COMP 414/514: Optimization – Algorithms, Complexity and Approximations

Lecture 1

Overview

$$\begin{aligned} \min_{x} \quad & f(x) \\ \text{s.t.} \quad & x \in \mathcal{C} \end{aligned}$$

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- Different objective classes
- Different strategies within each problem
- Different approaches based on computational capabilities
- Different approaches based on constraints

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$$\min_{x} \quad f(x)$$
$$\text{s.t.} \quad x \in \mathcal{C}$$

And, always having in mind applications in machine learning, AI and signal processing

Motivation (no fancy images included)

Provable efficiency

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Lots of data

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Harder problems

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More complicated models

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Provable efficiency

“What shall we do?”

Motivation (no fancy images included)

Provable efficiency

“What shall we do?”

Set up algo nicely

Use prior knowledge

Converge faster

Exploit resources

Topics

- Continuous **optimization** (in general)
  - See syllabus
  - Both theory and practice

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  - Both theory and practice
- Recent applications that drive research
- When no theory applies, some intuition

Examples

- Least squares / linear regression

(No, we will not re-define it)

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Examples

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$$\min_x \quad f(x) \qquad \text{s.t.} \quad x \in \mathcal{C} \quad \Rightarrow \quad \min_x \frac{1}{n} \sum_{i=1}^{n} \left(y_i - a_i^\top x \right)^2$$

Examples

- Quantum state tomography from limited samples

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$$\begin{aligned} \min_X \quad & f(X) \\ \text{s.t.} \quad & X \in \mathcal{C} \end{aligned} \quad \Rightarrow \quad \begin{aligned} \min_X \quad & \sum_{i=1}^{n} \big(y_i - \mathrm{Tr}(A_i^\top X)\big)^2 \\ \text{s.t.} \quad & \mathrm{Tr}(X) \leq 1 \\ & X \succeq 0 \end{aligned}$$

Examples

- Fleet management

$$\begin{aligned} \min_{X} \quad & f(X) \\ \text{s.t.} \quad & X \in \mathcal{C} \end{aligned}$$

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$$
\begin{aligned}
\min_{X} \quad & f(X) \\
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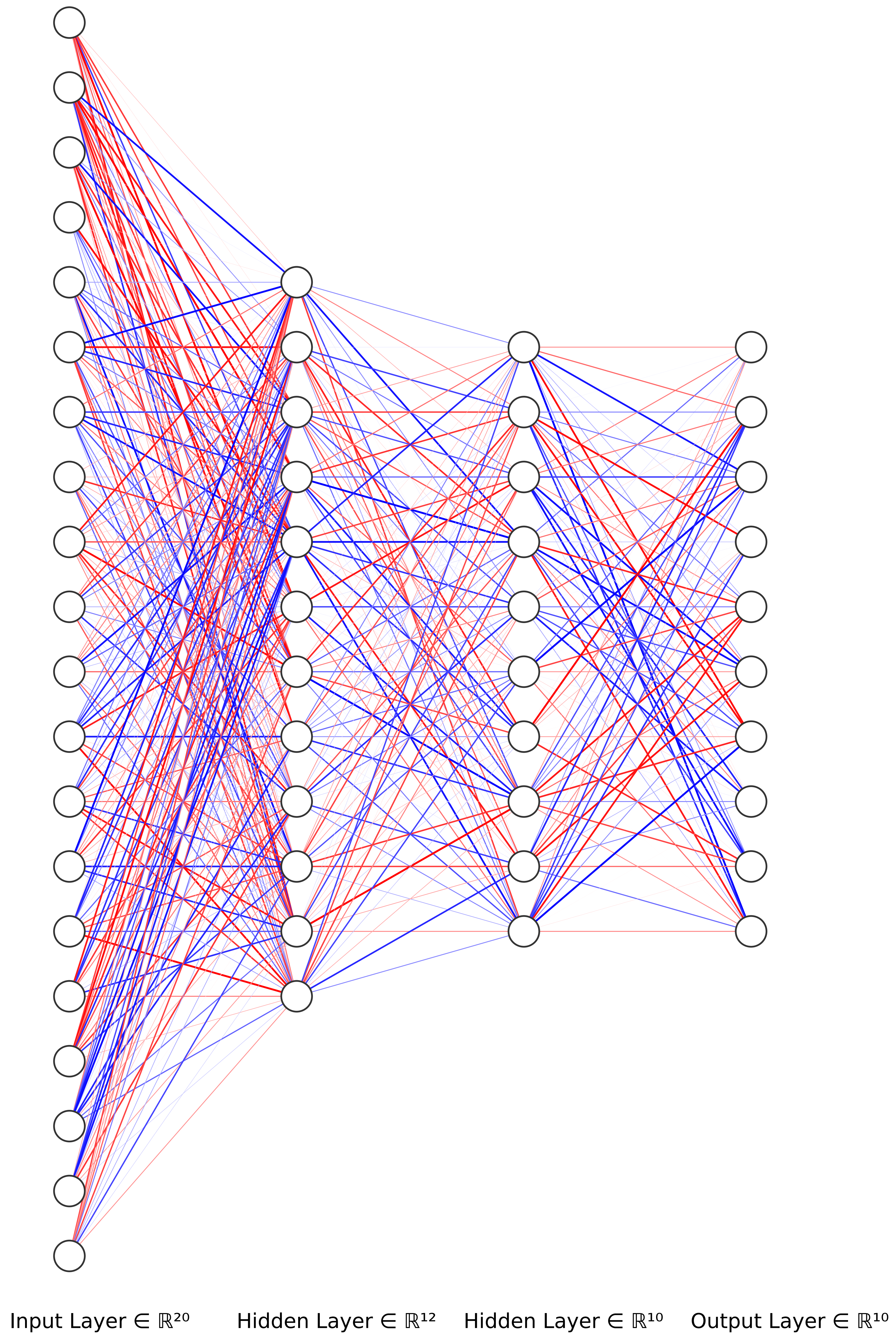
\end{aligned}
\quad \Rightarrow \quad
\begin{aligned}
\min_{x\in\{0,1\}^m, y\in\{0,1\}} \quad & f(y) = \sum_{i\in\mathcal{V}} \sum_{k=1}^{p} d_i (1-q) q^{k-1} y_{ik} \\
\text{s.t.} \quad & \sum_{j\in\mathcal{W}_i} x_j \geq \sum_{k=1}^{p} y_{ik}, i \in \mathcal{V} \\
& \sum_{j\in\mathcal{W}} x_j = p \\
& x_j \leq p_j
\end{aligned}
$$

Examples

- Neural networks



Examples

– Neural networks

$$\begin{aligned} \min_{X} \quad & f(X) \\ \text{s.t.} \quad & X \in \mathcal{C} \end{aligned} \quad \Rightarrow$$

$$\min_{W_i} \quad f(W_1, W_2) := \frac{1}{n} \sum_{i=1}^{n} \mathcal{L}\left(\widehat{y}_i, y_i\right).$$

where

$$\widehat{y}_i = \texttt{softmax}\left(\sigma\left(W_2 \cdot \sigma\left(W_1 \cdot x_i\right)\right)\right)$$

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- Bayesian algorithms

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- Just auditing is fine by me

What is the vision for this course?

- For starters, have in mind that this is a first–time taught course

(Any feedback is more than welcome)

- My purpose and vision is to introduce a series of optimization courses in the CS (and Duncan Hall's in general) curriculum

(Feedback on GoogleDoc)

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- The vision is for this course to be part of a sequence of courses that will focus on the theory+practice of methods

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Course format

- Lectures (slides) + whiteboard + in−class code running

(Some lectures have presentations, others will be handwritten – on purpose)

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- Your workload:

Graduate – HWs, final project or final exam
Undergraduate – HWs, final exam

Regarding assignments

(Just a comment)

Goals + outcomes

- Learn about research in related fields

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- Make connections between areas,
  understand how research advances in such areas..

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– Read and review recent papers

My goals

- Not to judge you whether you can solve HWs or not

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- Spark your interest in research
  where math and practice are combined together

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- A quiz will be given today for self-assessment

Grading policy

- 45% HWs
- 50% project/final exam
- 5%: participation, attendance
- 10%: scribing notes (**bonus**)

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Usually there is scaling in final grades. For me, a good grade is given based on the overall performance of the students: I value self-motivation, being proactive and enthusiasm.

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- Deliverable in LaTEX

  (template available online)

HWs

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Reviews (when applicable)

- Select papers from a pile of .pdfs that will be provided

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- Single page reviews, similar to NIPS/ICML standards:

(but not random as it usually is now)

  - Comment on novelty, clarity, importance
  - Main comments + your overall score

Presentations (for final projects)

(not certain yet)

- **When**: at the end of the course

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- Grading: slides quality, clarity of main ideas

Final Project

(see website)

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(see website)

Please come find me the earliest to discuss projects

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(see website)

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You should start reading papers soon, so that around mid-way you have a good project proposal

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- **HWs**: will be sent to you via email every week. (please do not distribute)

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- **Every week I will try to upload a chapter of the notes;** however I would appreciate any help with scribing throughout the semester

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- **In case I don't have the time to cover fully a session, I will decide whether you will read it yourself, or I will teach it the next time.**

Any questions?

Quiz
(15 min.)

Setting up the background

Vectors

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(Associative)

$$0 + x = x, \quad x \in \mathbb{R}^p$$

$$\alpha(x + y) = \alpha x + \alpha y, \quad x, y \in \mathbb{R}^p$$
(Distributive)

Vectors

- Span of a set of vectors:

$$\operatorname{span}\{x_1, x_2, \ldots, x_k\} = \{\alpha_1 x_1 + \alpha_2 x_2 + \cdots + \alpha_k x_k \mid \alpha_i \in \mathbb{R}, i = [1, k]\}$$

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- Inner product:

$$x^\top y = \langle x, y \rangle = \sum_{i=1}^{p} x_i \cdot y_i$$

Vectors

- Inner production interpretation:

$$\langle x, y \rangle = \|x\| \cdot \|y\| \cdot \cos\theta$$

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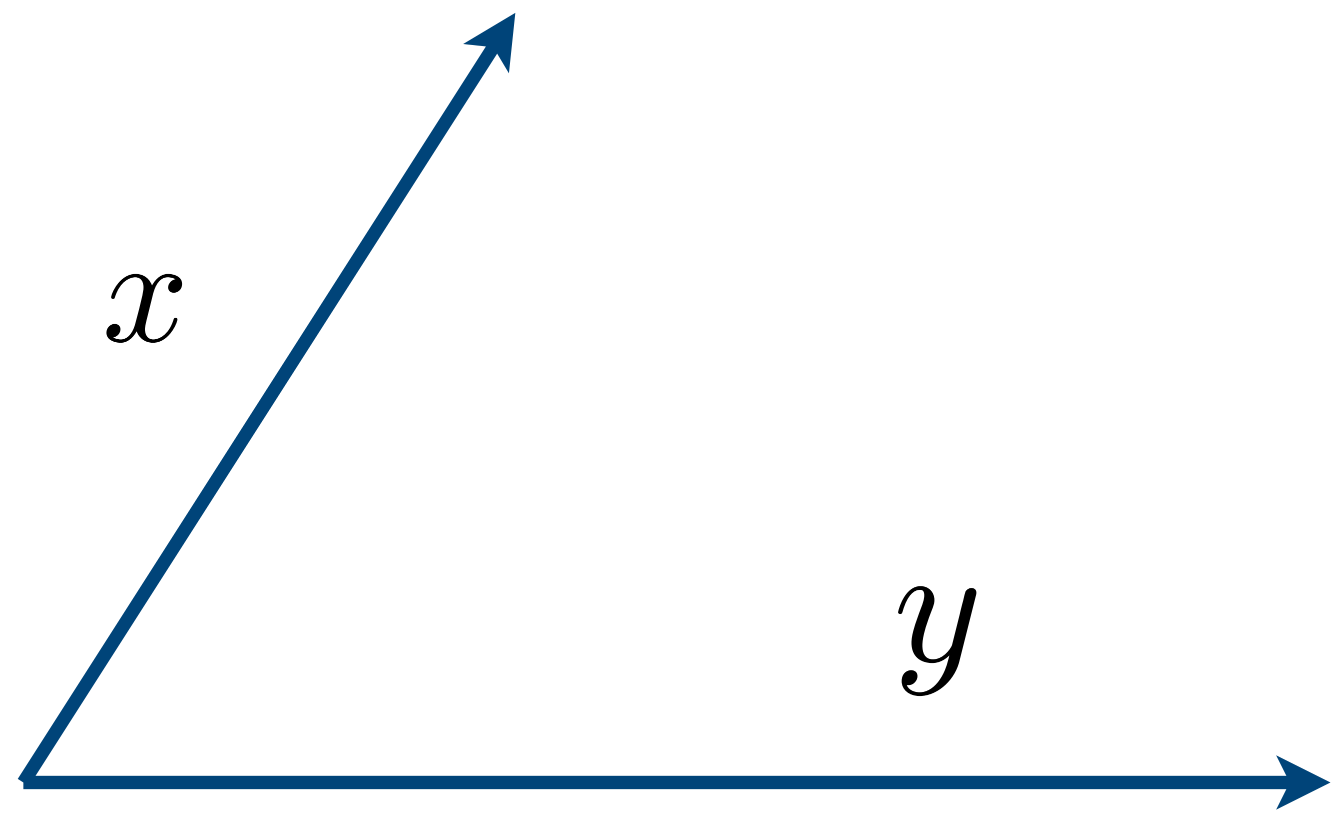

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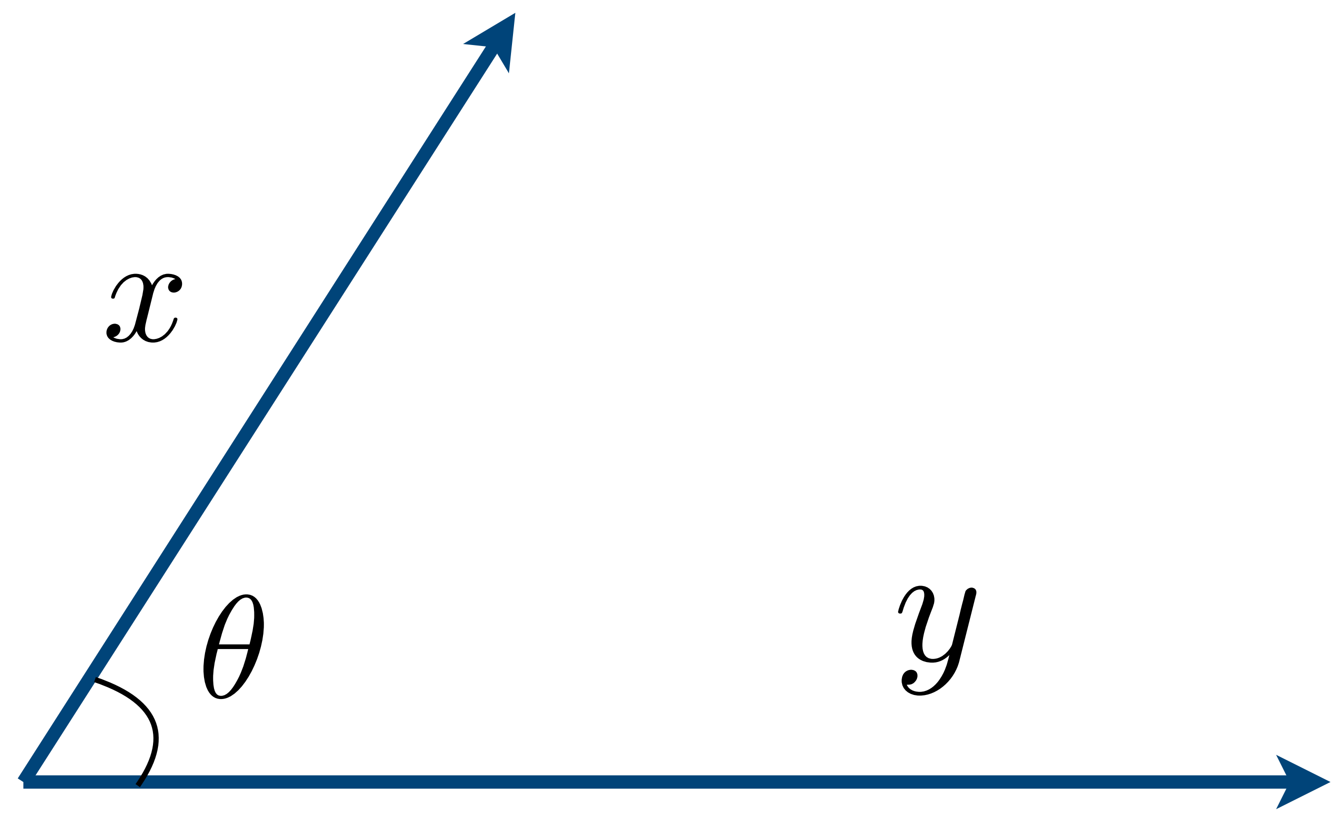

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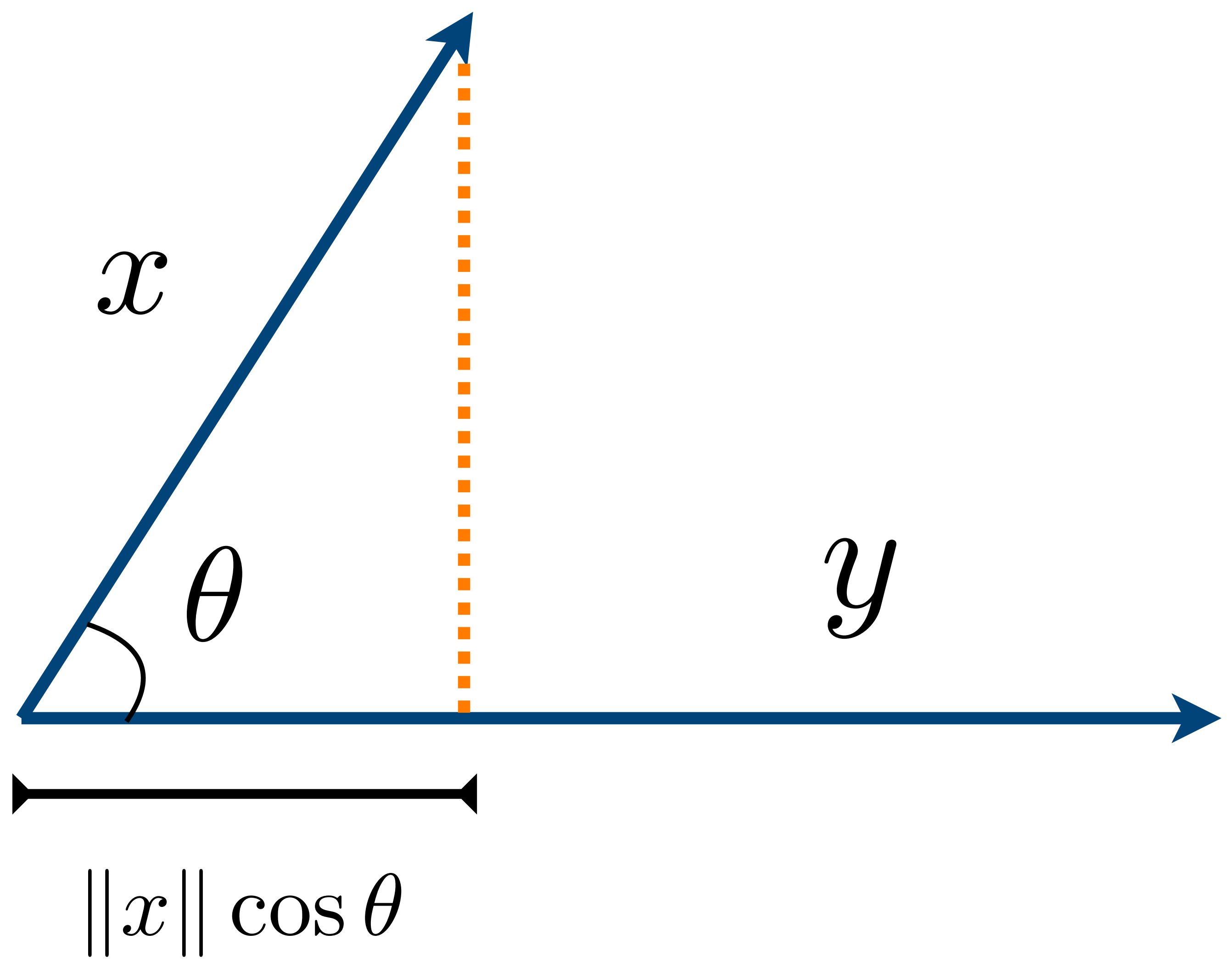

$$\langle x, y \rangle = \|x\| \cdot \|y\| \cdot \cos\theta$$

Vectors

– Norms = notion of distance in multiple dimensions

Properties:

$$\|x\| \geq 0, \forall x \in \mathbb{R}^p$$

$$\|x\| = 0, \text{ iff } x = 0$$

$$\|\alpha x\| = |\alpha| \|x\|, \forall \alpha \in \mathbb{R}$$

$$\|x + y\| \leq \|x\| + \|y\|$$ (Triangle inequality)

$$|x^\top y| \leq \|x\| \|y\|$$ (Cauchy–Schwarz)

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- Famous wanna−be norms: $\|x\|_0 = \mathrm{card}(x)$

Matrices

- Matrix in m, n dimensions: $A \in \mathbb{R}^{m \times n}$

$$A = \begin{bmatrix} A_{11} & A_{12} & \cdots & A_{1n} \\ A_{21} & A_{22} & \cdots & A_{2n} \\ \vdots & & \ddots & \vdots \\ A_{m1} & A_{m2} & \cdots & A_{mn} \end{bmatrix}$$

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- Names: Square, tall, fat, zero, identity, diagonal
- Properties:

$$A + B = B + A,\ \forall A, B \in \mathbb{R}^{m\times n}$$
$$(A + B) + C = A + (B + C),\ \forall A, B, C \in \mathbb{R}^{m\times n}$$
$$A + 0 = 0 + A,\ \forall A \in \mathbb{R}^{m\times n}$$
$$(A + B)^\top = A^\top + B^\top,\ \forall A, B \in \mathbb{R}^{m\times n}$$

Matrices

- Matrix multiplication： $C = AB$ where $C \in \mathbb{R}^{m\times p}$, $A \in \mathbb{R}^{m\times n}$, and $B \in \mathbb{R}^{n\times p}$

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- Special cases: vector inner product, matrix–vector mult., outer product

Matrices

– Matrix multiplication： $C = AB$ where $C \in \mathbb{R}^{m \times p}$, $A \in \mathbb{R}^{m \times n}$, and $B \in \mathbb{R}^{n \times p}$

$$AB = \begin{bmatrix} A_{11} & A_{12} & \cdots & A_{1n} \\ A_{21} & A_{22} & \cdots & A_{2n} \\ \vdots & & \ddots & \vdots \\ A_{m1} & A_{m2} & \cdots & A_{mn} \end{bmatrix} \cdot \begin{bmatrix} B_{11} & B_{12} & \cdots & B_{1p} \\ B_{21} & B_{22} & \cdots & B_{2p} \\ \vdots & & \ddots & \vdots \\ B_{n1} & B_{n2} & \cdots & B_{np} \end{bmatrix}$$

– Special cases： vector inner product, matrix–vector mult., outer product

– Properties：

$$(AB)C = A(BC),\ \forall A, B, C$$

$$\alpha(AB) = (\alpha A)B,\ \forall A, B$$

$$A(B + C) = AB + AC,\ \forall A, B, C$$

$$(AB)^\top = B^\top A^\top,\ \forall, A, B$$

$$AB \neq BA$$

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– Positive semi–definite matrices: $A \succeq 0$

1. $A \in \mathbb{R}^{n\times n}$
2. A is symmetric
3. $x^\top Ax \geq 0,\ \forall x \in \mathbb{R}^n,\ x \neq 0$

Matrices

- Matrix singular value decomposition: $A \in \mathbb{R}^{m \times n}$

$$A = U\Sigma V^\top = \sum_{i=1}^{r} \sigma_i u_i v_i^\top, \; U \in \mathbb{R}^{m\times r}, \Sigma \in \mathbb{R}^{r\times r}, V \in \mathbb{R}^{n\times r} \qquad r \leq \{m, n\}$$

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- Left and right singular vectors are orthogonal: $U^\top U = I$ and $V^\top V = I$

Matrices

- Norms:

$$\|A\|_F = \sqrt{\sum_{ij} A_{ij}^2}$$

(Frobenius norm)

$$\|A\|_* = \sum_{i}^{r} \sigma_i$$

(Nuclear norm)

$$\|A\|_2 = \max_i \sigma_i$$

(Spectral norm)

..there are more norms to worry about (e.g., operator norms)
but we will skip them here..

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  - There exists a square matrix, A^{-1}, such that $A^{-1}A = AA^{-1} = I$

Demo

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- We have set up background and notation w.r.t. linear algebra
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Next lecture

- Brief introduction to convex optimization and related topics

Intuition for inner product