

COMP 414/514:
Optimization – Algorithms, Complexity
and Approximations

Lecture 8

Overview

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 - Talked a little bit about **general smooth optimization** problems
 - This included both non-convex optimization, but also convex
 - The discussion was quite abstract (no particular application)

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 - Talked a little bit about **general smooth optimization** problems
 - This included both non-convex optimization, but also convex
 - The discussion was quite abstract (no particular application)
- We will consider (possibly) the simplest non-convex setting:
sparse model selection
 - We will provide motivation, background and alternative solutions
 - We will focus on how we can **provably and efficiently solve** such problems

Overview

$$\begin{array}{ll} \min_x & f(x) \\ \text{s.t.} & x \in \mathcal{C} \end{array}$$

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$$\min_x$$

s.t.

$$f(x)$$
$$x \in C$$

We will consider convex objectives..

..over non-convex constraints

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- We will focus on the cases of (structured) sparsity and low-rankness

Sparse linear regression

(Not again man!..)

- Generative model: $y_i = a_i^\top x^* + w_i$
 - $a_i \in \mathbb{R}^p$: features
 - $y_i \in \mathbb{R}$: responses
 - $w_i \in \mathbb{R}$: additive noise

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- Assuming data set $\{y_i, a_i\}_{i=1}^n$, $n < p$, find $x^* \in \mathbb{R}^p$

$$\begin{aligned}\min_{x \in \mathbb{R}^p} \quad & f(x) := \frac{1}{2} \|y - Ax\|_2^2 \\ \text{s.t.} \quad & \|x\|_0 \leq k\end{aligned}$$

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- Any suggestions how to solve this?

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- Solution #1: convexification + proj. gradient descent

LASSO

$$\begin{array}{ll} \min_{x \in \mathbb{R}^p} & f(x) := \frac{1}{2} \|y - Ax\|_2^2 \\ \text{s.t.} & \|x\|_1 \leq \lambda \end{array}$$



$$x_{t+1} = \Pi_{\|\cdot\|_1 \leq \lambda} (x_t - \eta \nabla f(x_t))$$

(Pros & Cons?)

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(Pros & Cons?)

Basis pursuit
(denoising)

- Solution #2: convexification + **proximal** gradient descent

$$\min_{x \in \mathbb{R}^p} f(x) := \frac{1}{2} \|y - Ax\|_2^2 + \rho \|x\|_1$$



$$x_{t+1} = \text{Prox}_{\rho \|\cdot\|_1} (x_t - \eta \nabla f(x_t))$$

(Pros & Cons?)

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(Pros & Cons?)

Hard-thresholding

- Solution #3: keep non-convexity + non-convex projected gradient descent

$$\begin{array}{ll} \min_{x \in \mathbb{R}^p} & f(x) := \frac{1}{2} \|y - Ax\|_2^2 \\ \text{s.t.} & \|x\|_0 \leq k \end{array} \longrightarrow x_{t+1} = H_k (x_t - \eta \nabla f(x_t))$$

(Pros & Cons?)

But before we proceed..

- Some questions:

$$\begin{aligned} \min_{x \in \mathbb{R}^p} \quad & f(x) := \frac{1}{2} \|y - Ax\|_2^2 \\ \text{s.t.} \quad & \|x\|_0 \leq k \end{aligned}$$

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- A: "Sparsity makes problems exponentially hard to solve"

(This assumes the most general case)

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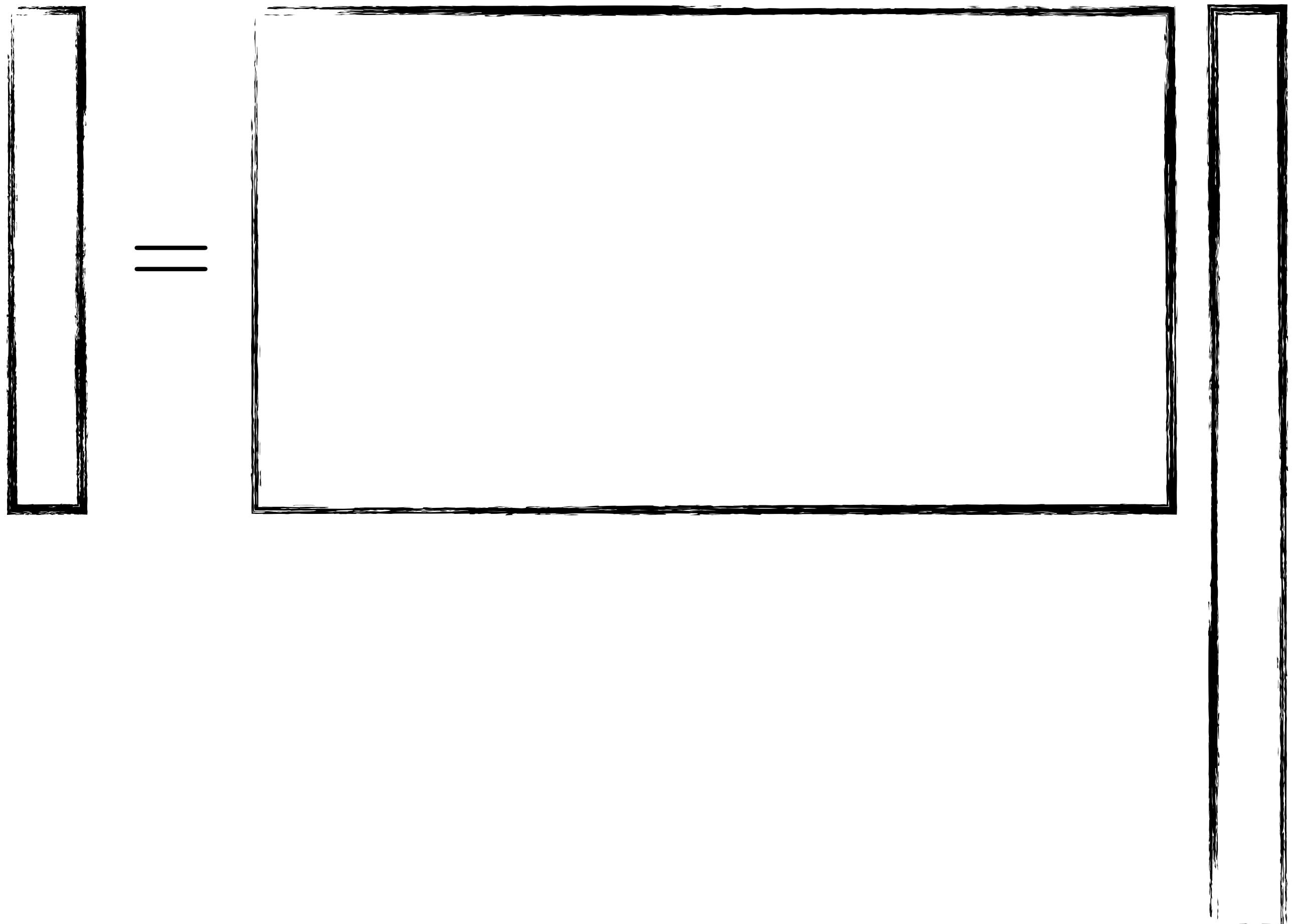
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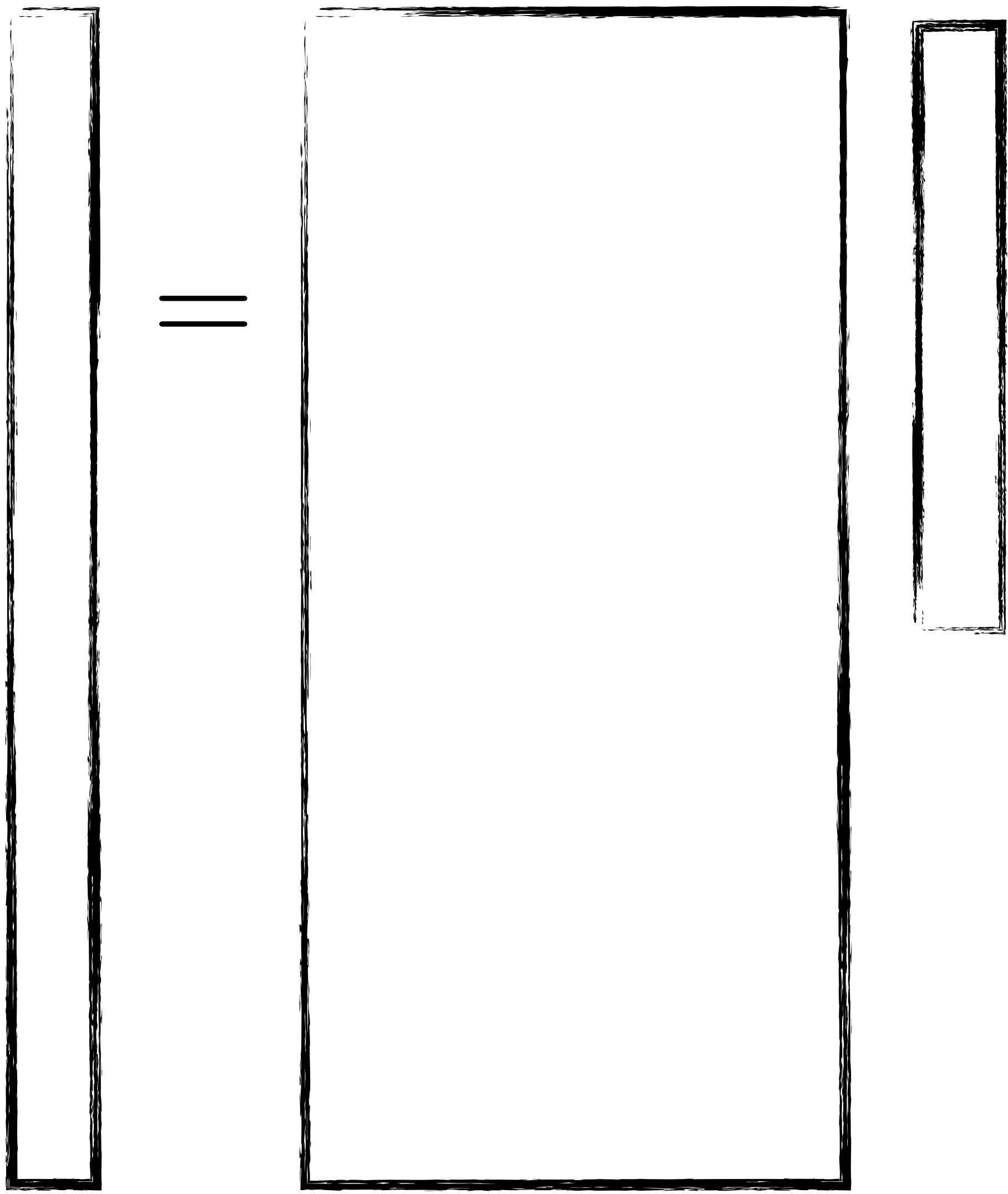
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 - Q: "But isn't the problem underdetermined? ($n \ll p$)"
 - A: "Yes, without any constraints, the problem has infinite solutions"
 - Q: "Why then do we have hopes solving this problem?"
 - A: "Under assumptions on A , and the relation between (n, p, k) , we Will see that on average this problem can be solved in polynomial time"

Over- vs. under-parameterized



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Under-parameterized

Iterative hard thresholding (IHT)

(It is just projected gradient descent on sparsity constraints)

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- Now, imagine yourself implementing this.. What are the hyper-parameters?
 - "How do we set the step size?"
 - "How do we select the initial point? (it is non-convex after all)"
 - "What if we don't know the sparsity level?"
 - "Are there any other tricks we can use?"

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- Property of I : isometry

$$(1 - \delta) \|x_1 - x_2\|_2^2 \leq \|I(x_1 - x_2)\|_2^2 \leq (1 + \delta) \|x_1 - x_2\|_2^2, \quad \text{for some } \delta \in [0, 1], \forall x_1, x_2 \in \mathbb{R}^p$$

(Interpretation?)

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- Other properties in literature: Nullspace property,
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- How can we use this property in proving convergence of IHT?

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- We get linear convergence to the global optimum!

How does it perform in practice?

Demo

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(In our case, we generate it as Gaussian so with high probability we are fine)

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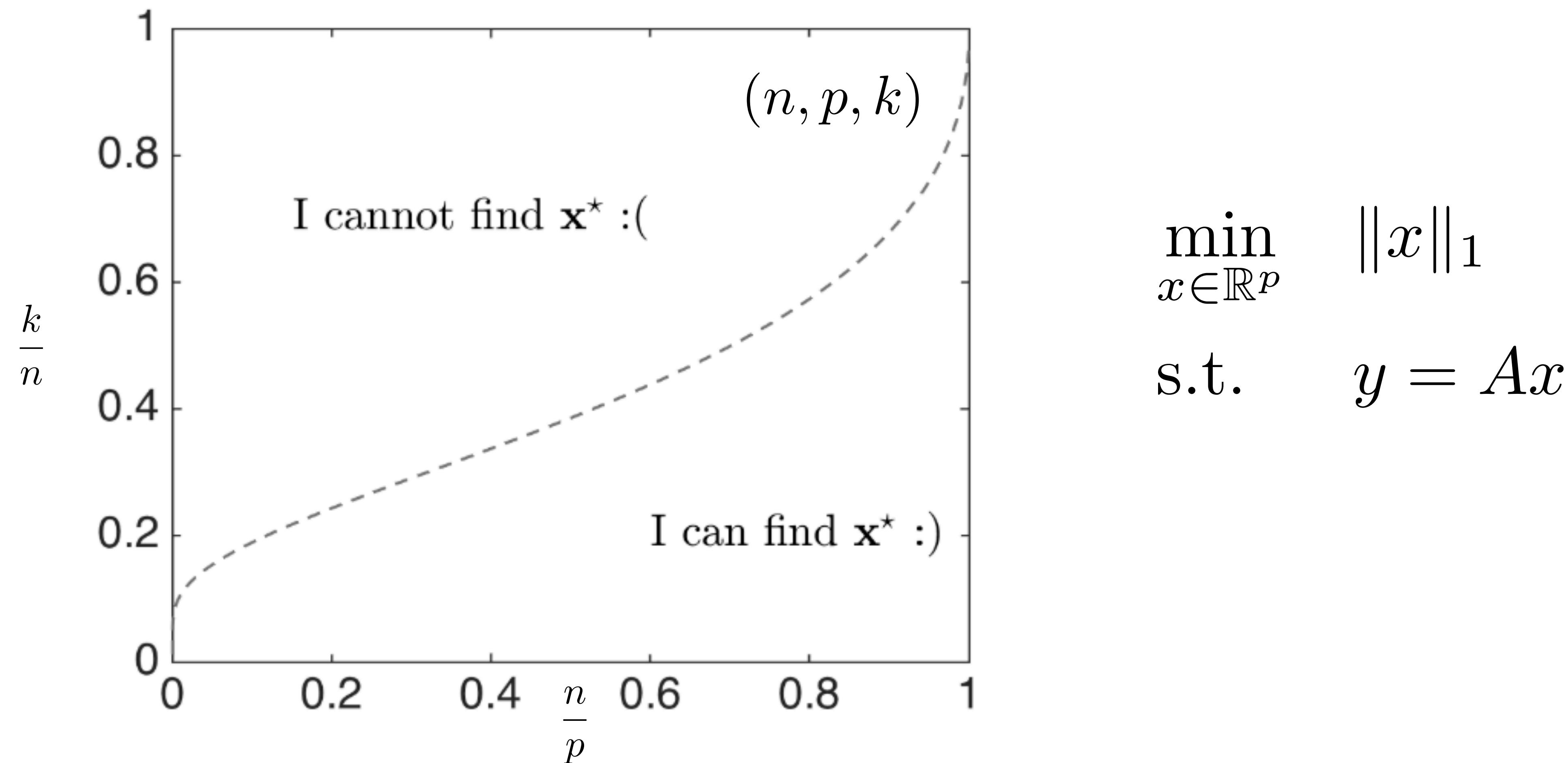
Note: Checking whether a fixed matrix actually satisfies RIP is NP-hard..

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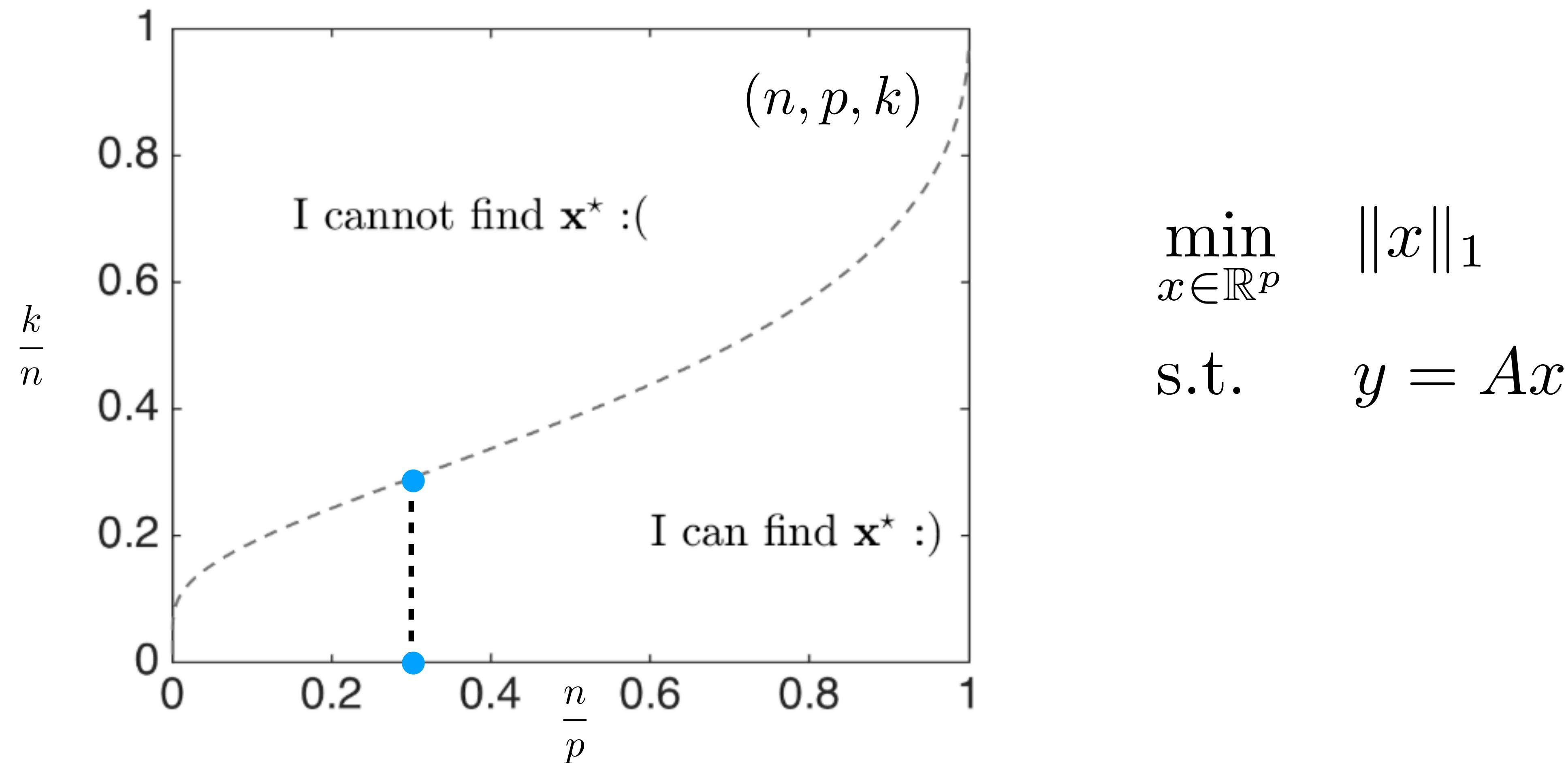
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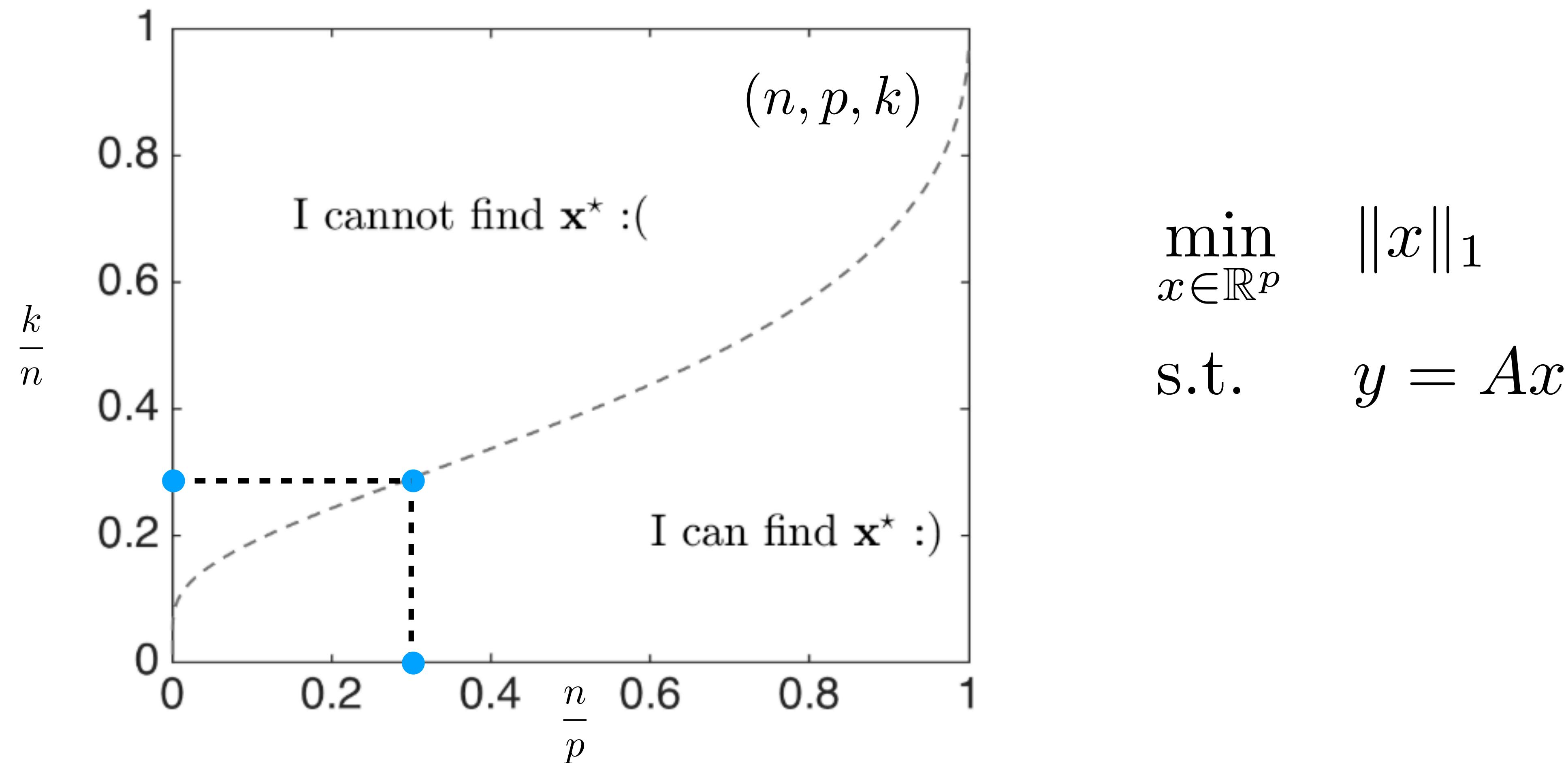
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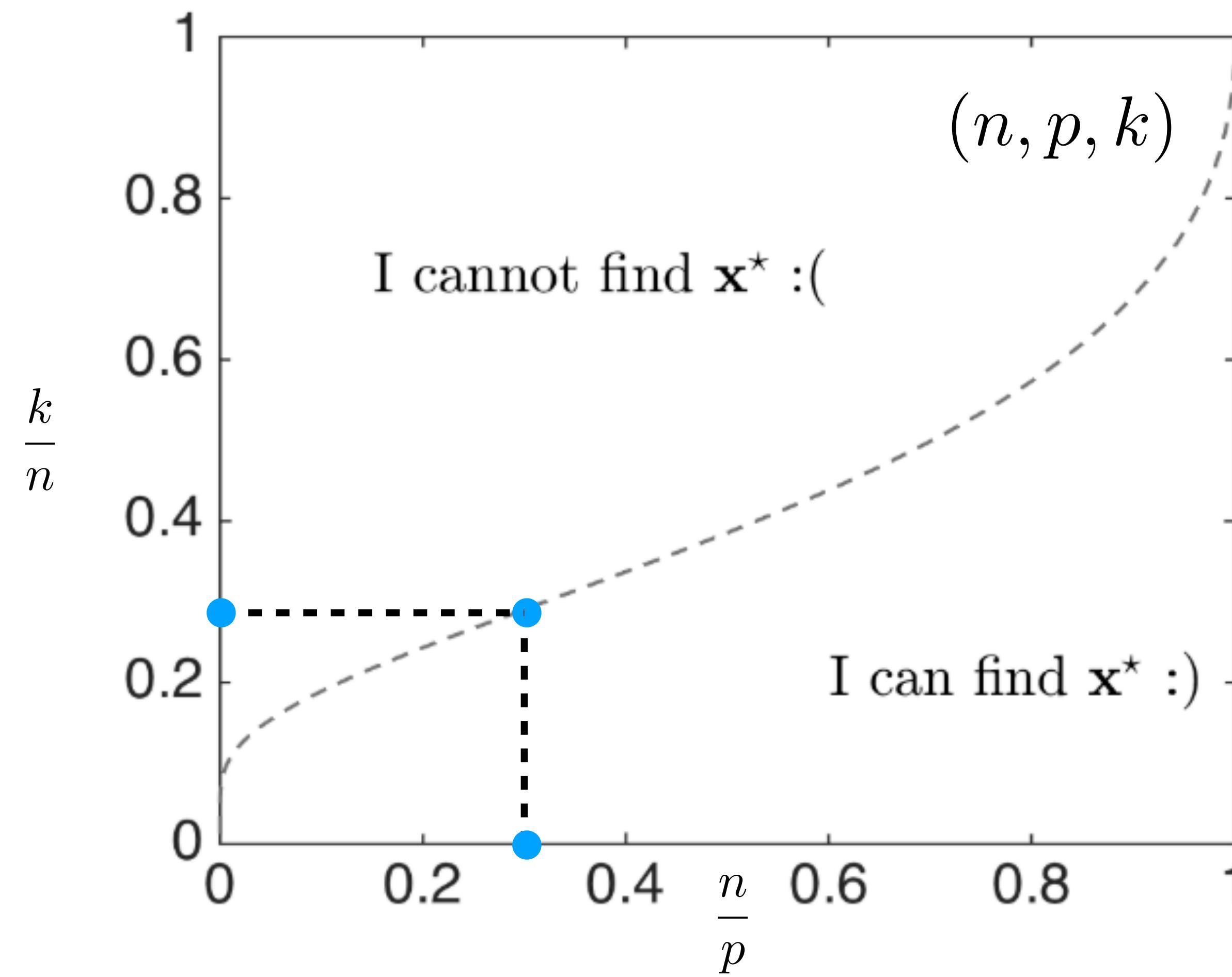
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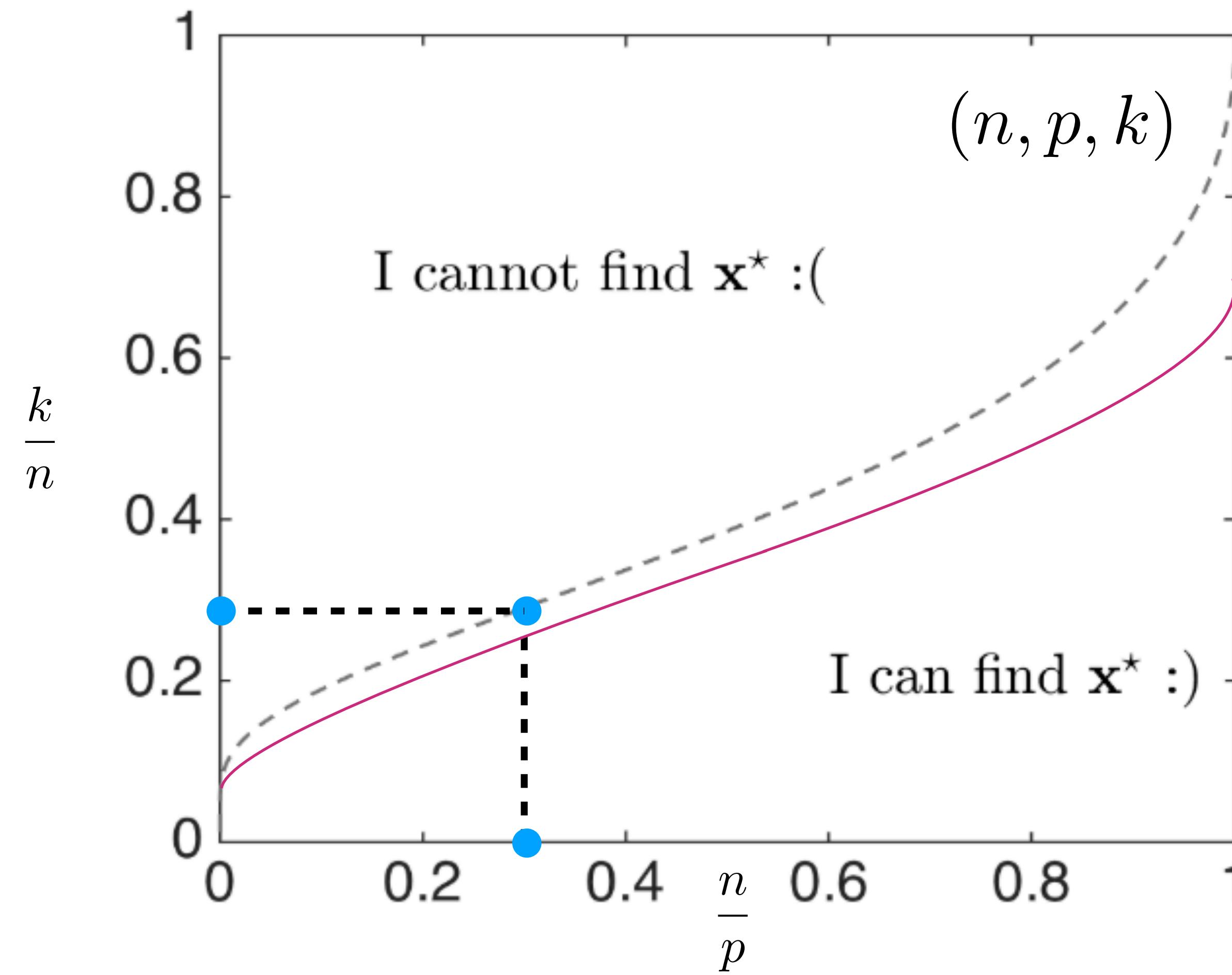


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- But, is this practical? Generally, no!

(Value of δ is NP-hard to find)

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Exact line search

- What we performed is exact line search:

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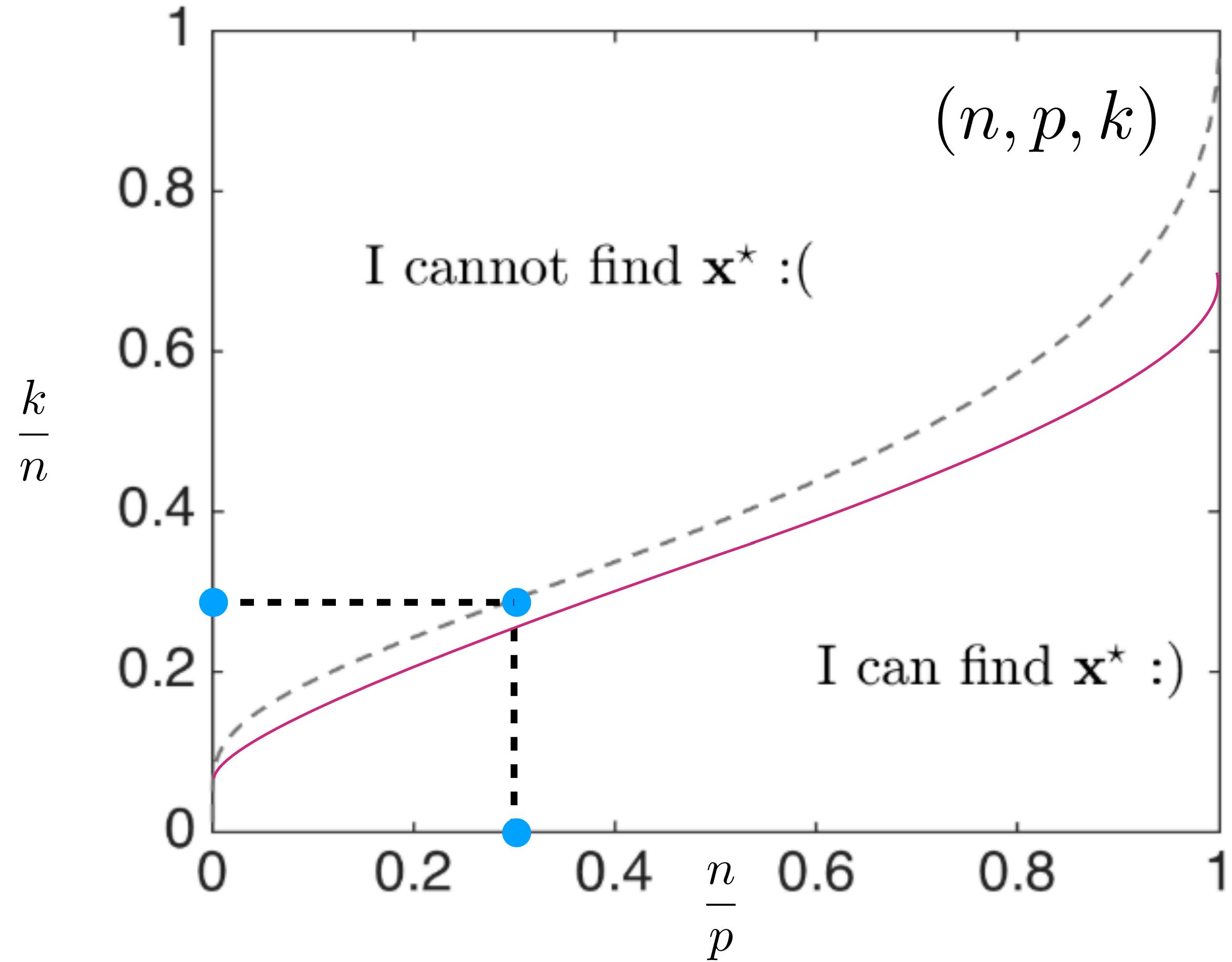
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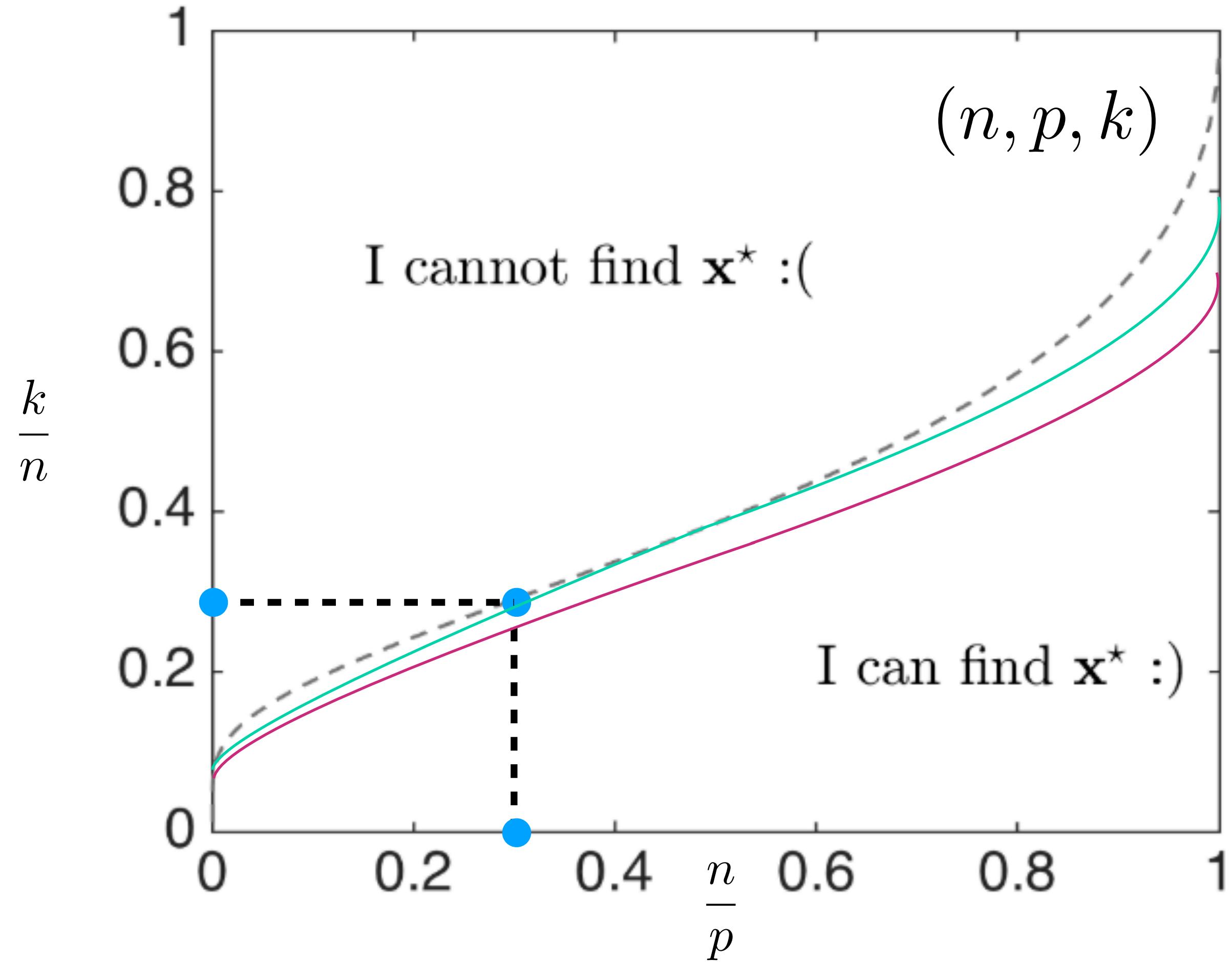
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$$\eta = \arg \min_{\eta \in \mathbb{R}_+} f(x_t - \eta \nabla_{Q_t} f(x_t))$$
- Q: “Great! Why don’t we use that all the time?”
- A: “Because, moving beyond least squares, solving this might be as difficult as the original problem”

Phase transition update



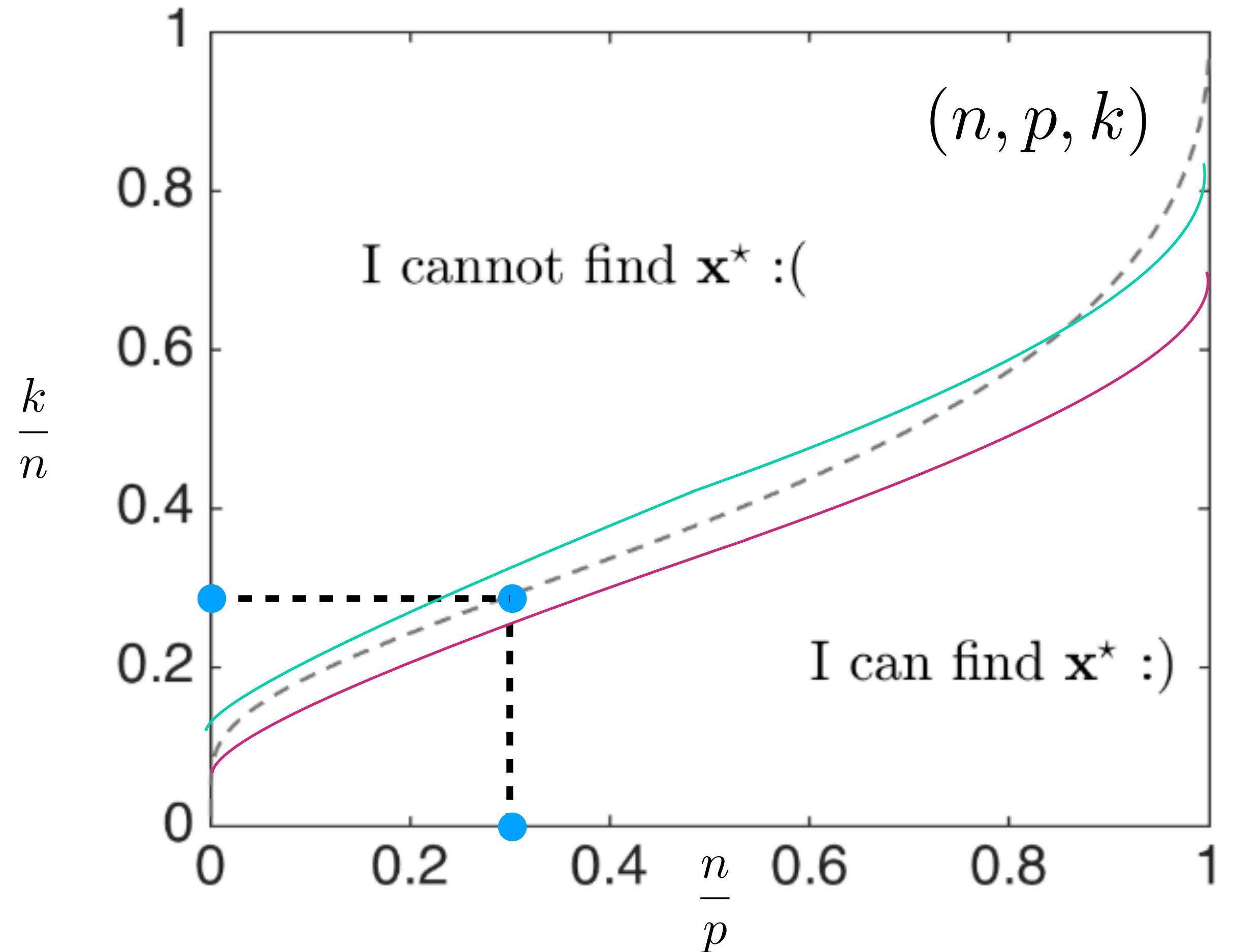
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(Some of these methods can
be found in the Review part)

But does this step size selection work in theory?

Whiteboard

What about the sparsity level k ?

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- Q: “What if we set $k = p$?“

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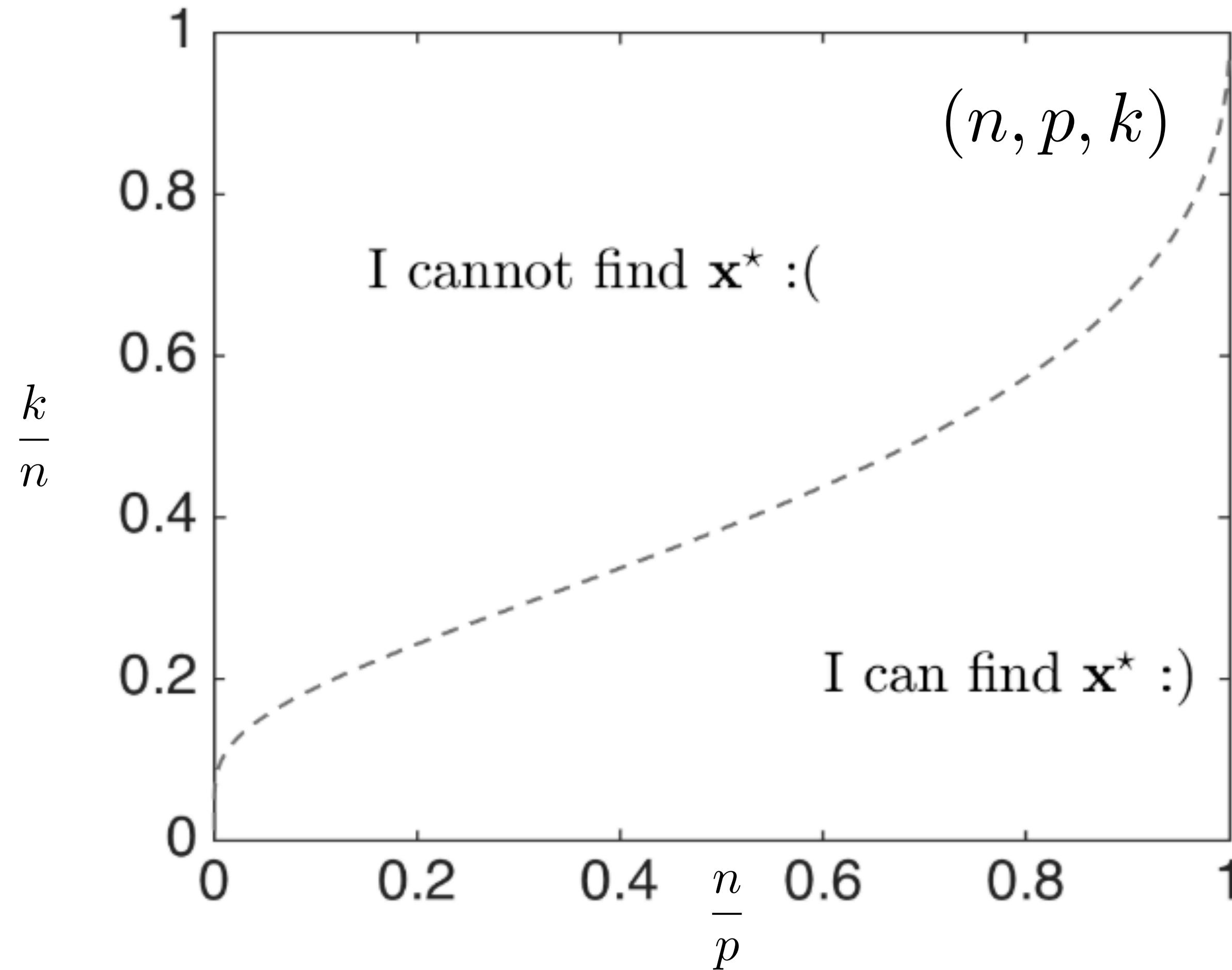
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- A: “We actually get denser and denser solutions”
- Q: “Is there any non-provable tweaks? ”
- A: “Problem-dependent strategies”

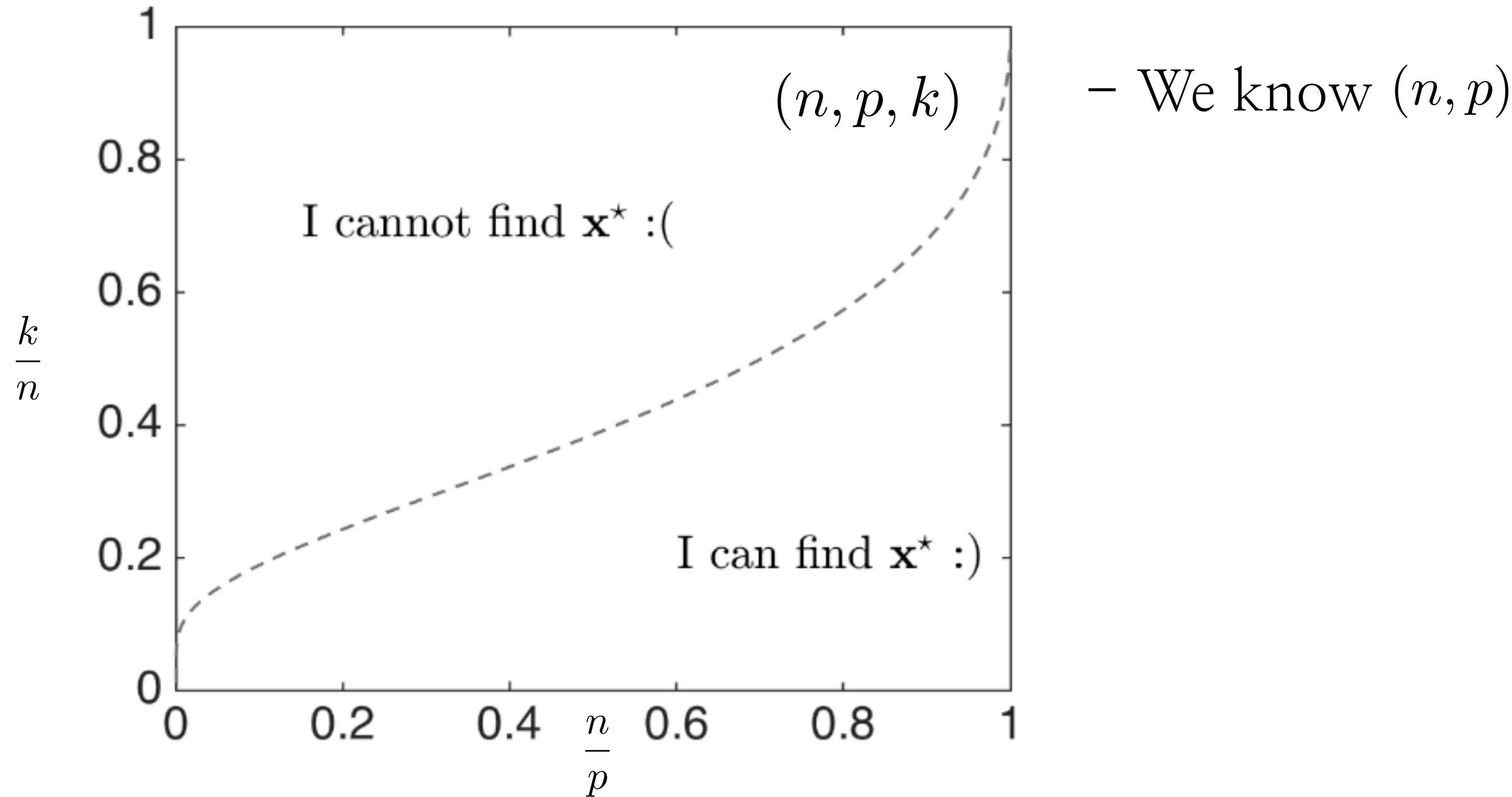
Heuristic for k on linear regression

(More like an upper bound for sparsity level)



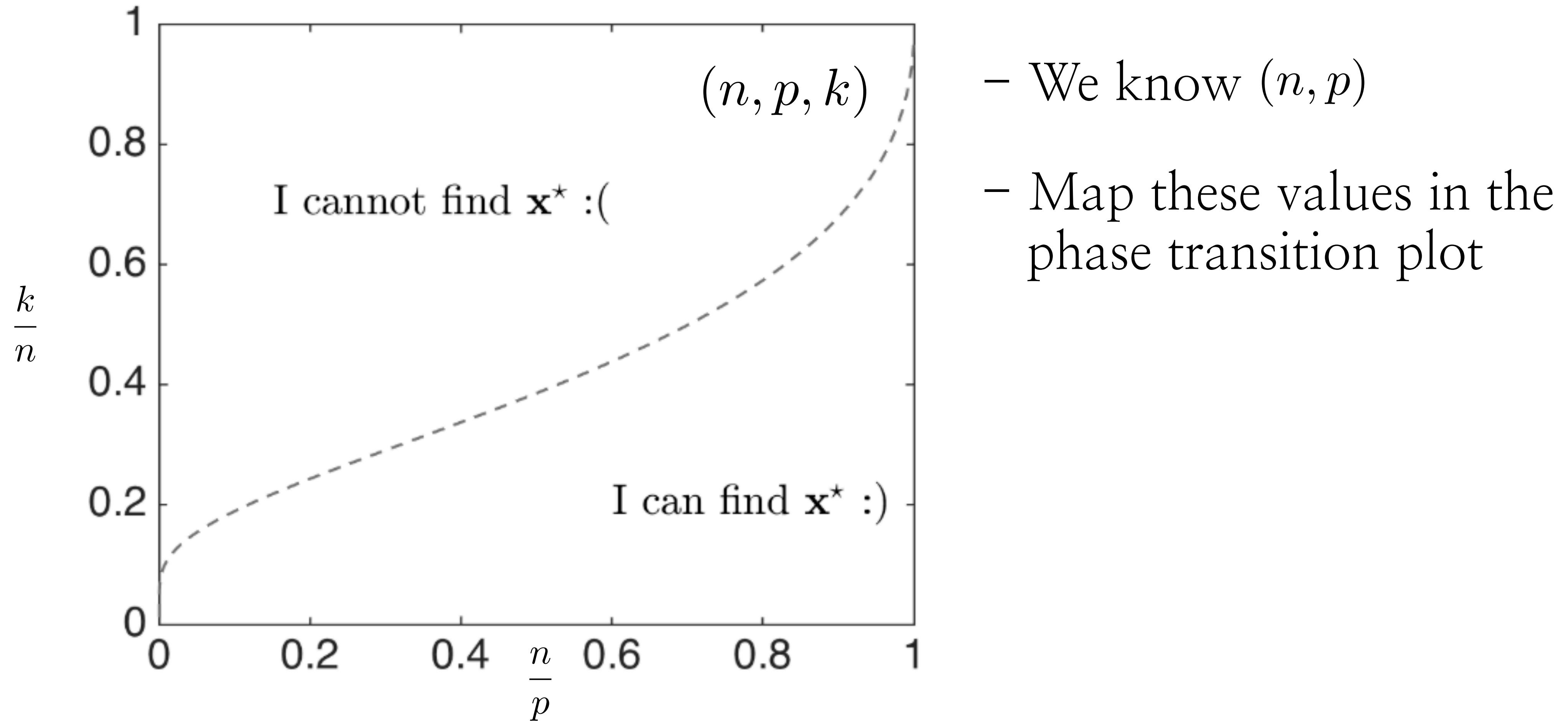
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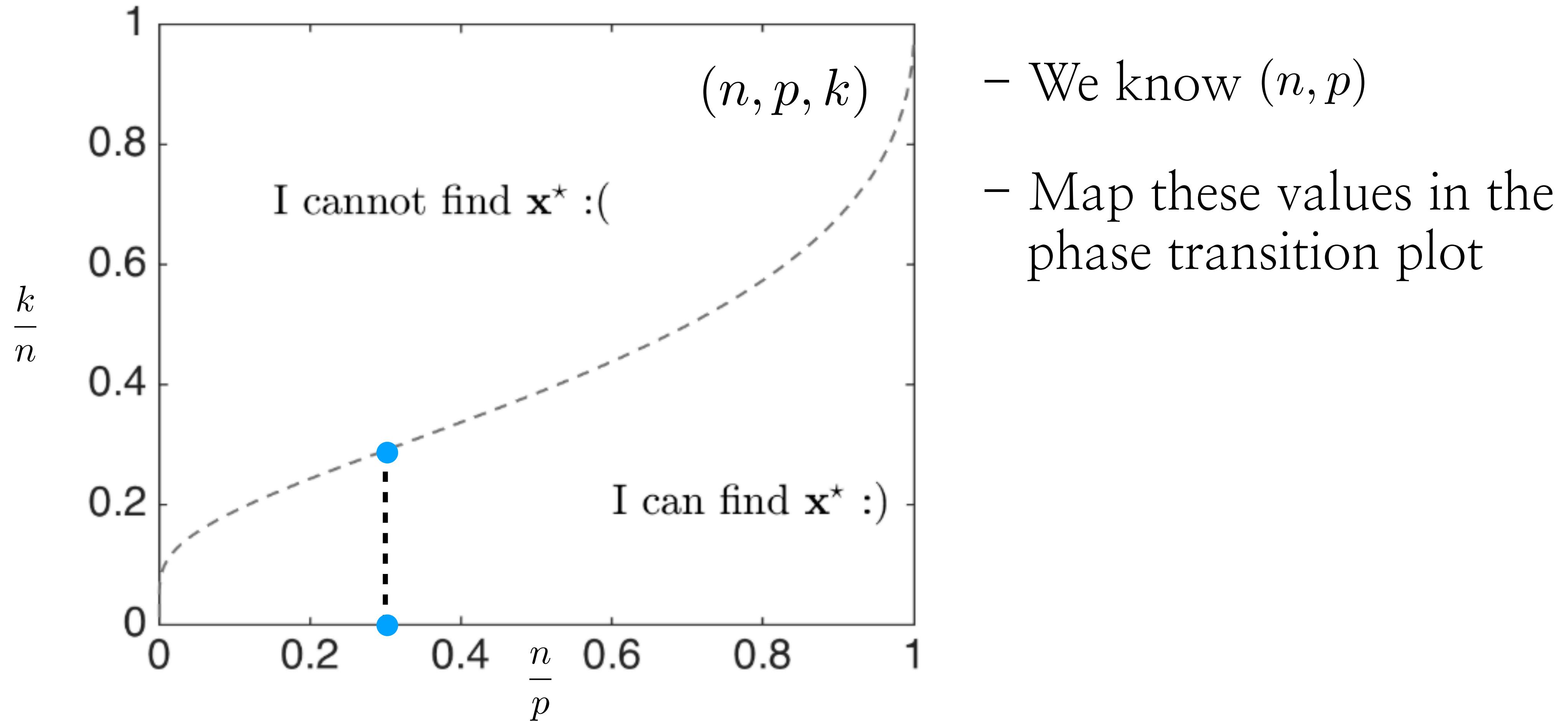
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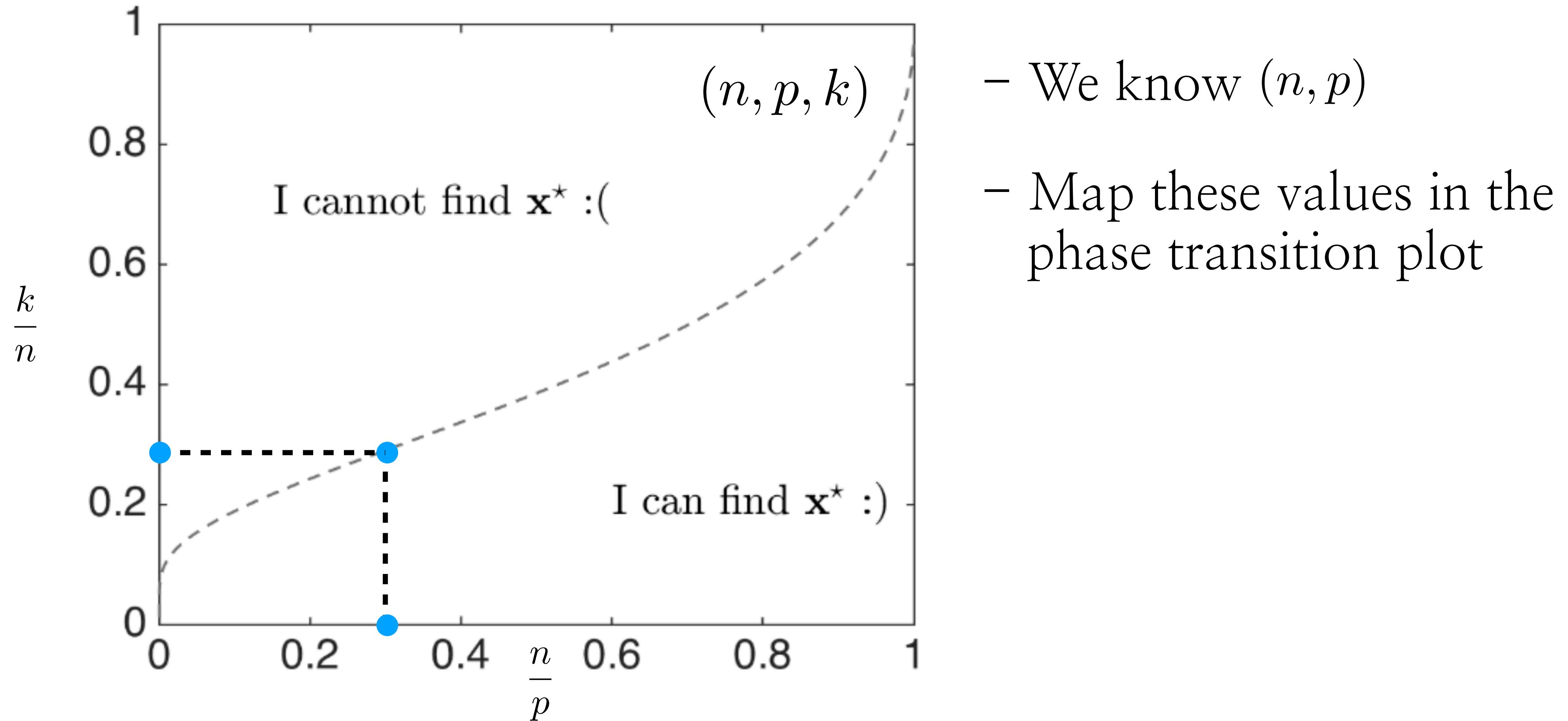
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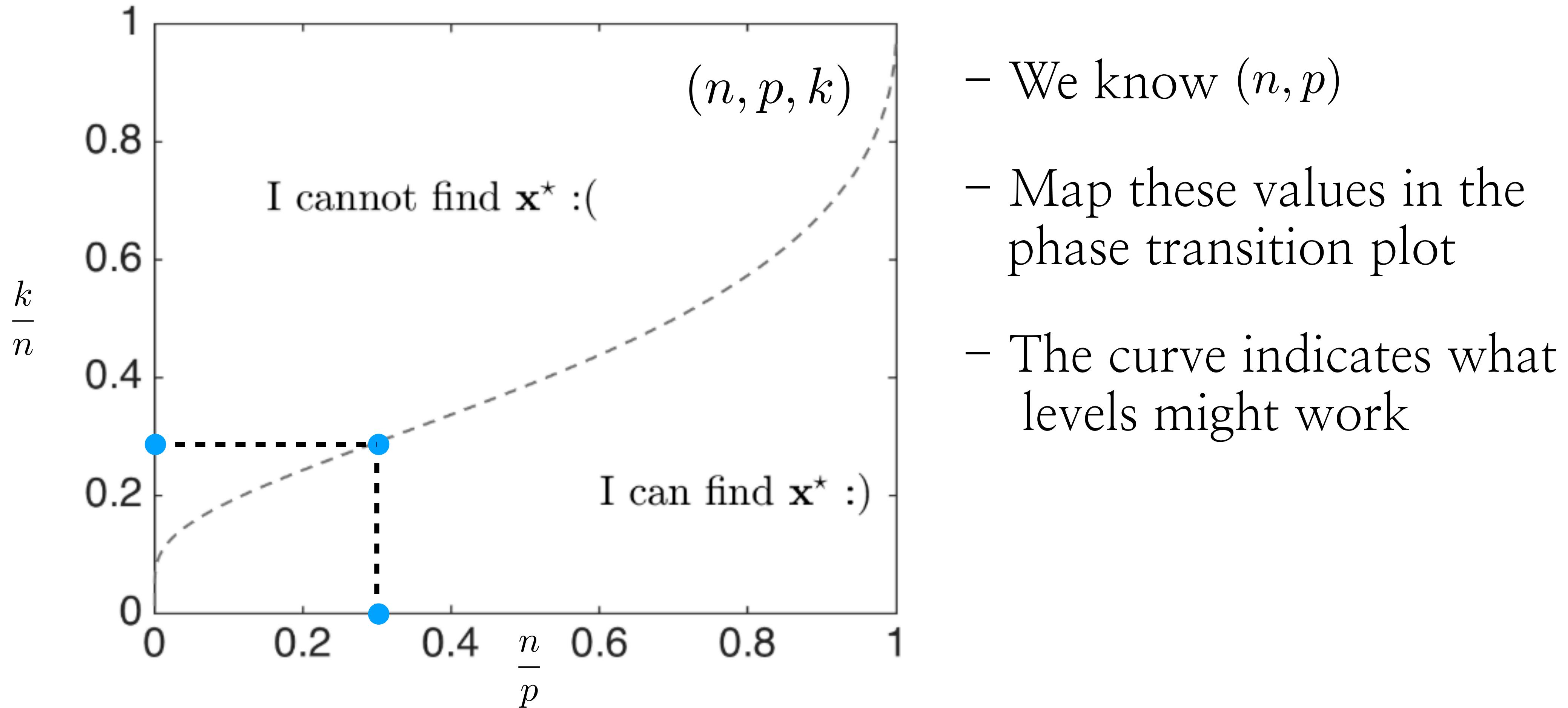
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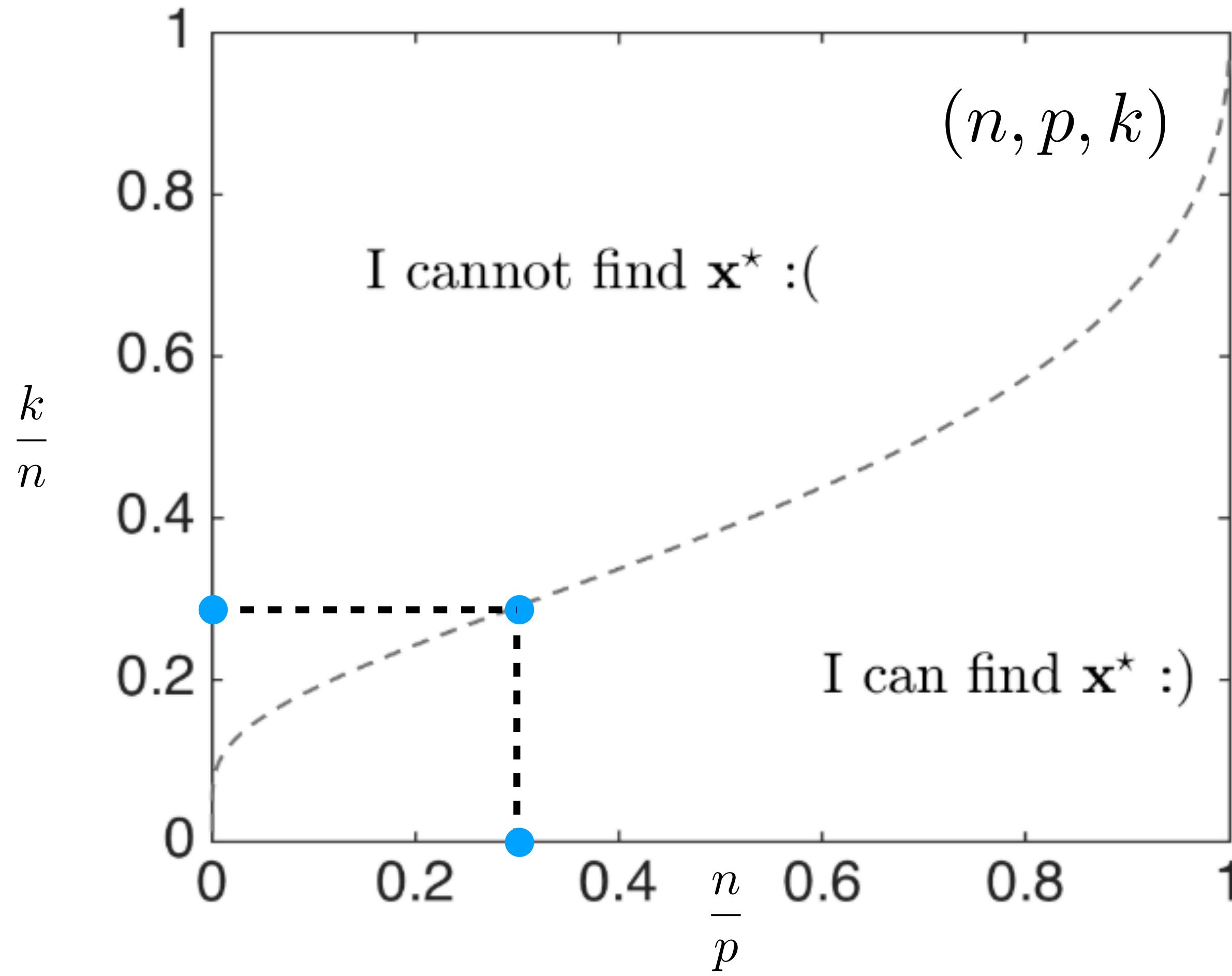
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- We know (n, p)
- Map these values in the phase transition plot
- The curve indicates what levels might work
- Run algorithm, take results, interpret: if it works, good; otherwise, do binary search over sparsity level

“All these sound interesting.. but do they extend to other objectives? And how are they related with what we discussed so far?”

(Lipschitz gradient continuity, strong convexity, Hessians, etc..)

A different view of RIP

- Reminder

$$(1 - \delta) \|x_1 - x_2\|_2^2 \leq \|A(x_1 - x_2)\|_2^2 \leq (1 + \delta) \|x_1 - x_2\|_2^2,$$

for some $\delta \in (0, 1)$, $\forall k$ -sparse $x_1, x_2 \in \mathbb{R}^p$

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- When objective has Lipschitz continuous gradients and is strongly convex:

$$\mu I \preceq \nabla^2 f(x) \preceq L I$$

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– “Restricted”: the properties hold over a subset of \mathbb{R}^p

Let’s say $x \in \mathcal{C} \subseteq \mathbb{R}^p$

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(Note that this L is different from
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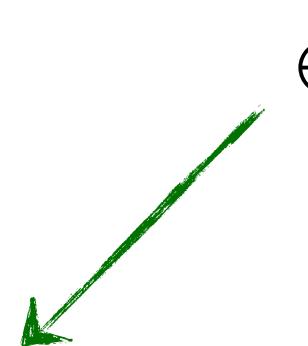
- Restricted strong convexity

$$f(y) \geq f(x) + \langle \nabla f(x), y - x \rangle + \frac{\mu}{2} \|x - y\|_2^2, \quad \forall x, y \in \mathcal{C}$$

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Examples

- Sparse logistic regression

$$\begin{aligned} \min_{x \in \mathbb{R}^p} \quad & \frac{1}{n} \sum_{i=1}^n \log (1 + \exp (-y_i a_i^\top x)) + \frac{\lambda}{2} \|x\|_2^2 \\ \text{s.t.} \quad & \|x\|_0 \leq k \end{aligned}$$


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and restricted smoothness with constant:

$$L = (\lambda_{\max}(A^\top A, k) + \lambda)$$

(For more information, see
“Gradient hard thresholding pursuit”)

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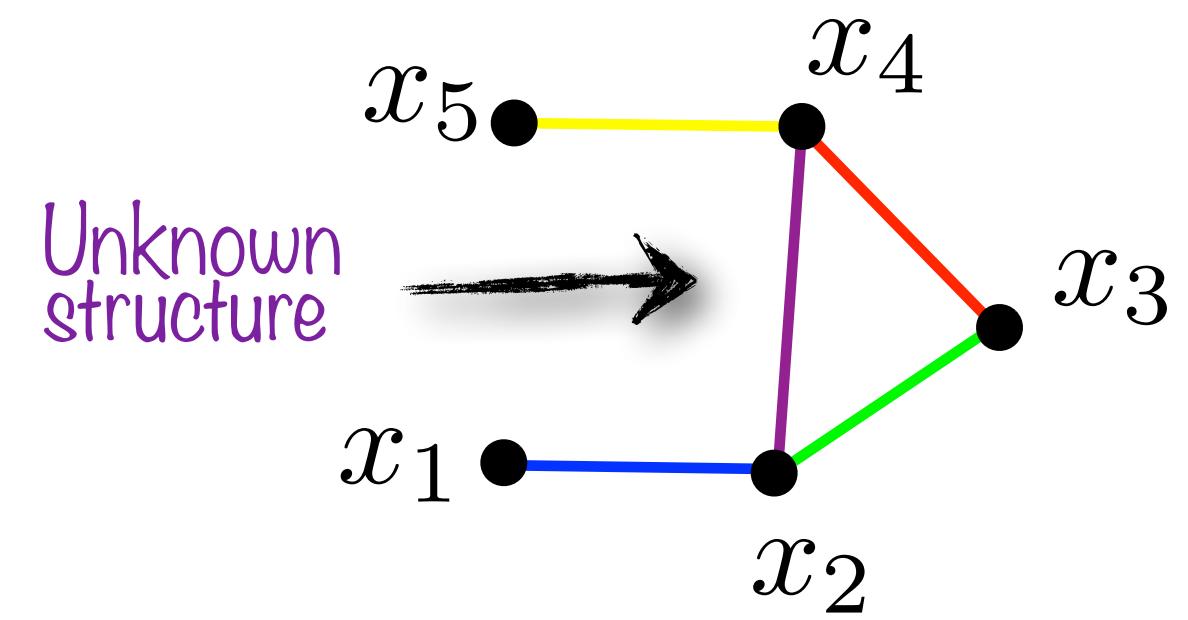
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Whiteboard

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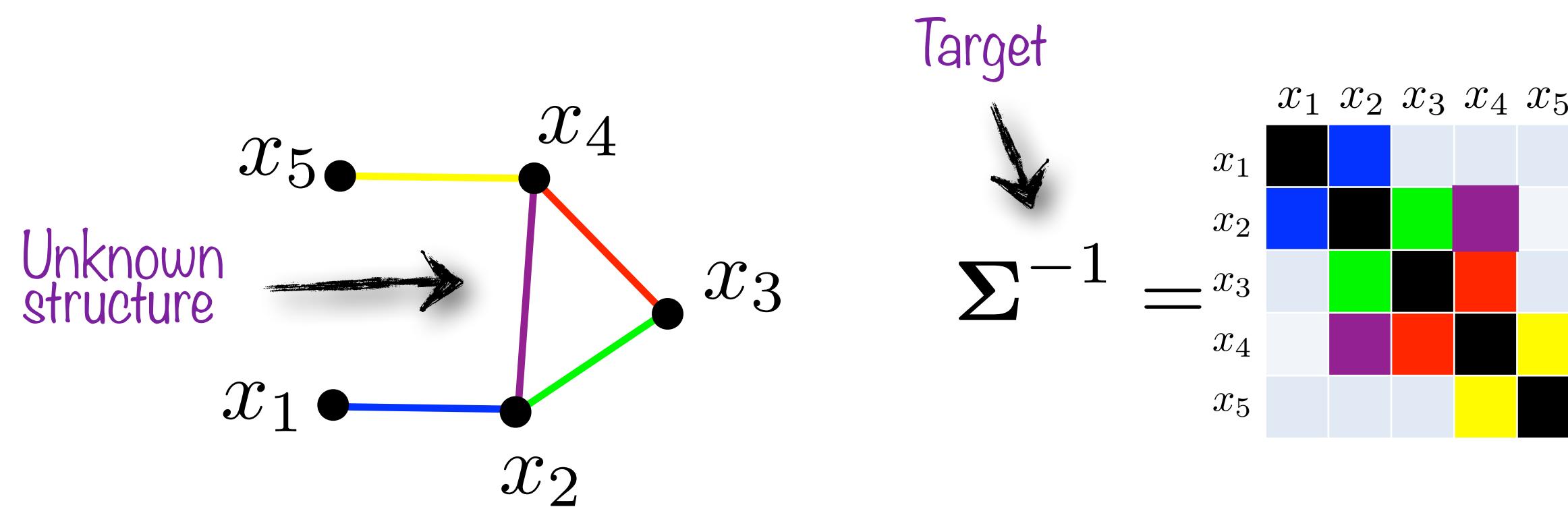
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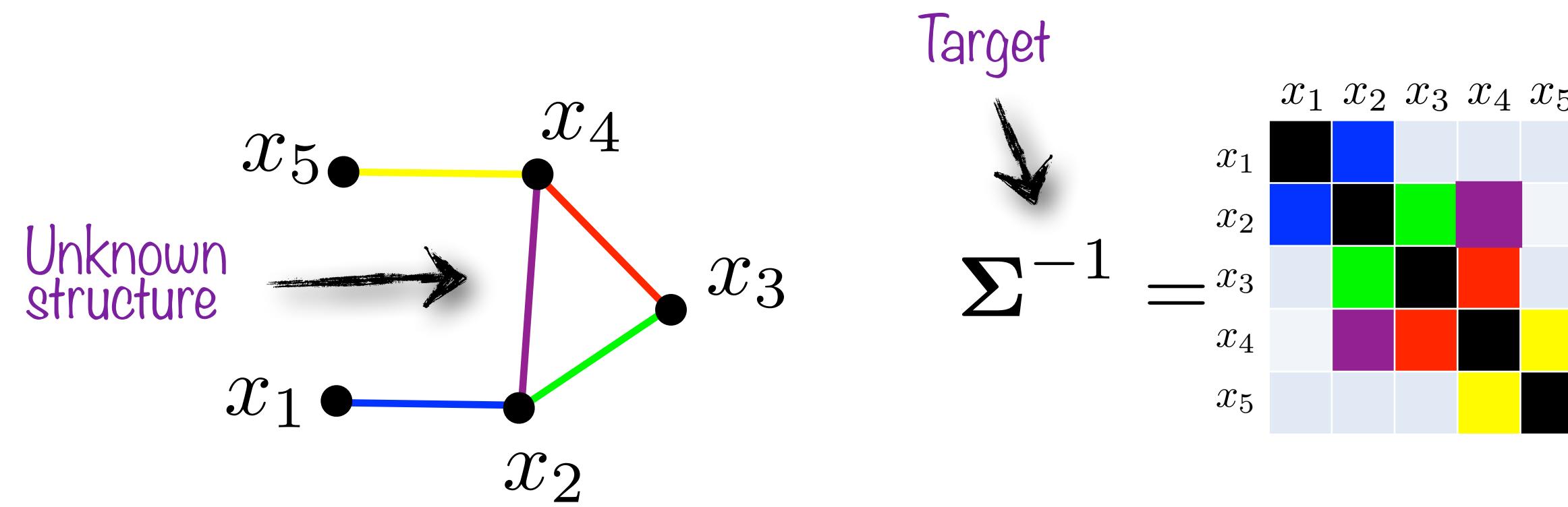
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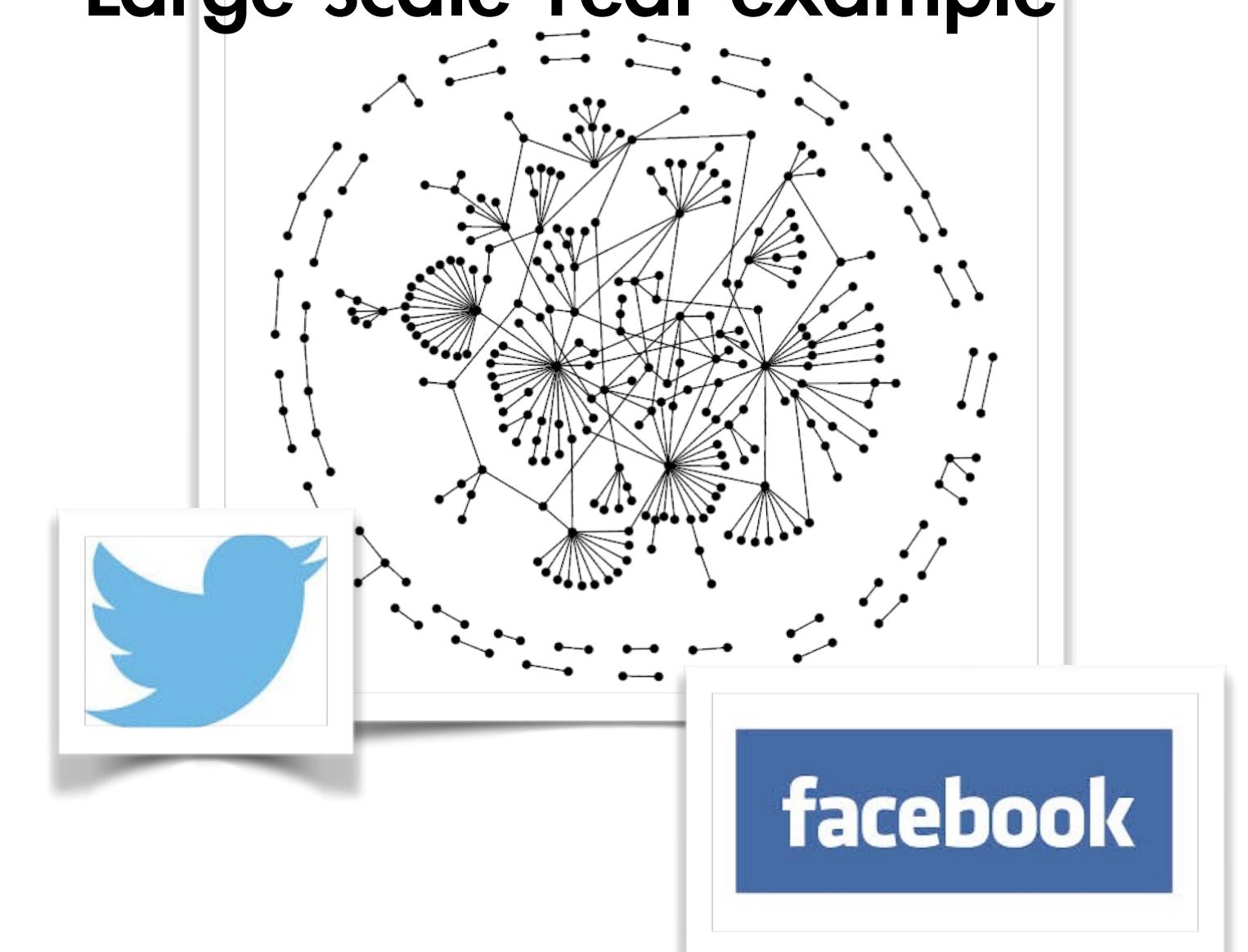
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Whiteboard



Large-scale real example



Examples

- Graphical model selection (under Gaussian assumptions)

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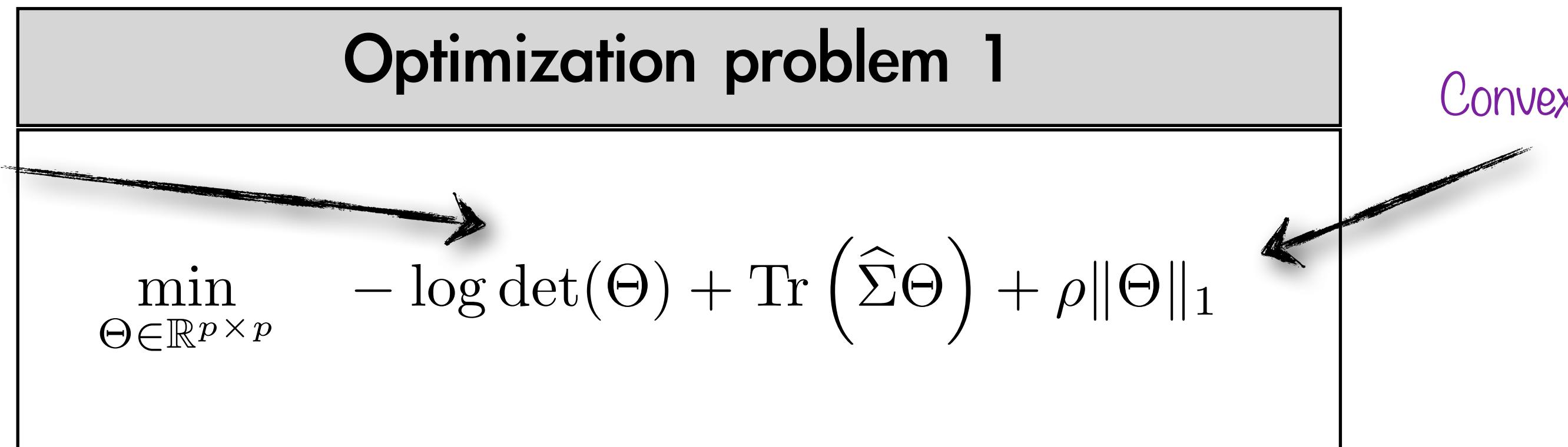
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$$\min_{\Theta \in \mathbb{R}^{p \times p}} -\log \det(\Theta) + \text{Tr}(\widehat{\Sigma}\Theta) + \rho\|\Theta\|_1$$

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Optimization problem 2

$$\begin{aligned} \min_{\Theta \in \mathbb{R}^{p \times p}} & -\log \det(\Theta) + \text{Tr}(\widehat{\Sigma}\Theta) \\ \text{s.t.} & \|\Theta\|_0 \leq k \end{aligned}$$

Non-convex

Beyond plain sparsity

- Our discussion so far holds for discrete structures beyond sparsity:
Block-sparsity, overlapping block-sparsity, dispersive models, tree sparsity,
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- Our discussion so far holds for discrete structures beyond sparsity:
Block-sparsity, overlapping block-sparsity, dispersive models, tree sparsity, graph-sparsity, etc..
- As long as the projection onto the combinatorial constraint can be computed efficiently:

$$\begin{aligned} \min_{x \in \mathbb{R}^p} \quad & \|x - y\|_2^2 \\ \text{s.t.} \quad & x \in \mathcal{C} \end{aligned}$$

- Various extensions include **inexact projections**, **greedy approaches**, and there are connections with (sub/super)modular optimization

Interlude: Statistics in Data Science

- We will use the example of RIP
- Disclaimer: this is not a complete introduction to concentration inequalities

(How many would be interested in learning about concentration inequalities (as a course)?)

Conclusion

- This lecture considers **sparse model selection** in Data Science applications
- While there are rigorous and efficient methods also in the convex domain we followed the **non-convex path** of hard thresholding methods
- We discussed some global convergence guarantees, and highlighted the importance of hyper-parameter tuning

Next lecture

- We will consider the case of **low-rank recovery**, natural extension of sparsity – there, we have different ways to exploit non-convexity