COMP 414/514: Optimization – Algorithms, Complexity and Approximations

Lecture 8

Overview

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  - Talked a little bit about **general smooth optimization** problems
  - This included both non-convex optimization, but also convex
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  - Talked a little bit about **general smooth optimization** problems
  - This included both non-convex optimization, but also convex
  - The discussion was quite abstract (no particular application)
- We will consider (possibly) the simplest non-convex setting: **sparse model selection**
  - We will provide motivation, background and alternative solutions
  - We will focus on how we can **provably and efficiently solve** such problems

Overview

$$\begin{aligned} \min_{x} \quad & f(x) \\ \text{s.t.} \quad & x \in \mathcal{C} \end{aligned}$$

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..over non−convex constraints

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$$\min_{x} \quad f(x)$$

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$$\text{s.t.} \quad \boxed{x \in \mathcal{C}}$$

− We will focus on the cases of (structured) sparsity and low−rankness

Sparse linear regression

(Not again man!..)

- Generative model: $y_i = a_i^\top x^\star + w_i$
  - $a_i \in \mathbb{R}^p$: features
  - $y_i \in \mathbb{R}$: responses
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- Assuming data set $\{y_i, a_i\}_{i=1}^n$, $n < p$, find $x^\star \in \mathbb{R}^p$

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- Any suggestions how to solve this?

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– Solution #1: convexification + proj. gradient descent

LASSO

$$\min_{x \in \mathbb{R}^p} \quad f(x) := \frac{1}{2}\|y - Ax\|_2^2$$
$$\text{s.t.} \quad \|x\|_1 \leq \lambda$$

$$\longrightarrow \quad x_{t+1} = \Pi_{\|\cdot\|_1 \leq \lambda}\left(x_t - \eta \nabla f(x_t)\right)$$

(Pros & Cons?)

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– Solution #2: convexification + **proximal** gradient descent

Basis pursuit (denoising)

$$\min_{x \in \mathbb{R}^p} \quad f(x) := \frac{1}{2}\|y - Ax\|_2^2 + \rho\|x\|_1 \quad \longrightarrow \quad x_{t+1} = \text{Prox}_{\rho\|\cdot\|_1}\left(x_t - \eta \nabla f(x_t)\right)$$

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(Pros & Cons?)

Hard-thresholding

– Solution #3: keep non-convexity + **non-convex projected gradient descent**

$$\min_{x\in\mathbb{R}^p} \quad f(x) := \frac{1}{2}\|y - Ax\|_2^2 \qquad \text{s.t.} \quad \|x\|_0 \leq k \quad \longrightarrow \quad x_{t+1} = H_k\left(x_t - \eta\nabla f(x_t)\right)$$

(Pros & Cons?)

But before we proceed..

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  - Q: "How easy it is to solve ℓ_0−pseudo norm problems?"
  - A: "Sparsity makes problems exponentially hard to solve"

(This assumes the most general case)

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  - A: ``Yes, without any constraints, the problem has infinite solutions``

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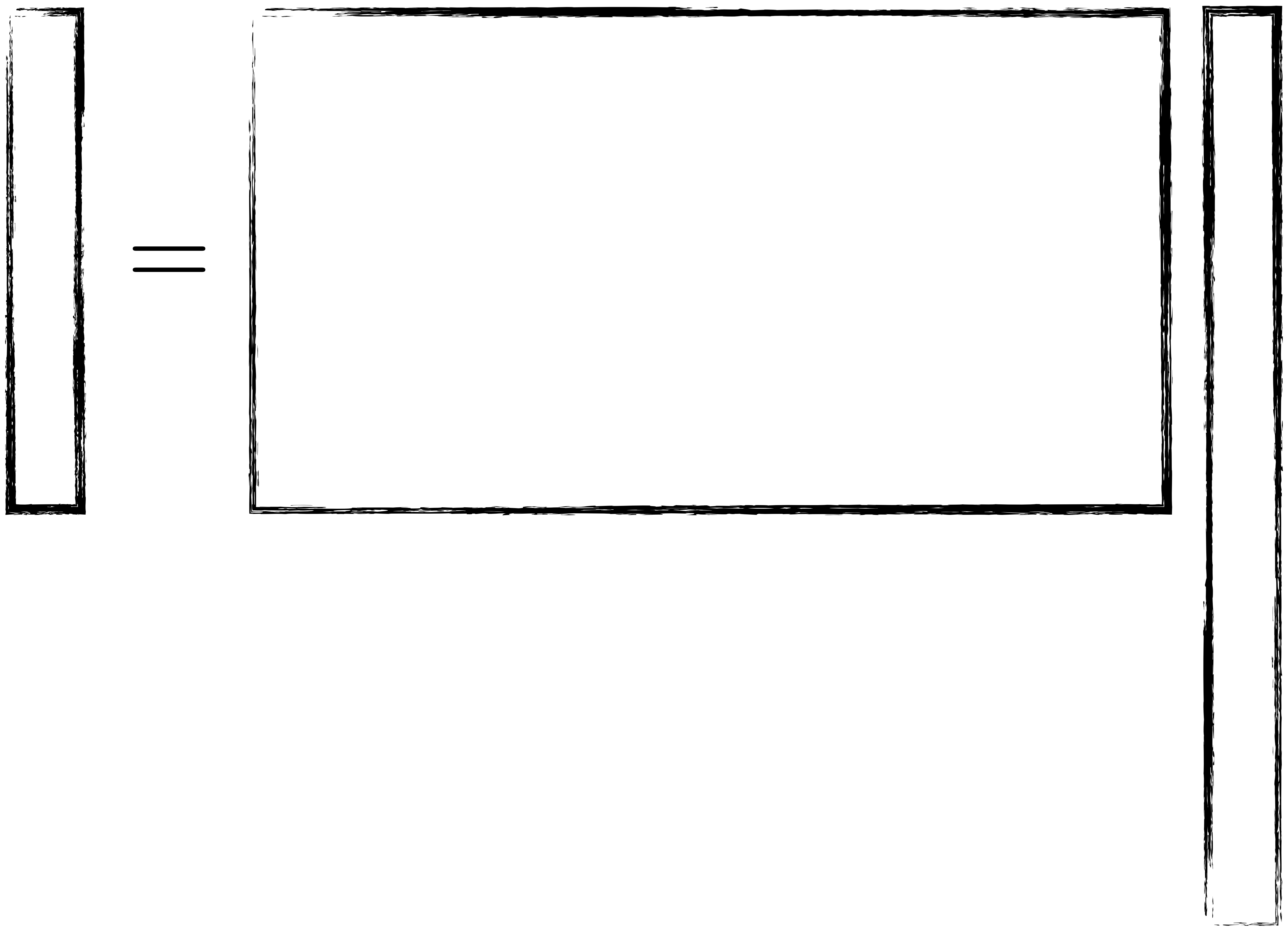
  - Q: ``But isn't the problem underdetermined? $(n \ll p)$``
  - A: ``Yes, without any constraints, the problem has infinite solutions``
  - Q: ``Why then do we have hopes solving this problem?``
  - A: ``Under assumptions on $A$, and the relation between $(n, p, k)$, we Will see that on average this problem can be solved in polynomial time``

Over− vs. under−parameterized



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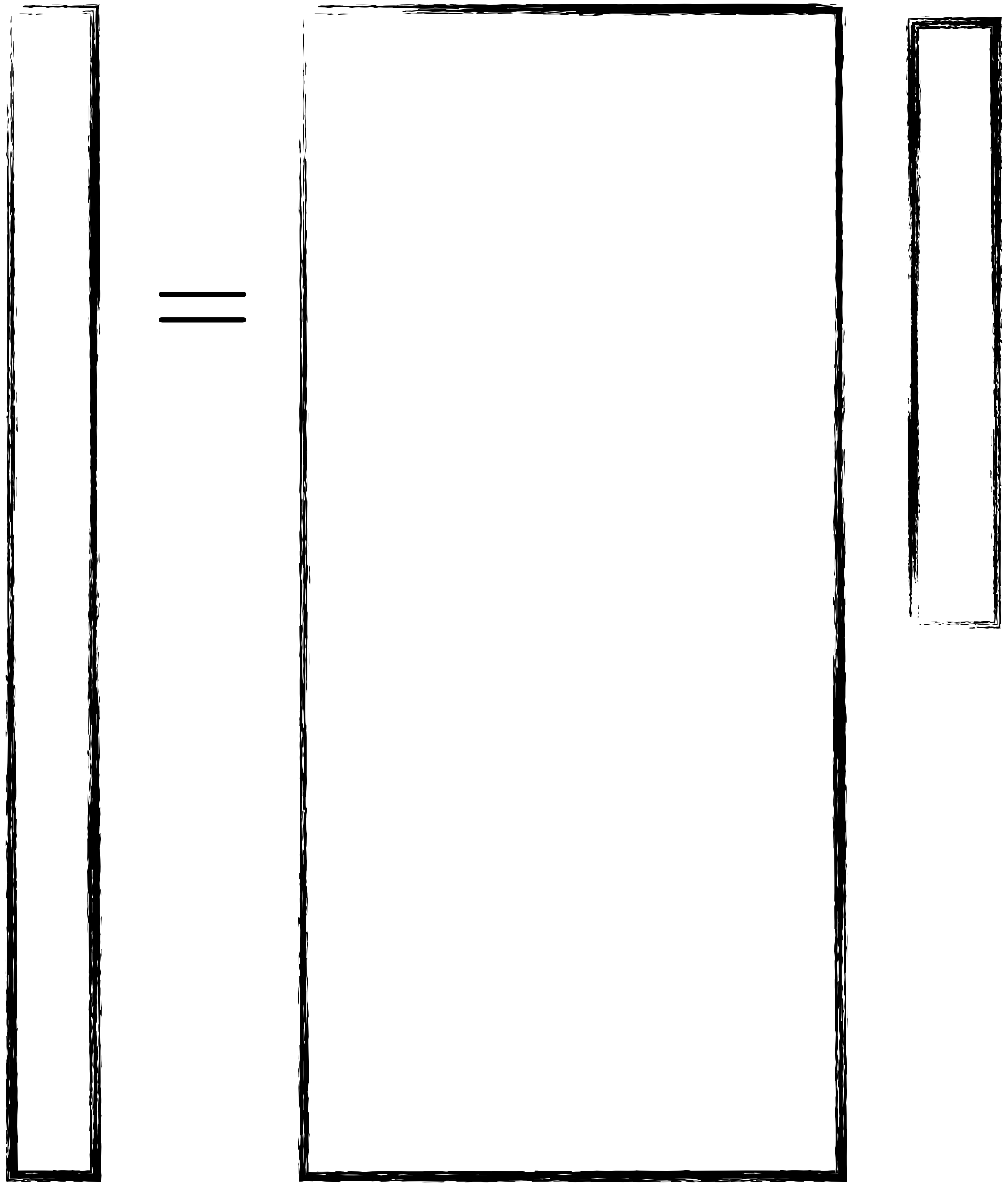

Under−parameterized

Iterative hard thresholding (IHT)

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- IHT:

$$x_{t+1} = H_k\left(x_t - \eta \nabla f(x_t)\right)$$

where $H_k(z) \in$

$$\begin{aligned} \operatorname*{argmin}_{x \in \mathbb{R}^p} \quad & \|x - z\|_2^2 \\ \text{s.t.} \quad & \|x\|_0 \leq k \end{aligned}$$

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- Now, imagine yourself implementing this.. What are the hyper−parameters?
  - \`\`How do we set the step size?\`\`
  - \`\`How do we select the initial point? (it is non−convex after all)\`\`
  - \`\`What if we don't know the sparsity level?\`\`
  - \`\`Are there any other tricks we can use?\`\`

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- Property of I: isometry

$$(1-\delta)\|x_1 - x_2\|_2^2 \leq \|I(x_1 - x_2)\|_2^2 \leq (1-\delta)\|x_1 - x_2\|_2^2, \quad \text{for some } \delta \in [0,1], \ \forall x_1, x_2 \in \mathbb{R}^p$$

(Interpretation?)

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- How can we use this property in proving convergence of IHT?

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Whiteboard

- We get linear convergence to the global optimum!

How does it perform in practice?

Demo

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- It turns out that matrices that satisfy:

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(In our case, we generate it as Gaussian so with high probability we are fine)

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Note: Checking whether a fixed matrix actually satisfies RIP is NP-hard..

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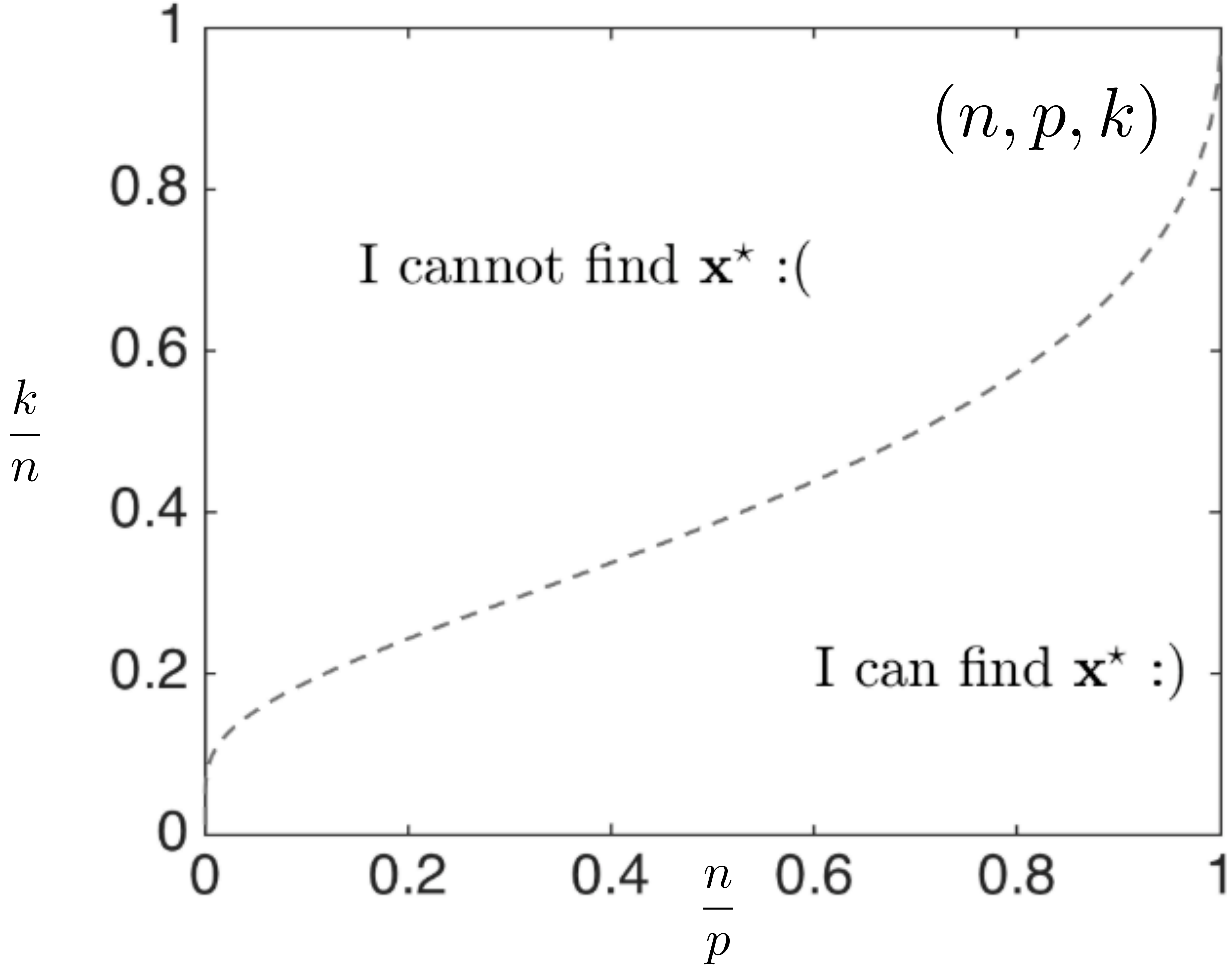

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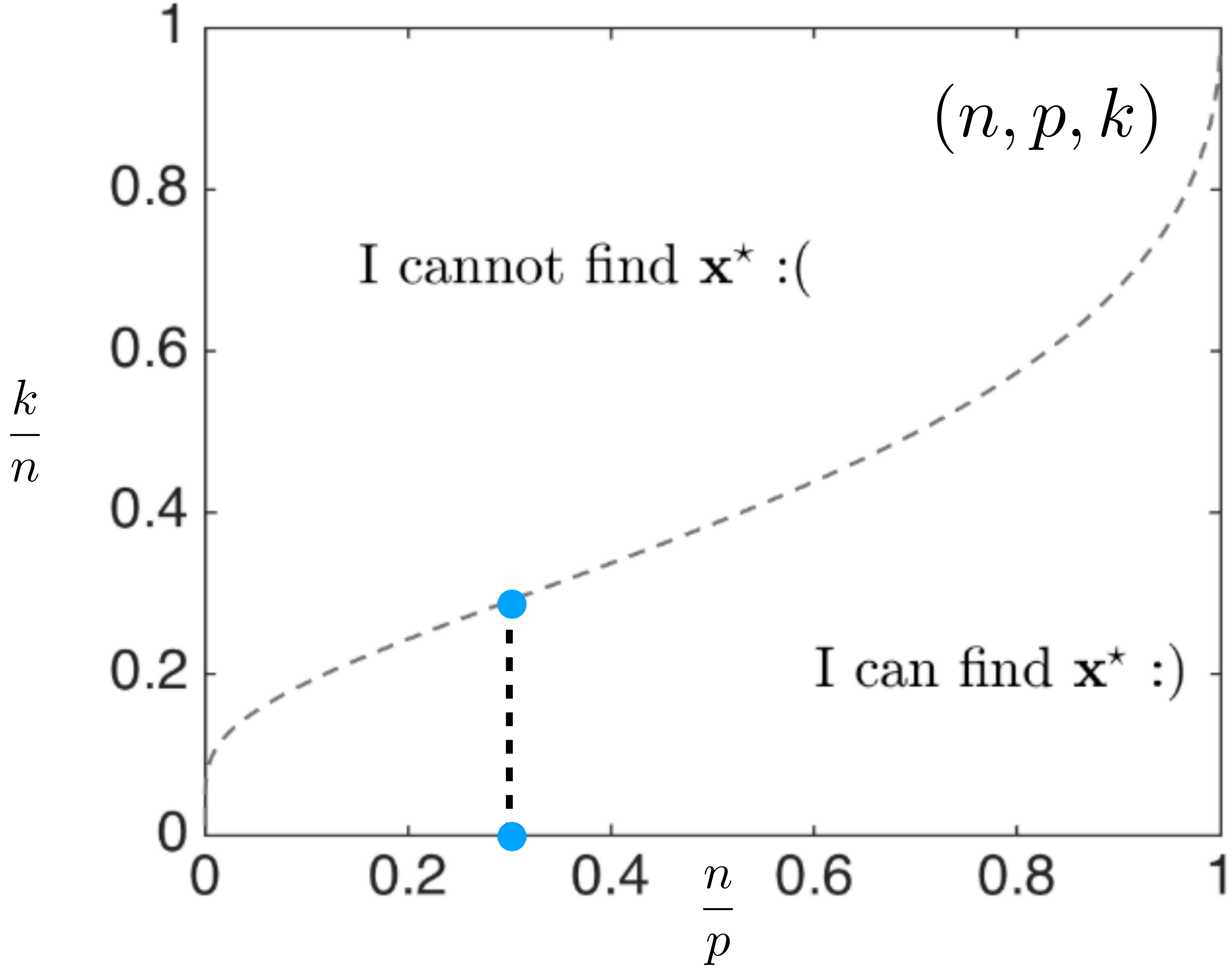

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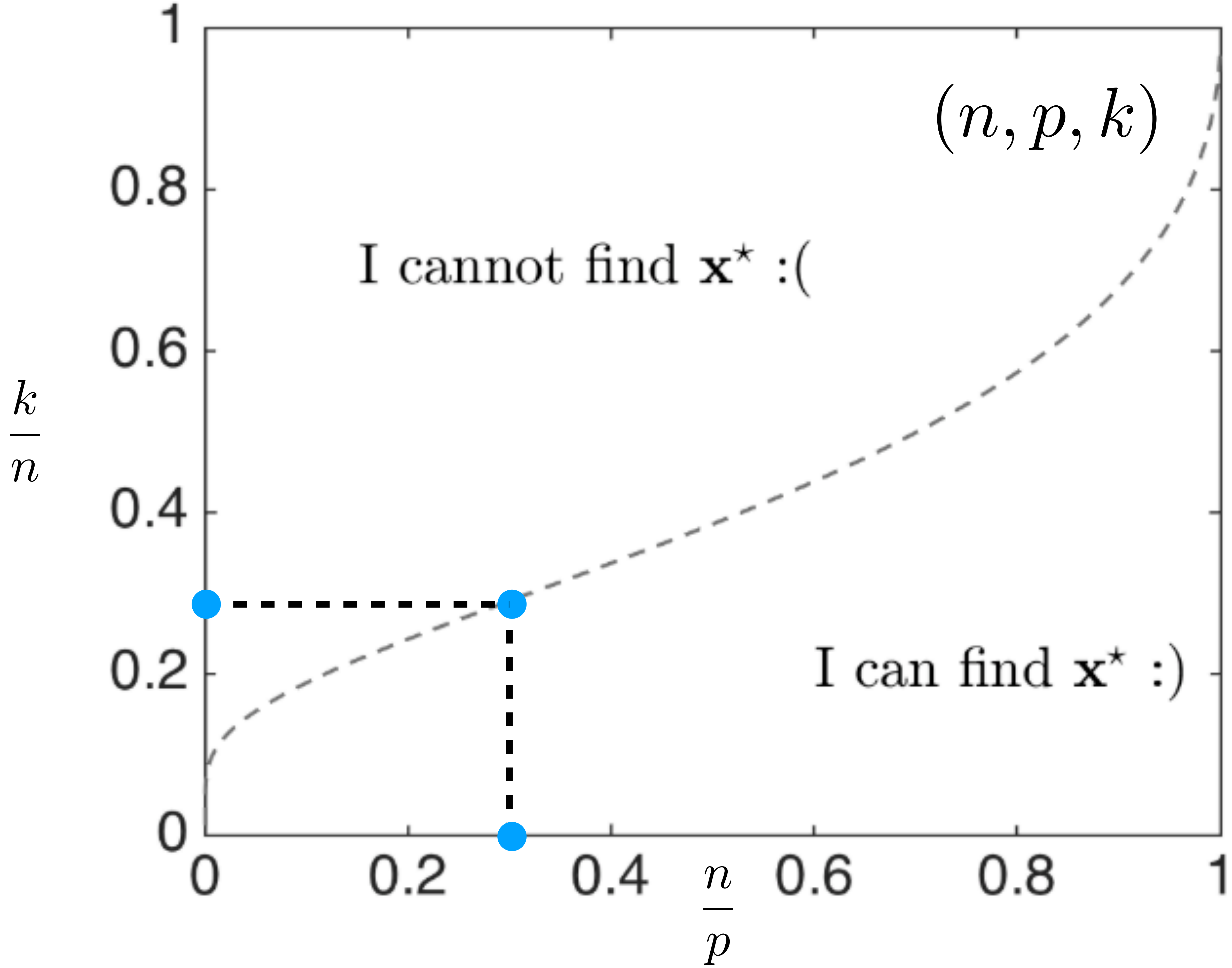

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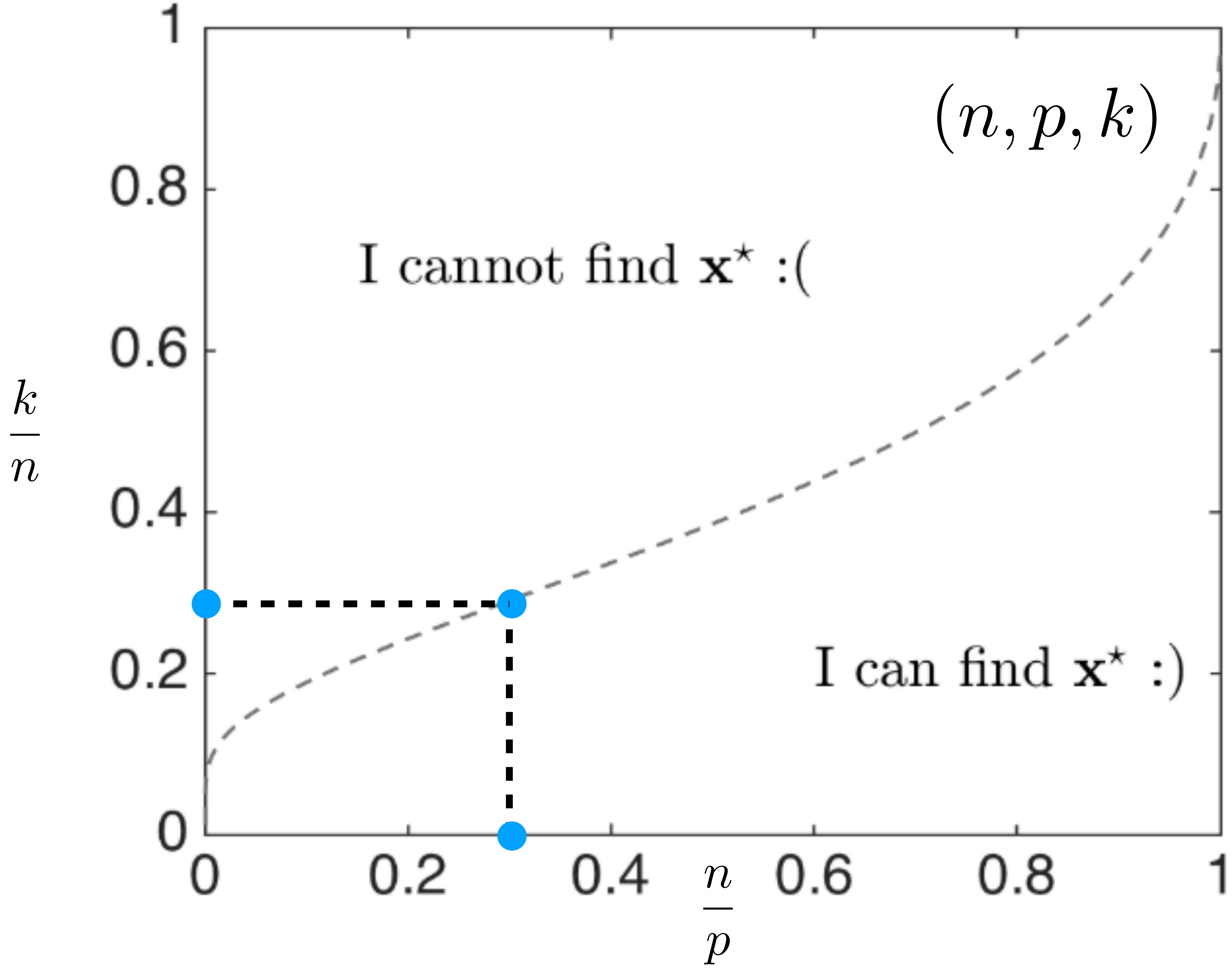

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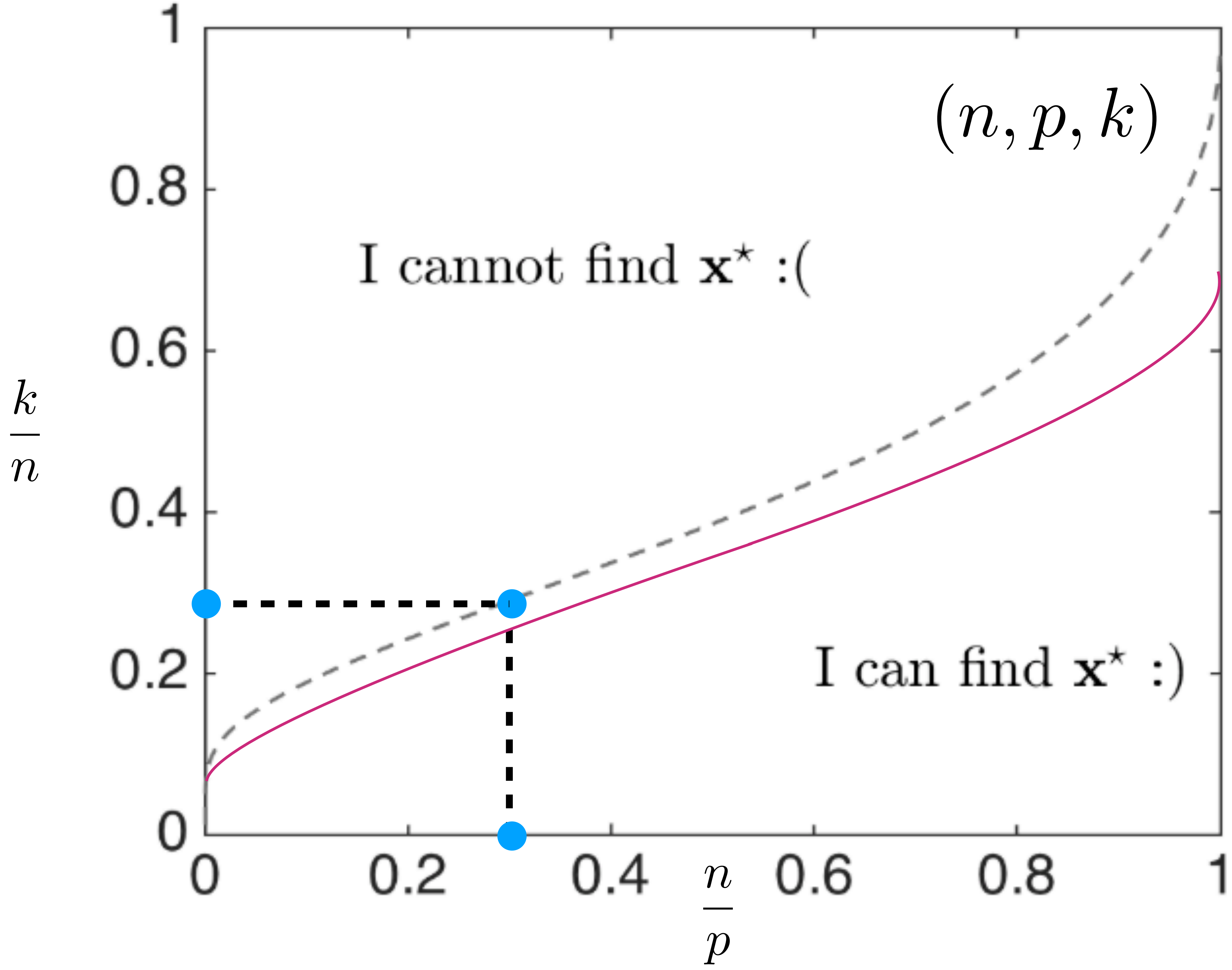

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Whiteboard

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(In literature section, there is work that validates theoretically this selection)

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- But, is this practical? Generally, no!

(Value of δ is NP−hard to find)

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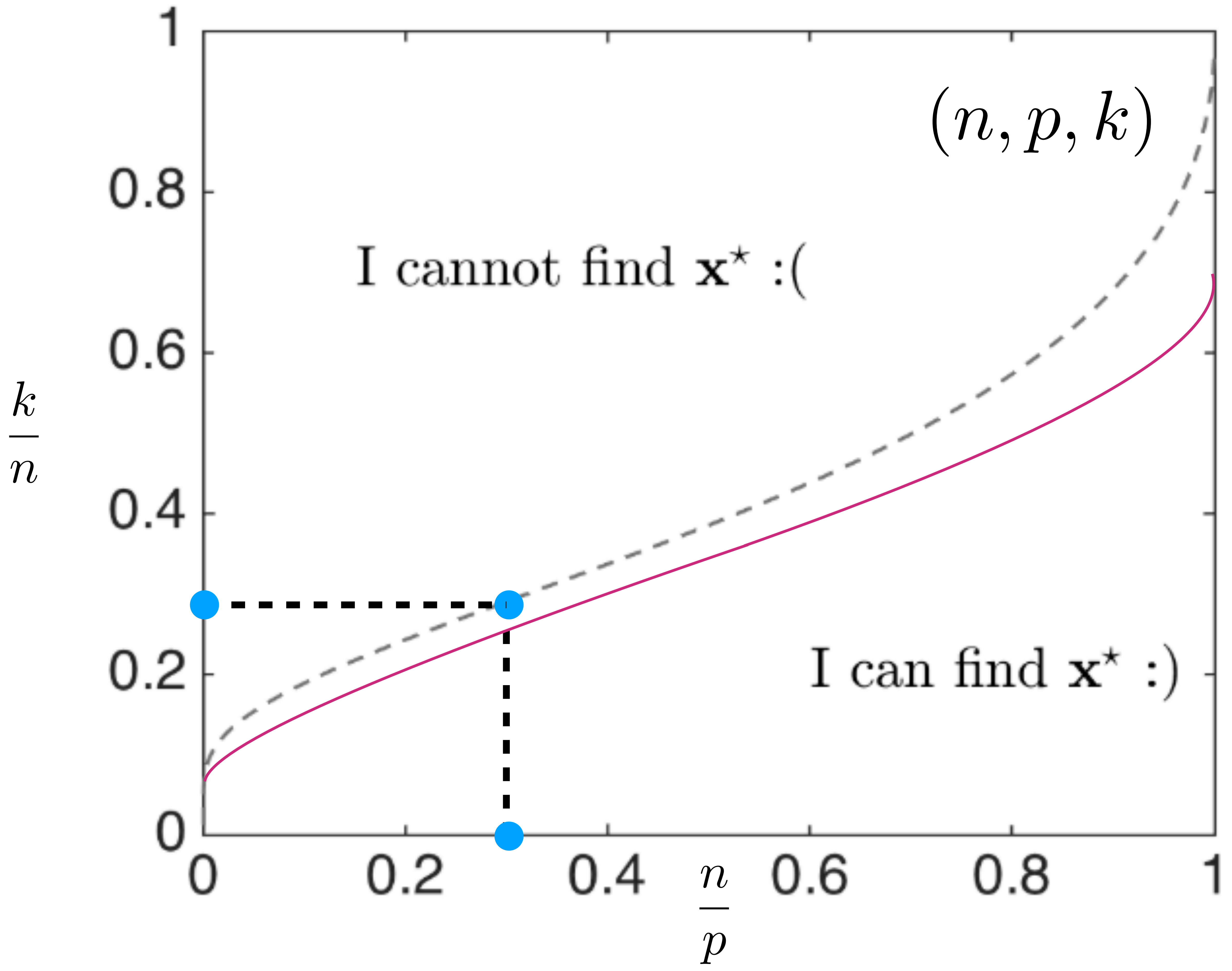
$$\eta = \arg\min_{\eta \in \mathbb{R}_+} f\left(x_t - \eta \nabla_{Q_t} f(x_t)\right)$$

- Q: ‵‵Great! Why don't we use that all the time?‵‵
- A: ‵‵Because, moving beyond least squares, solving this might be as difficult as the original problem‵‵

Phase transition update



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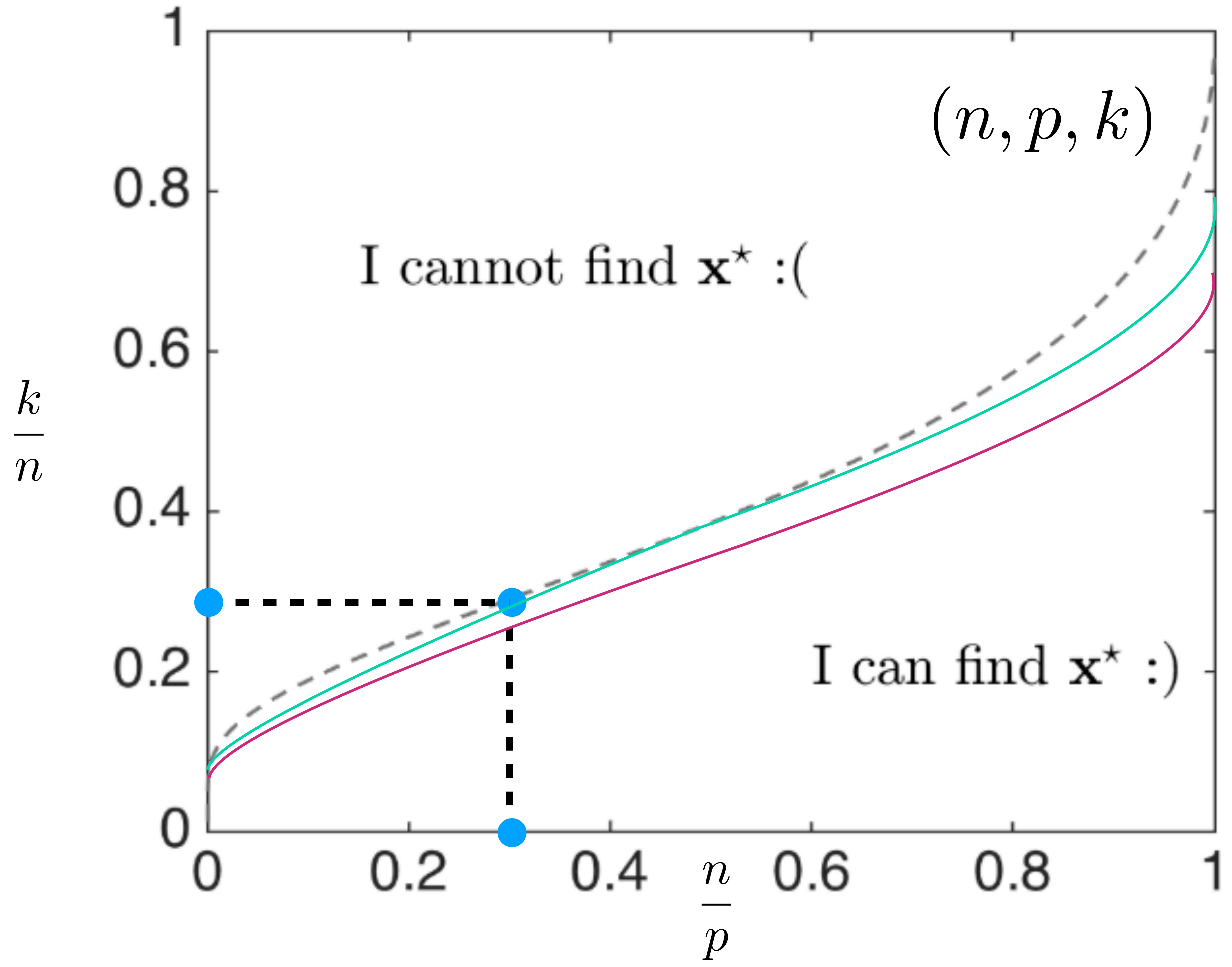

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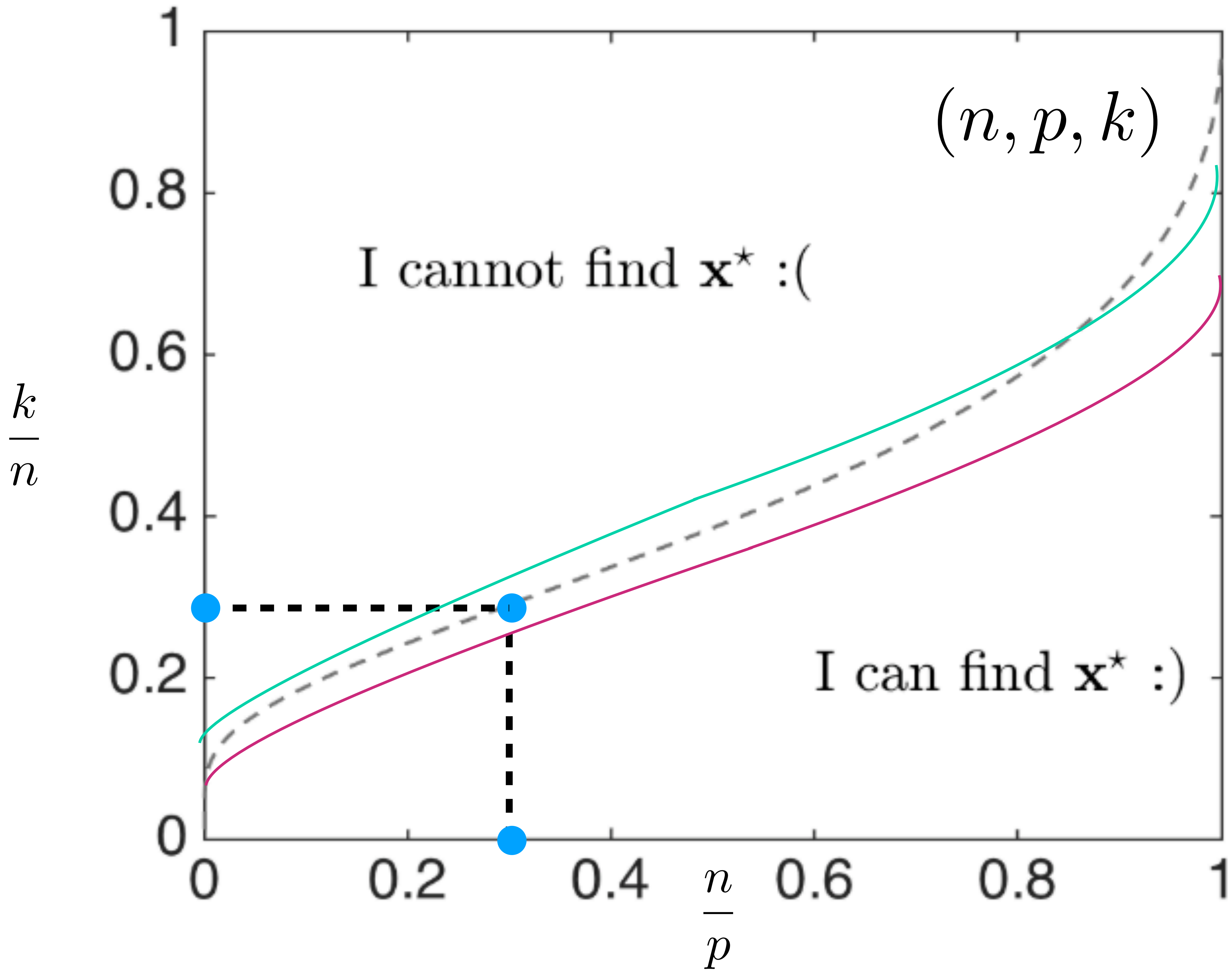

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(Some of these methods can be found in the Review part)

But does this step size selection work in theory?

Whiteboard

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- Q: ``What happens if we overshoot sparsity level?``
- A: ``We actually get denser and denser solutions``
- Q: ``Is there any non−provable tweaks?``
- A: ``Problem−dependent strategies``

``All these sound interesting.. but do they extend to other objectives? And how are they related with what we discussed so far?``

(Lipschitz gradient continuity, strong convexity, Hessians, etc..)

A different view of RIP

- Reminder

$$(1-\delta)\|x_1 - x_2\|_2^2 \leq \|A(x_1 - x_2)\|_2^2 \leq (1+\delta)\|x_1 - x_2\|_2^2,$$

$$\text{for some } \delta \in (0,1),\ \forall\ k\text{-sparse } x_1, x_2 \in \mathbb{R}^p$$

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- When objective has Lipschitz continuous gradients and is strongly convex:

$$\mu I \preceq \nabla^2 f(x) \preceq LI$$

Restricted smoothness and strong convexity

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  Let's say $x \in \mathcal{C} \subseteq \mathbb{R}^p$

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- Restricted strong convexity

$$f(y) \geq f(x) + \langle \nabla f(x), y - x \rangle + \frac{\mu}{2}\|x - y\|_2^2, \quad \forall x, y \in \mathcal{C}$$

(Note that this μ is different from that of general strong convexity)

Examples

- Sparse logistic regression

$\in \{\pm 1\}$

$$\begin{aligned} \min_{x \in \mathbb{R}^p} \quad & \frac{1}{n} \sum_{i=1}^{n} \log\left(1 + \exp\left(-y_i a_i^\top x\right)\right) + \frac{\lambda}{2} \|x\|_2^2 \\ \text{s.t.} \quad & \|x\|_0 \le k \end{aligned}$$

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Satisfies restricted strong convexity with constant:

$$\mu = (\gamma_k + \lambda), \ \text{where } \gamma_k = \lambda_{\min}\left(A^\top \Lambda A, k\right)$$

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and restricted smoothness with constant:

$$L = \left(\lambda_{\max}\left(A^\top A, k\right) + \lambda\right)$$

(For more information, see
"Gradient hard thresholding pursuit")

Examples

- Graphical model selection (under Gaussian assumptions)

Whiteboard

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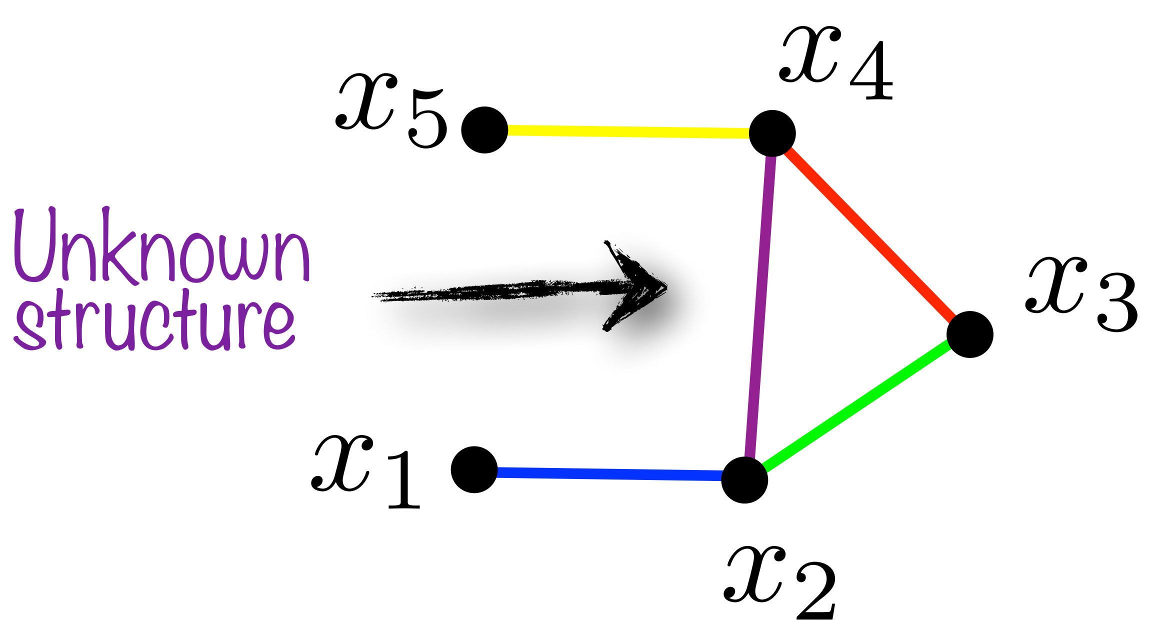

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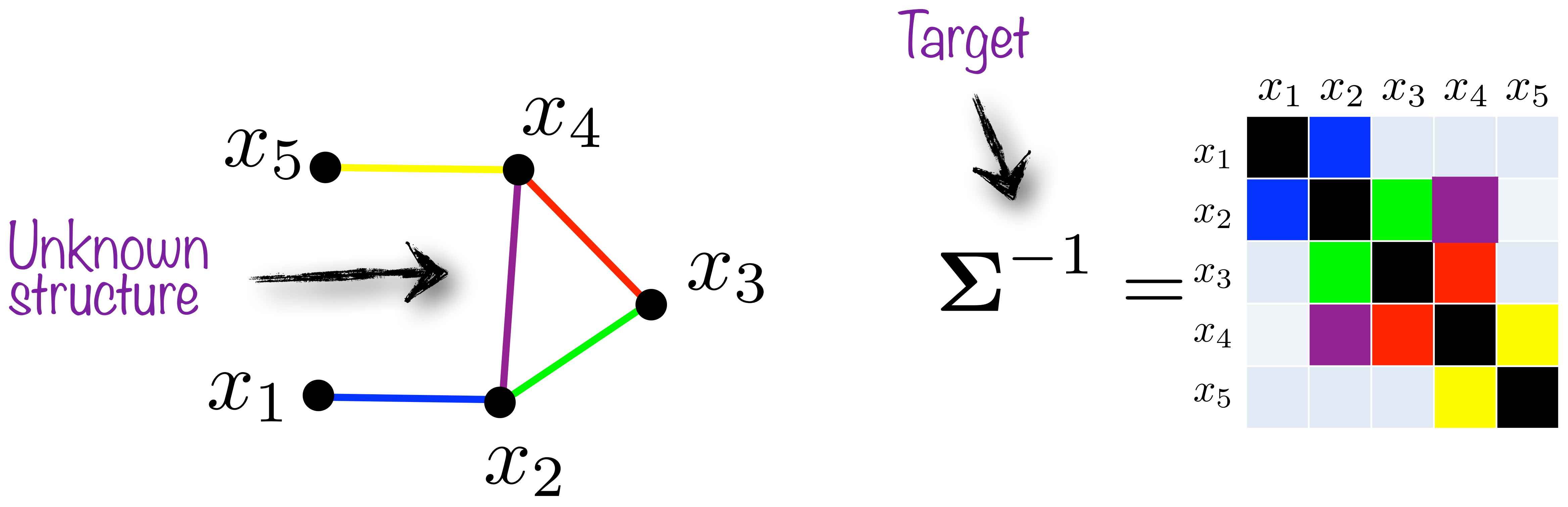

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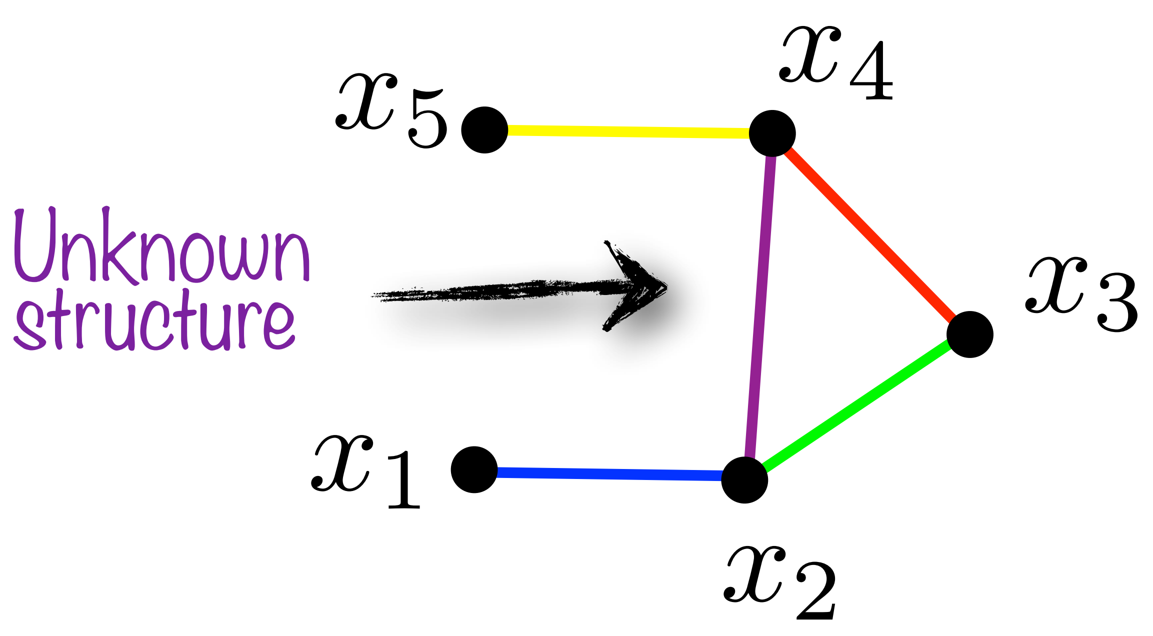
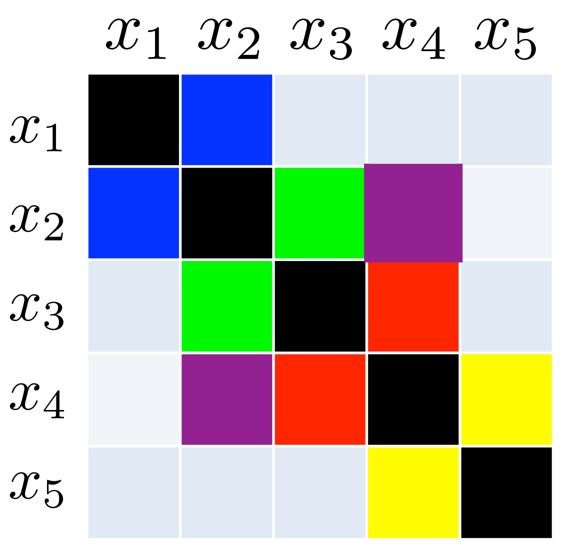
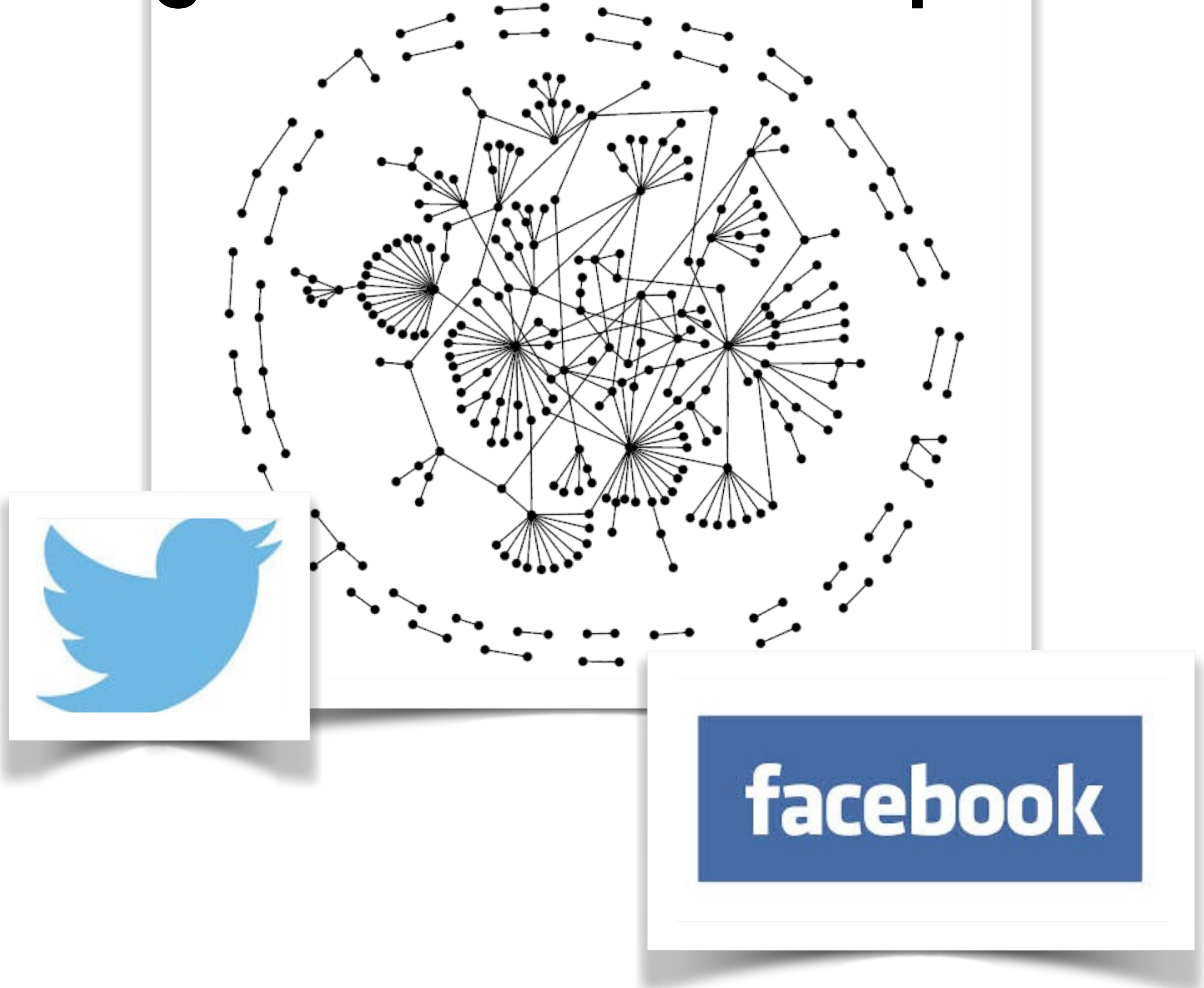


Target

$$\Sigma^{-1} =$$



Large-scale real example



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- Graphical model selection (under Gaussian assumptions)
  - Given a data set $\mathcal{D}$, drawn from a joint pdf with unknown covariance $\mathbf{\Sigma}$, the aim is to learn a sparse matrix $\mathbf{\Theta}$ that approximates $\mathbf{\Sigma}^{-1}$.

    Input: sample covariance $\widehat{\mathbf{\Sigma}}$ calculated usually from limited samples

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Convex

Optimization problem 2
$\min_{\Theta \in \mathbb{R}^{p\times p}} \quad -\log\det(\Theta) + \mathrm{Tr}\left(\widehat{\Sigma}\Theta\right)$ $\text{s.t.} \quad \|\Theta\|_0 \le k$

Non-convex

Beyond plain sparsity

- Our discussion so far holds for discrete structures beyond sparsity:

  Block–sparsity, overlapping block–sparsity, dispersive models, tree sparsity, graph–sparsity, etc..

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- Our discussion so far holds for discrete structures beyond sparsity:

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- As long as the projection onto the combinatorial constraint can be computed efficiently:

$$\begin{array}{ll} \min_{x \in \mathbb{R}^p} & \|x - y\|_2^2 \\ \text{s.t.} & x \in \mathcal{C} \end{array}$$

- Various extensions include inexact projections, greedy approaches, and there are connections with (sub/super)modular optimization

Interlude: Statistics in Data Science

- We will use the example of RIP

- Disclaimer: this is not a complete introduction to concentration inequalities

  (How many would be interested in learning about concentration inequalities (as a course)?)

Conclusion

- This lecture considers **sparse model selection** in Data Science applications
- While there are rigorous and efficient methods also in the convex domain we followed the **non-convex path** of hard thresholding methods
- We discussed some global convergence guarantees, and highlighted the importance of hyper-parameter tuning

Next lecture

- We will consider the case of **low-rank recovery**, natural extension of sparsity – there, we have different ways to exploit non-convexity