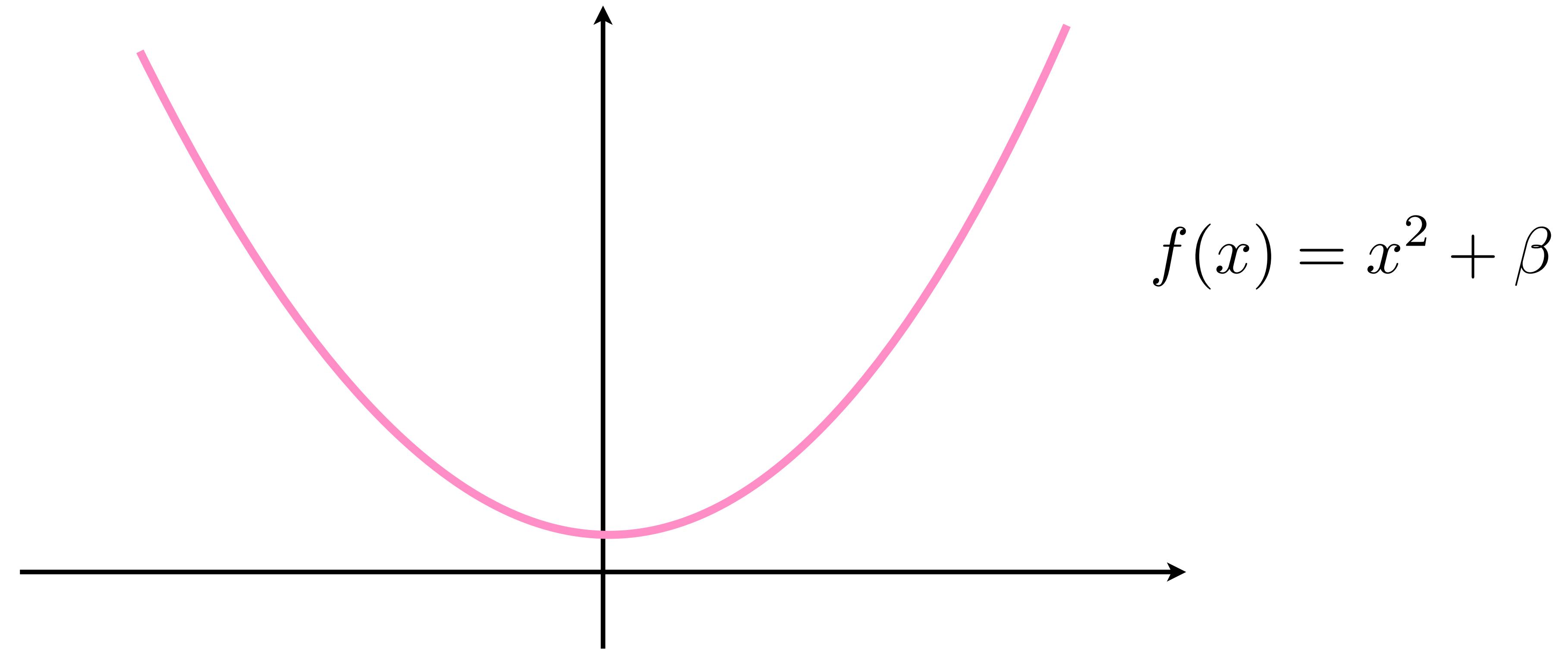
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$$f(\alpha x + (1 - \alpha)y) \leq \alpha f(x) + (1 - \alpha)f(y), \quad \forall a \in [0, 1]$$

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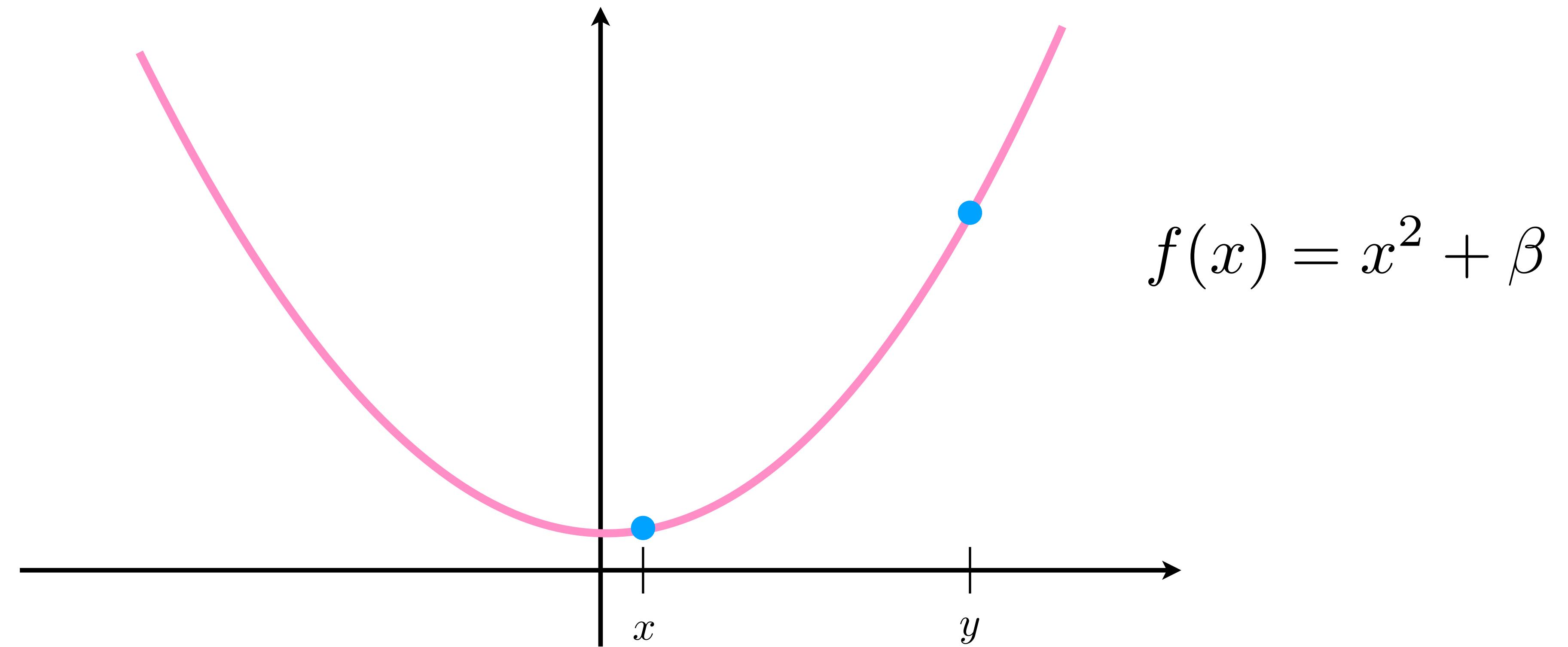
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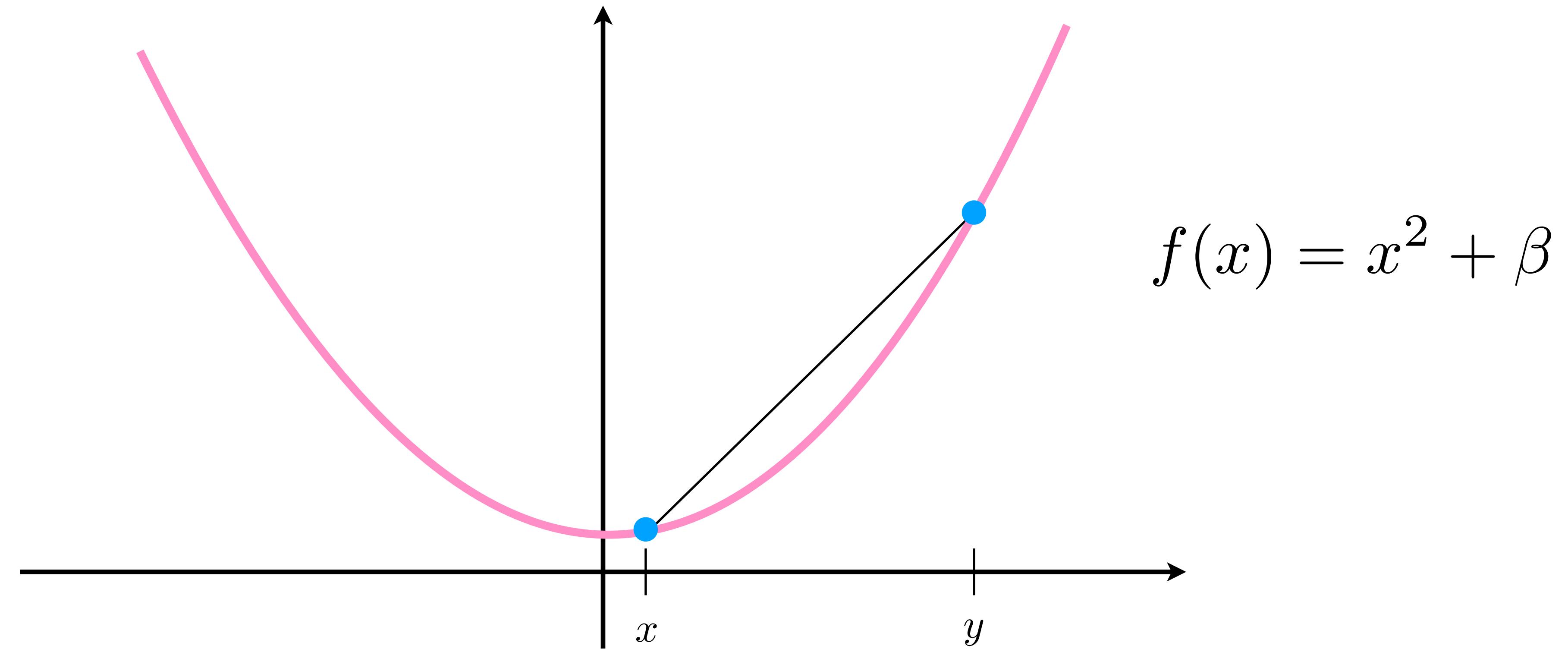
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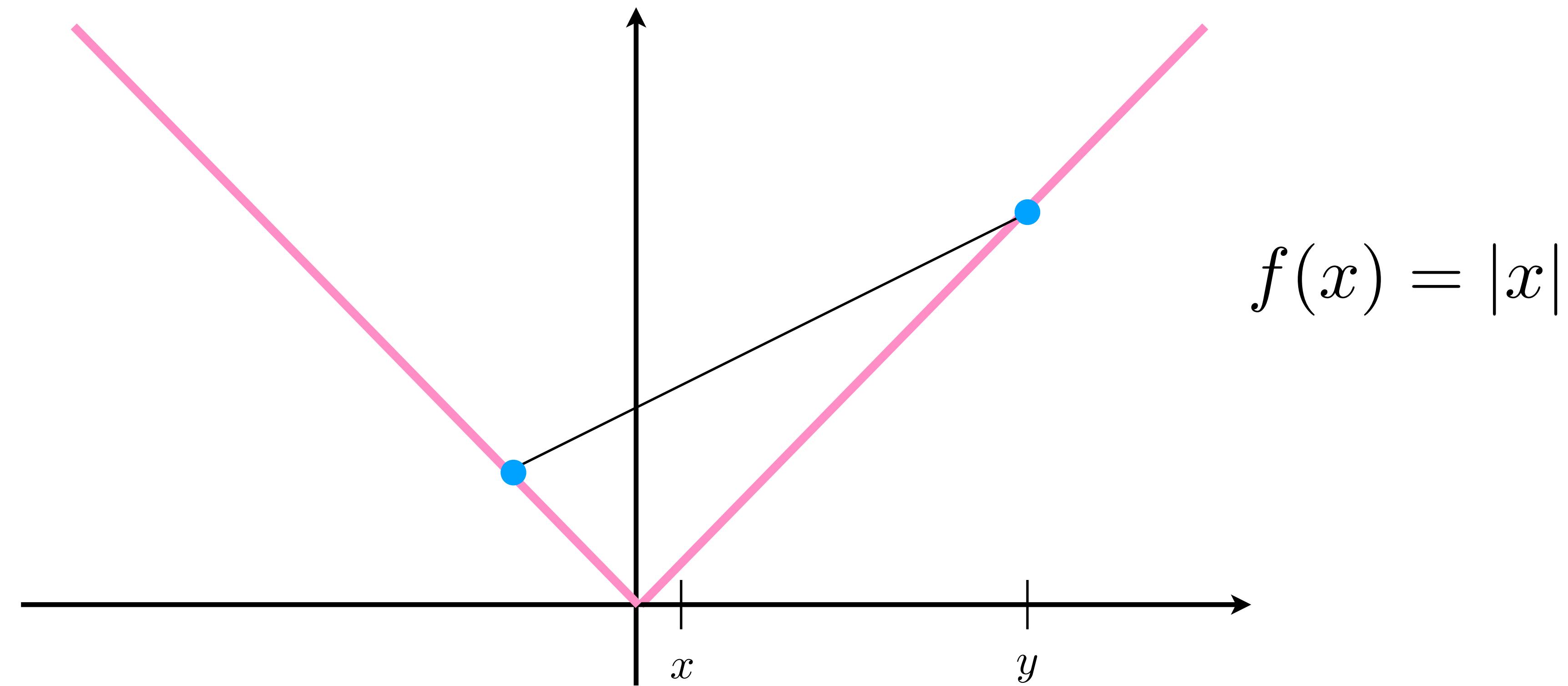
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Convex functions

- Examples:

Function	Example	Attributes
ℓ_p vector norms, $p \geq 1$	$\ \mathbf{x}\ _2, \ \mathbf{x}\ _1, \ \mathbf{x}\ _\infty$	convex
ℓ_p matrix norms, $p \geq 1$	$\ \mathbf{X}\ _* = \sum_{i=1}^{\text{rank}(\mathbf{X})} \sigma_i$	convex
Square root function	\sqrt{x}	concave, nondecreasing
Maximum of functions	$\max\{x_1, \dots, x_n\}$	convex, nondecreasing
Minimum of functions	$\min\{x_1, \dots, x_n\}$	concave, nondecreasing
Sum of convex functions	$\sum_{i=1}^n f_i, f_i$ convex	convex
Logarithmic functions	$\log(\det(\mathbf{X}))$	concave, assumes $\mathbf{X} \succ 0$
Affine/linear functions	$\sum_{i=1}^n X_{ii}$	both convex and concave
Eigenvalue functions	$\lambda_{\max}(\mathbf{X})$	convex, assumes $\mathbf{X} = \mathbf{X}^T$

Convex functions

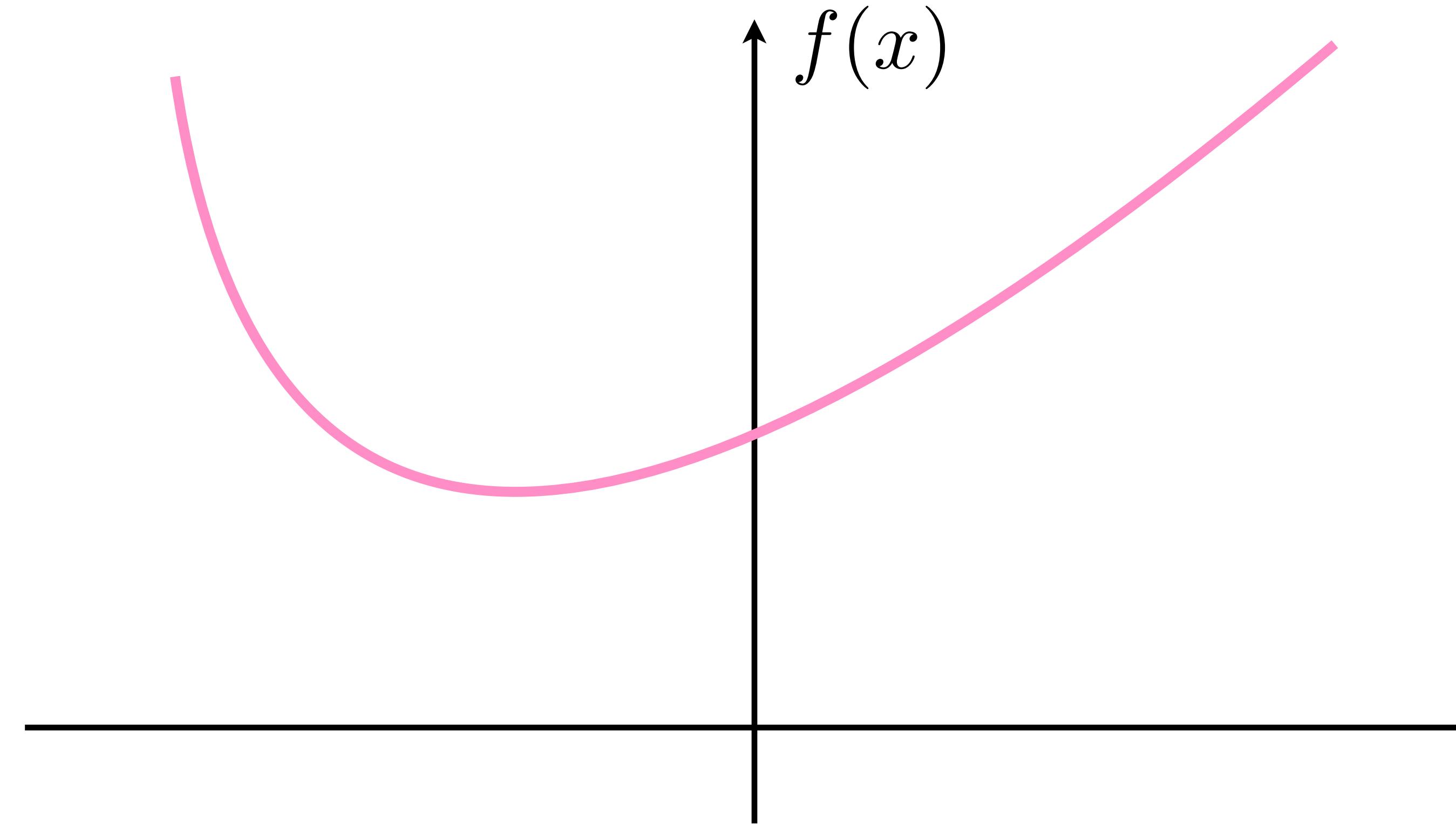
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Convex functions

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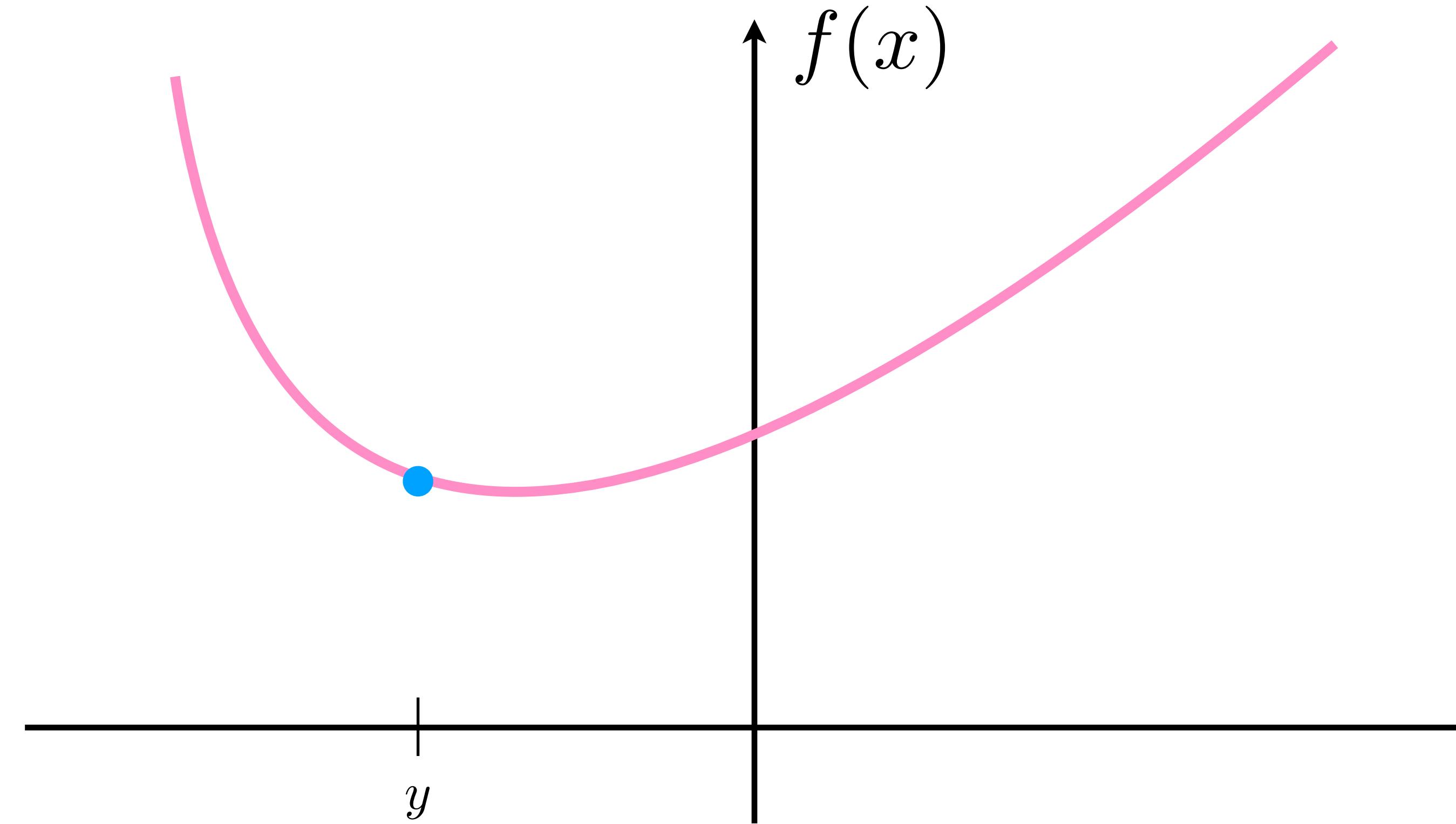
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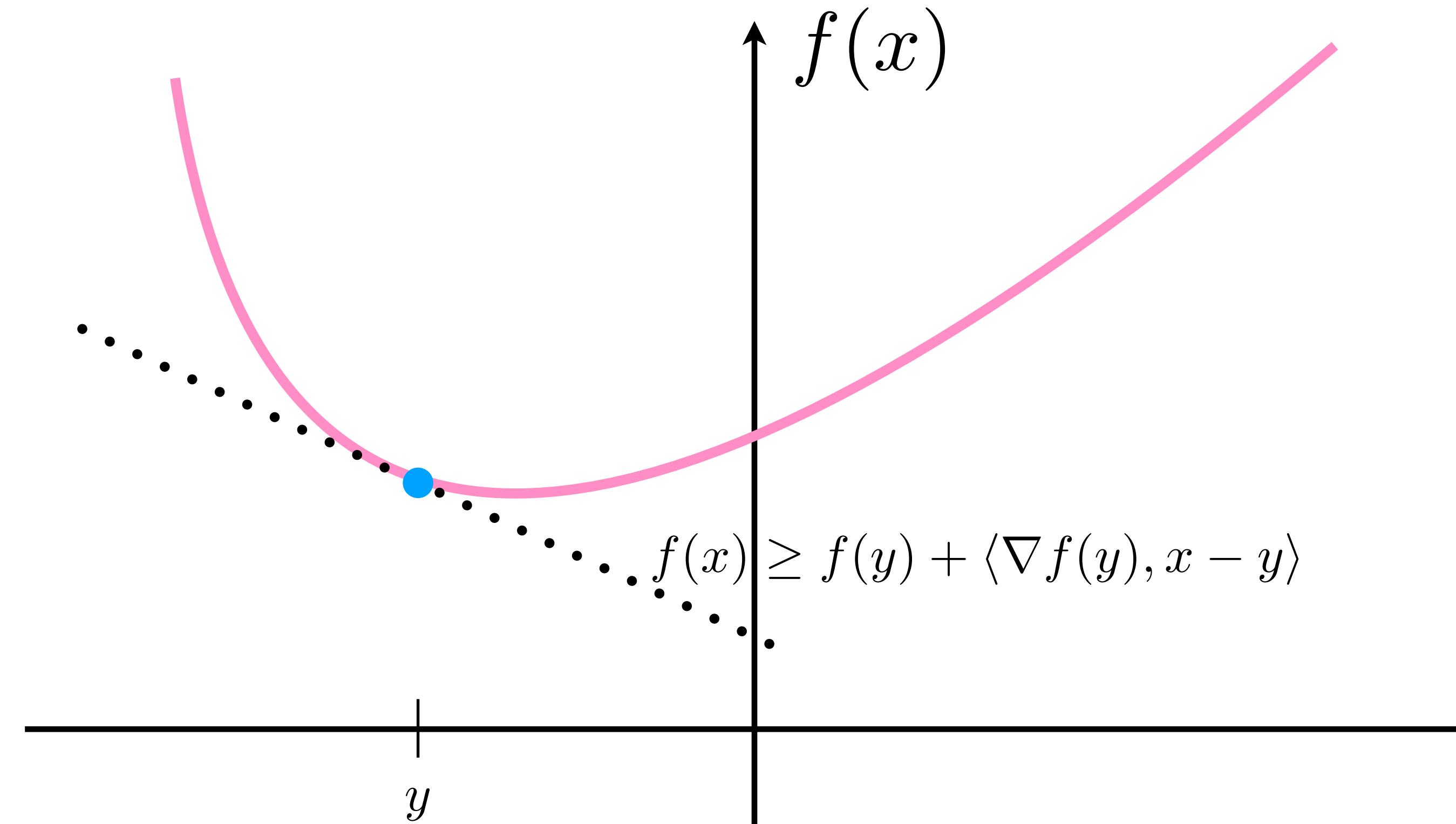
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$$\nabla^2 f(x) \succeq 0, \quad \forall x$$

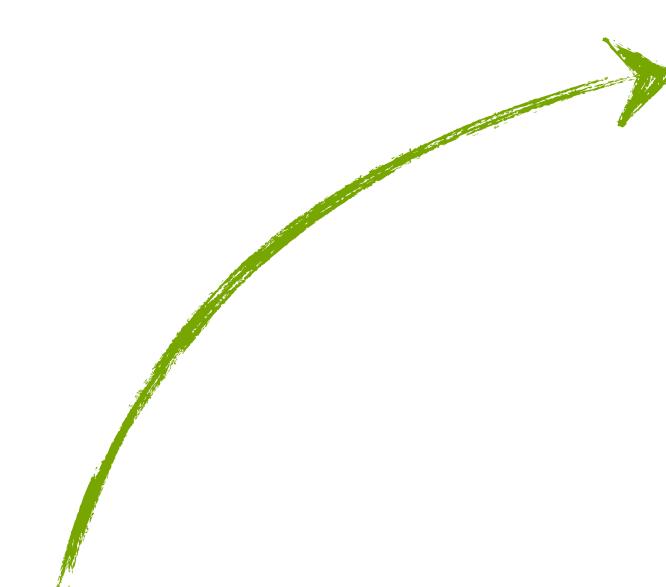
(Assuming the function is twice differentiable)

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Any interpretations?

Convex functions

- Key consequences of convexity

“Any stationary point is a global minimum”

Proof:

Convex functions

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Proof: Assume a stationary point x^* . This implies $\nabla f(x^*) = 0$

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Proof: Assume a stationary point x^* . This implies $\nabla f(x^*) = 0$

By convexity:

$$f(x) \geq f(x^*) + \langle \nabla f(x^*), x - x^* \rangle = f(x^*), \quad \forall x$$

- This is what makes convex optimization preferable.

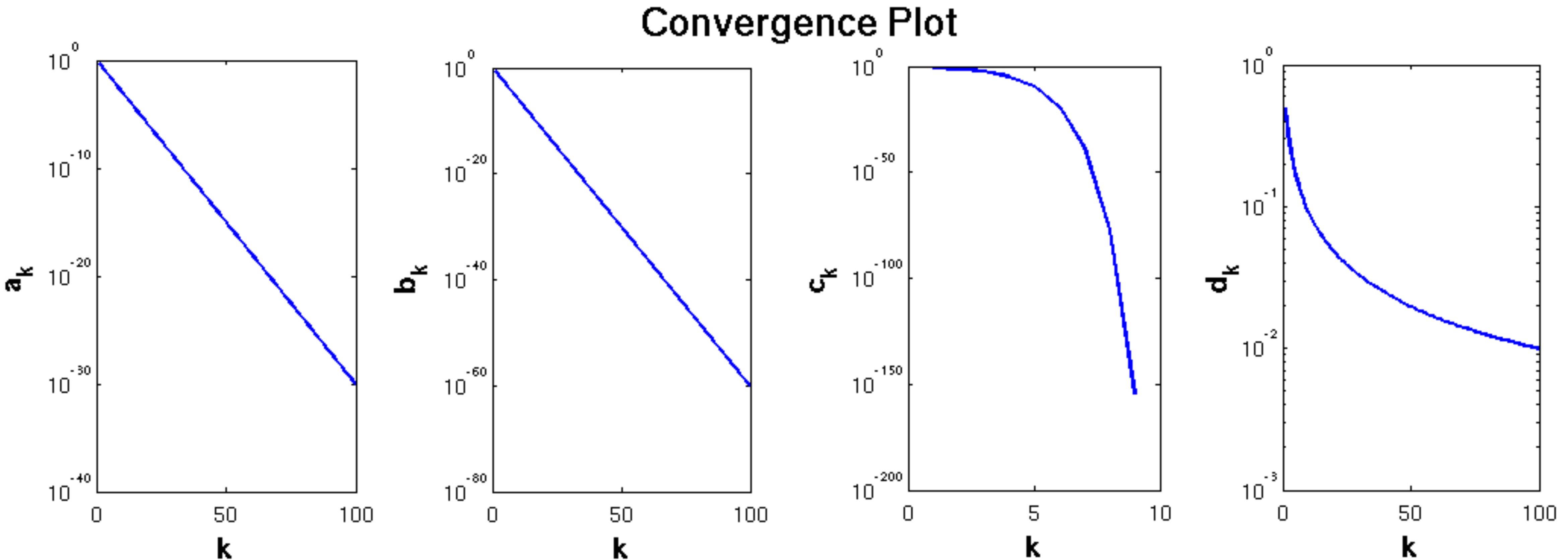
(Any local solution is actually global –
this does not mean that convex optimization is necessarily tractable!)

Does convexity improve guarantees?

Whiteboard

Convergence rates 101

(Source: Wikipedia)



$$O(\log 1/\varepsilon)$$

$$q^k, \quad q \in (0, 1)$$

$$O(\log \log(1/\varepsilon))$$

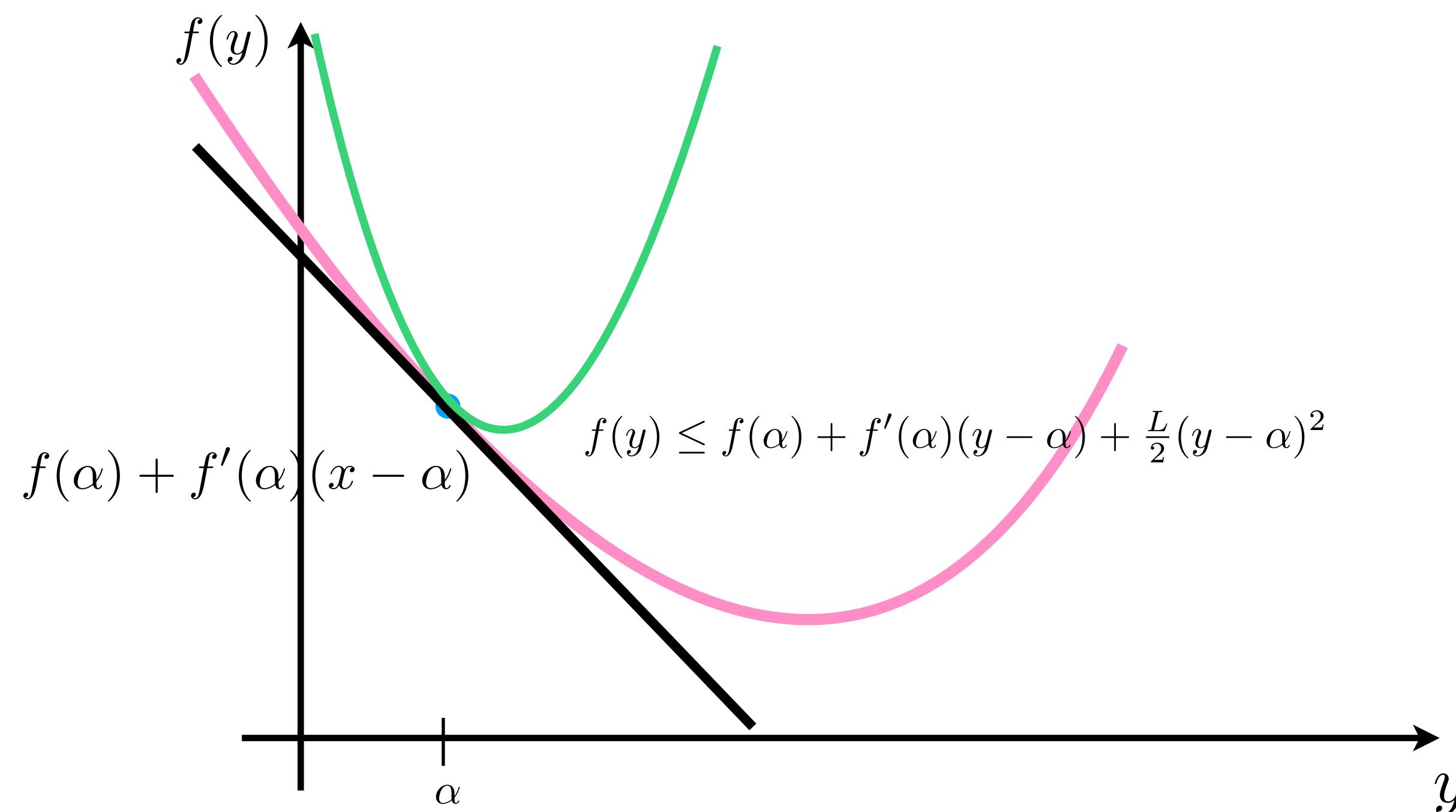
$$O(1/\varepsilon^2), O(1/\varepsilon), O(1/\sqrt{\varepsilon})$$

$$O(1/k^2), O(1/k), O(\sqrt{k})$$

Can we achieve a better performance?

- Strong convexity: $f(y) \geq f(x) + \langle \nabla f(x), y - x \rangle + \frac{\mu}{2} \|x - y\|_2^2, \forall x, y$
- Strong convexity parameter: $\mu > 0$

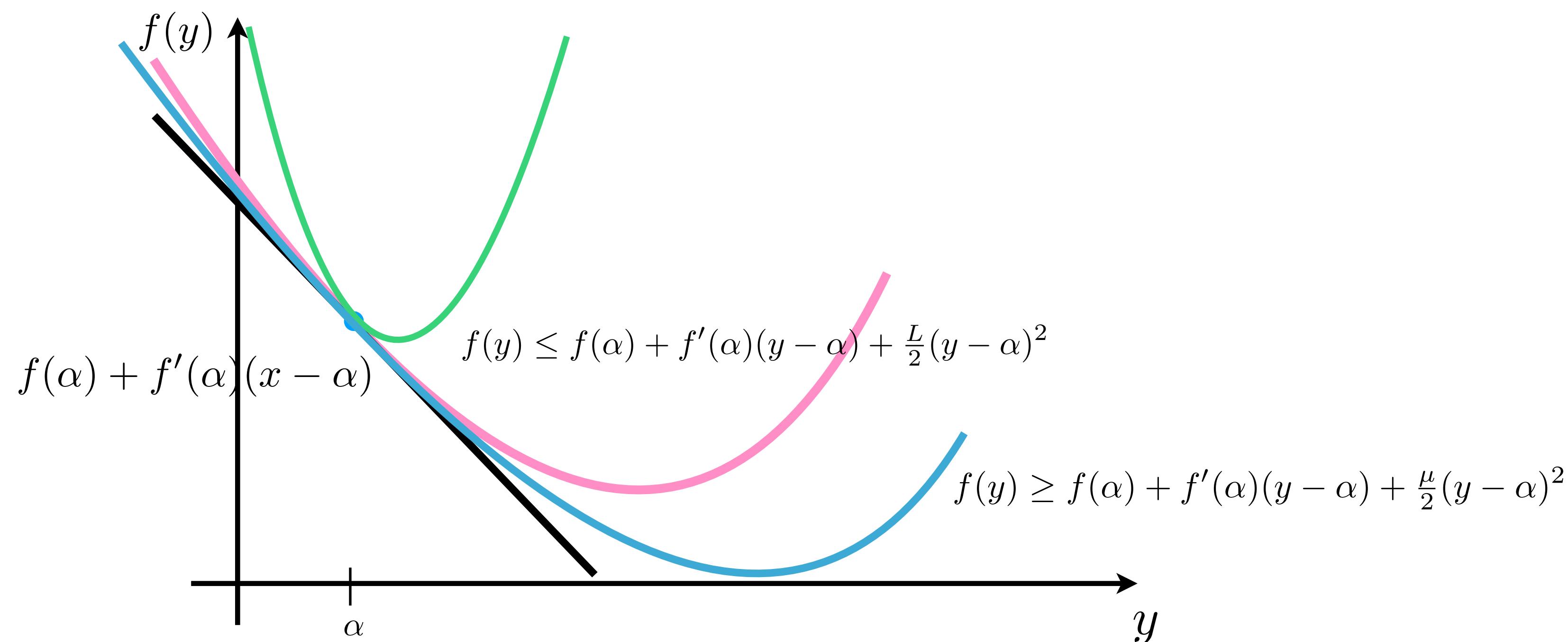
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Strong convexity

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$$f(y) \leq f(x) + \langle \nabla f(x), y - x \rangle + \frac{1}{2\mu} \|\nabla f(x) - \nabla f(y)\|_2^2$$

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- Another important one:

Interpretation?

$$\nabla^2 f(x) \succeq \mu I$$

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- Another important one: Interpretation?

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- What if we also have Lipschitz continuous gradients?

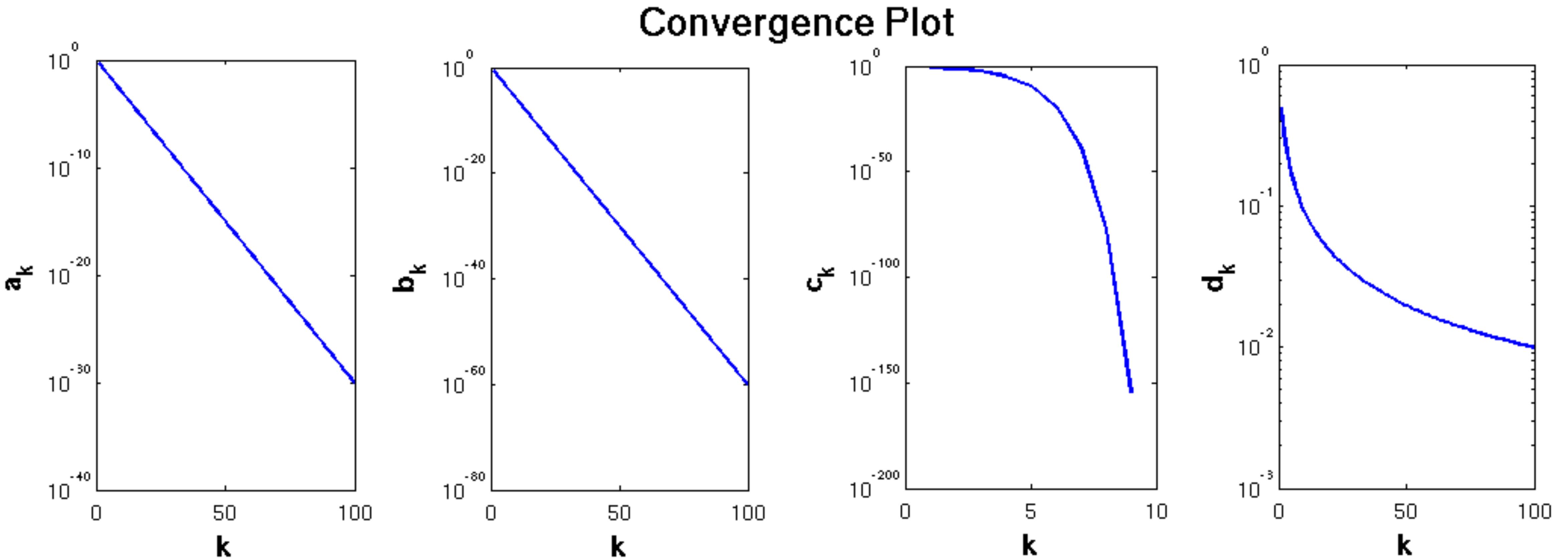
$$\langle \nabla f(x) - \nabla f(y), x - y \rangle \geq \frac{\mu L}{\mu+L} \|x - y\|_2^2 + \frac{1}{\mu+L} \|\nabla f(x) - \nabla f(y)\|_2^2$$

What is the gain?

Whiteboard

Convergence rates 101

(Source: Wikipedia)



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$$O(1/k^2), \quad O(1/k), \quad O(\sqrt(k))$$

What should be our expectations: Lower bounds

- For objectives with Lipschitz continuous gradients:

$$f(x_t) - f(x^*) \geq \frac{3L\|x_0 - x^*\|_2^2}{32(t + 1)^2}$$

(Under these assumptions, and using only gradients, we cannot achieve better than $O\left(\frac{1}{t^2}\right)$)

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- In addition, for objectives that are strongly convex:

$$\|x_t - x^*\|_2^2 \geq \left(\frac{\sqrt{\kappa} - 1}{\sqrt{\kappa} + 1}\right)^{2t} \|x_0 - x^*\|_2^2$$

$$\kappa := \frac{L}{\mu}$$

(The case we described has near optimal exponent, but does not involve the square root of κ)

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- In future lectures: acceleration techniques that achieves these rates

Convex optimization

Demo

Are there other, more powerful, global assumptions?

- Remember, our analysis is based on: $\|\nabla f(x) - \nabla f(y)\|_2 \leq L\|x - y\|_2, \quad \forall x, y$
(and on $f(y) \geq f(x) + \langle \nabla f(x), y - x \rangle + \frac{\mu}{2}\|x - y\|_2^2$ when we talk about convex functions)

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- Polyak–Łojasiewicz (PL) inequality

$$\frac{1}{2}\|\nabla f(x)\|_2^2 \geq \xi(f(x) - f(x^\star)), \quad \forall x, \quad \text{for some } \xi > 0$$

(Any thoughts about what this implies?)

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- Using PL inequality + Lipschitz gradient continuity:

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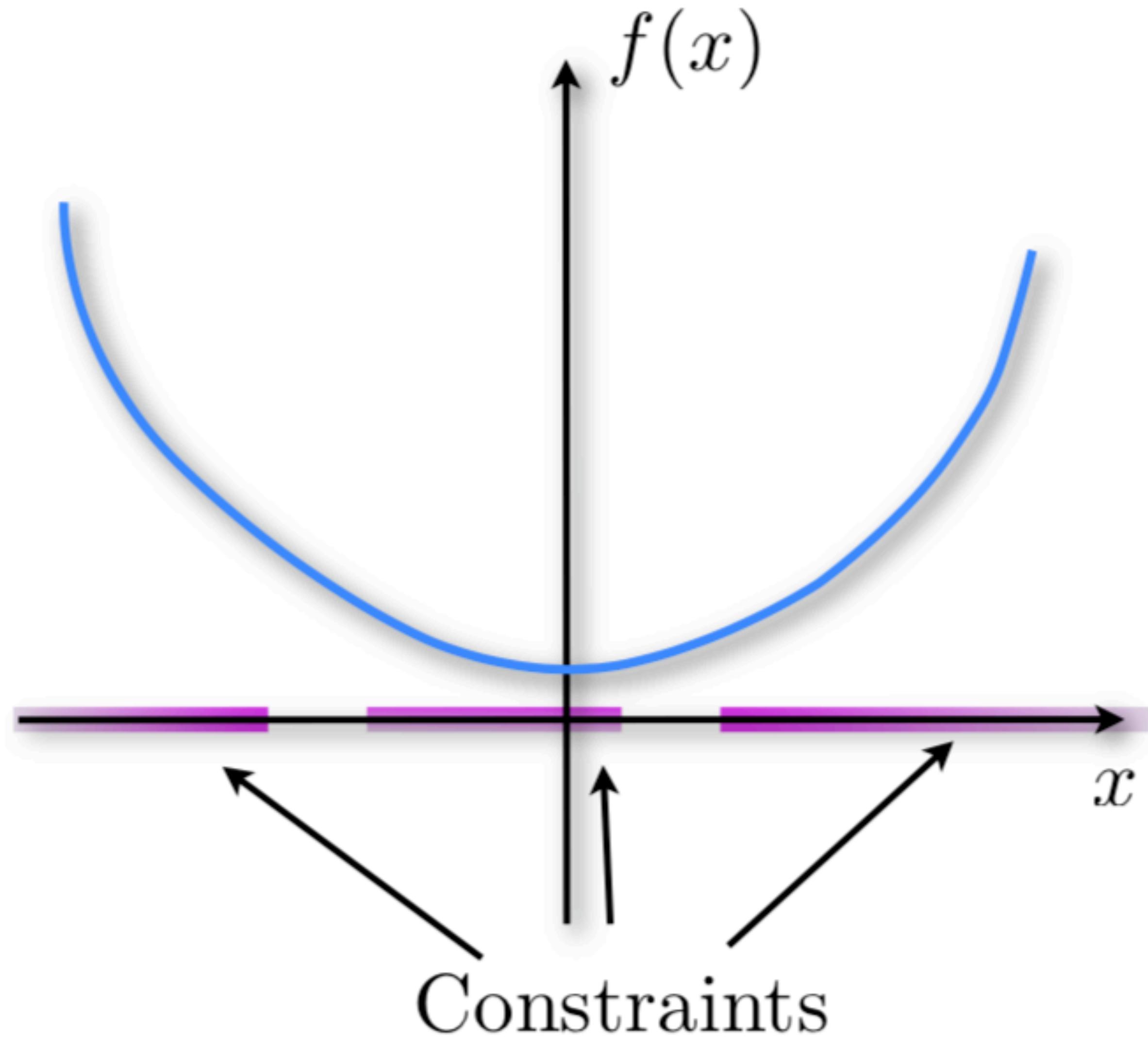
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- Does not use **convexity**: holds for invex functions (stationary = global)

Convex optimization is not only about the objective



– Back to the first slide:

$$\begin{aligned} & \min_x f(x) \\ \text{s.t. } & x \in \mathcal{C} \end{aligned}$$

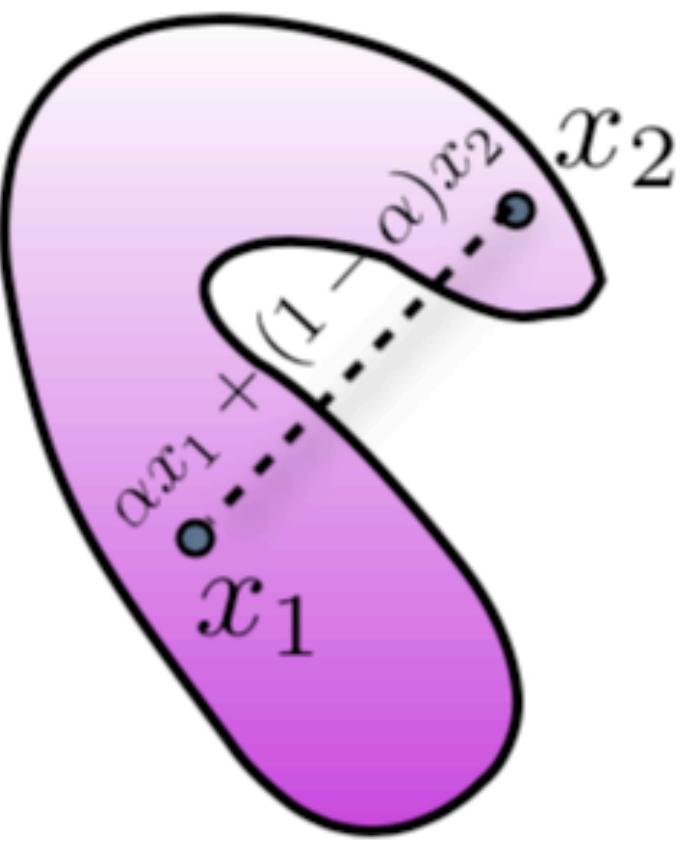
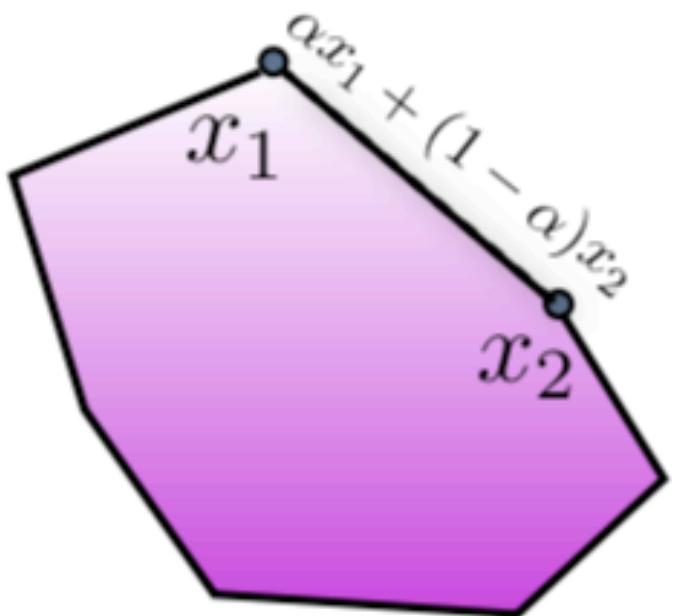
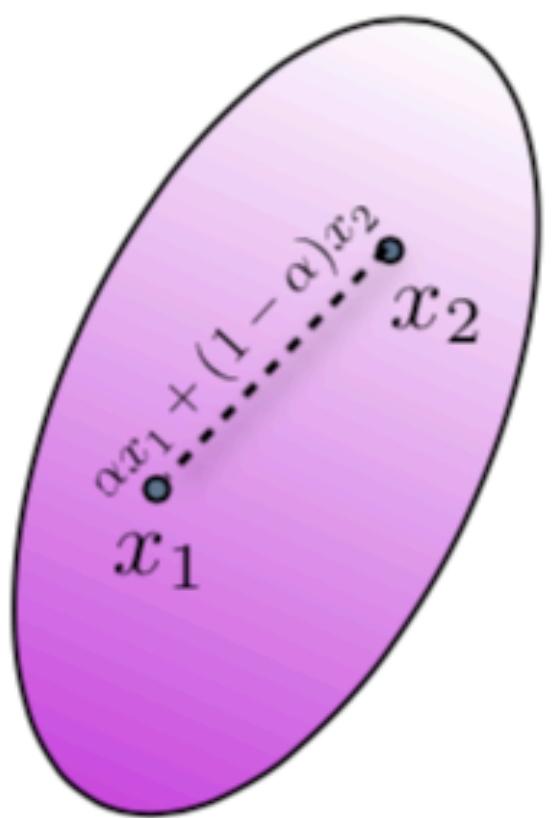
(We will worry about this in the lectures to follow!)

Convex sets

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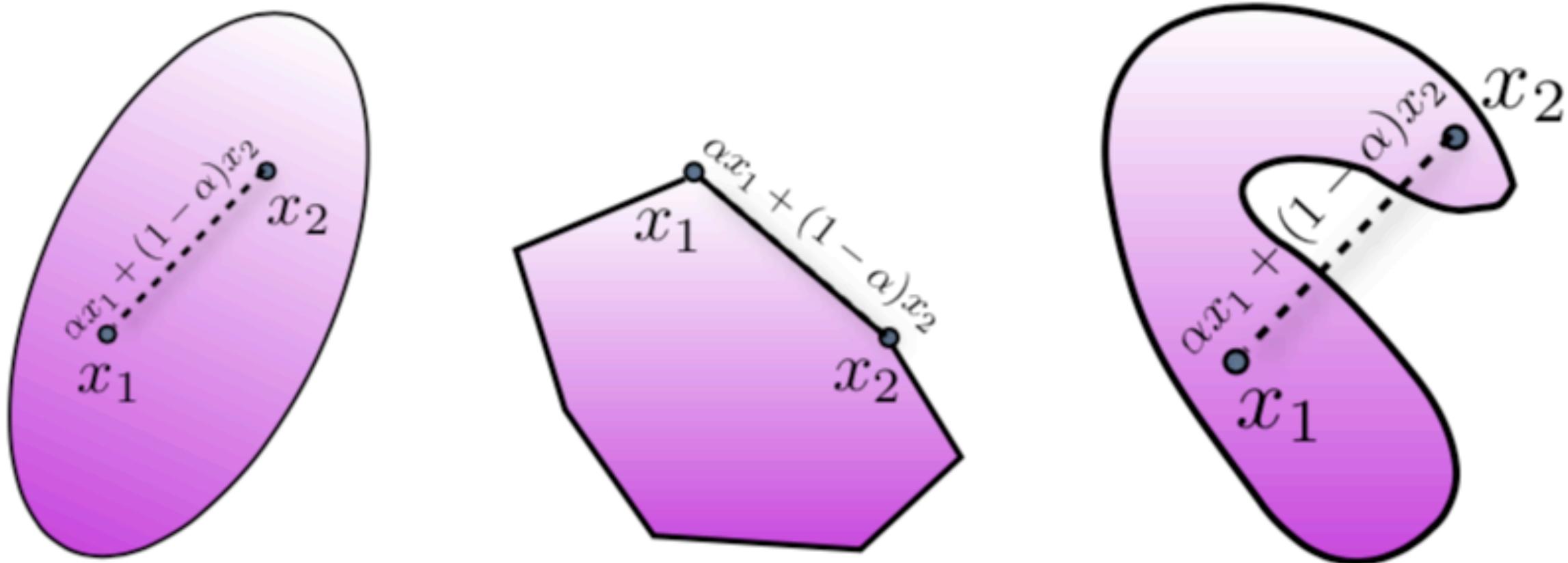
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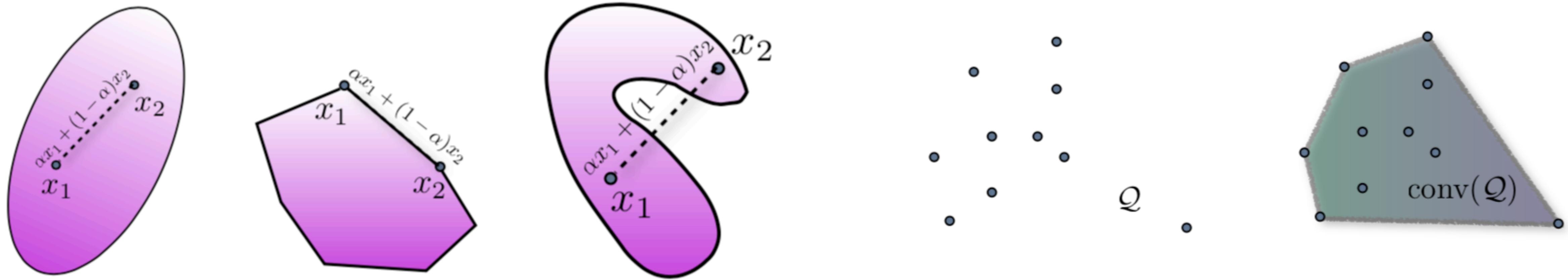
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- Convex hull of points: $\text{conv}(\mathcal{V}) = \left\{ \sum_{i=1}^{|\mathcal{V}|} \alpha_i x_i : \sum_{i=1}^{|\mathcal{V}|} \alpha_i = 1, \alpha_i \geq 0, x_i \in \mathcal{V} \right\}$

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Projections onto convex sets

$$\Pi_{\mathcal{C}}(x) = \arg \min_{y \in \mathcal{C}} \|x - y\|_2^2$$

(The use of Euclidean norm is arbitrary
and often depends on the application)

$$\Pi_{\mathcal{C}}(x) = \arg \min_{y \in \mathcal{C}} \|x - y\|_1$$

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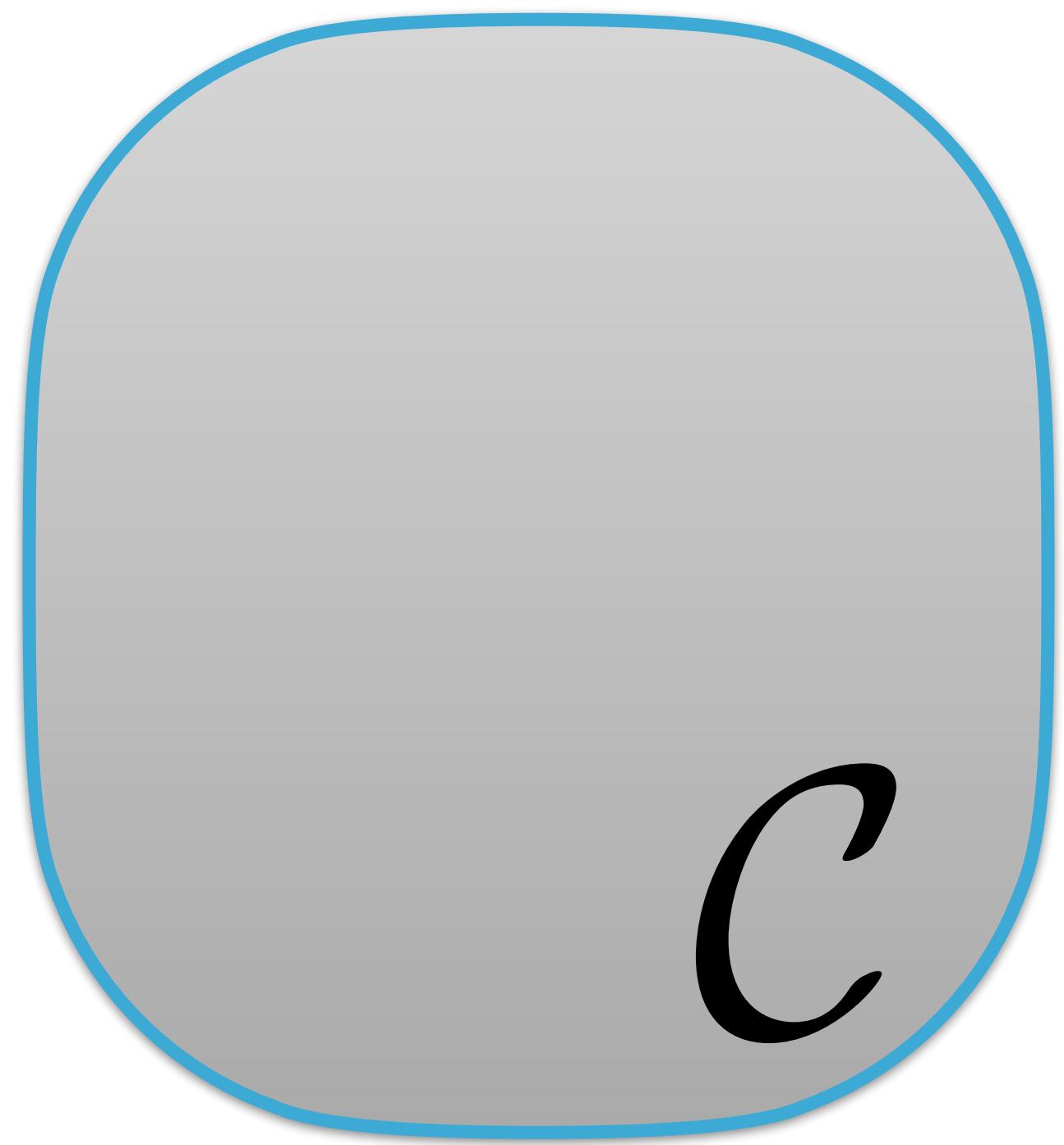
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Projections onto convex sets

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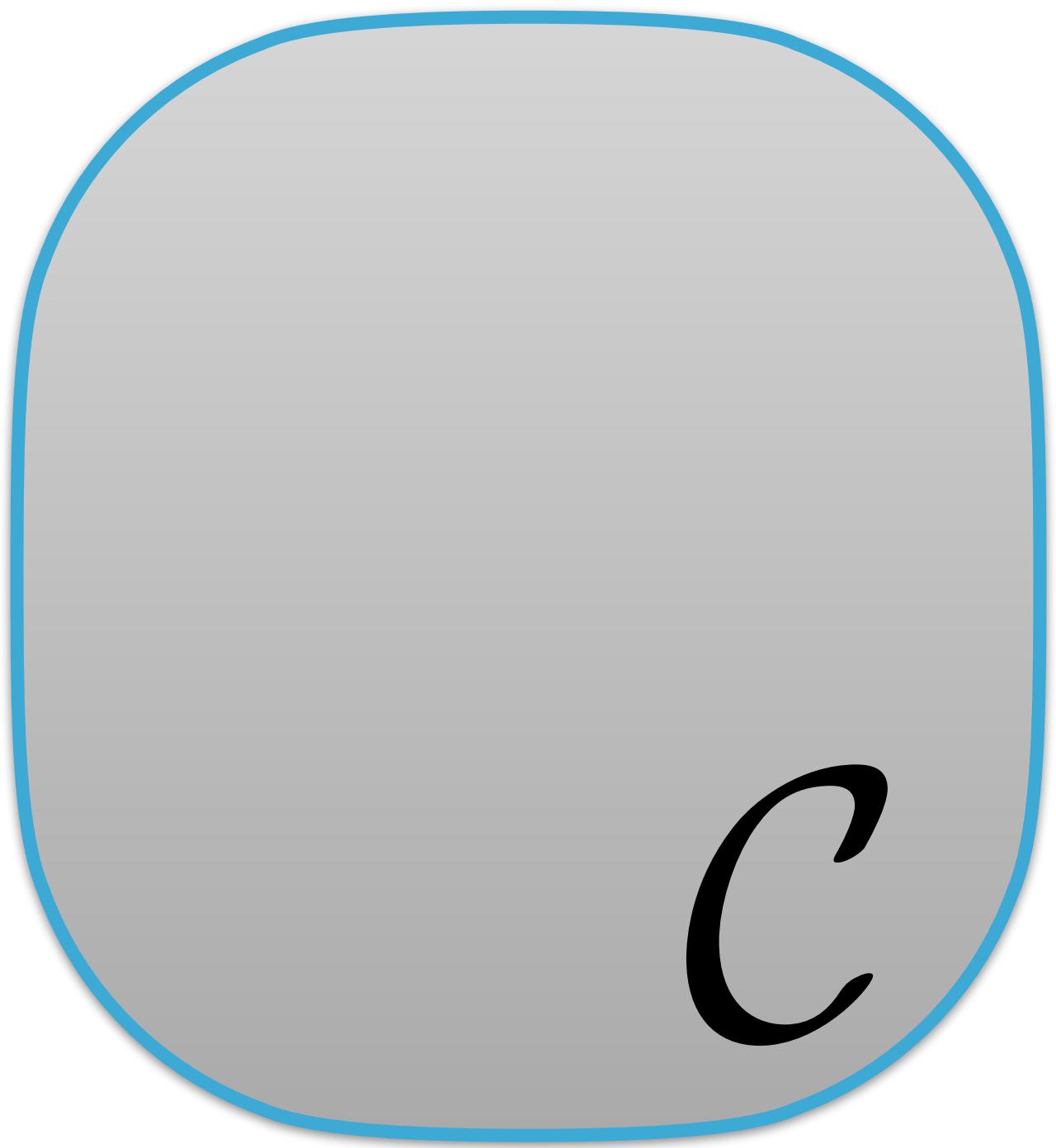
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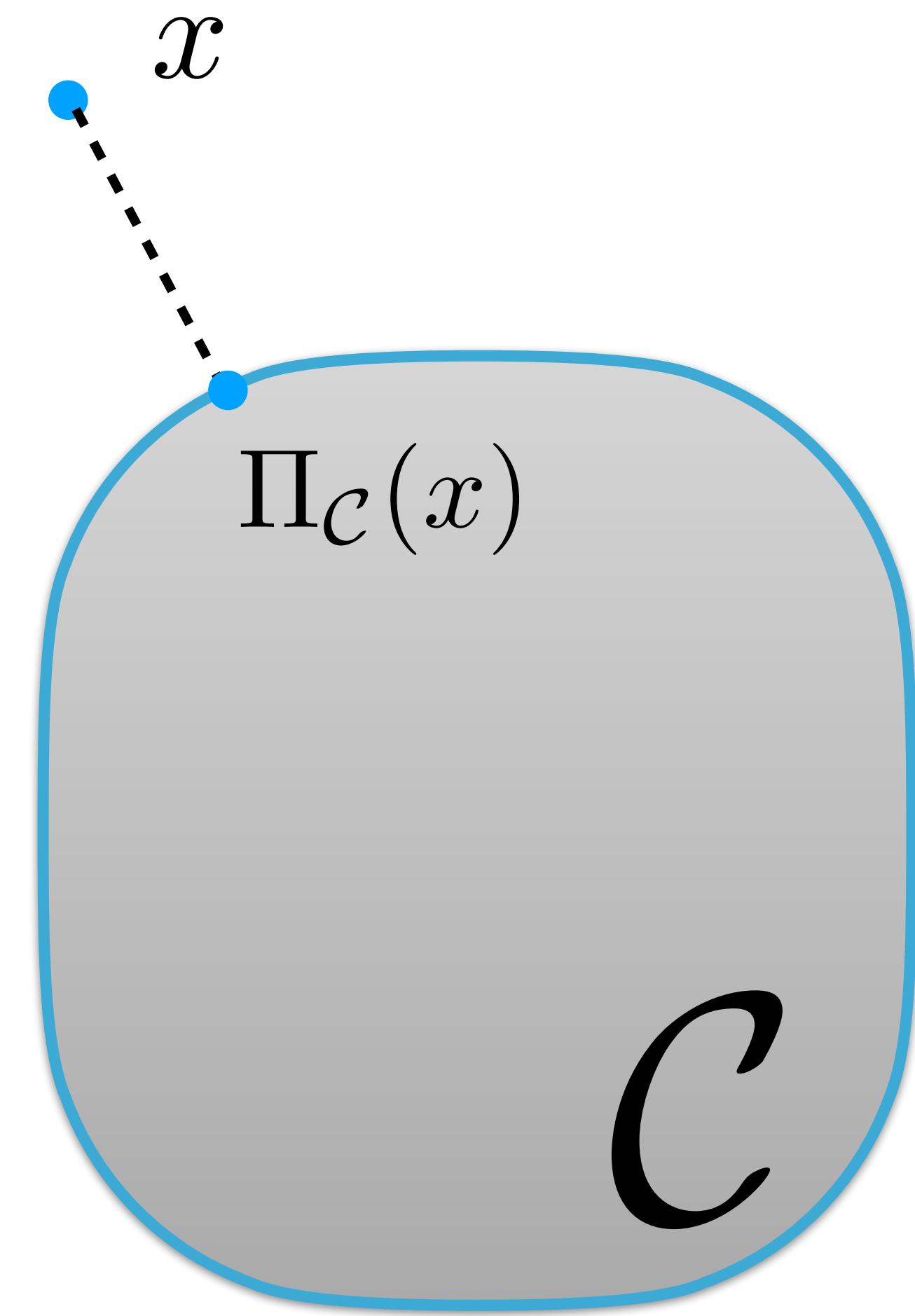
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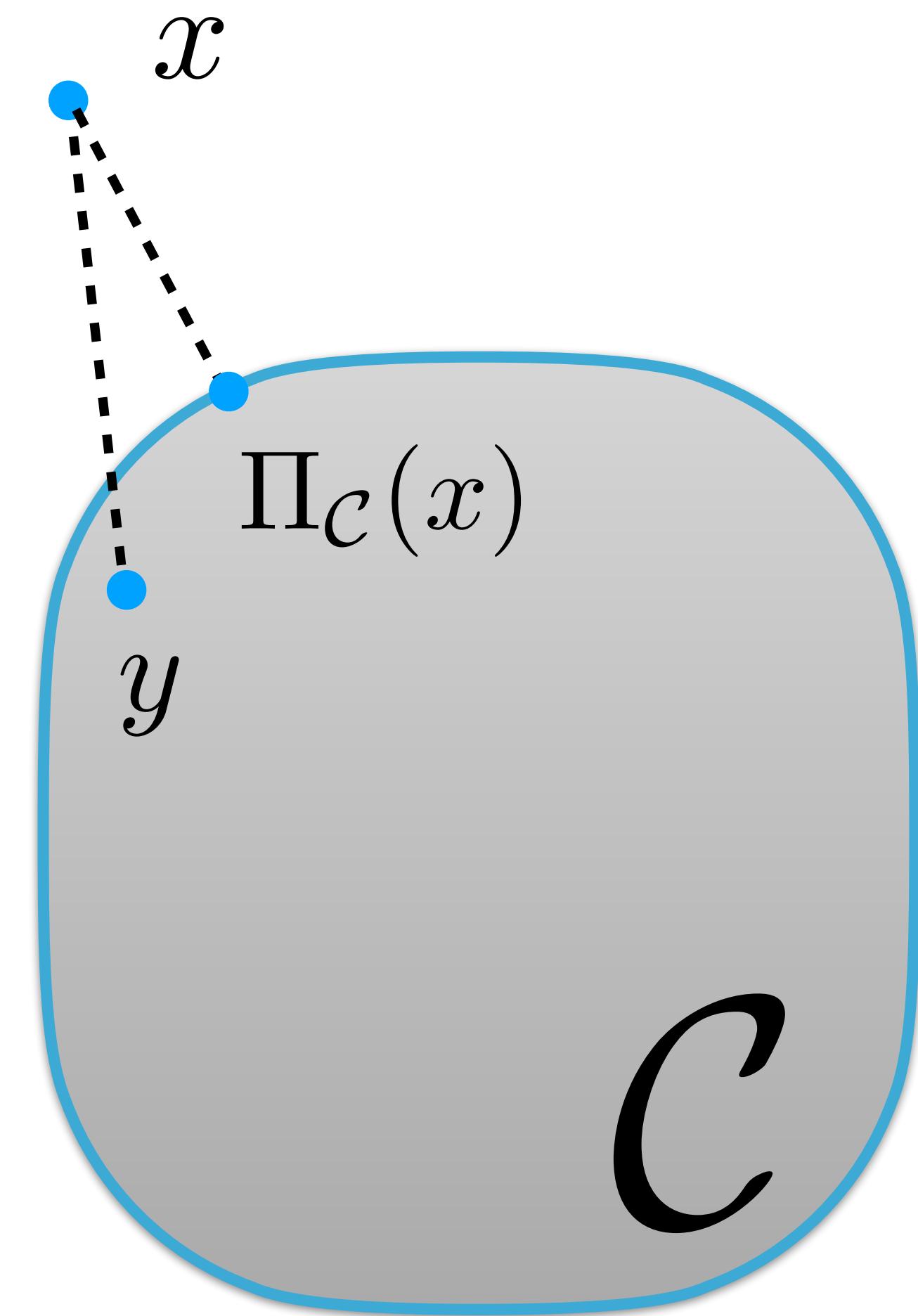
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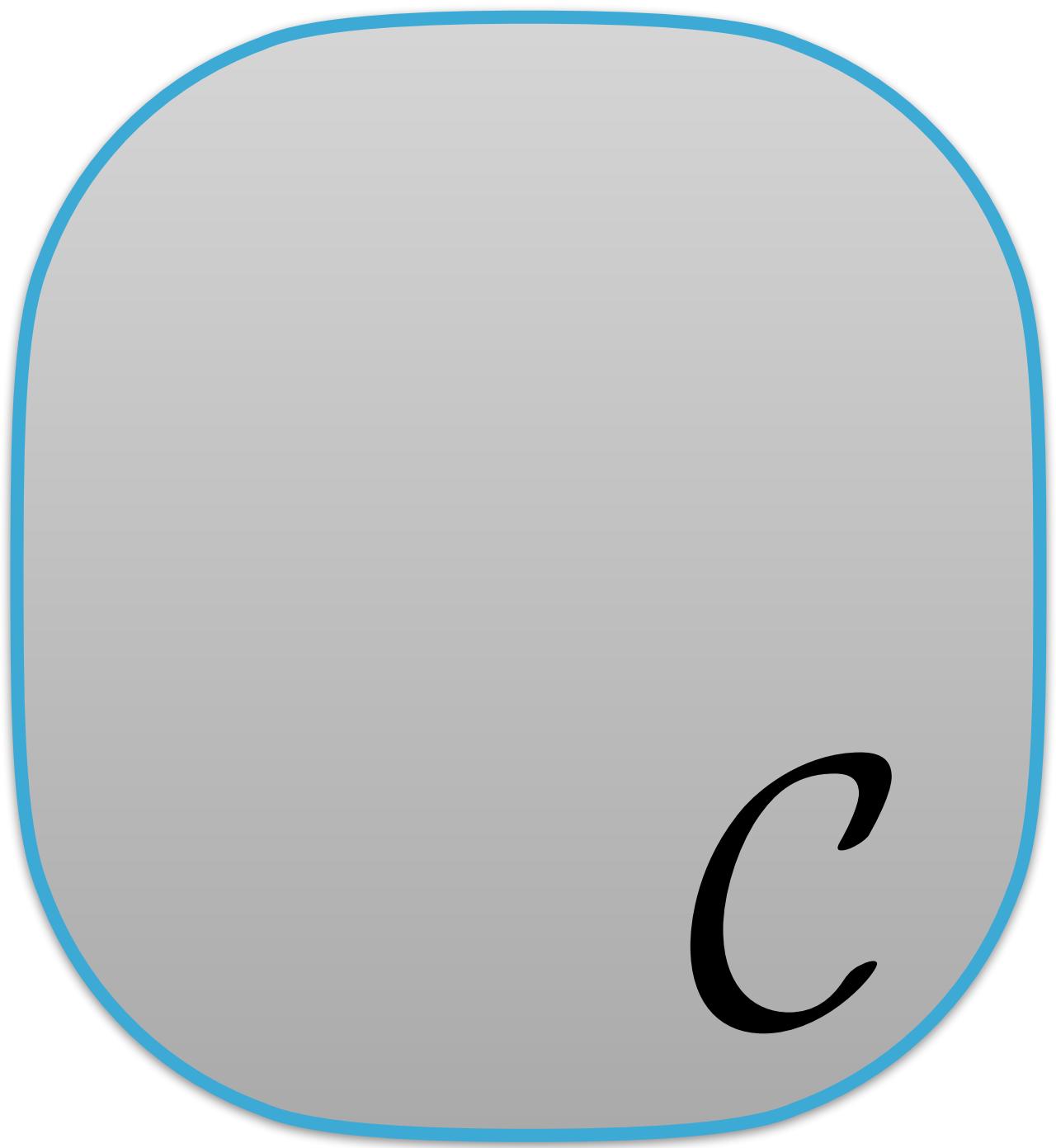
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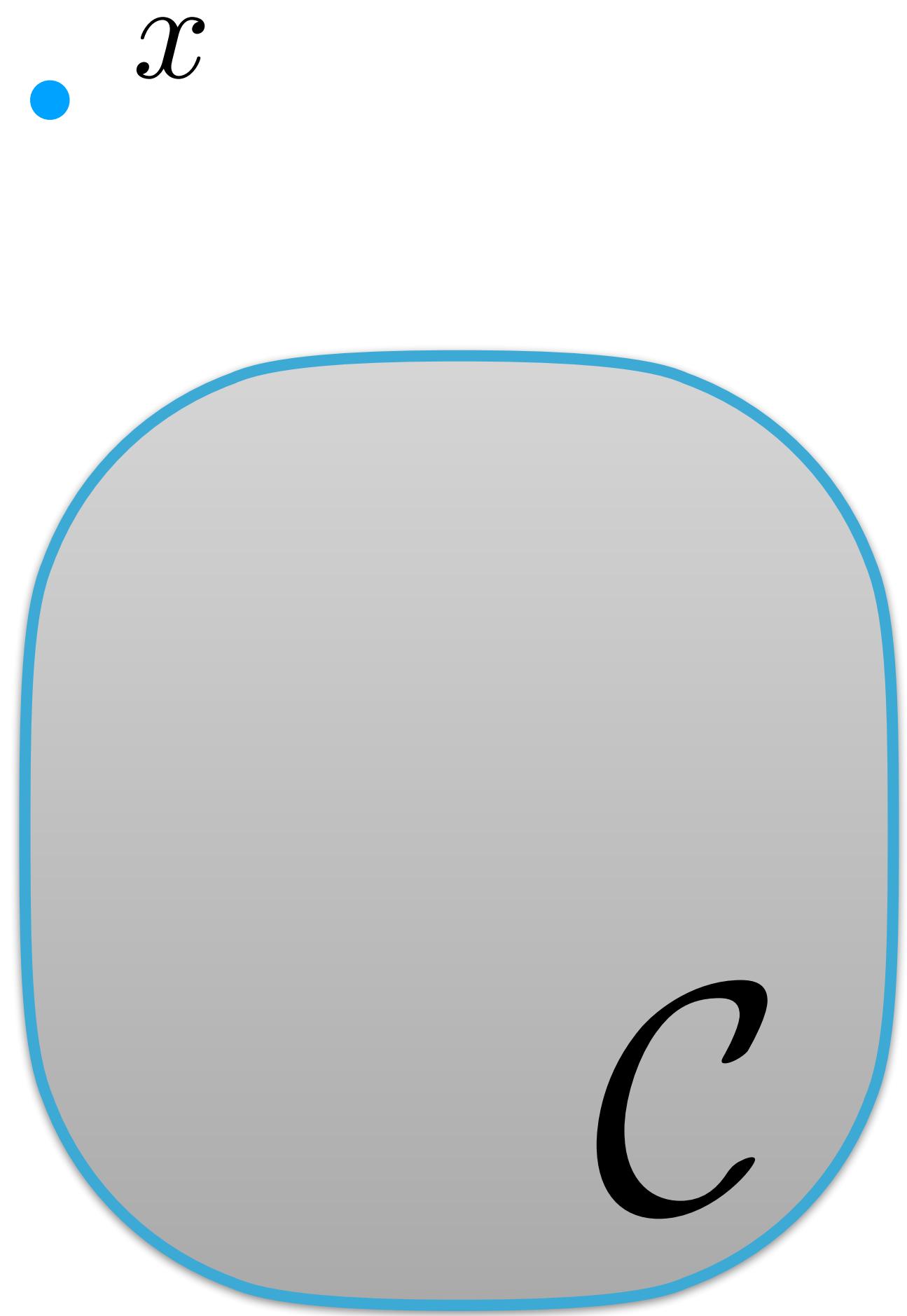
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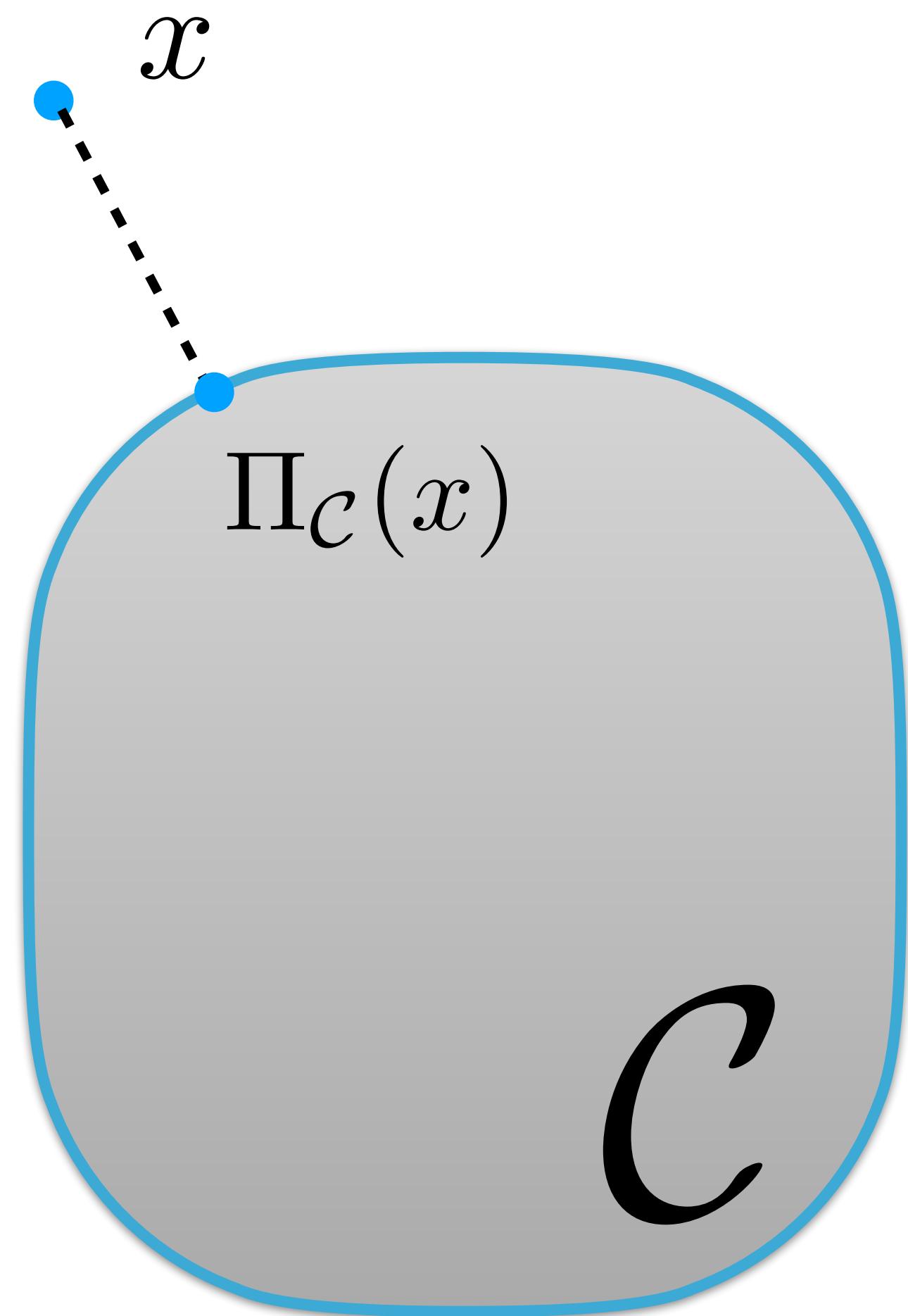
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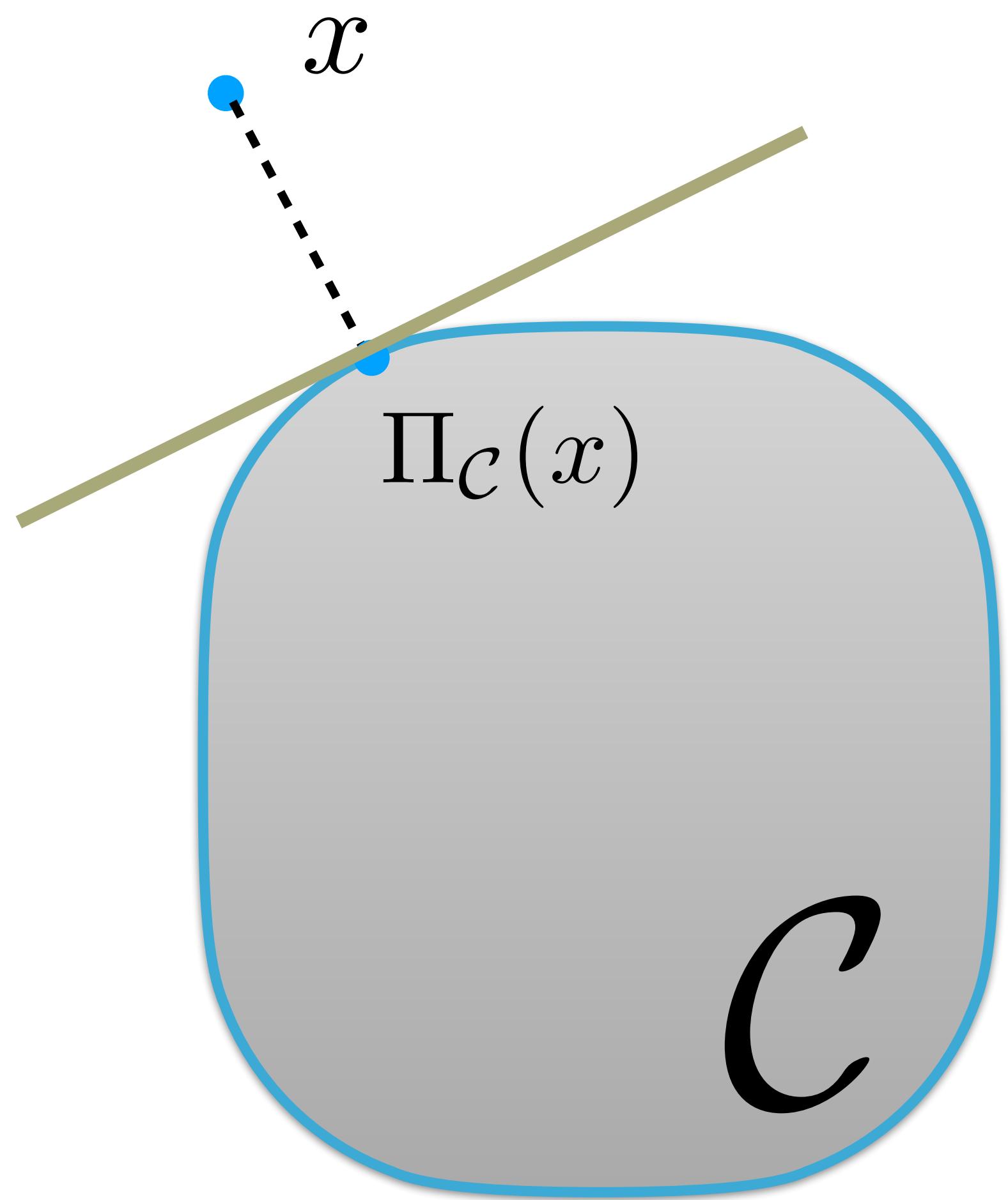
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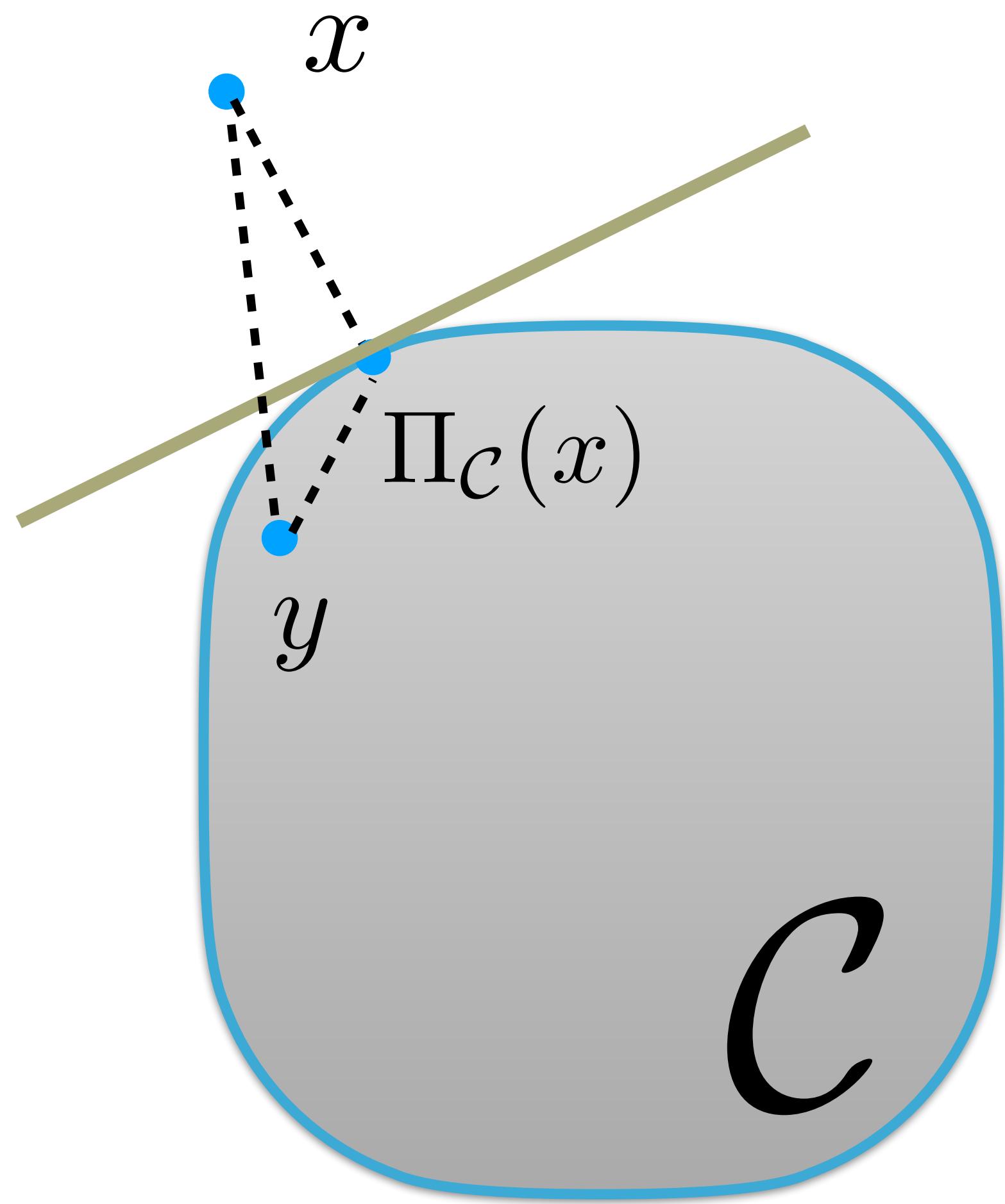
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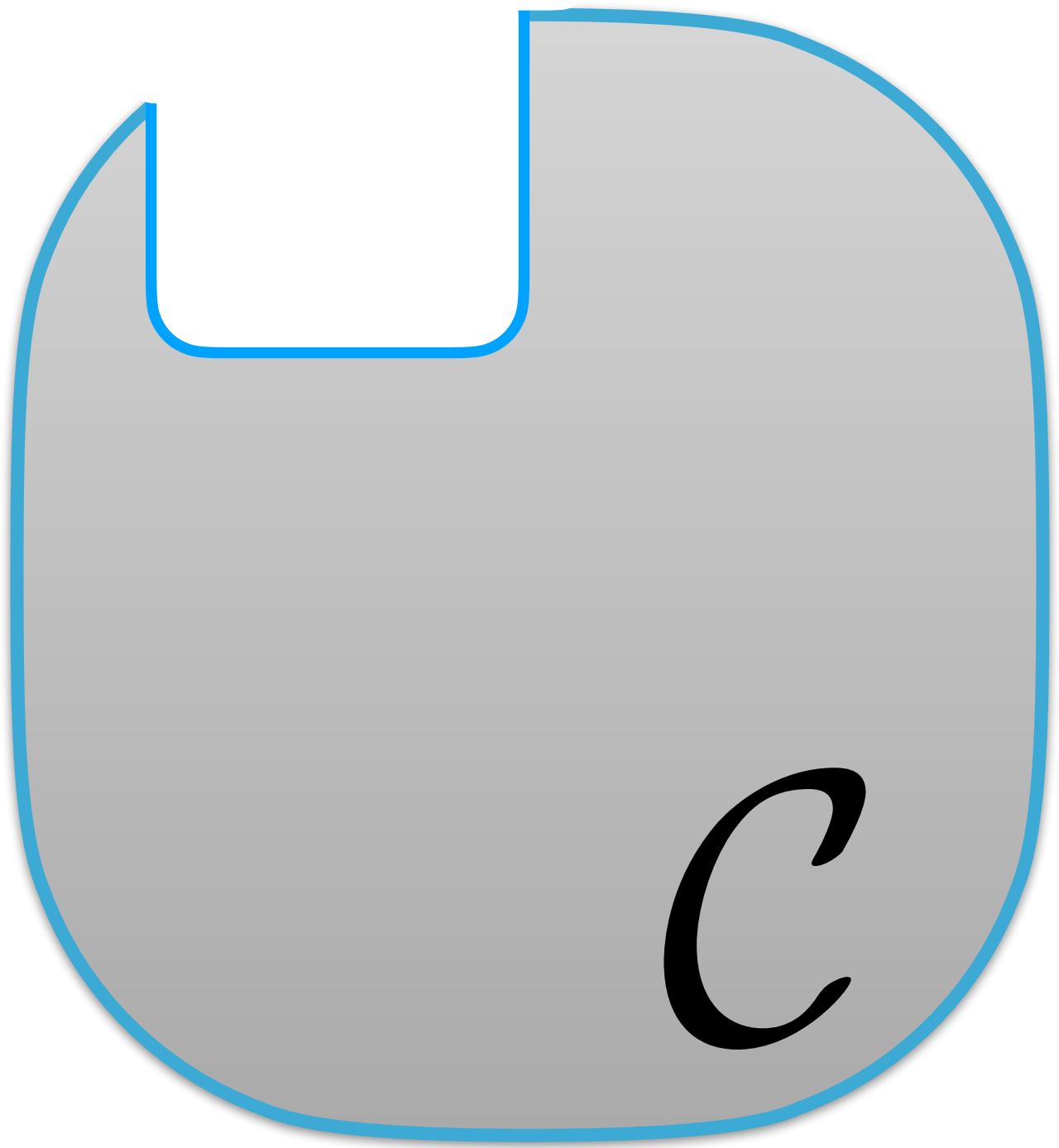
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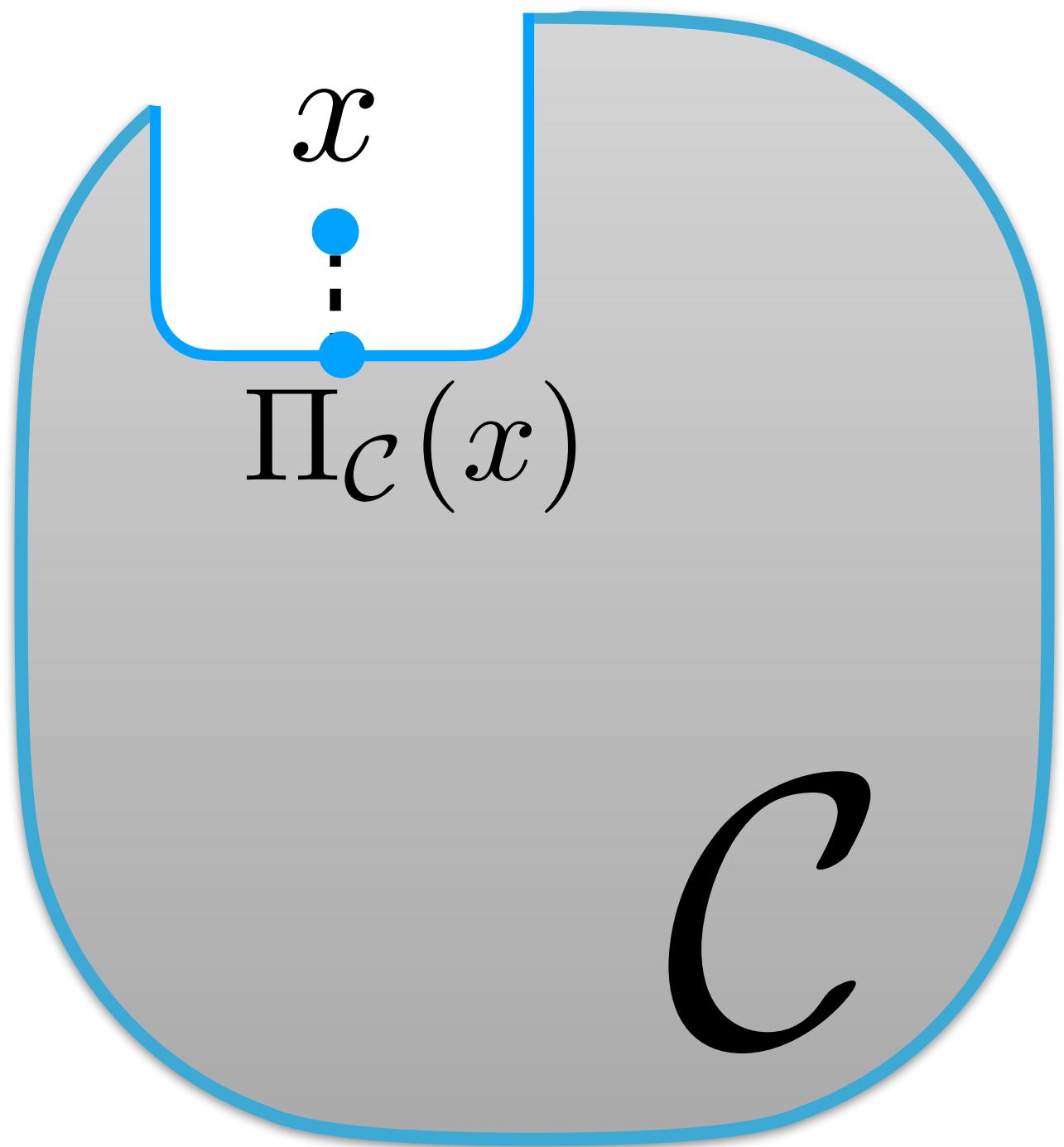
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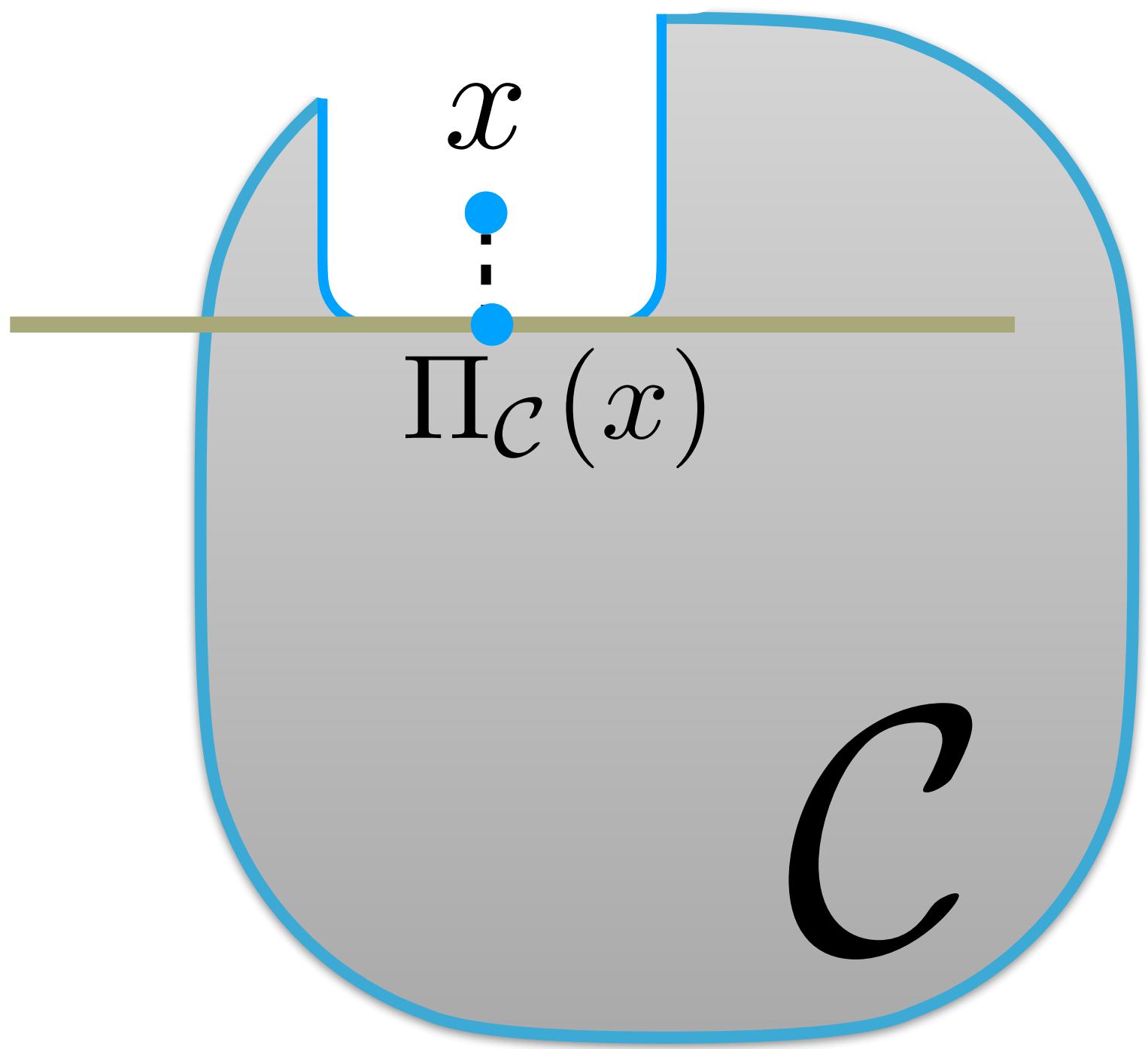
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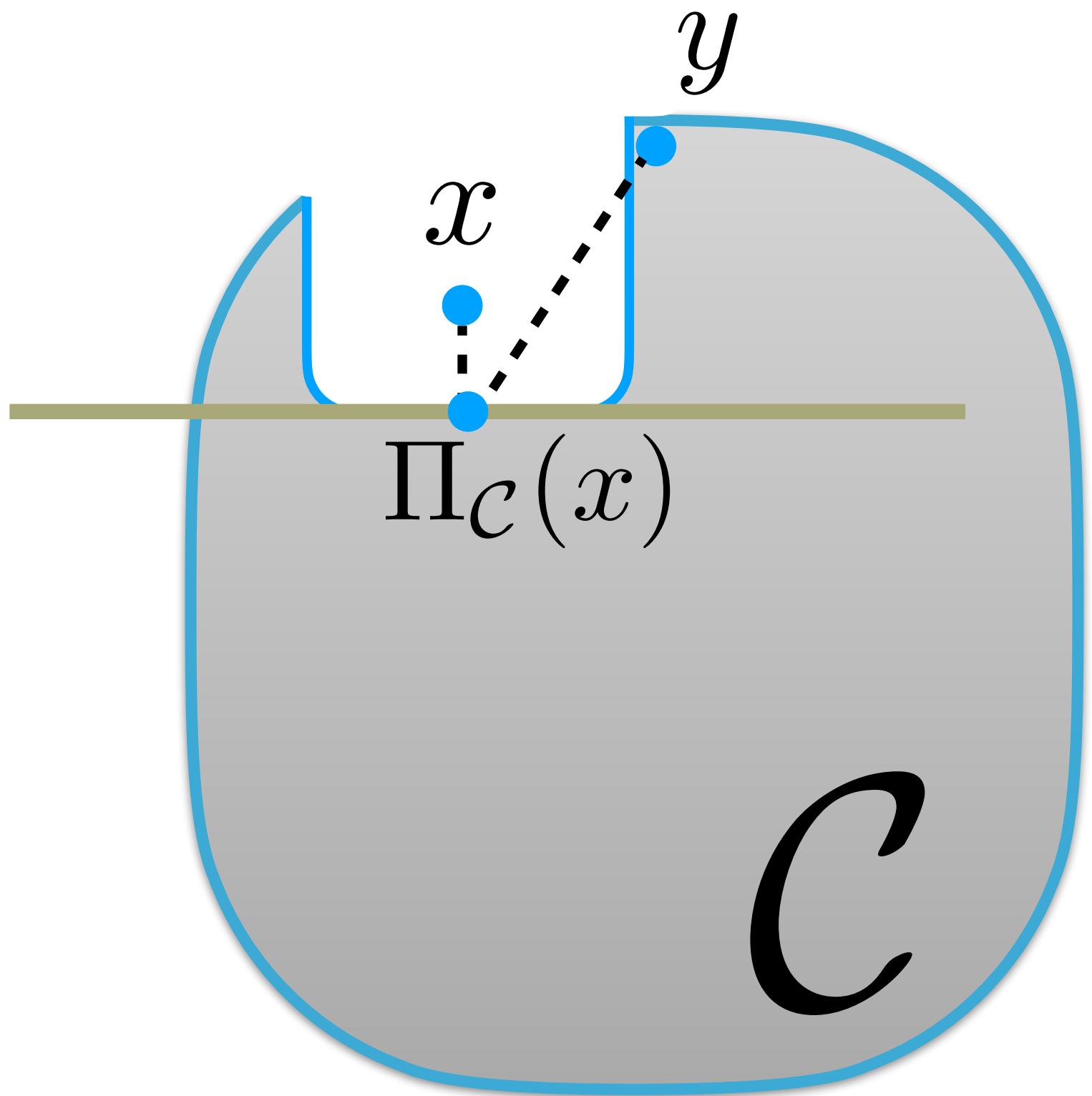
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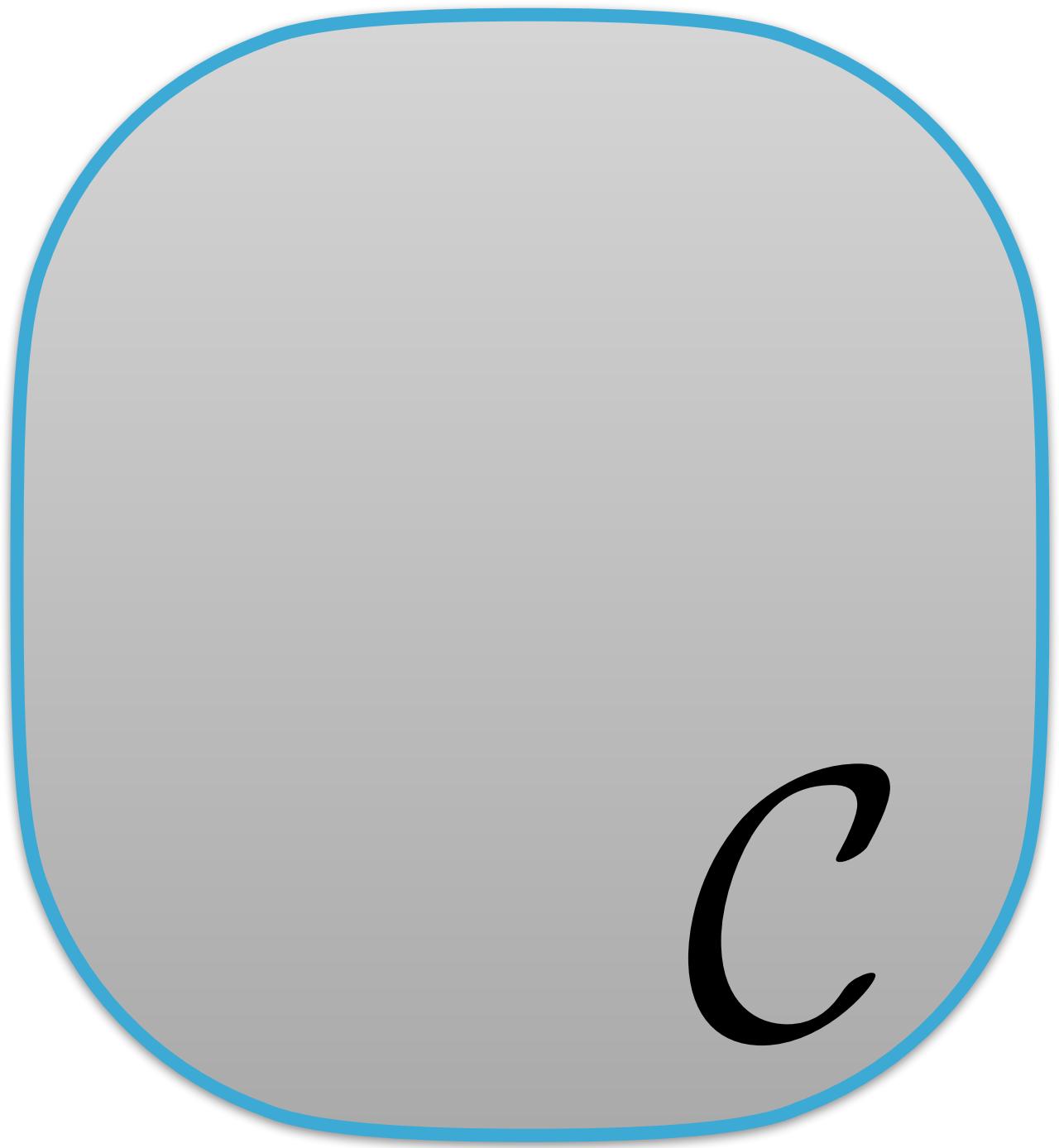
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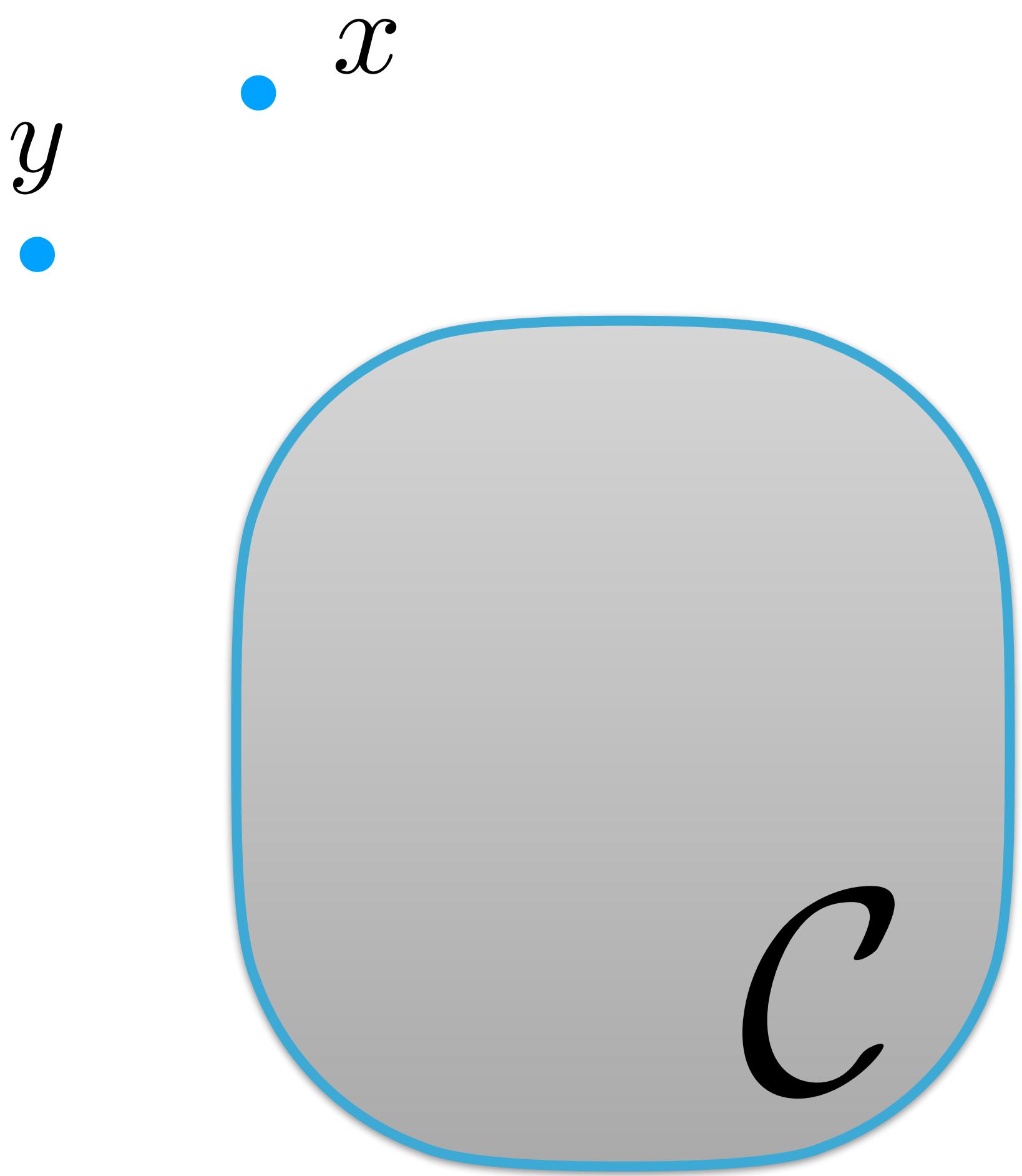
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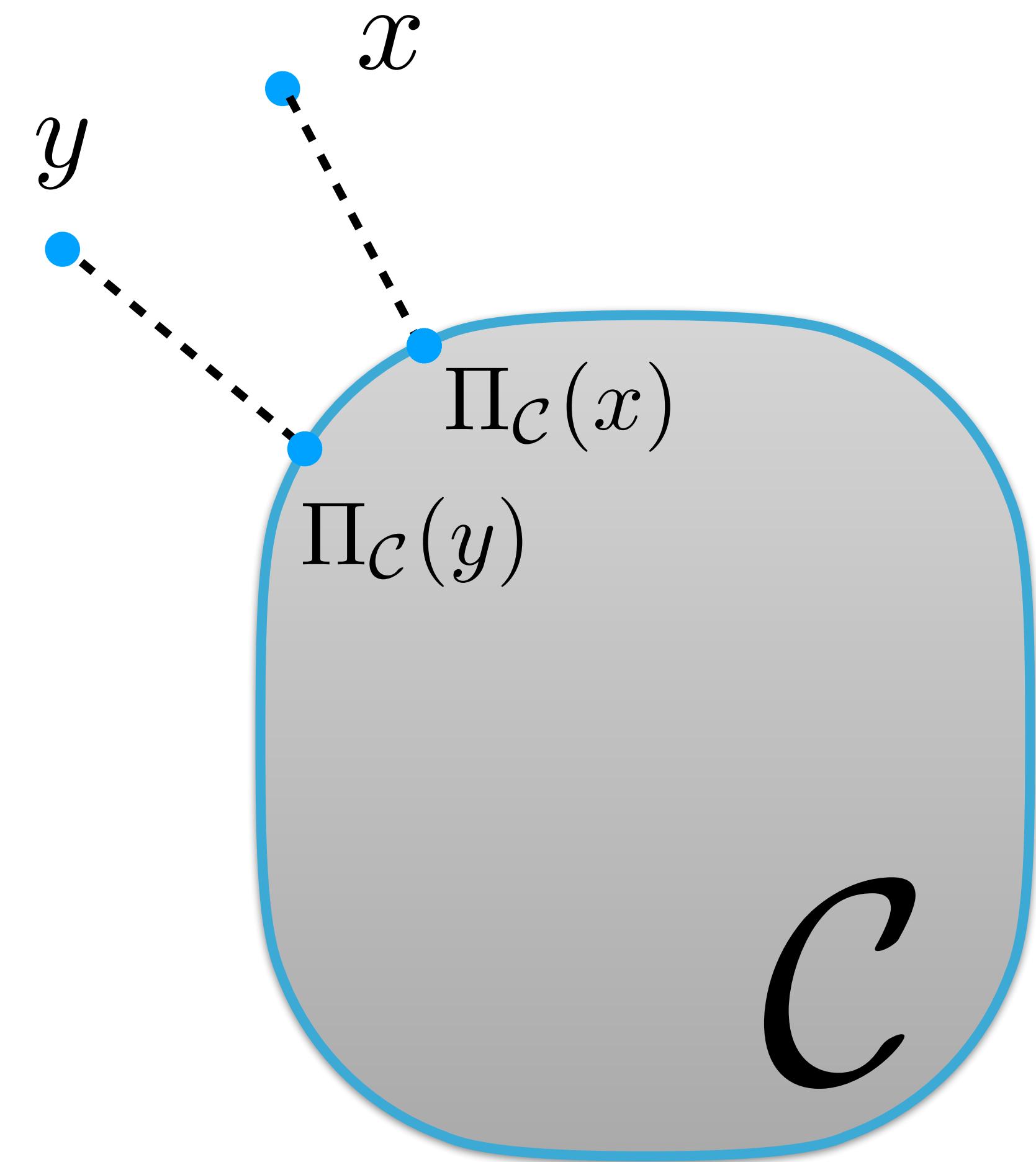
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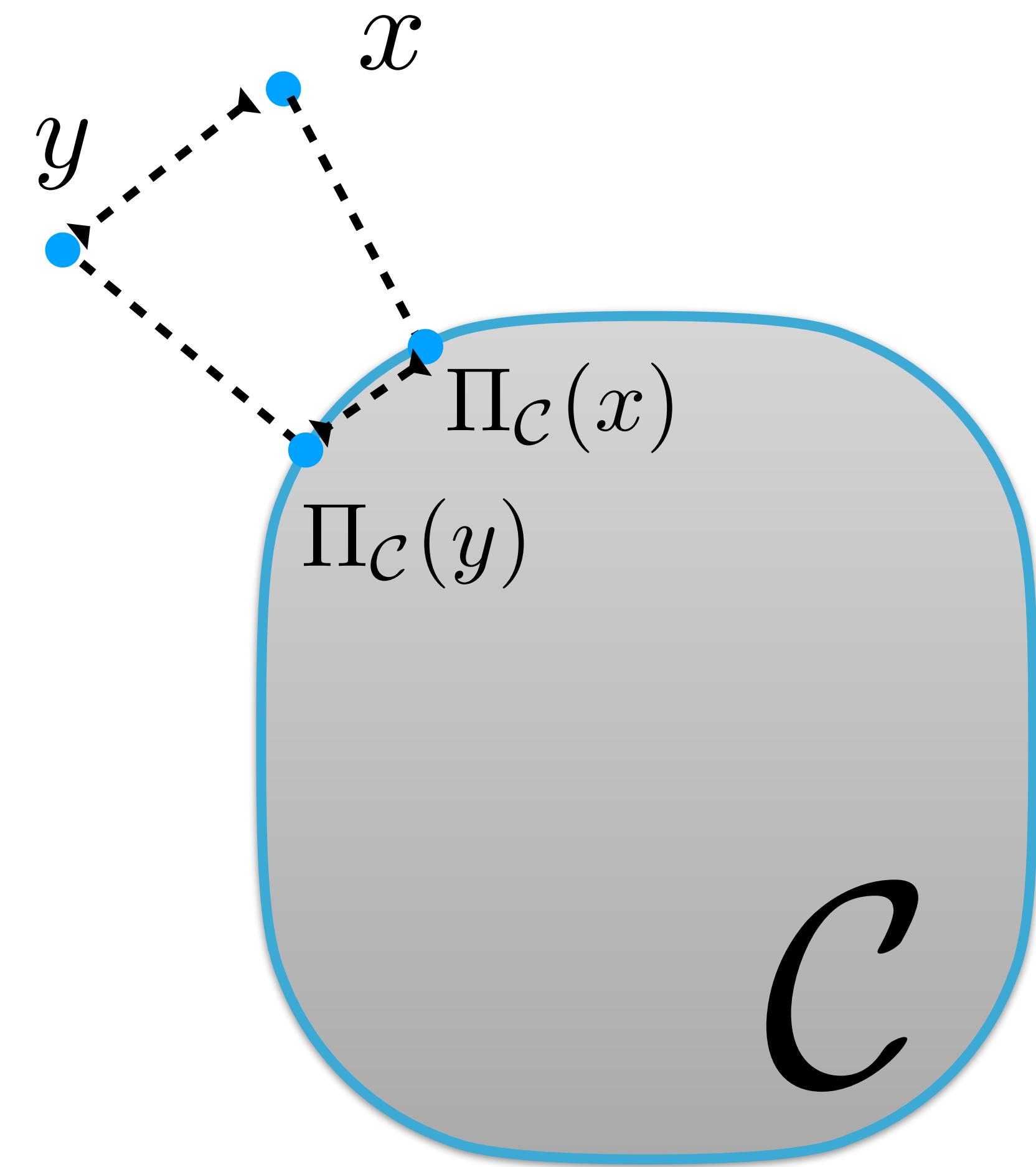
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Whiteboard

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- But we observed that, despite non-convexity, it works just fine..

(Thus, a different analysis is needed, depending on the problem at hand)

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Operations Research is an area where multiple, difficult constraints appear
- Prof. Richard Tapia is teaching a course on constrained convex opt.

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- Why should we still care about convex optimization?

Several practical problems are actually convex

Many practical problems can be approximated by convex ones

If one doesn't understand convex opt., why even try understanding non-convex opt.?

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Next lecture

- We will consider an important variant for convex optimization
for large-scale computing: Frank–Wolfe (conditional gradient) algorithm