

COMP 414/514:
Optimization – Algorithms, Complexity
and Approximations

Lecture 10

Overview

- In the previous lecture, we:
 - Considered **low-rank model selection** in Data Science applications
 - Followed the **non-convex path**, beyond hard thresholding methods
 - Discussed some global convergence guarantees (under proper initialization assumptions) and mentioned some open questions

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 - Followed the **non-convex path**, beyond hard thresholding methods
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- For the next 2–3 lectures, we will worry about the **landscape** of such non-convex scenarios:
 - We will discuss about **types of stationary points**, focus on **saddle points** and study some of their properties
 - We will introduce conditions that allow **escaping from saddle points**
 - We will study matrix sensing as a test case, and how to prove “no spurious local minima” arguments

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(Specifically can be polynomially solvable)

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- Example: Homogeneous quartics

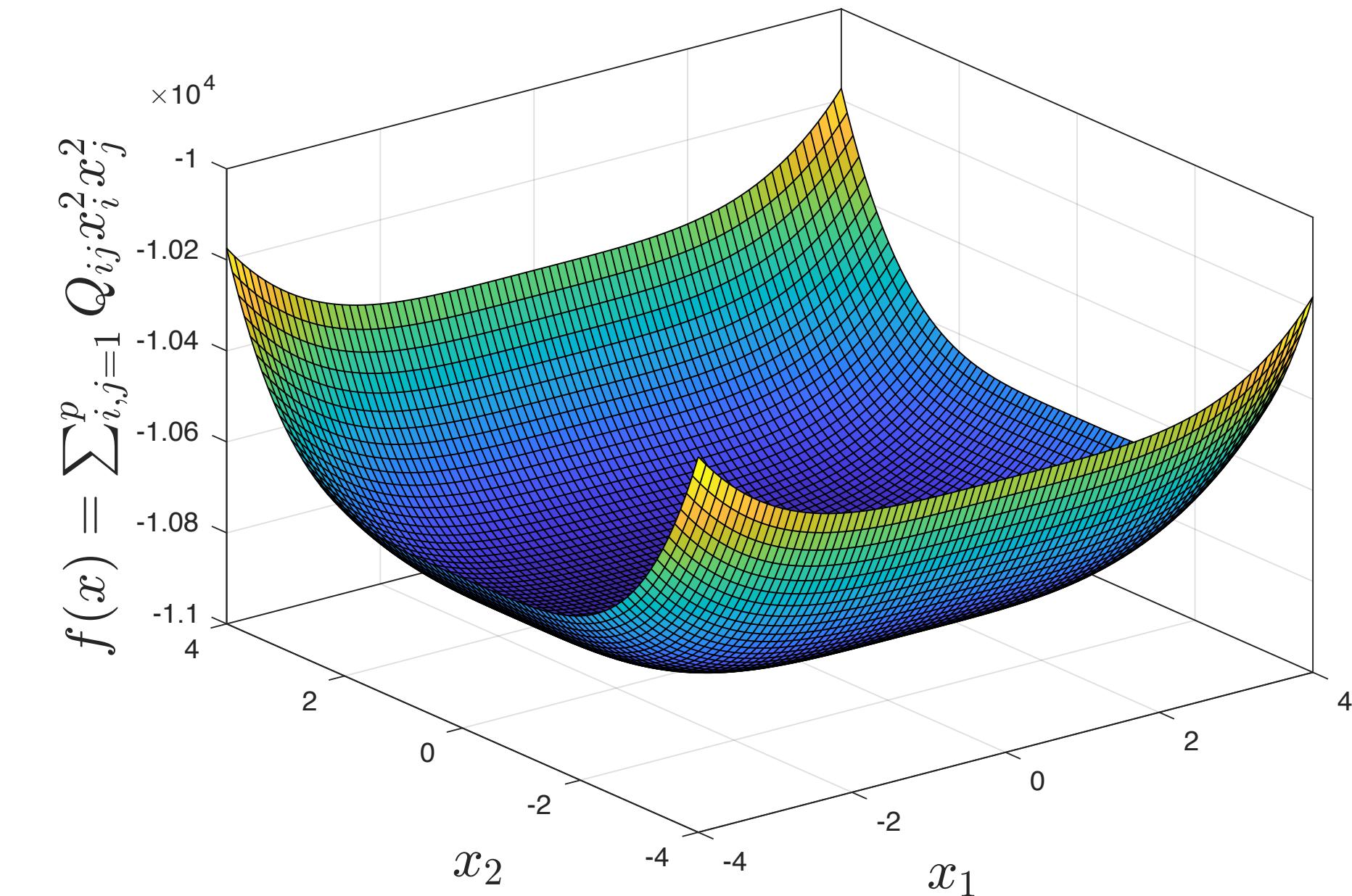
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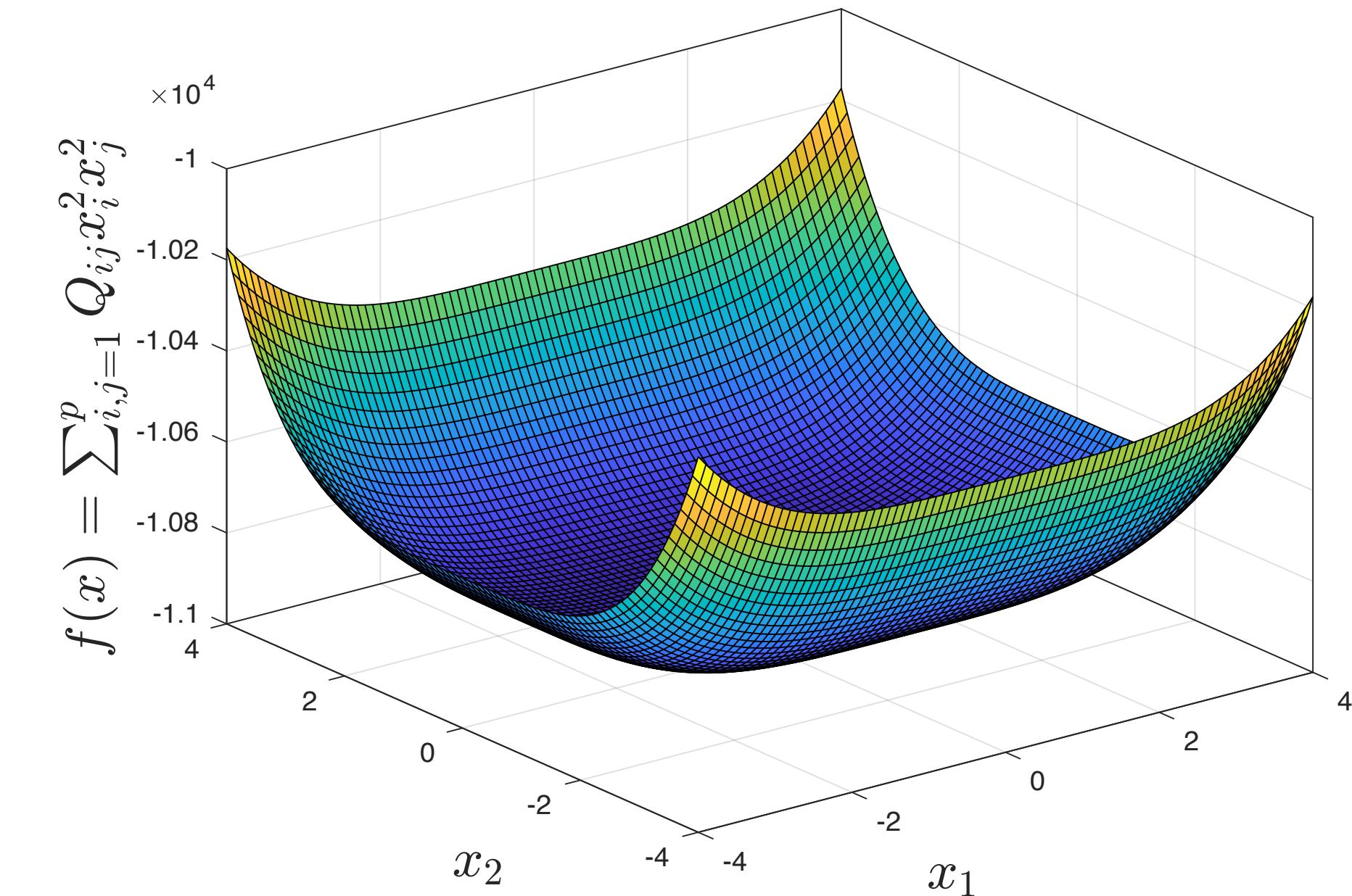


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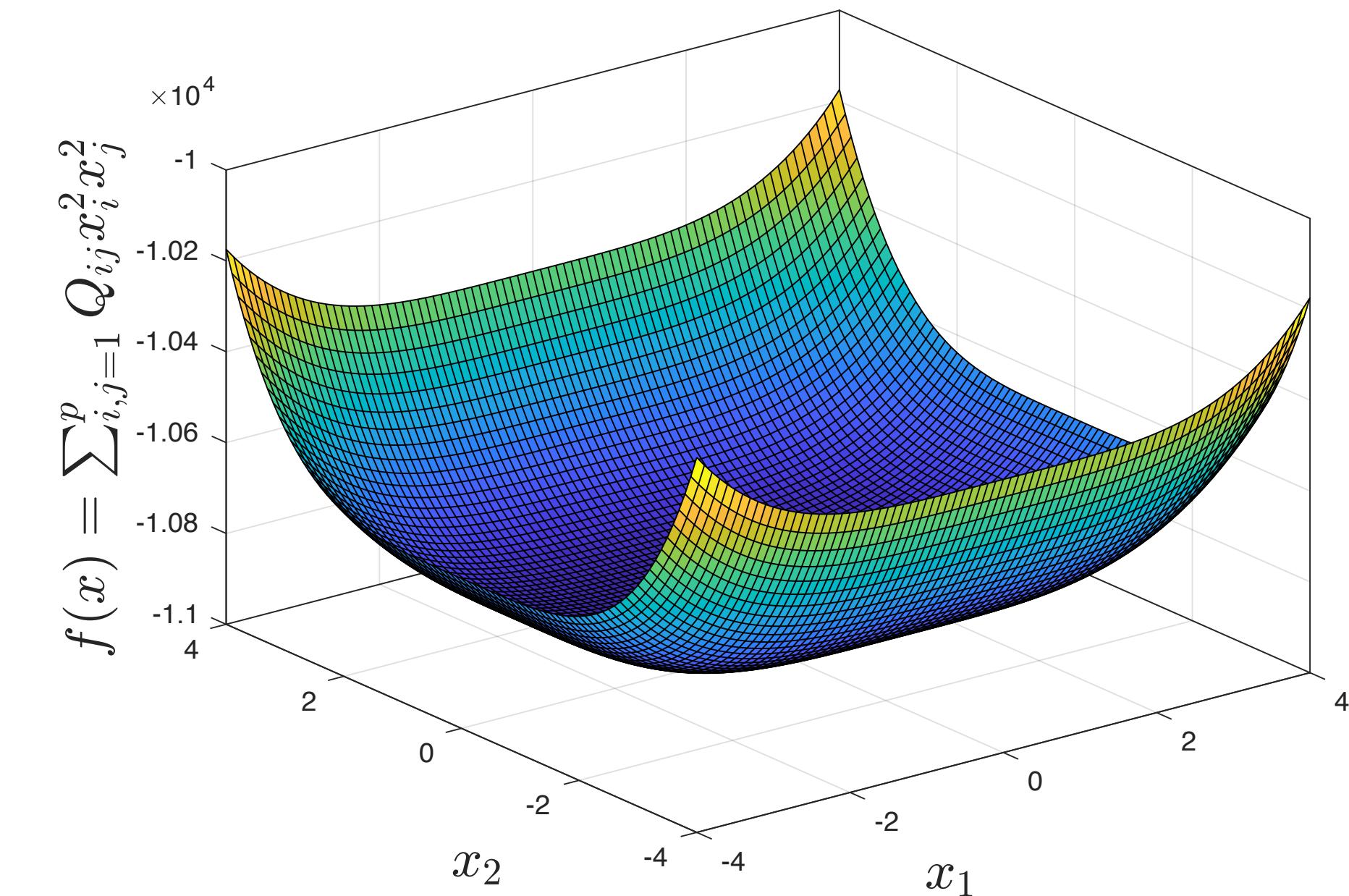


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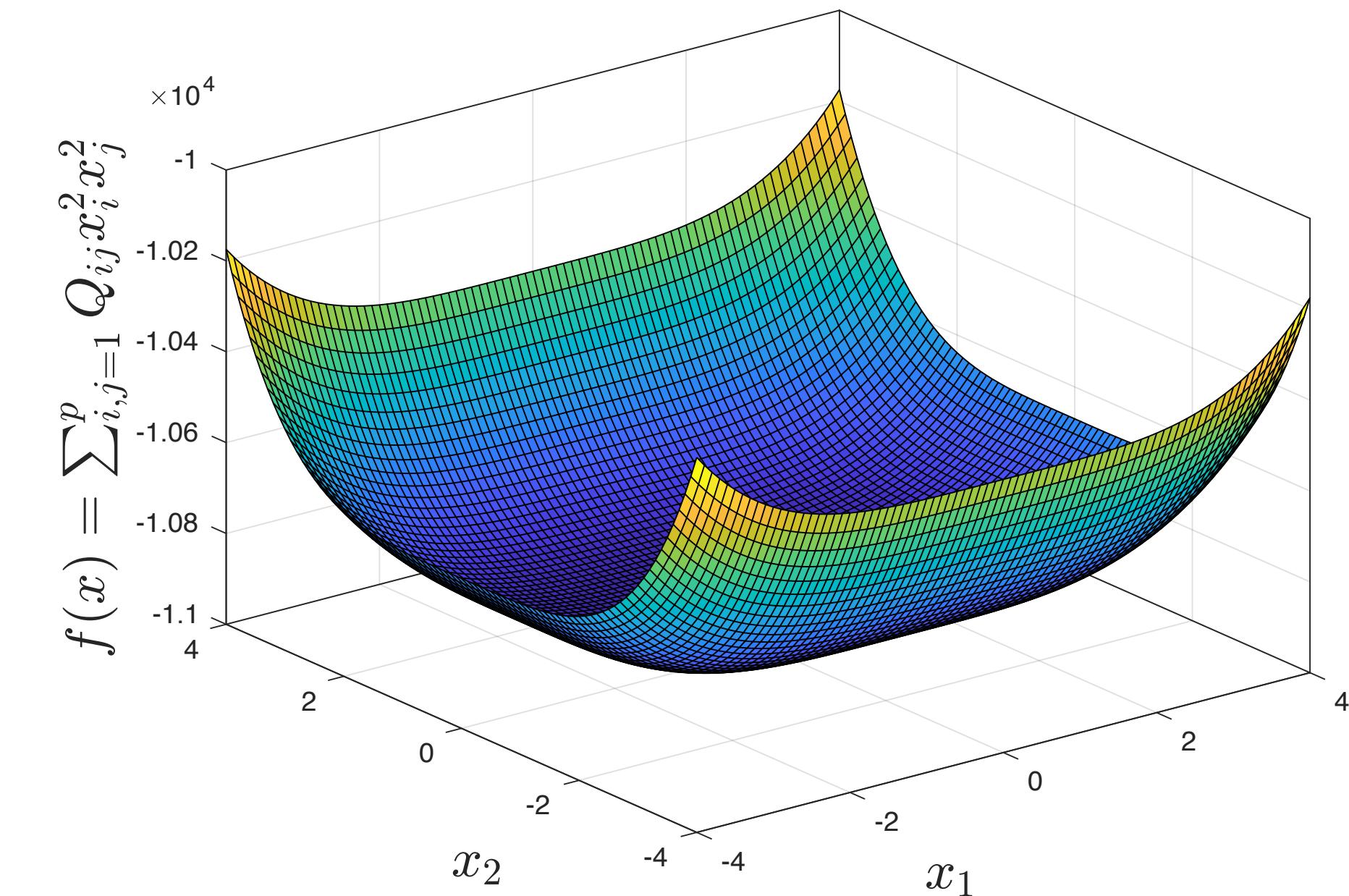
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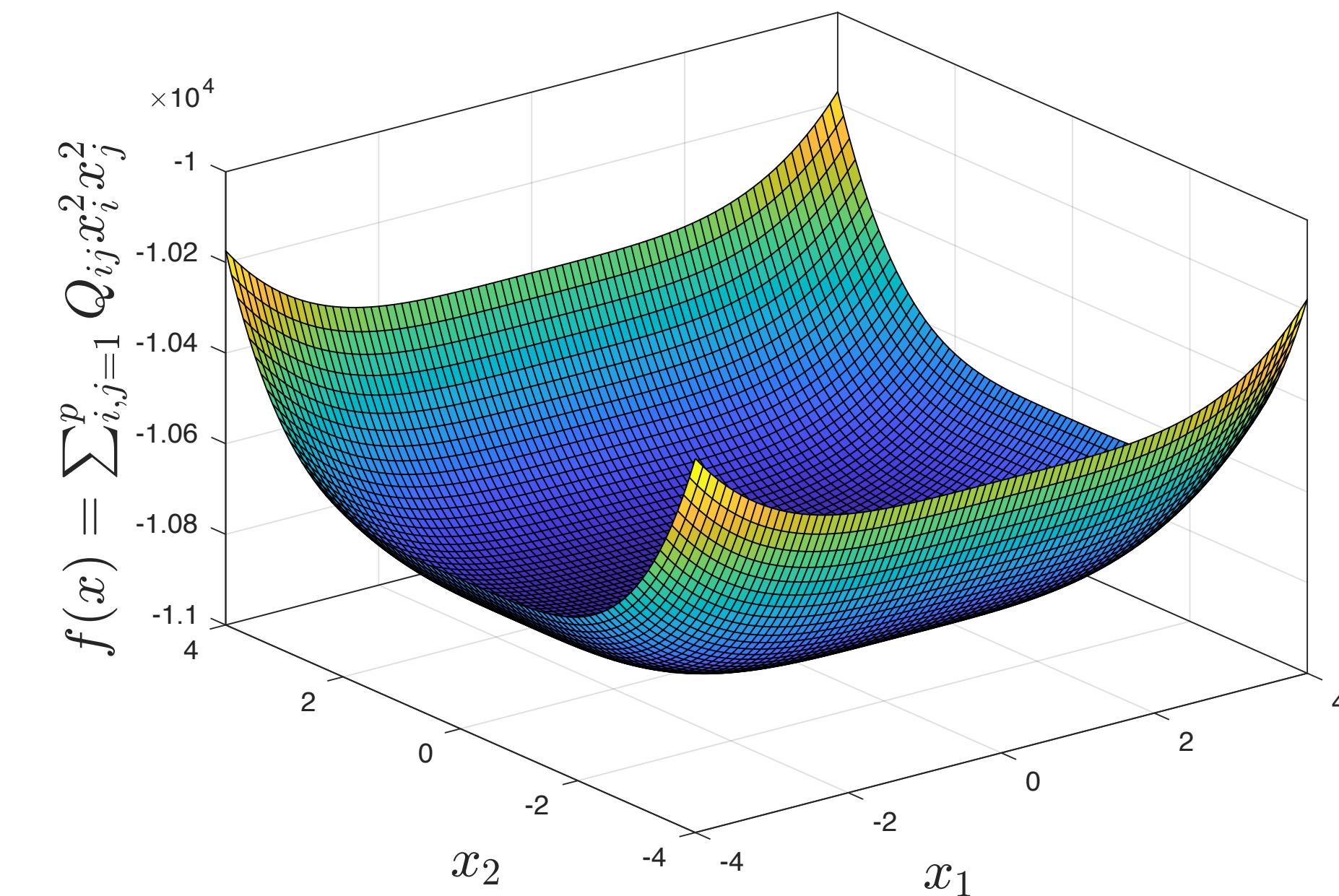
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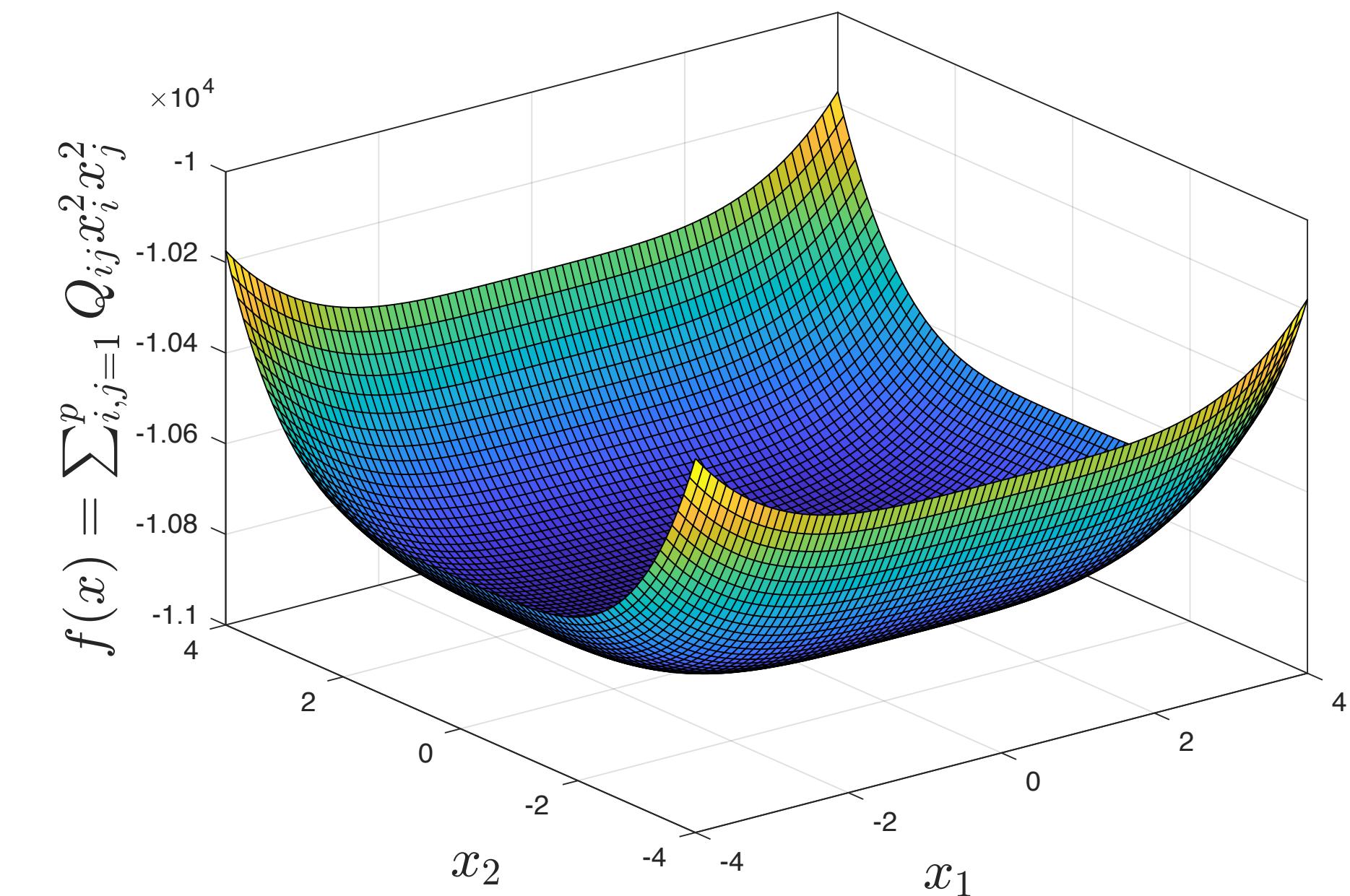
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 - Thus, $x = 0$ is global min.

$$- p = 2, Q \succeq 0$$

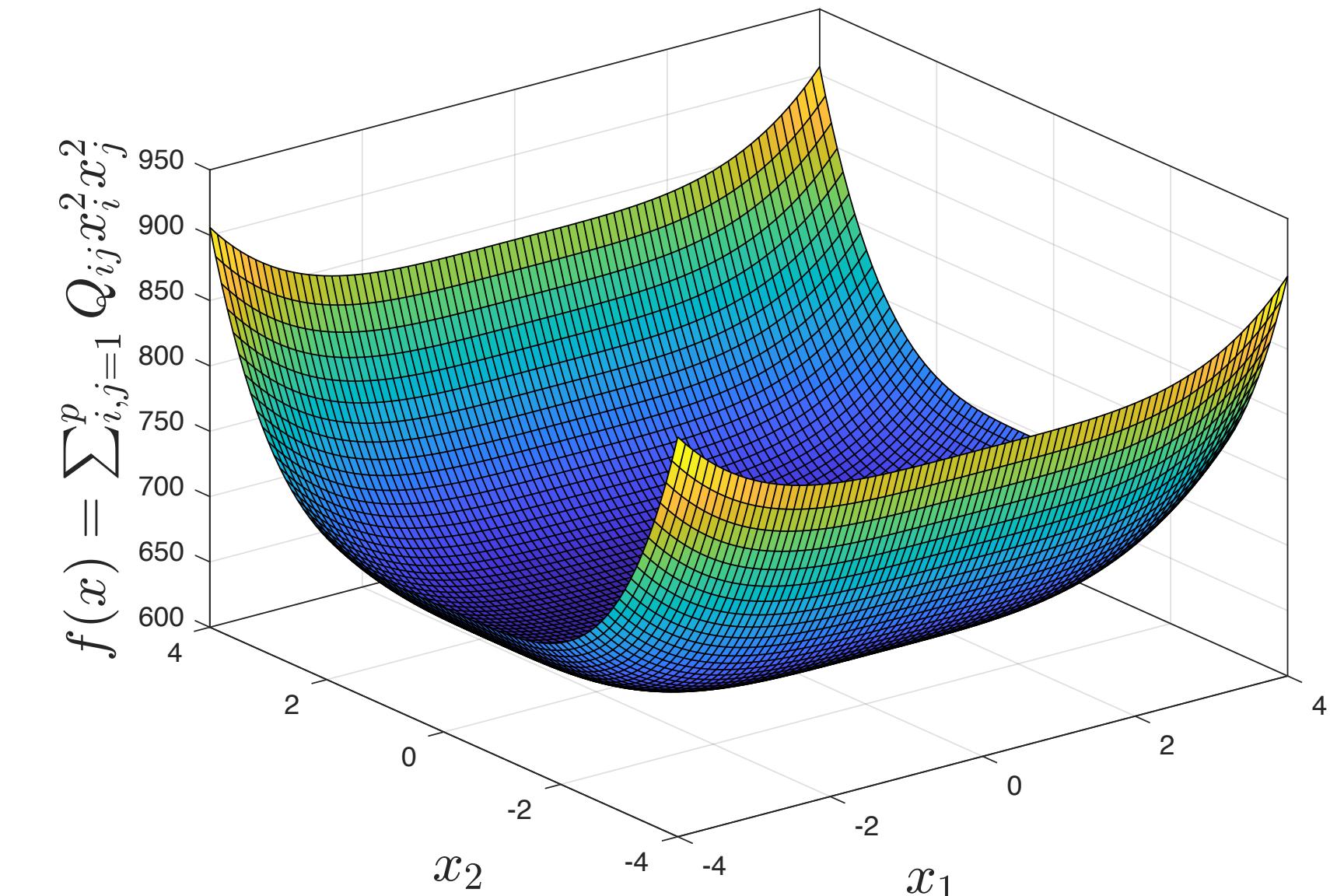


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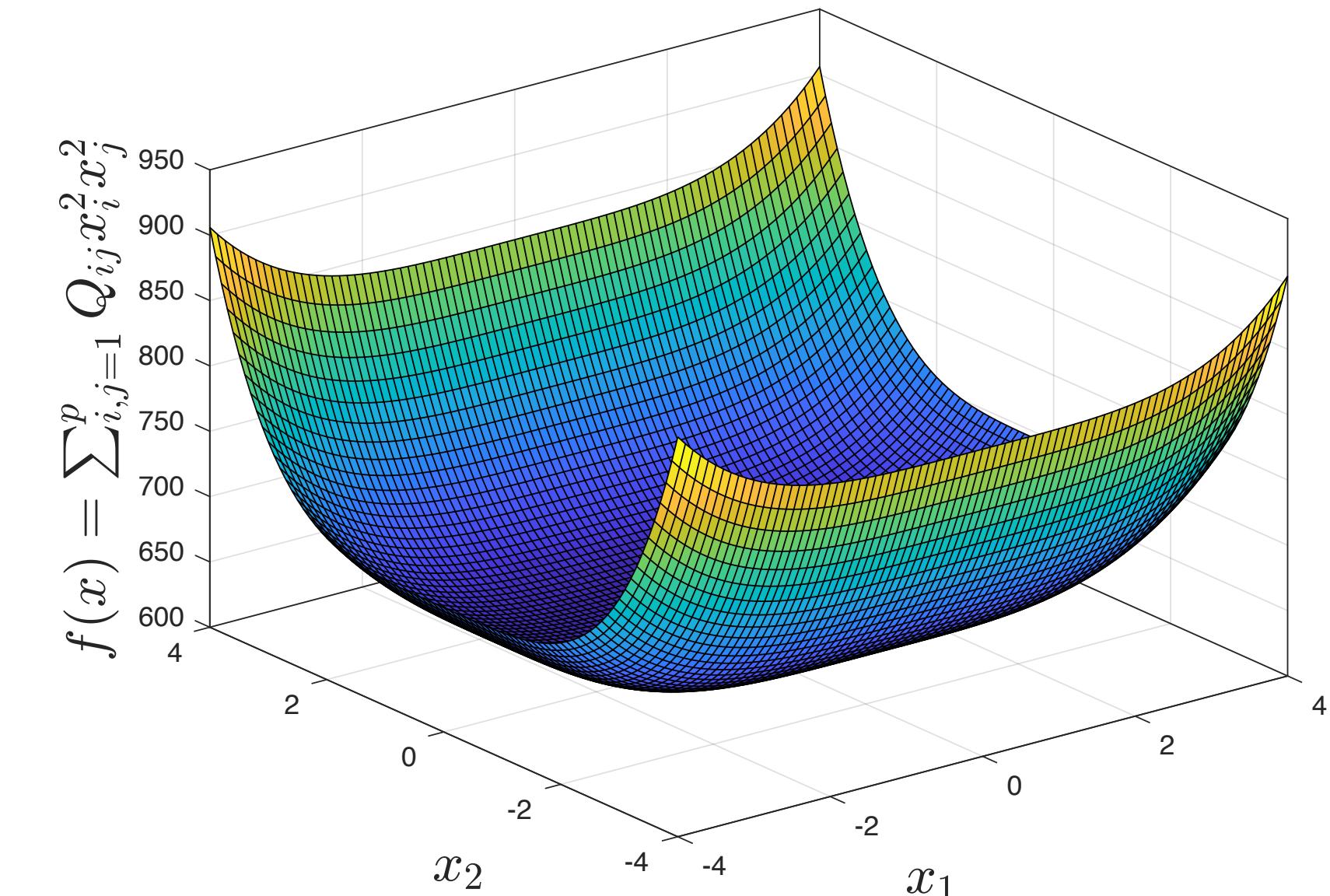


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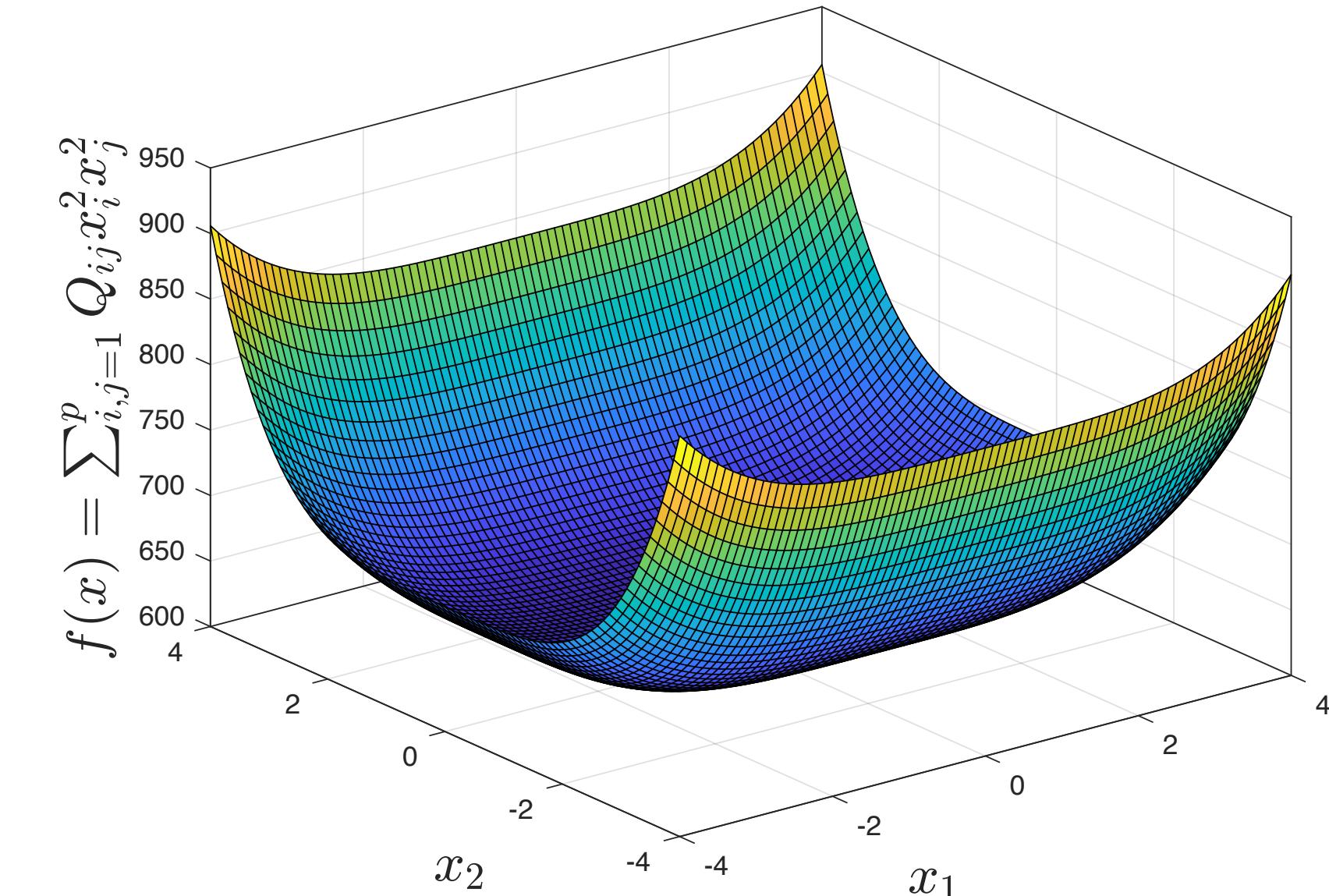


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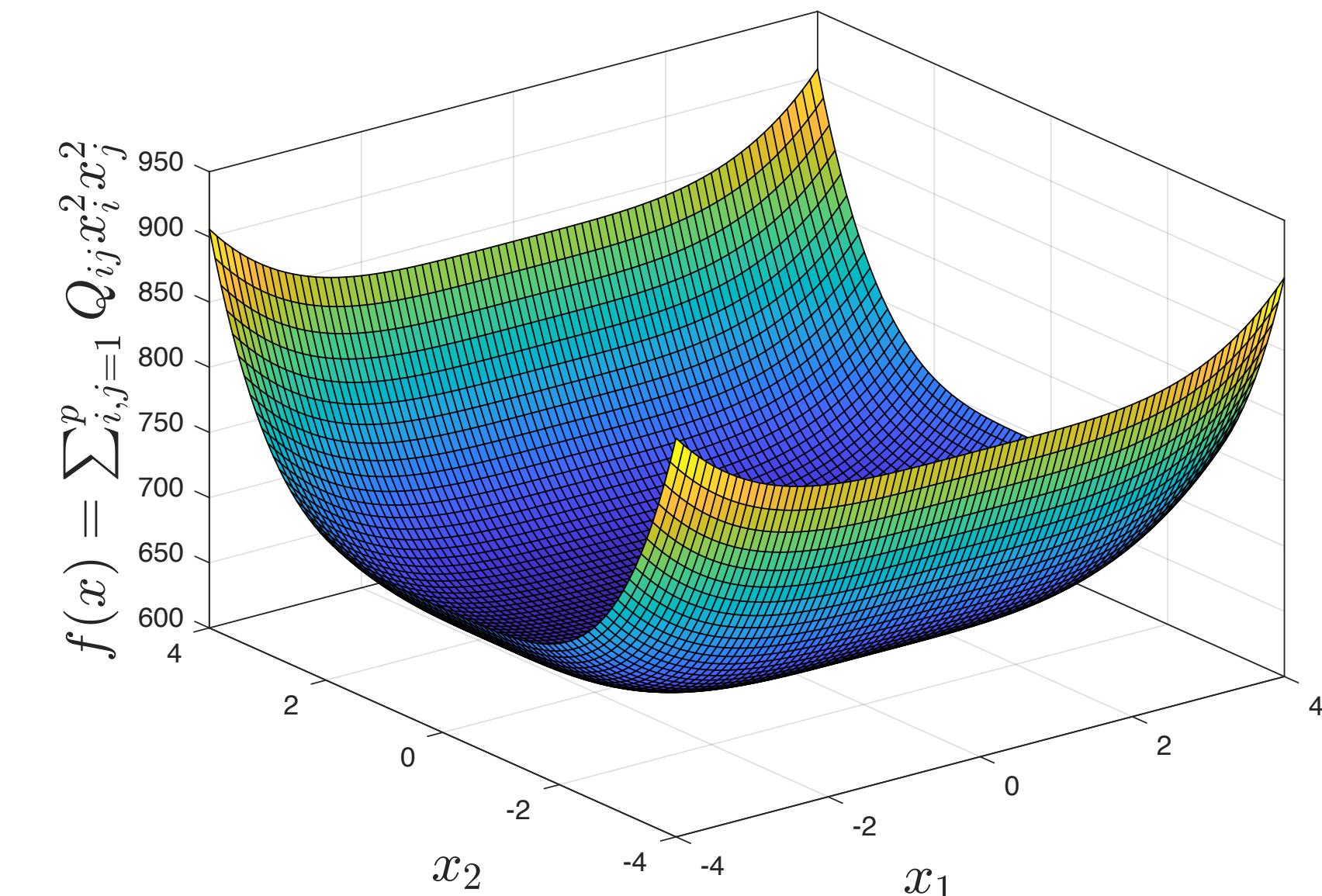
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- Same observations apply

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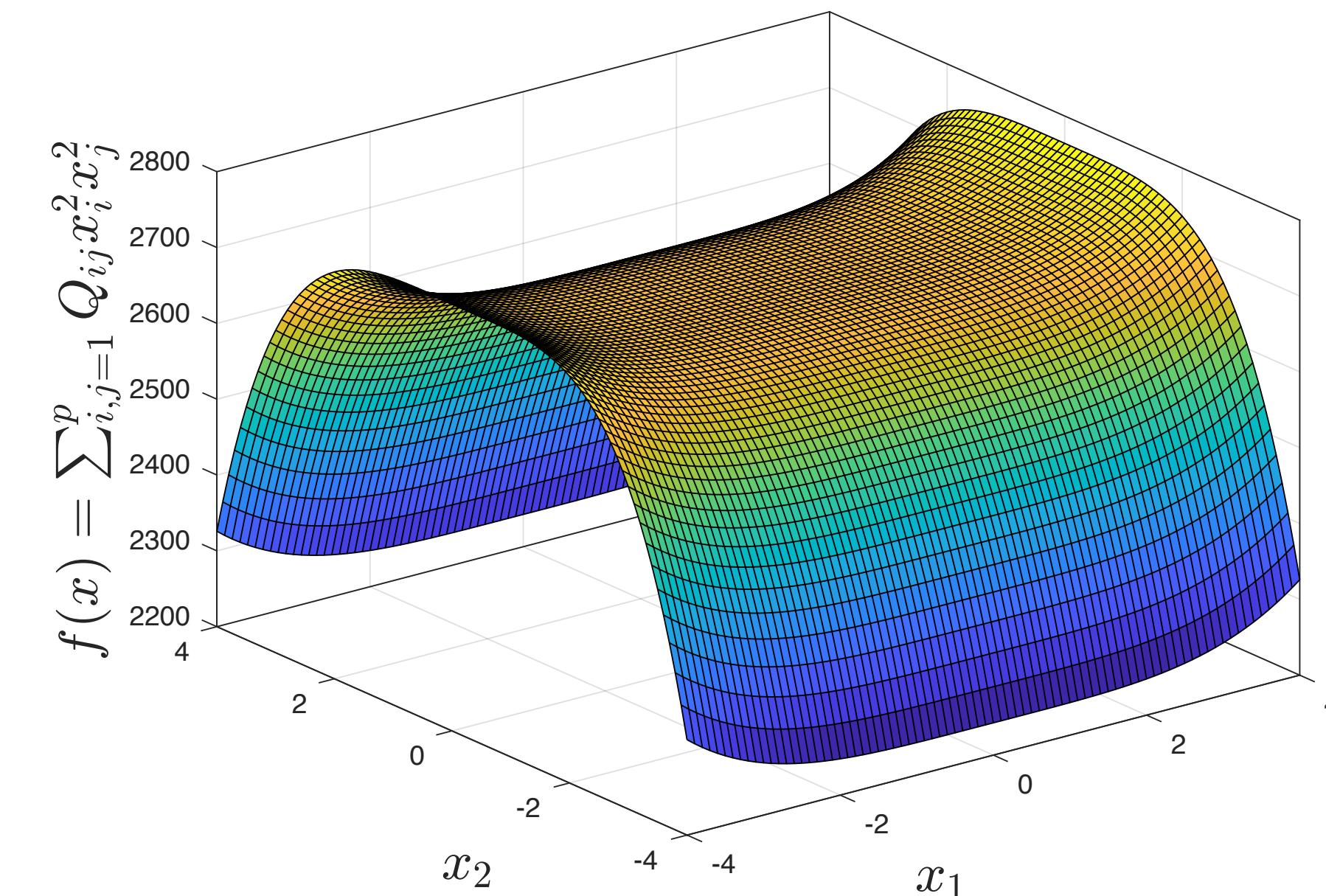


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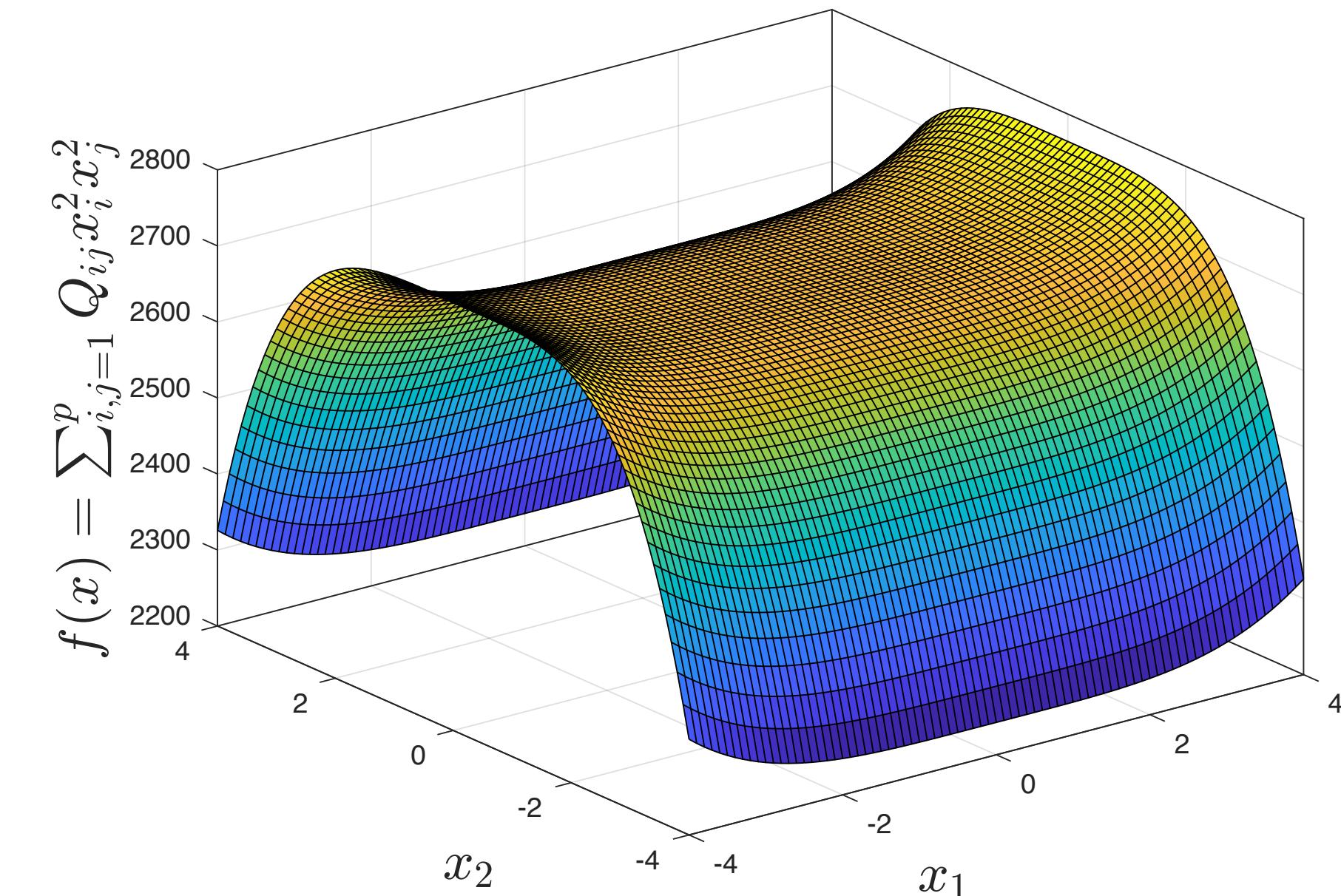


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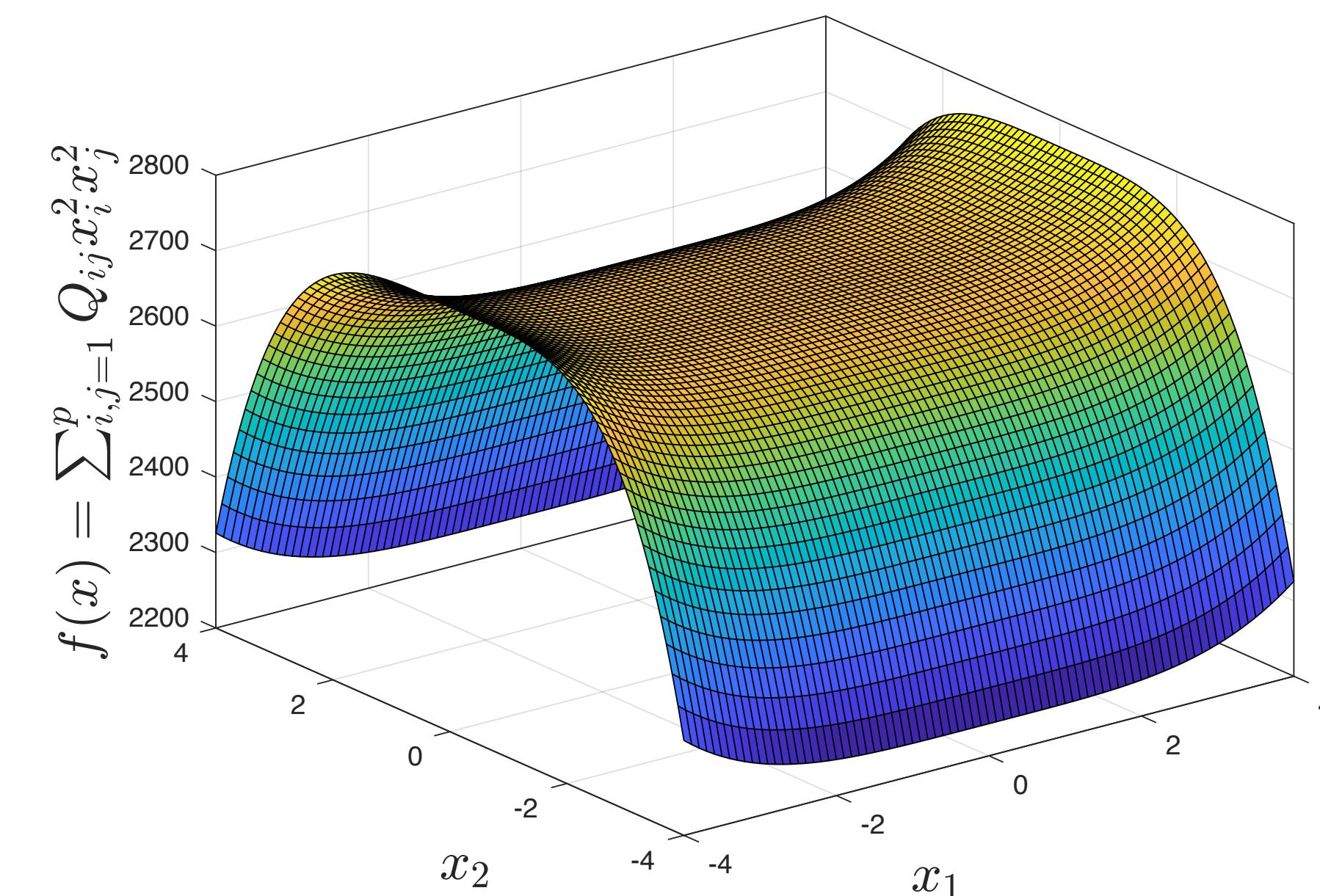


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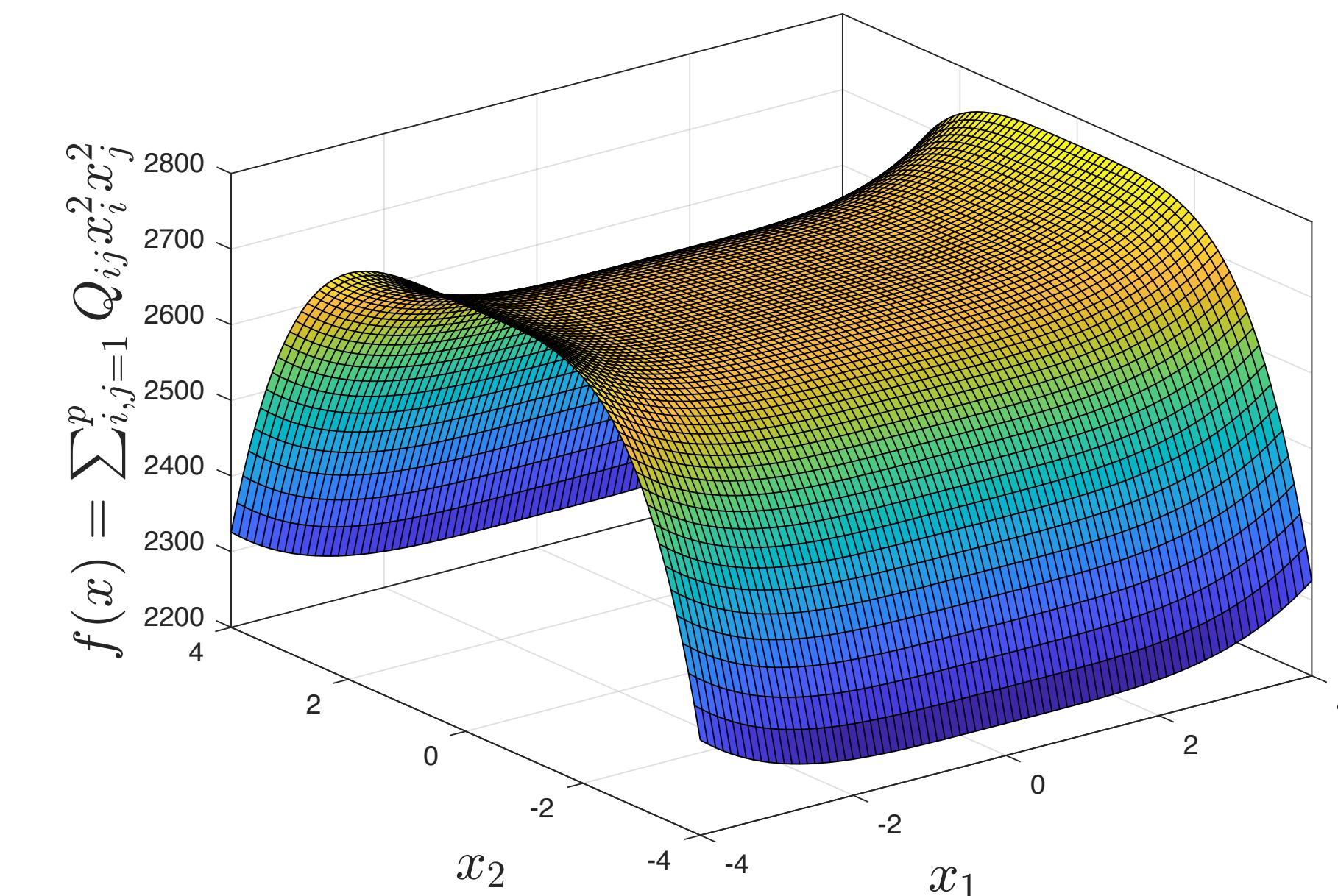
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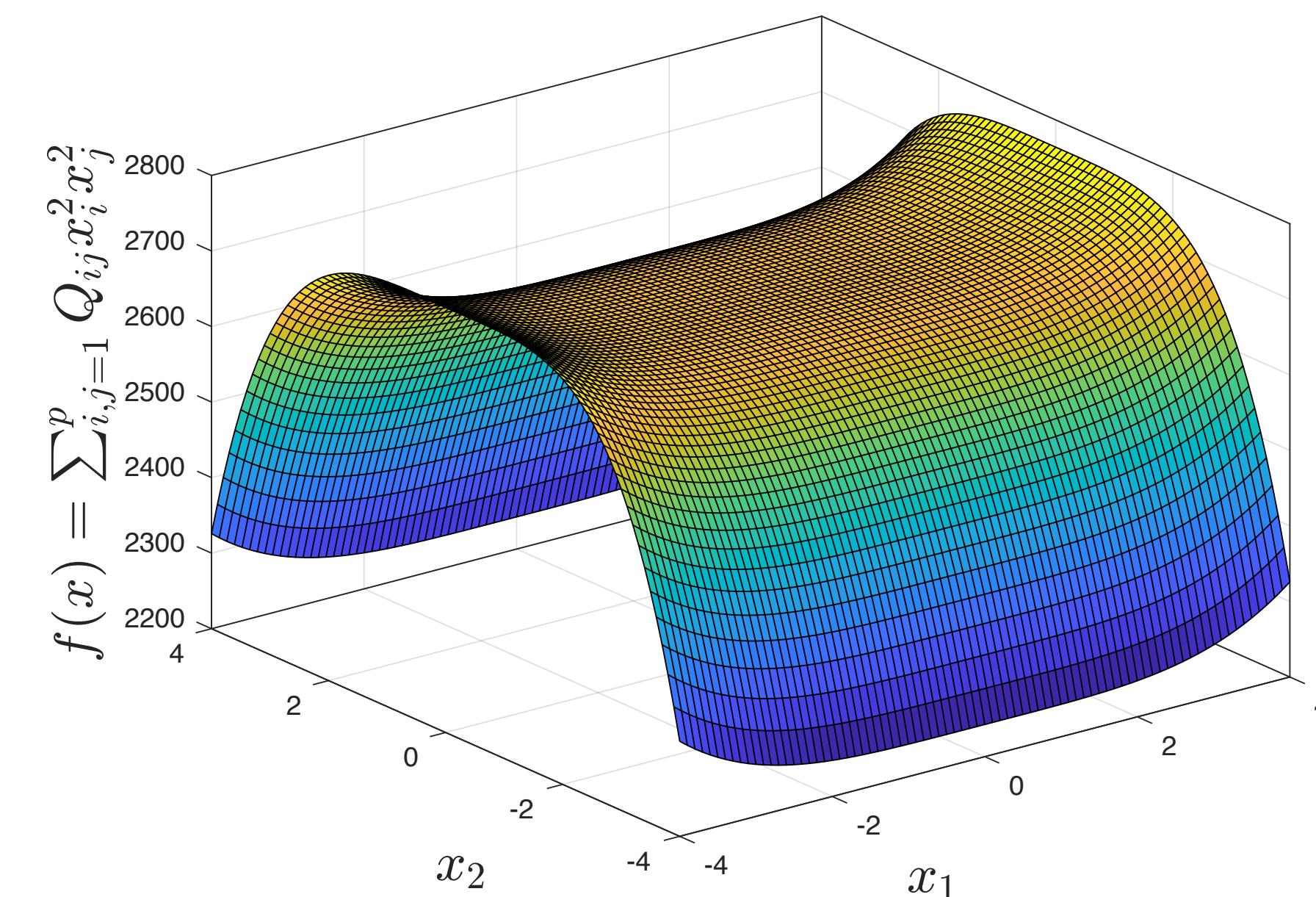
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 - Thus, zero is a minimum, maximum or saddle point

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- 0 is not a global minimizer if there exists non-negative u such that $u^\top Q u < 0$
- This is equivalent to checking if Q is not co-positive: NP-hard!
(equivalent to finding plant cliques in graphs)

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- Not the only example: QCQP, matrix completion/matrix sensing, etc.
tensor (matrix) decompositions

Flash back: GD and types of critical points

(also called stationary points)

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- Gradient descent for generic smooth functions:

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As t increases, the objective function decreases,
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- No guarantees on the type of critical point we converge to

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Desirable but not easily attainable!

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Often the next best thing

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- “The loss surfaces of multilinear networks”, Chromanska et al., 2014

“We conjecture that both simulated annealing and SGD converge to the band of low critical points, and that all critical points found there are local minima of high quality measured by the test error. This emphasizes a major difference between large- and small-size networks where for the latter poor quality local minima have non-zero probability of being recovered.”

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- For larger models, **a local minima is "good enough"**, since its loss value is roughly similar.
(We do not know the exact value of the global minima, so we can only conjecture)
- Why would this be true in practice?
 - Different random seeds lead to different models with similar performance

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Often stall the convergence to
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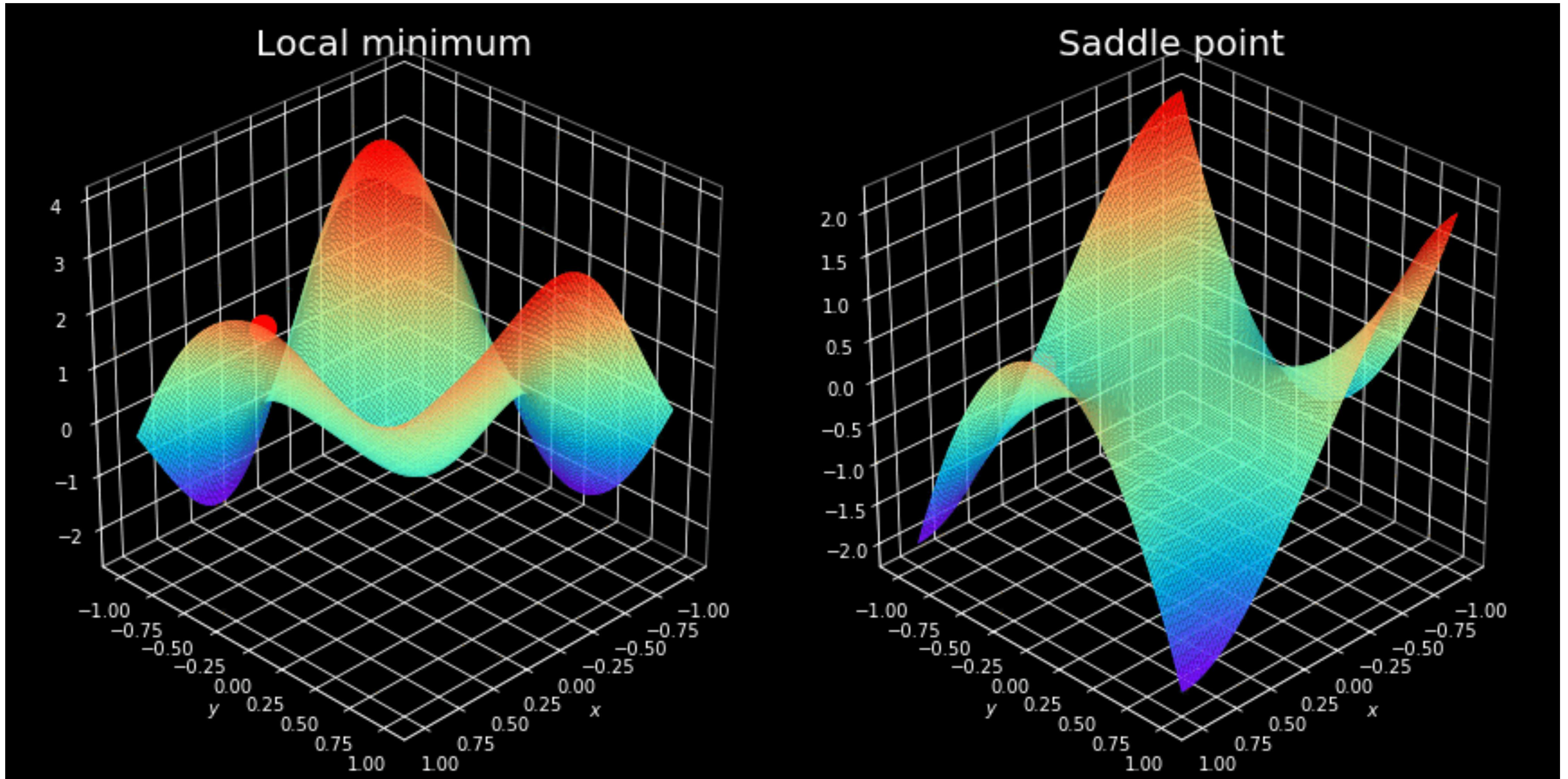
What's the deal with saddle points?

- “Identifying and attacking the saddle point problem in high-dimensional non-convex optimization.”, Dauphin et al., 2014

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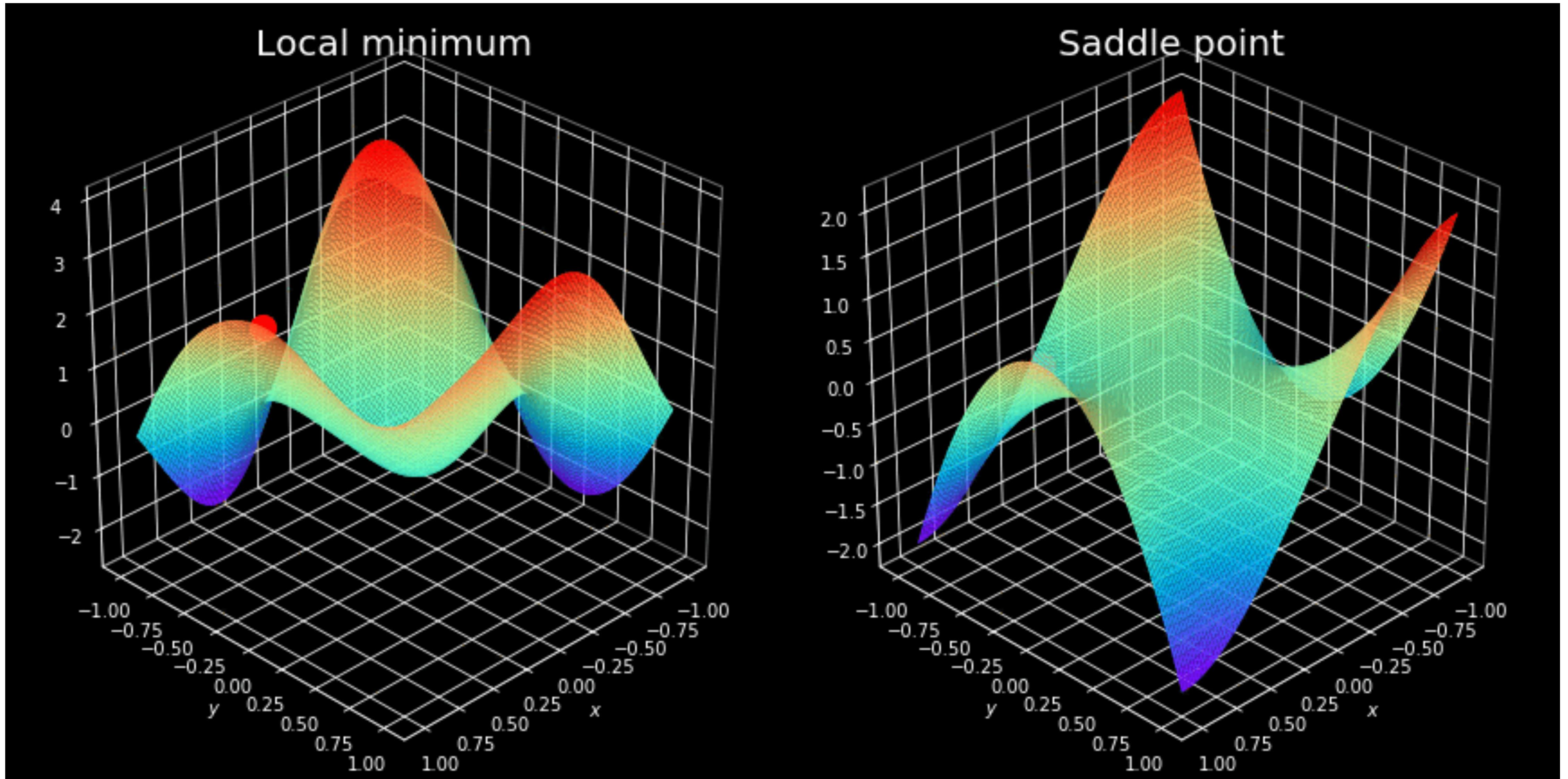
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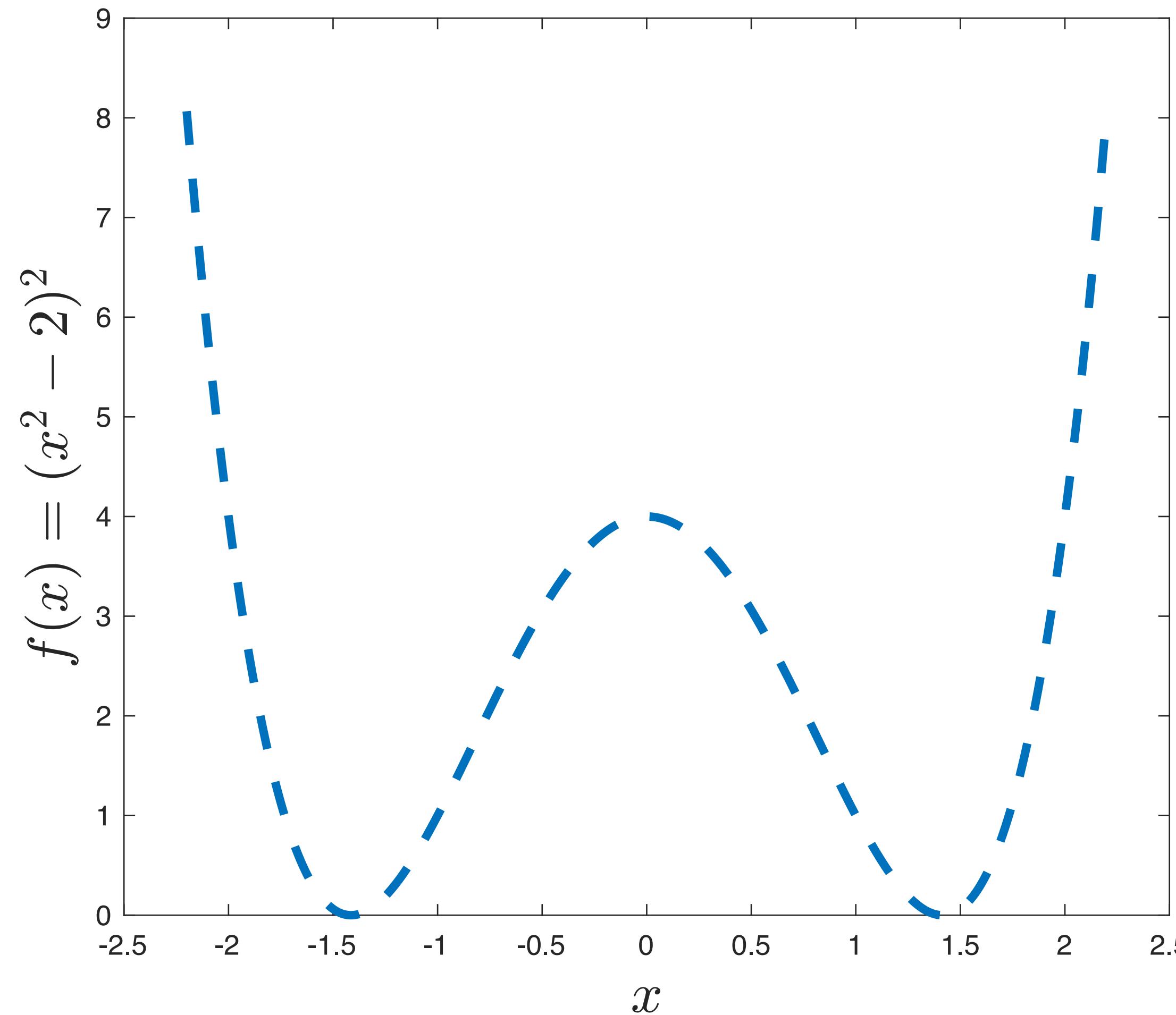
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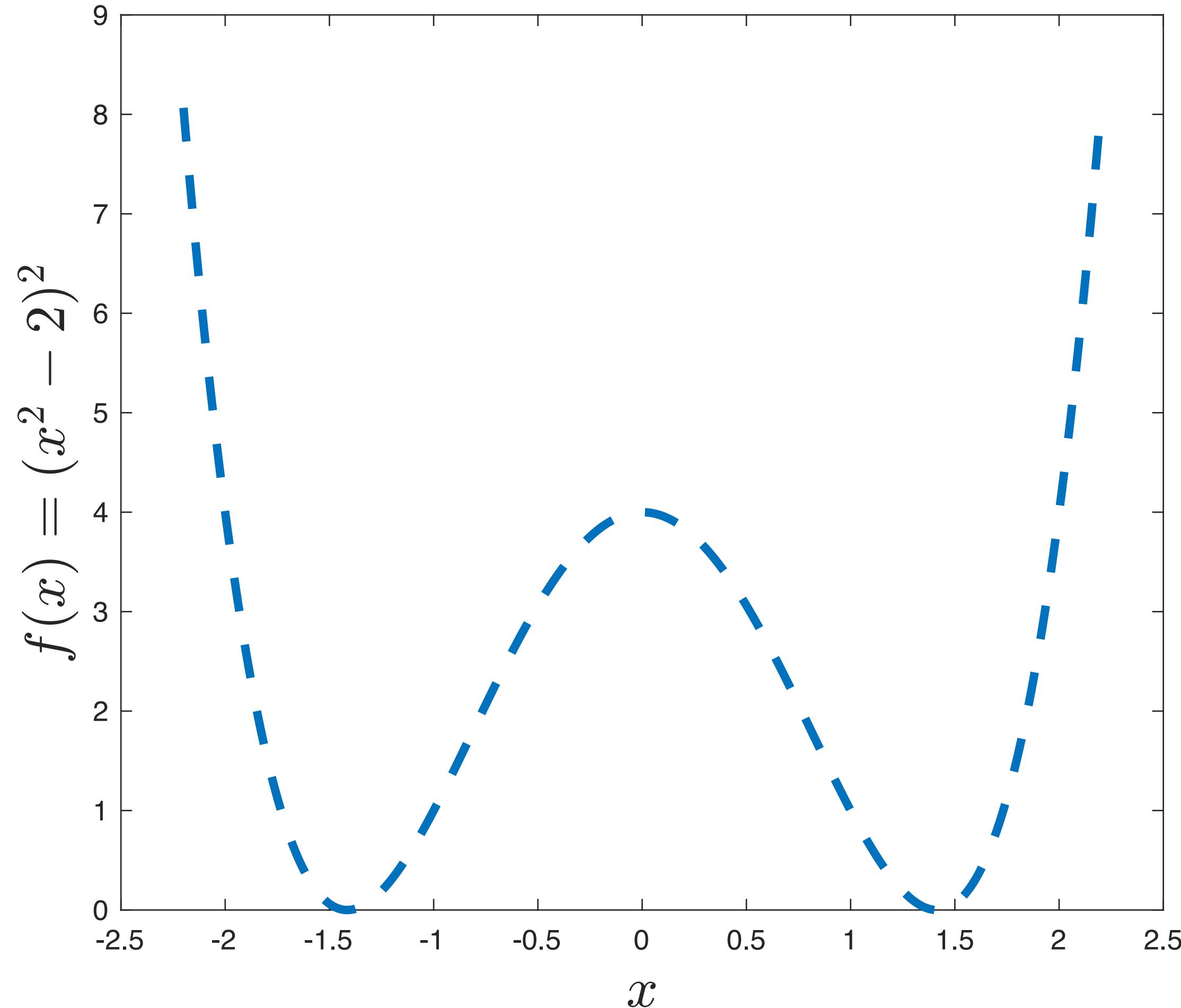
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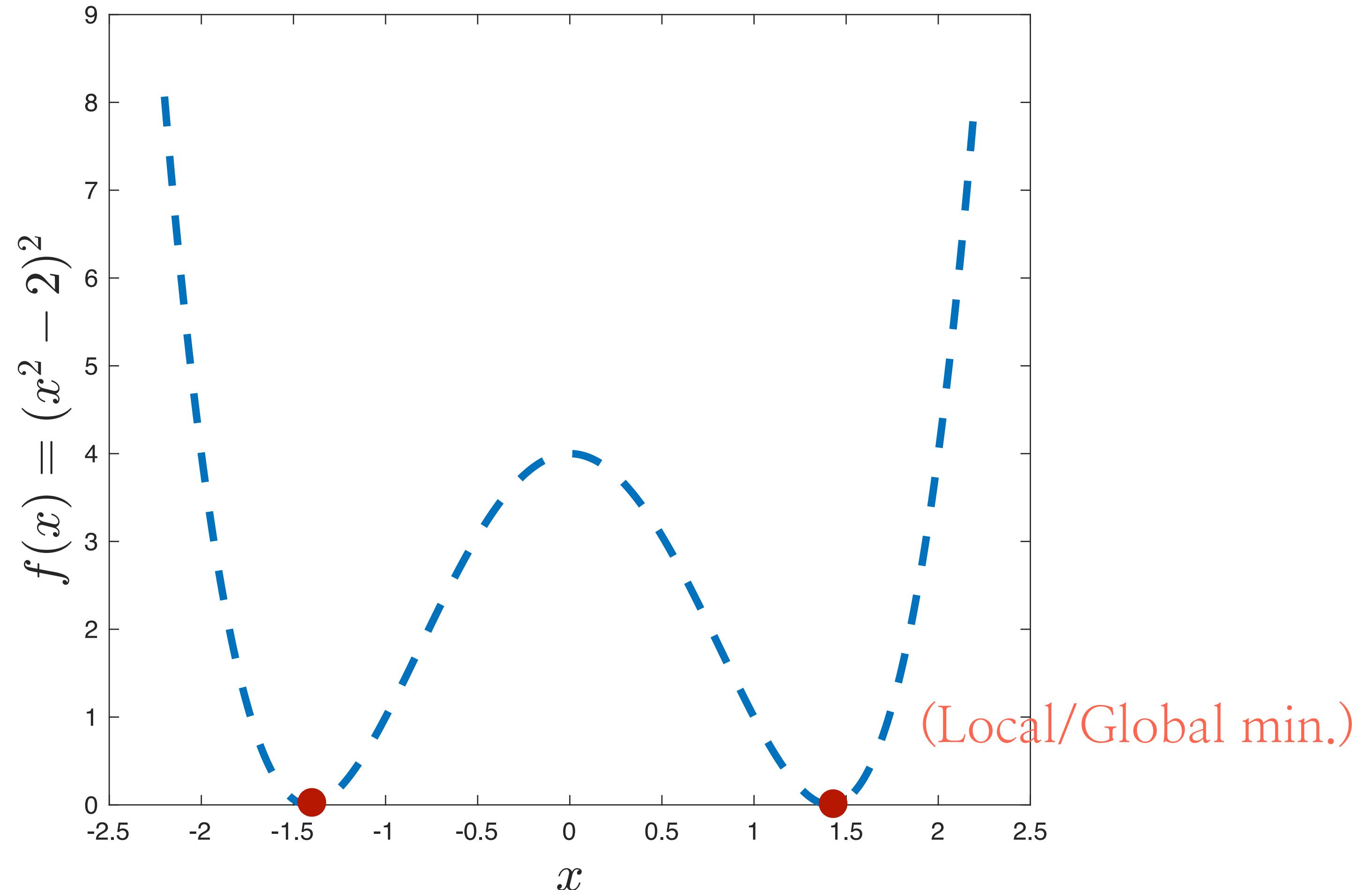
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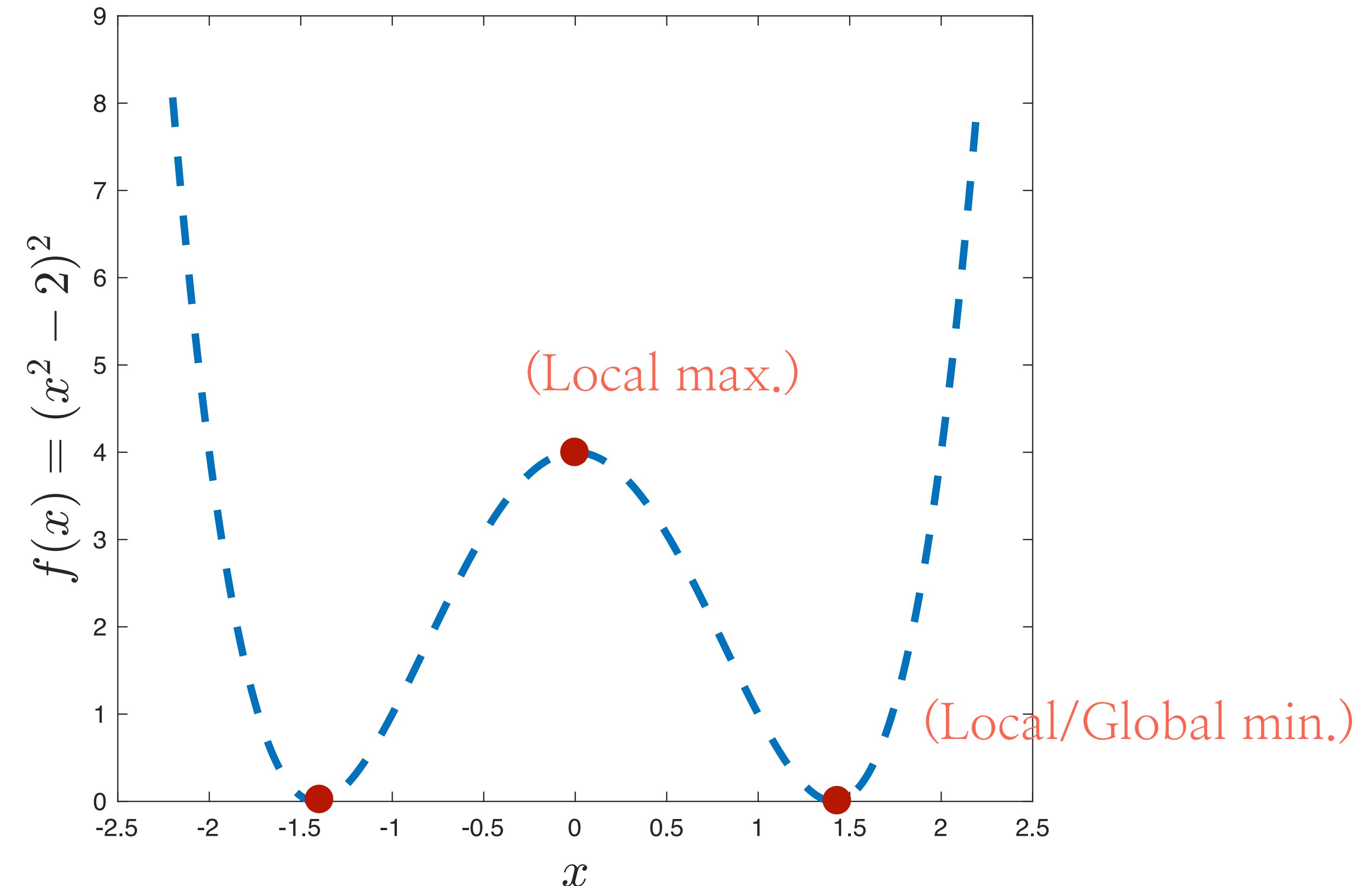
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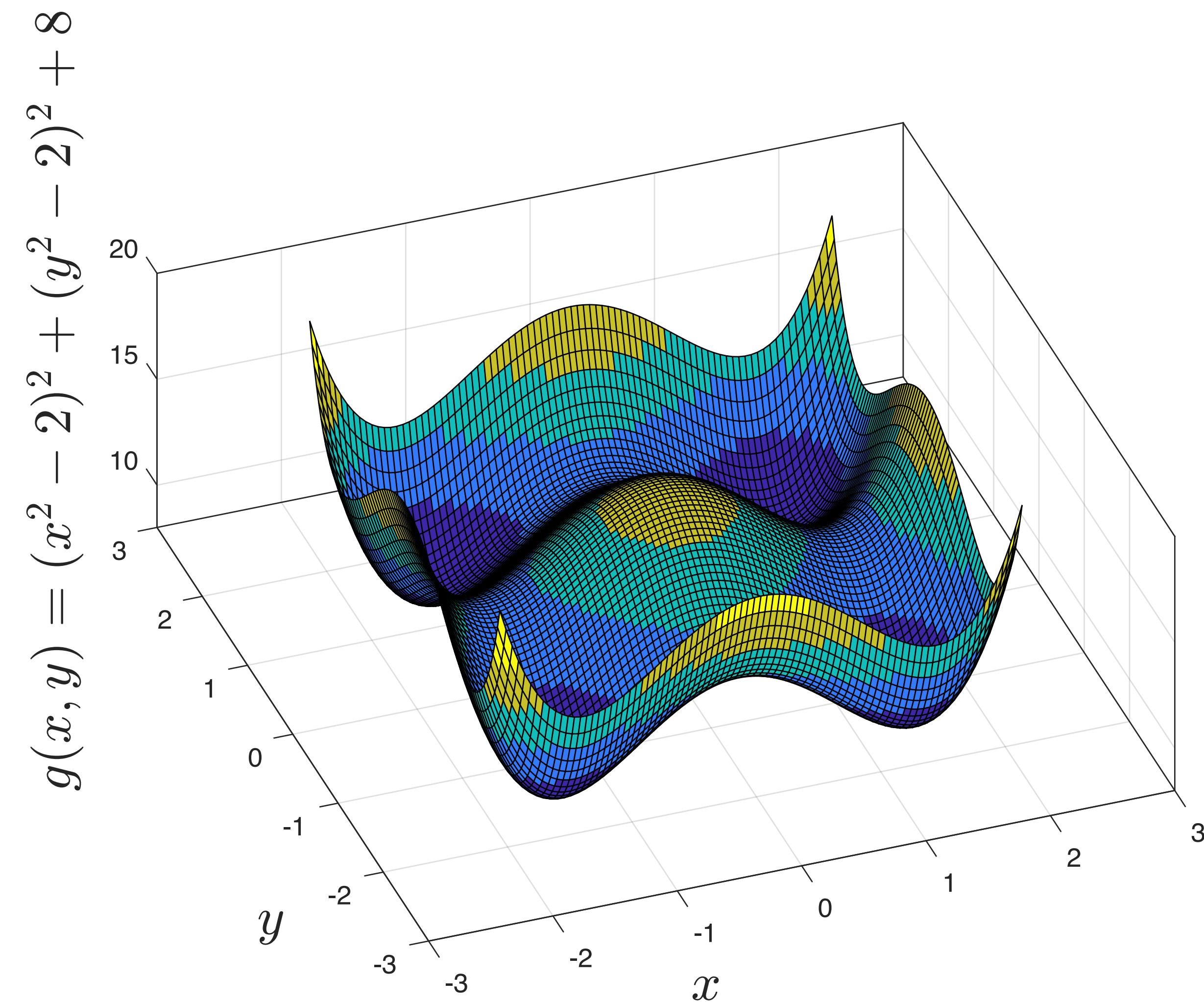
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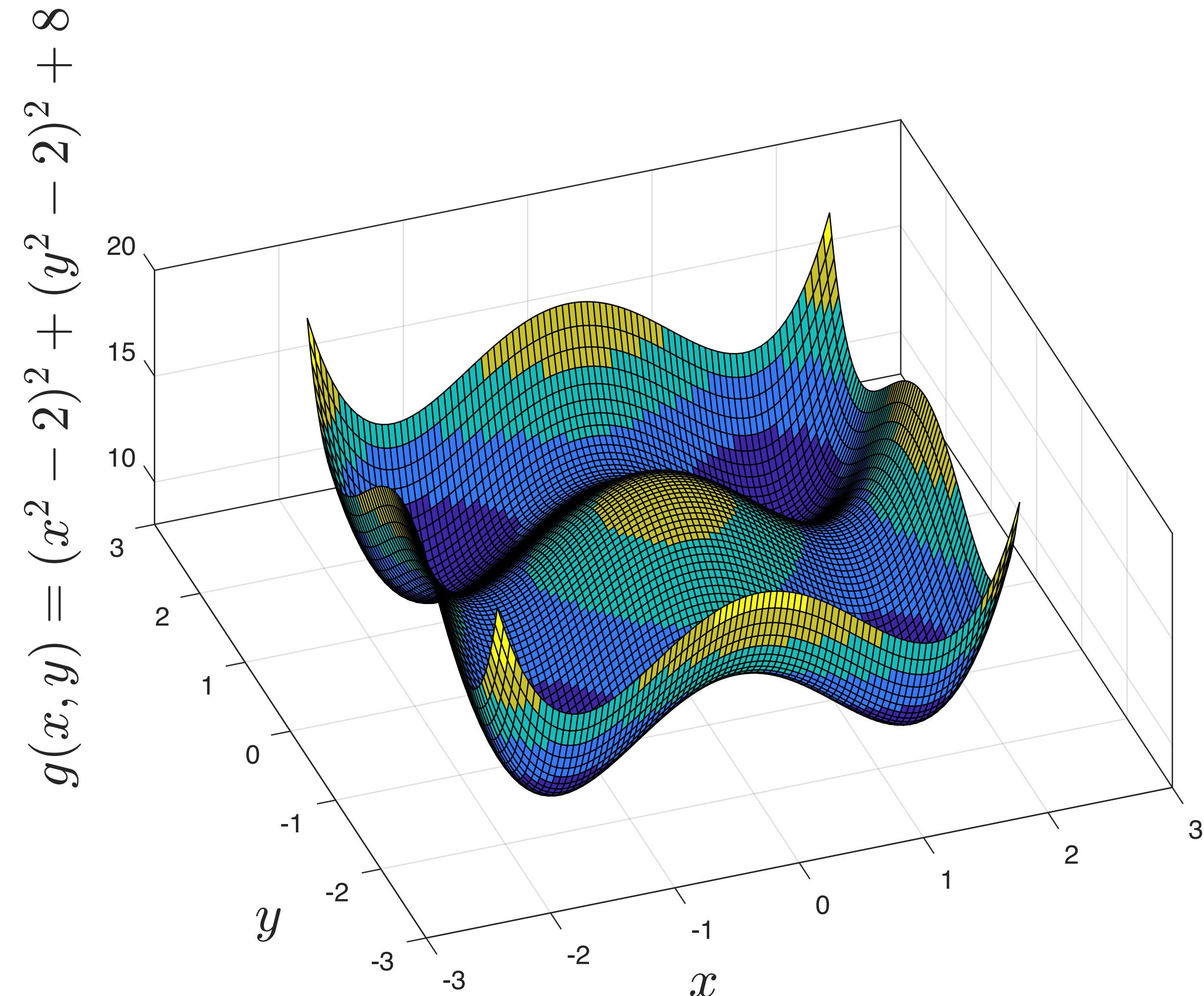
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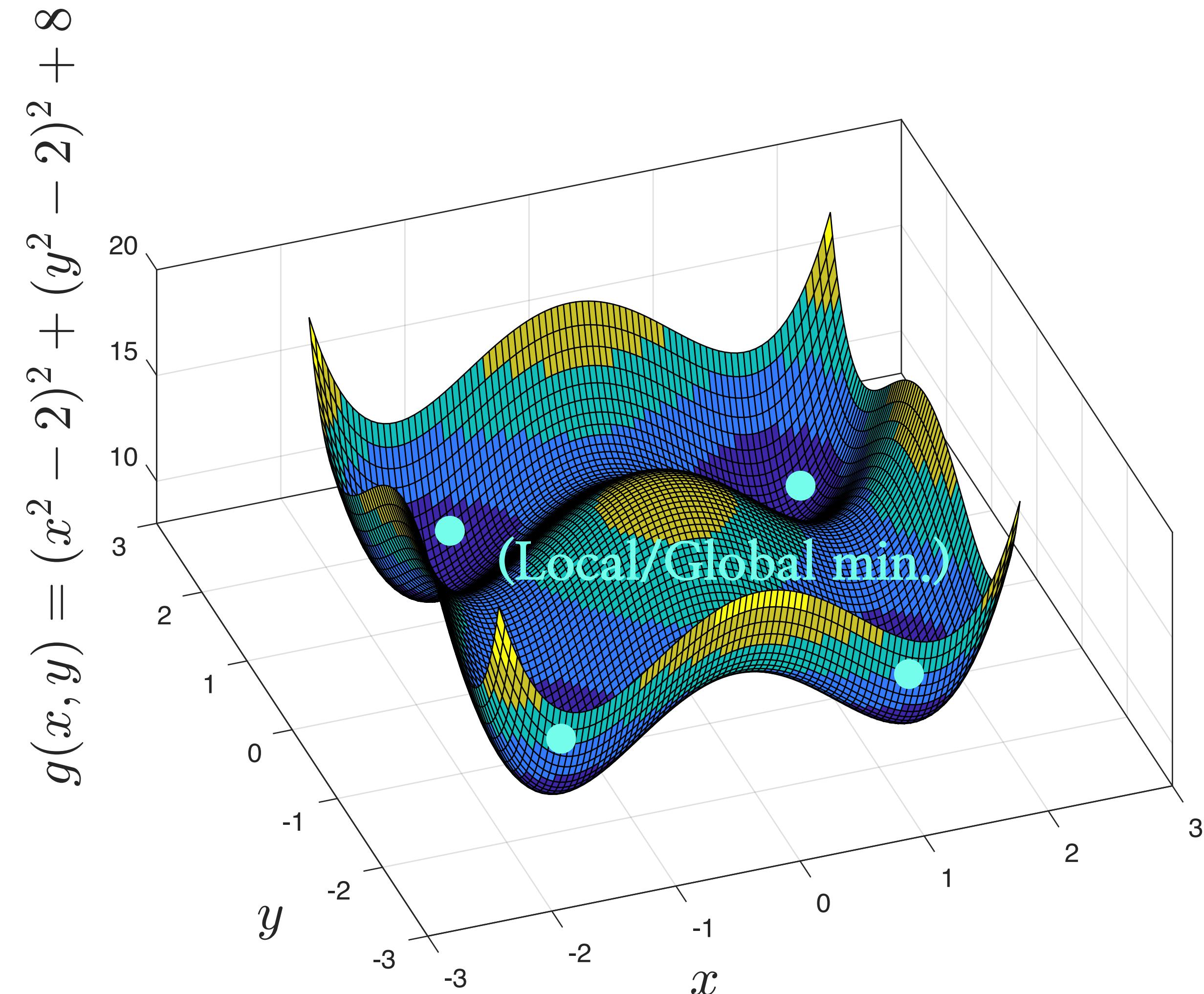
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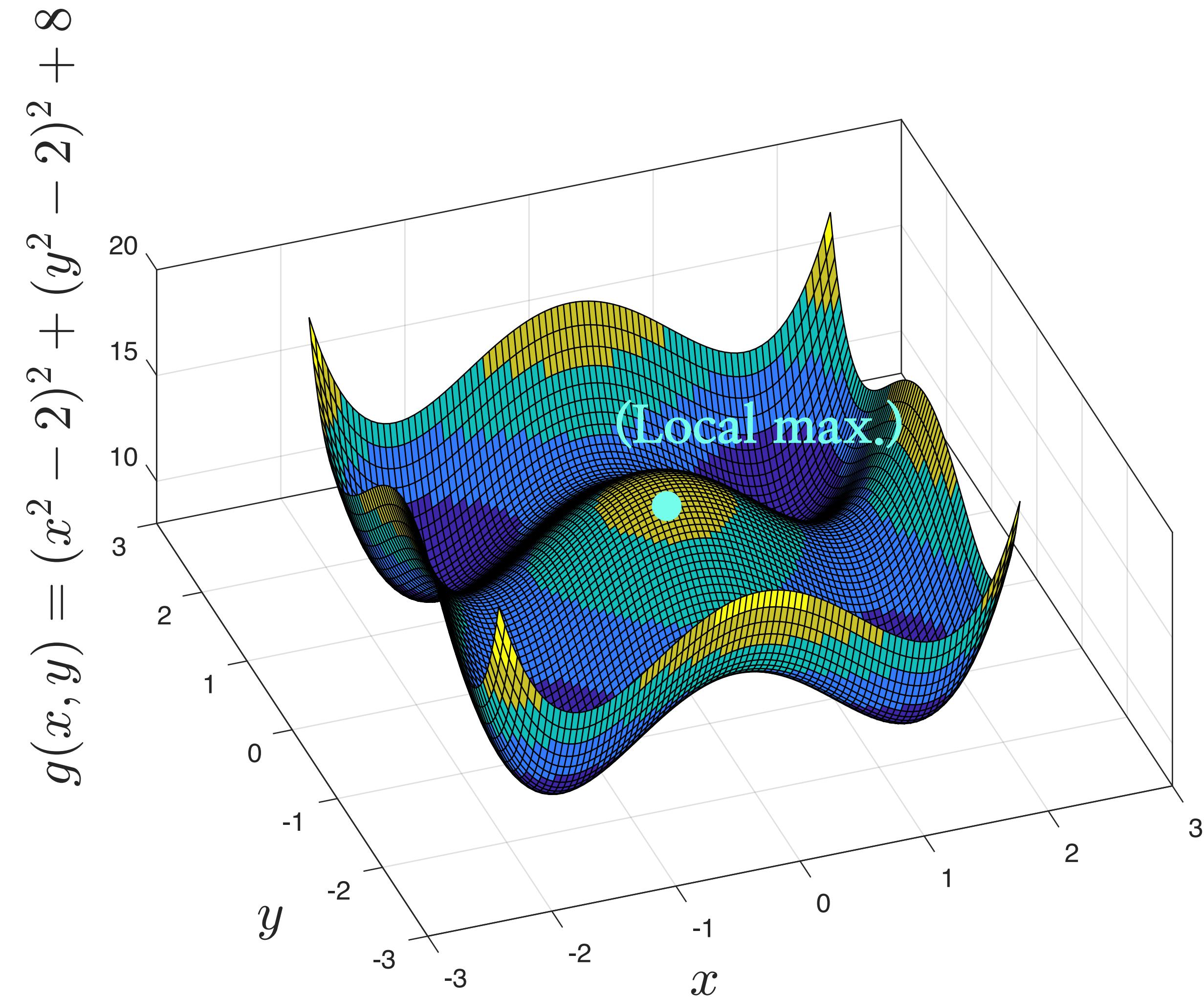
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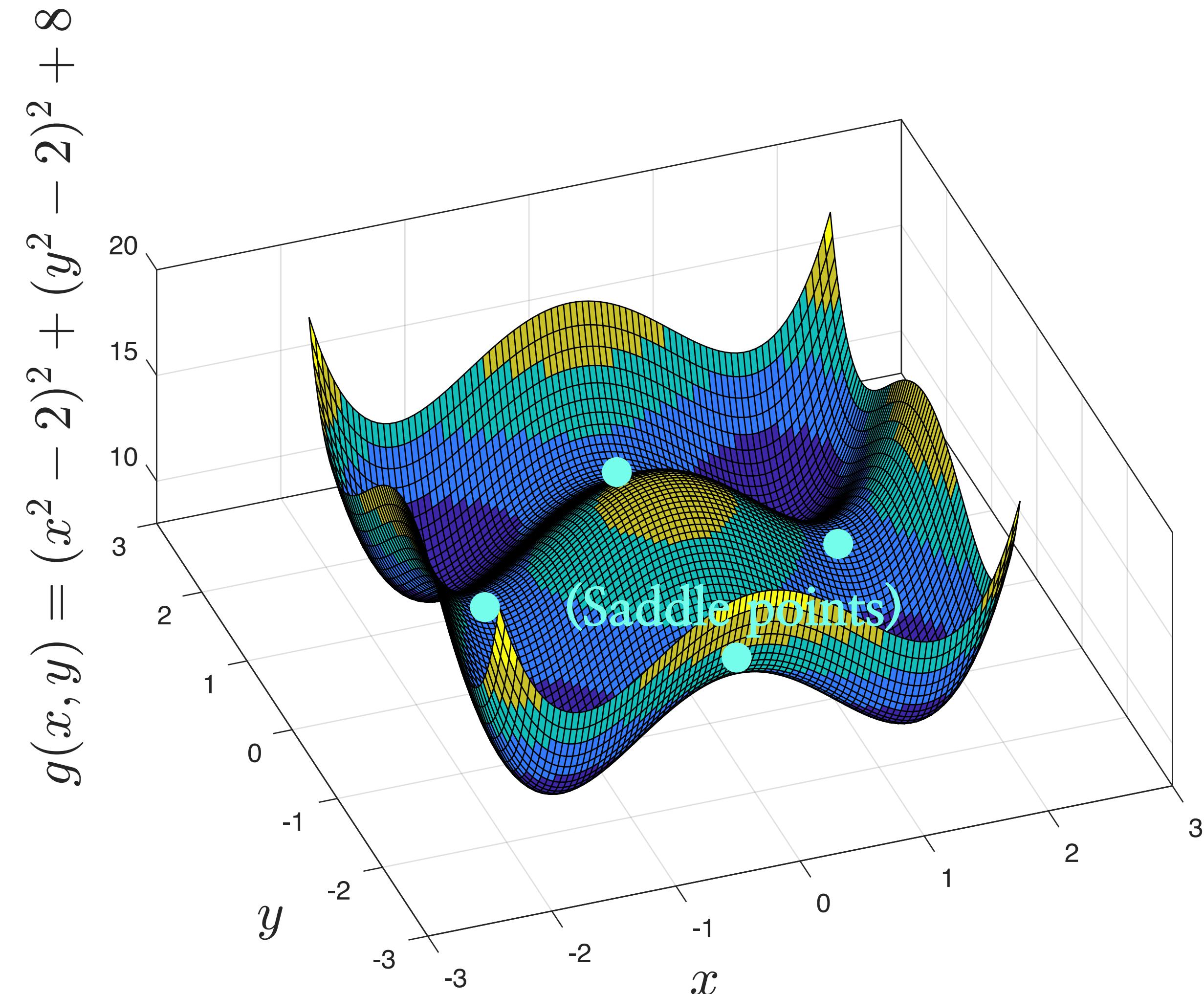
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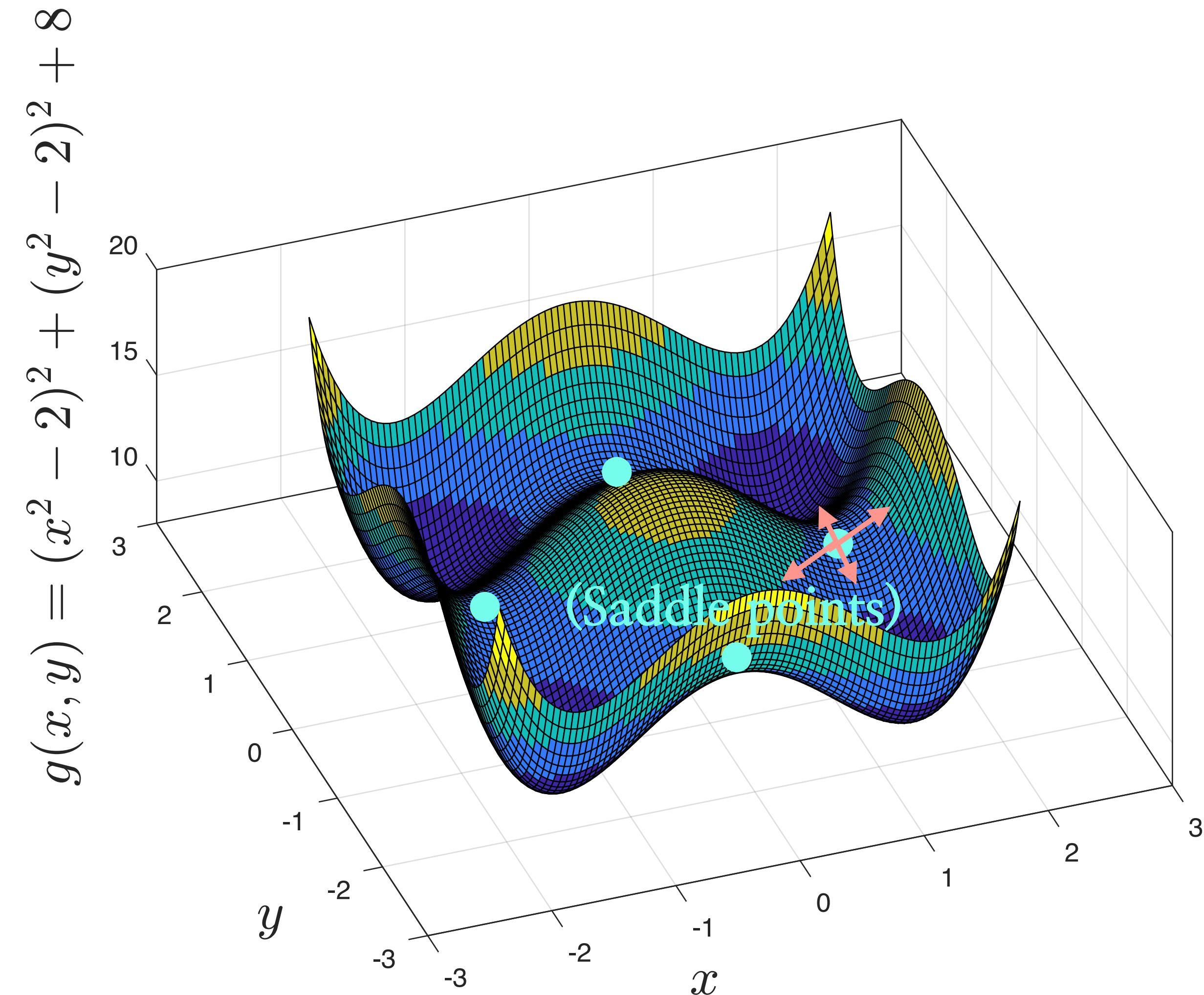
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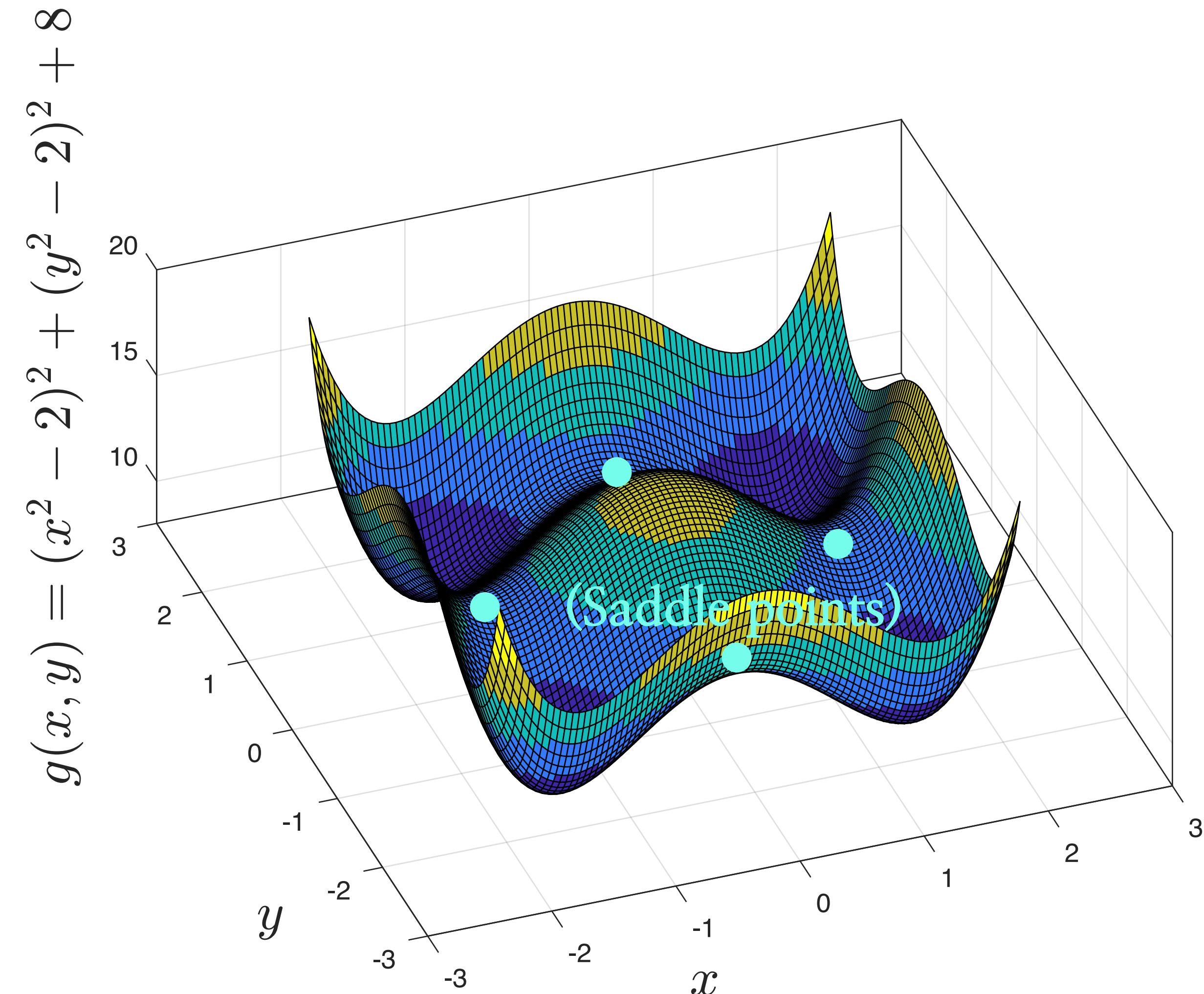
- Toy example #2: $g(x, y) = f(x) + f(y) + 8 = (x^2 - 2)^2 + (y^2 - 2)^2 + 8$
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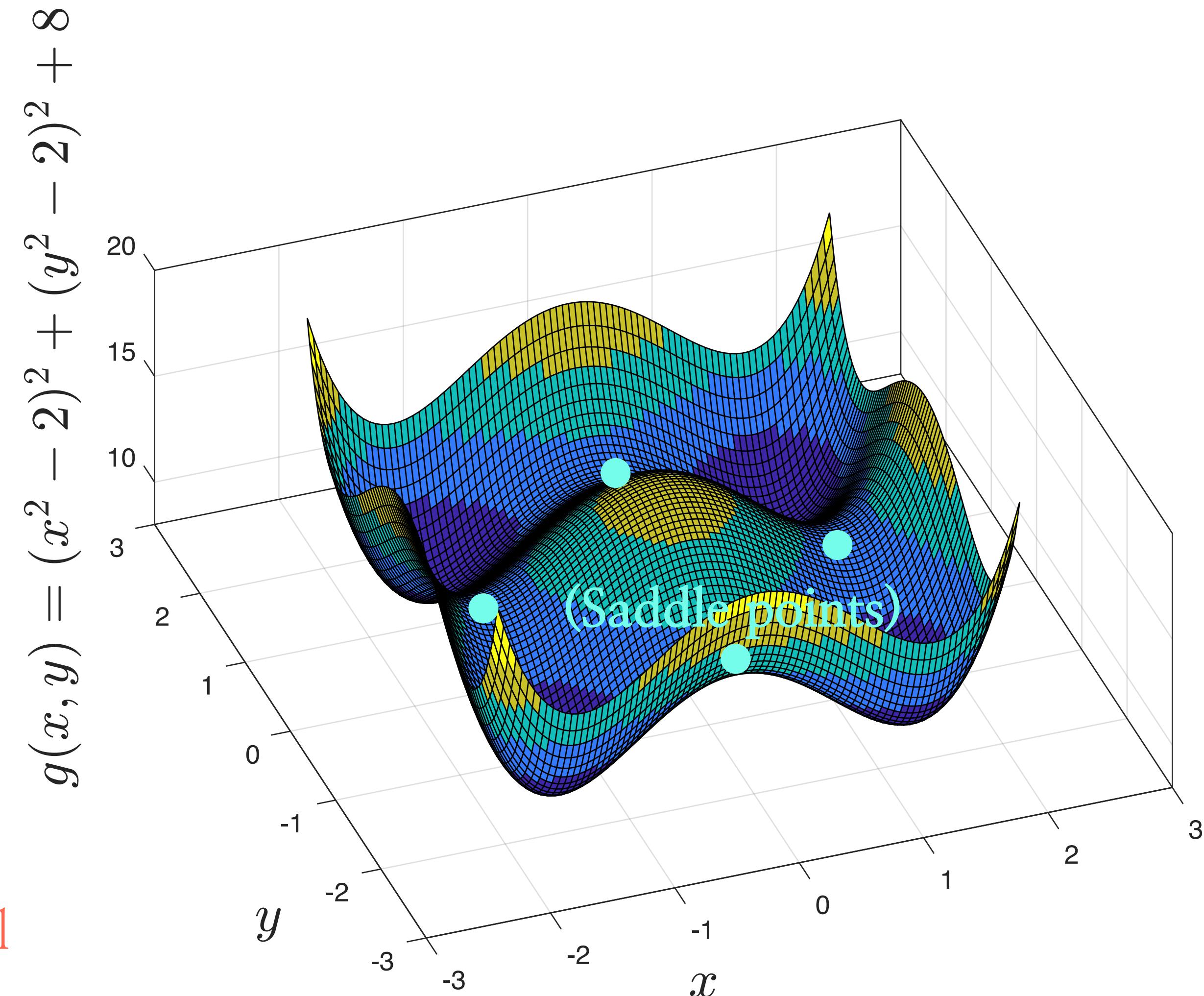
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- “Does it generalize?”

Yes! From 2D to 3D, we get 4 local minima, and 8 saddle points!



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1. Can we identify saddle points?

2. How does methods such as gradient descent behave in practice?

3. Are there conditions that indicate that always there is a way-out of saddle points?

Second-order derivative test

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By second-order Taylor's expansion:

$$f(x + \eta u) \approx f(x) + \frac{\eta^2}{2} \langle \nabla^2 f(x)u, u \rangle > f(x) \quad (\text{Local min.})$$

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 4. Positive, negative and zero eigenvalues: general saddle point

What can we hope for at saddle points?

- One can use intuition from **second-order** Taylor expansion:

$$f(y) \approx f(x) + \langle \nabla f(x), y - x \rangle + \frac{1}{2} \langle \nabla^2 f(x)(y - x), y - x \rangle$$

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- Thus, we need to characterize the # of steps we might require to escape

Strict saddle property and functions

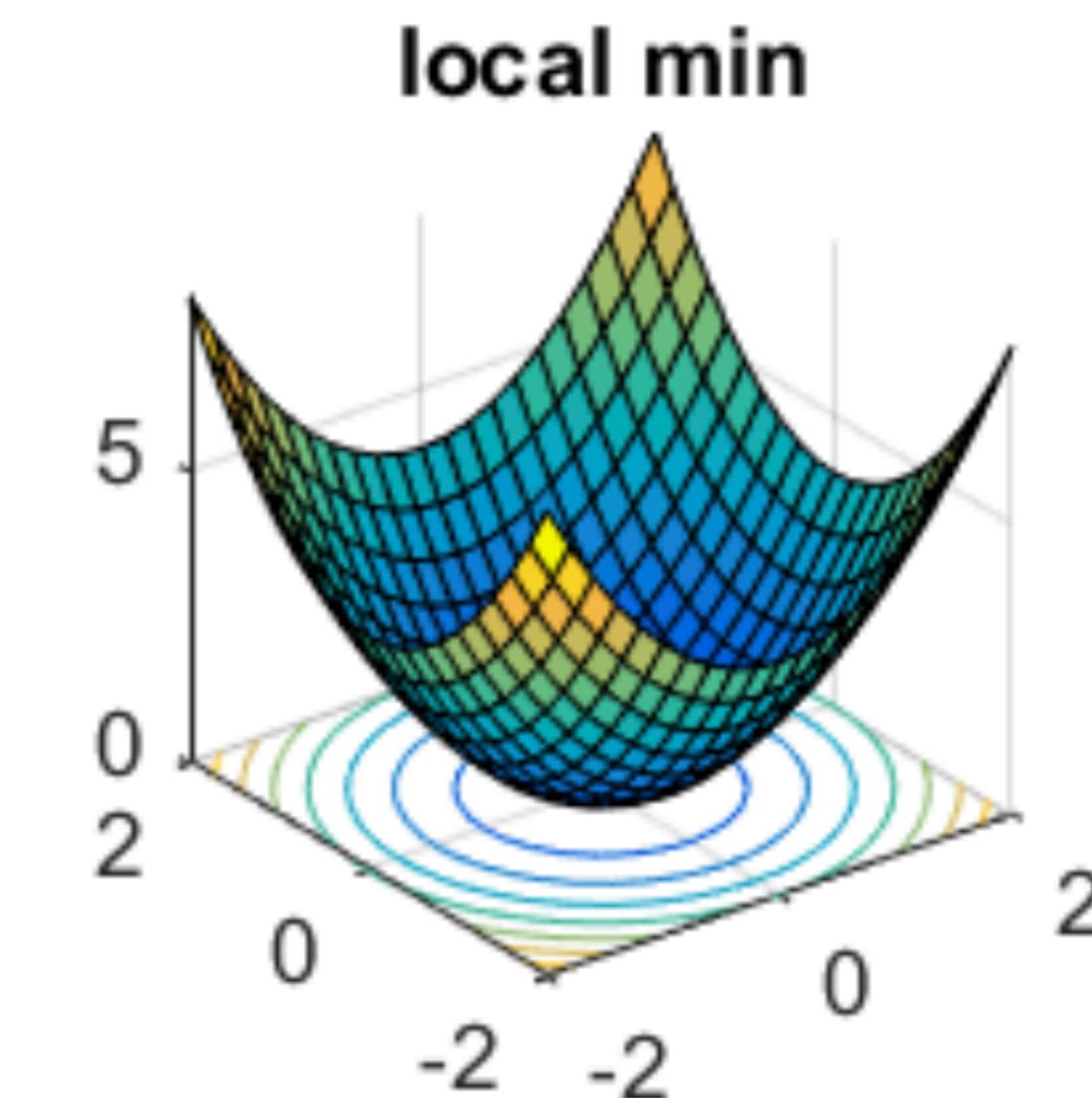
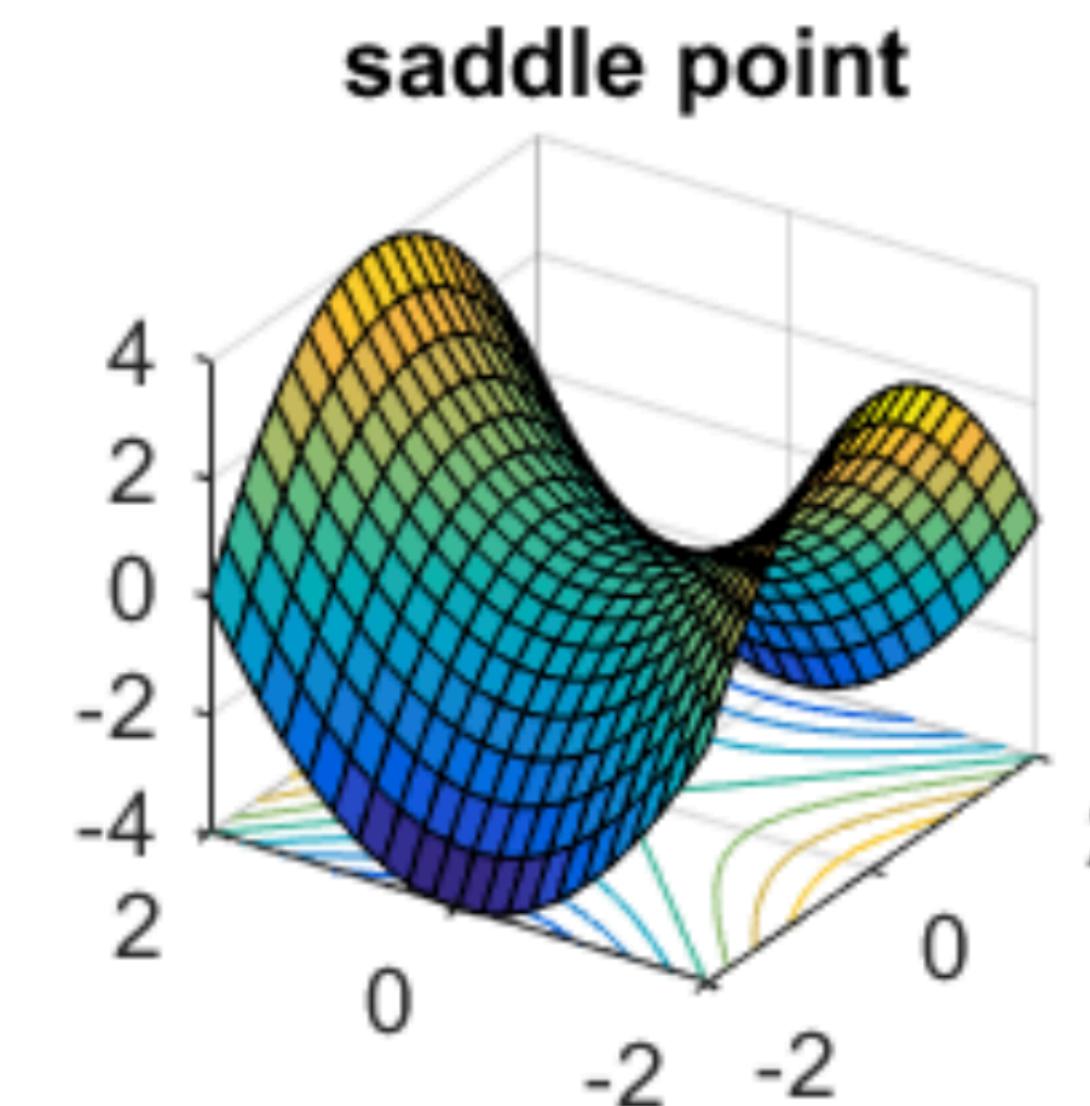
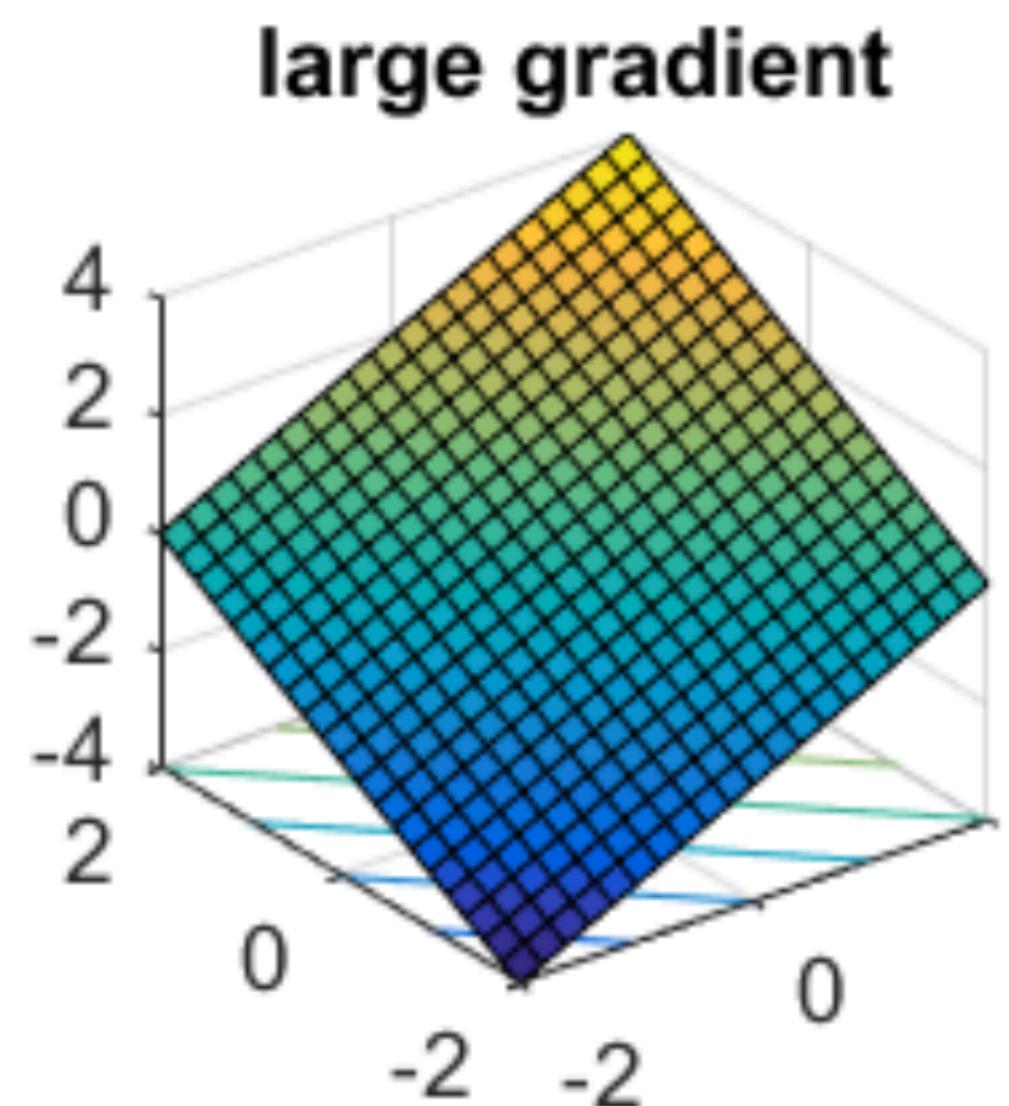
“A function $f(x)$ is strict saddle or satisfies the strict saddle property, if all points x in its domain satisfy at least one of the following:

- i. The gradient is large, i.e., $\|\nabla f(x)\|_2 \geq \alpha$
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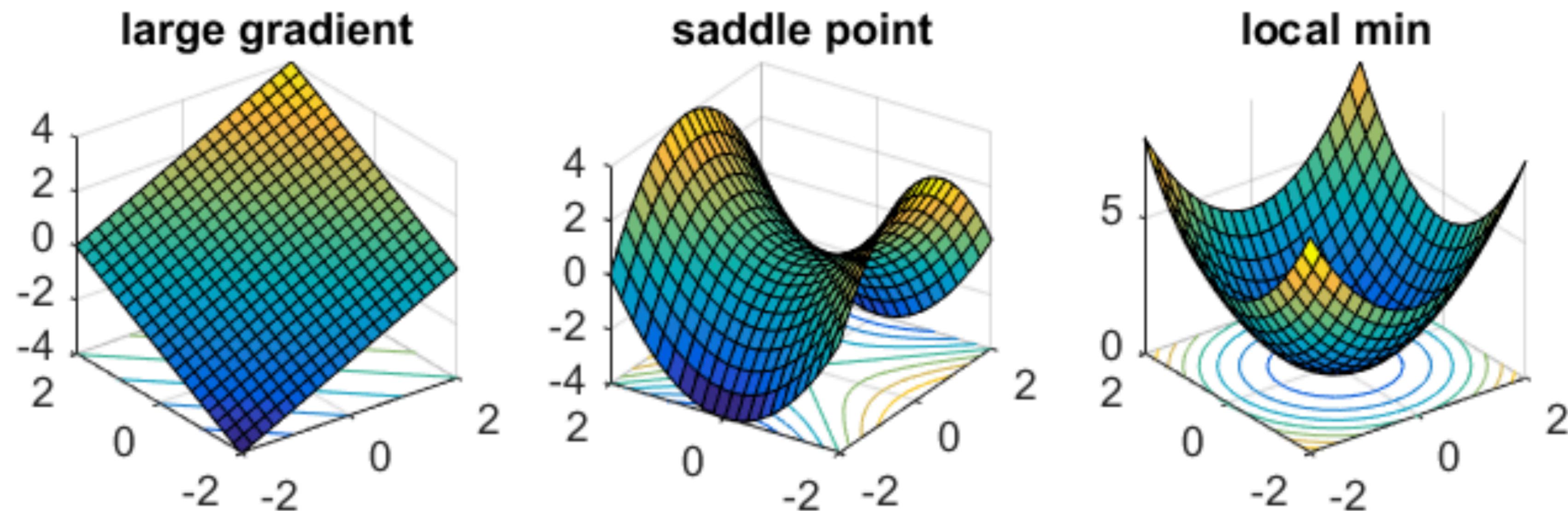
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– Tensor decomposition, dictionary learning, phase retrieval, matrix sensing..

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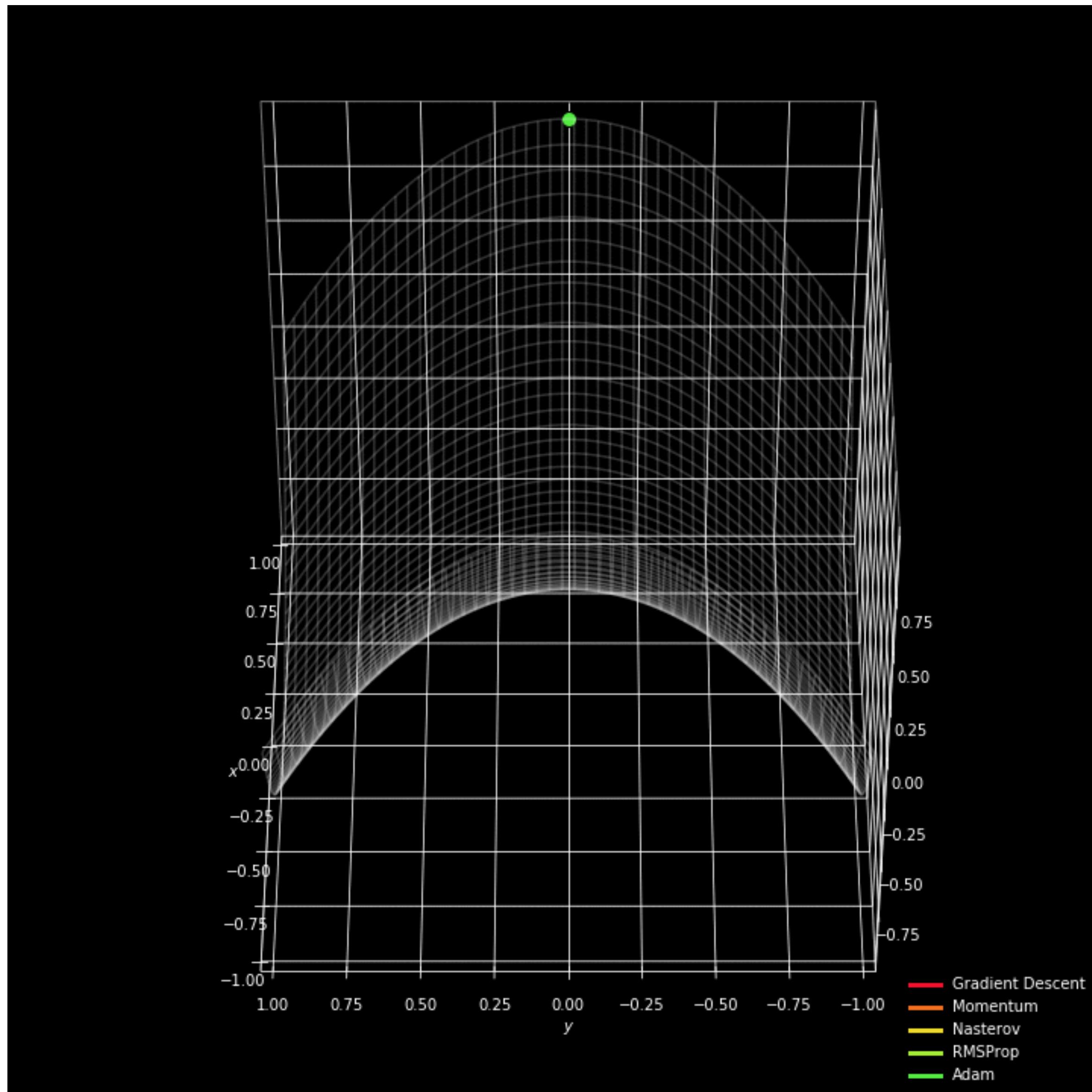
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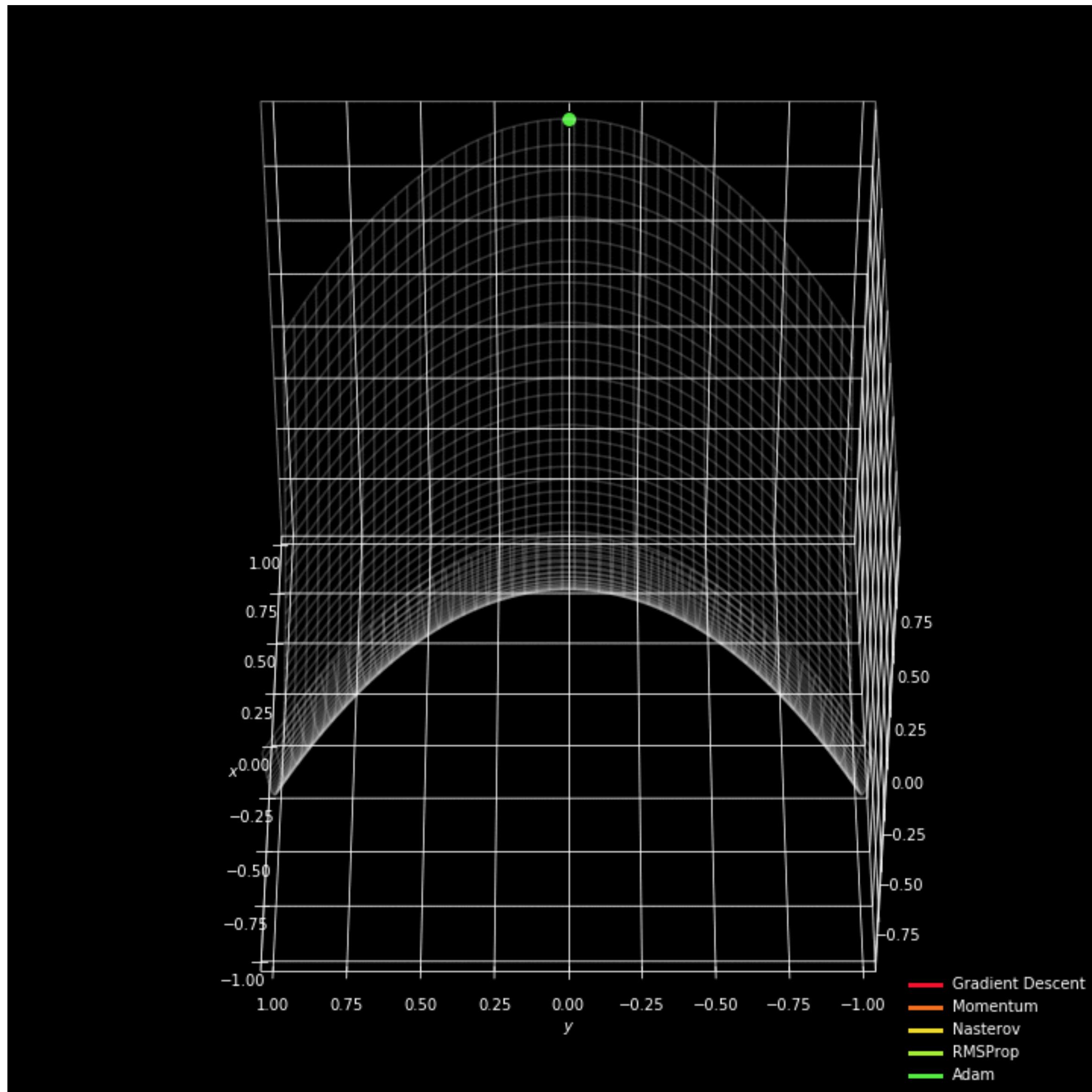
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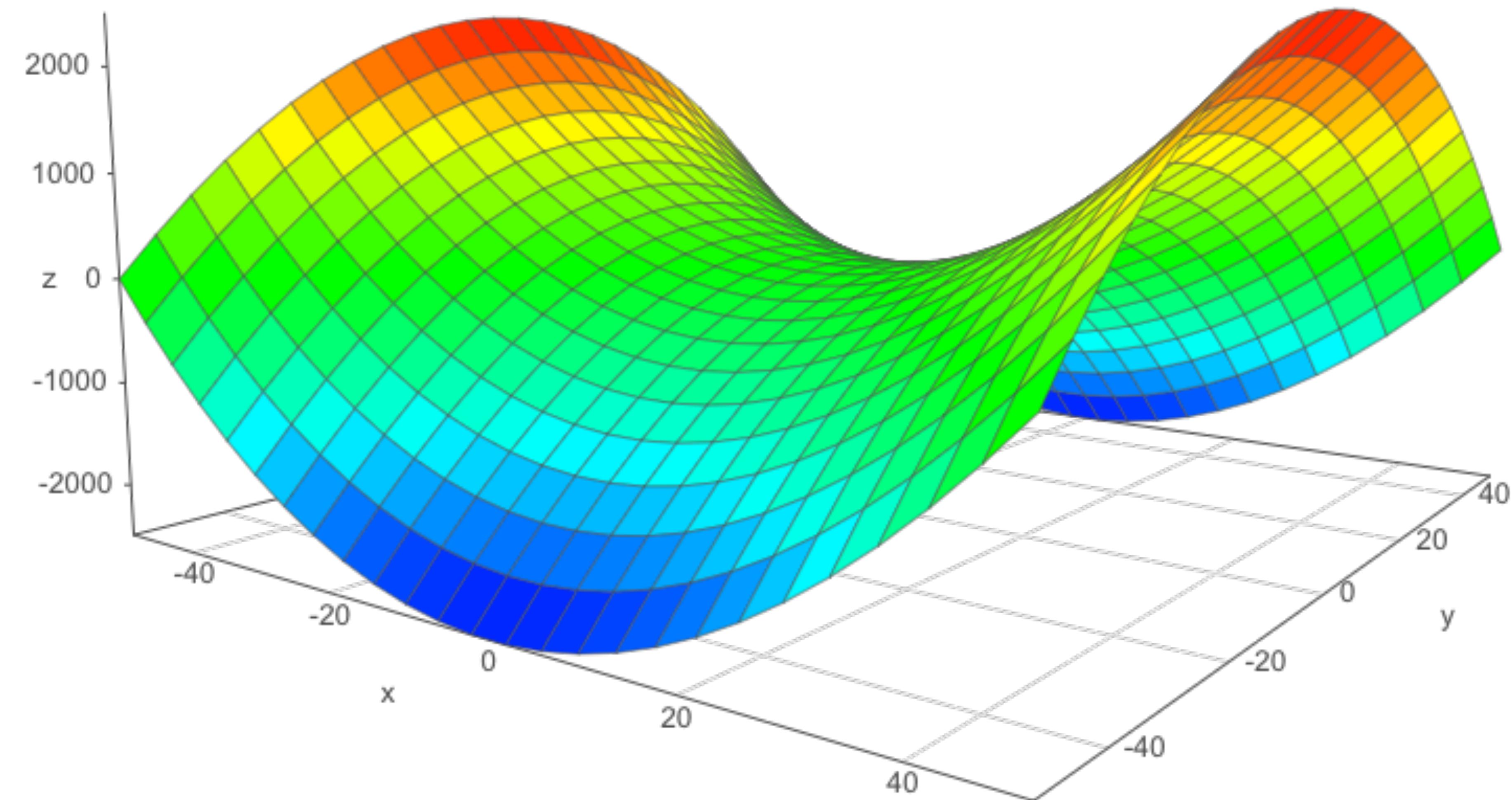
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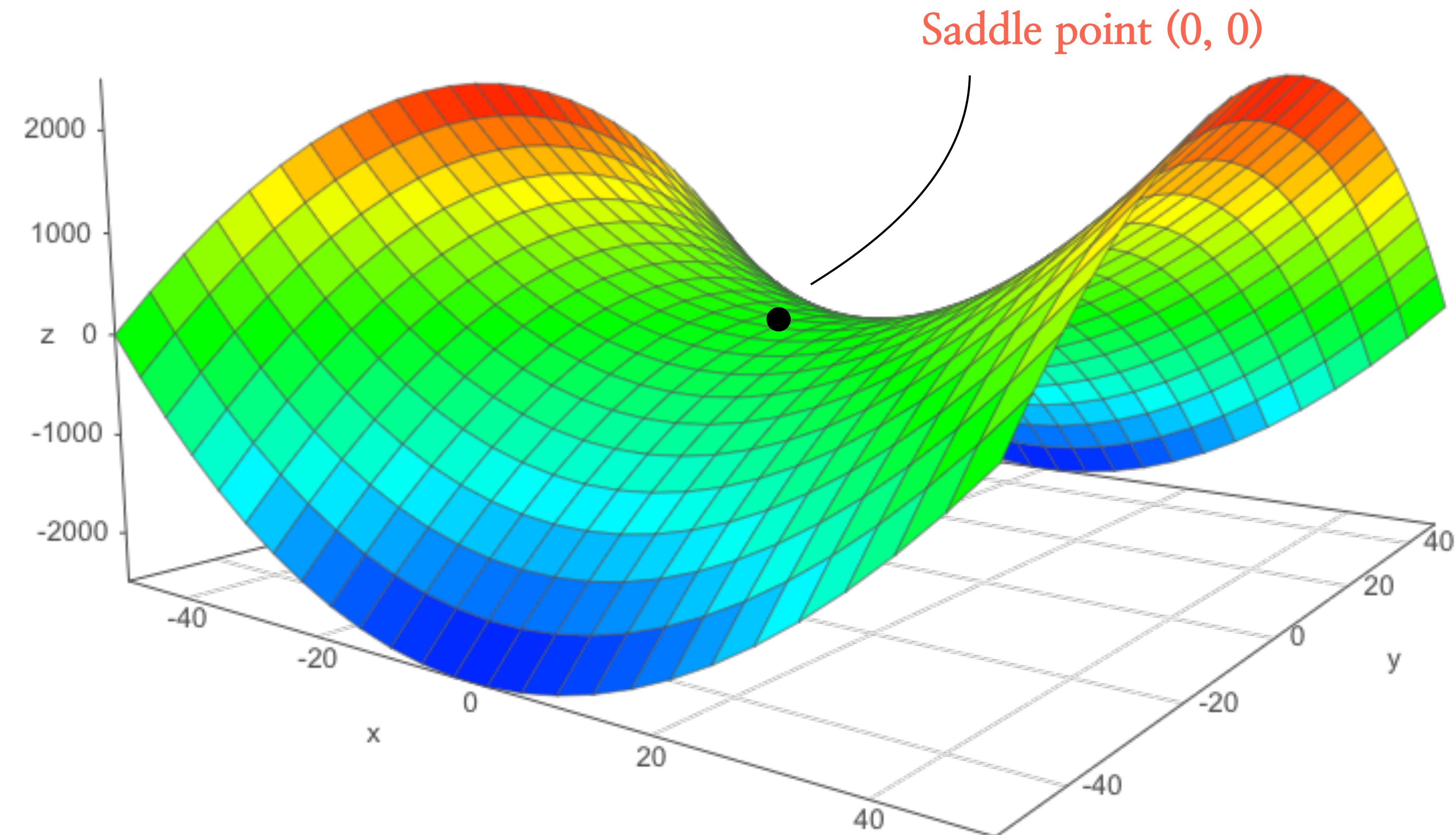
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(Noisy gradient descent)

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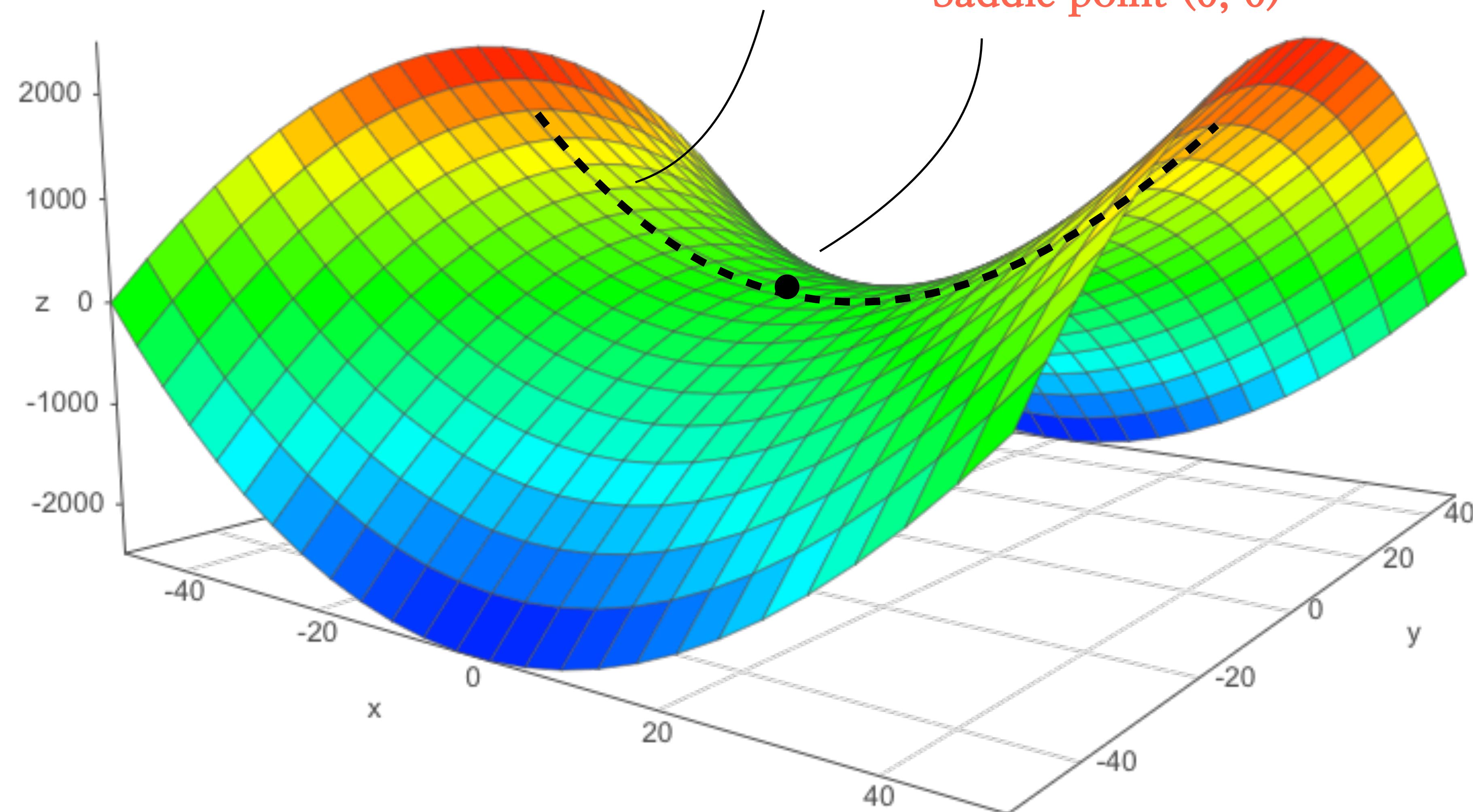
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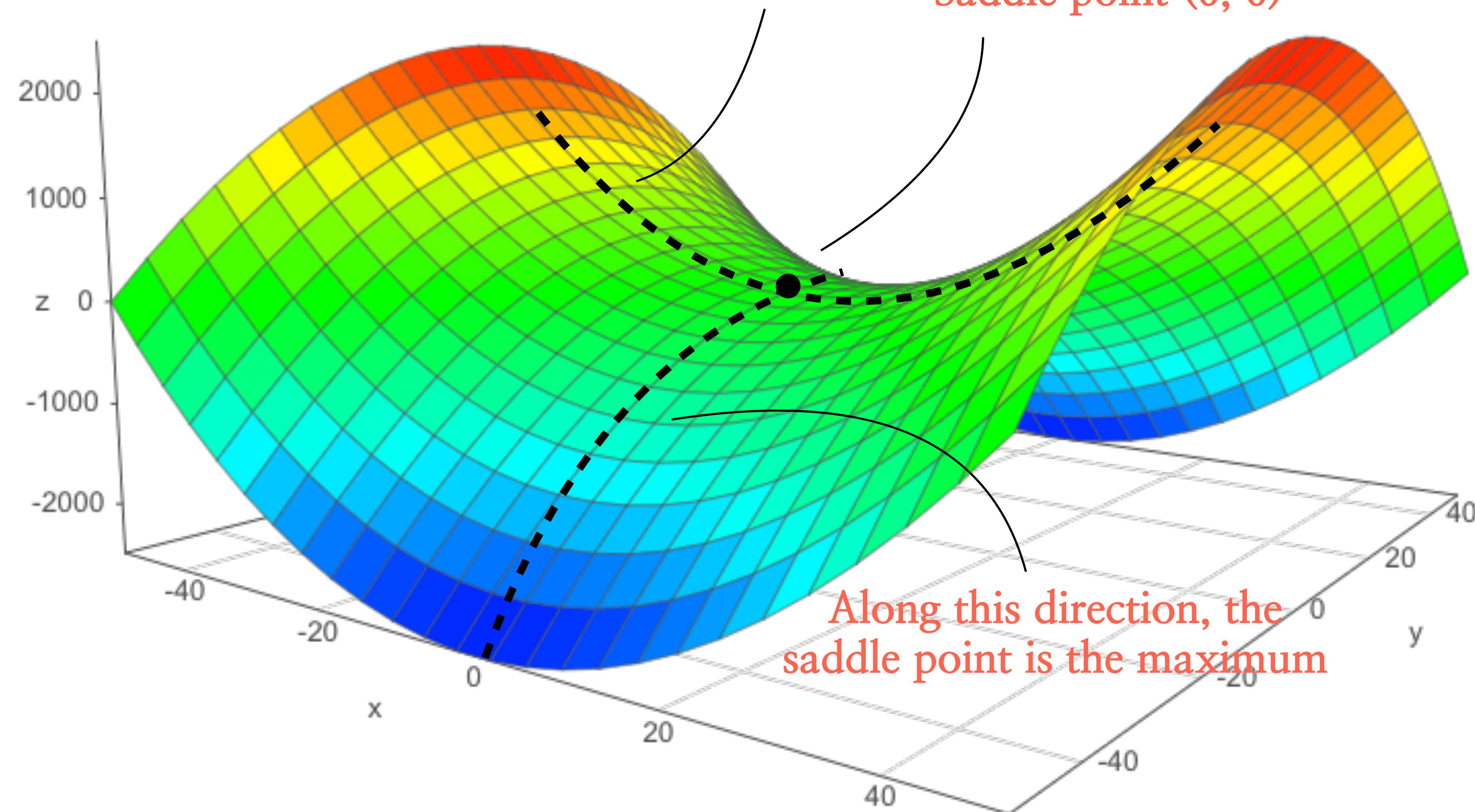
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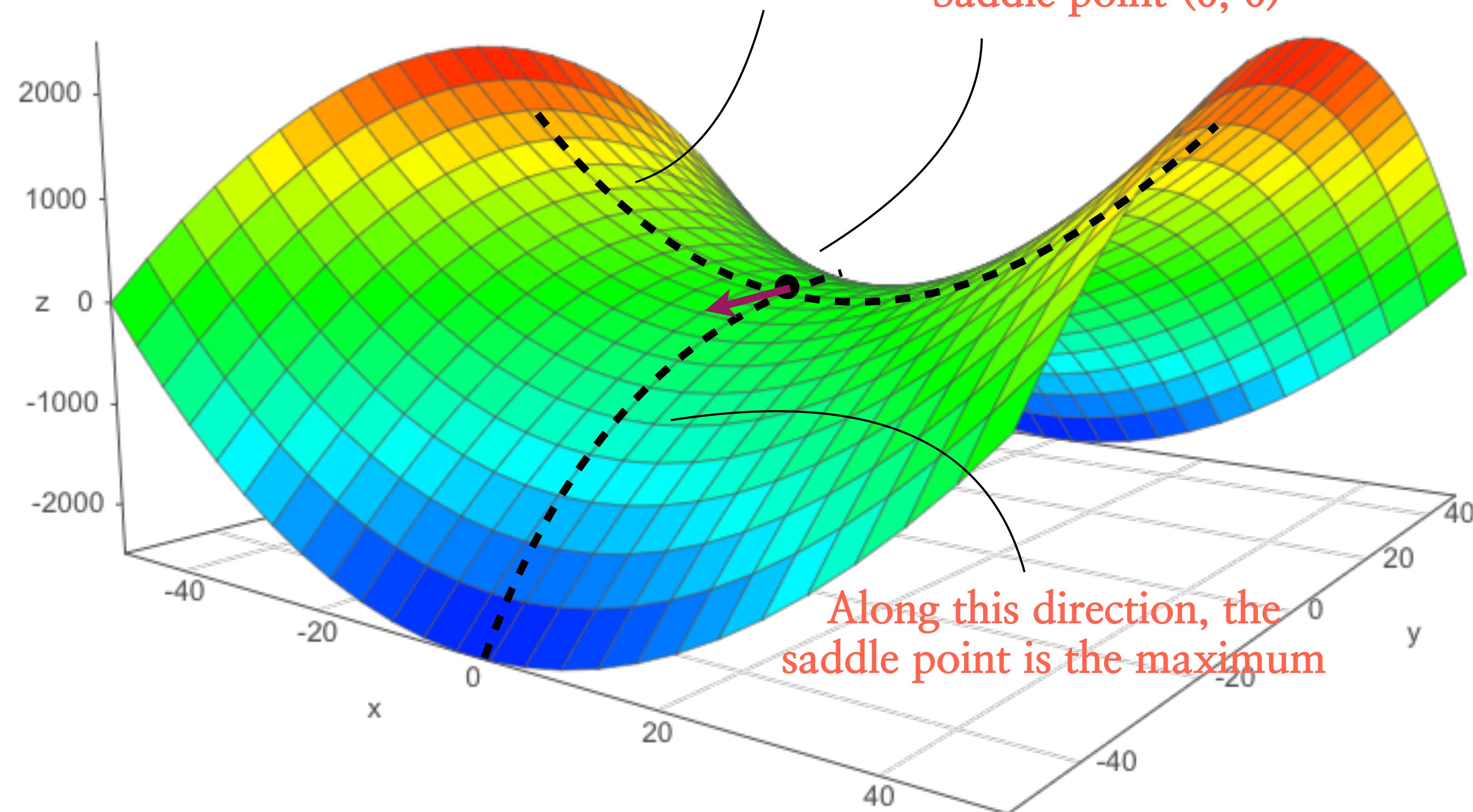
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(“Noisy” stochastic gradient descent – stochasticity is not a problem, but a feature)

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- Overall, total runtime could be up to $O(p^3)$ – differently, $\tilde{O}(1/\epsilon^4)$ iters.

Should we worry about saddle points? A different perspective

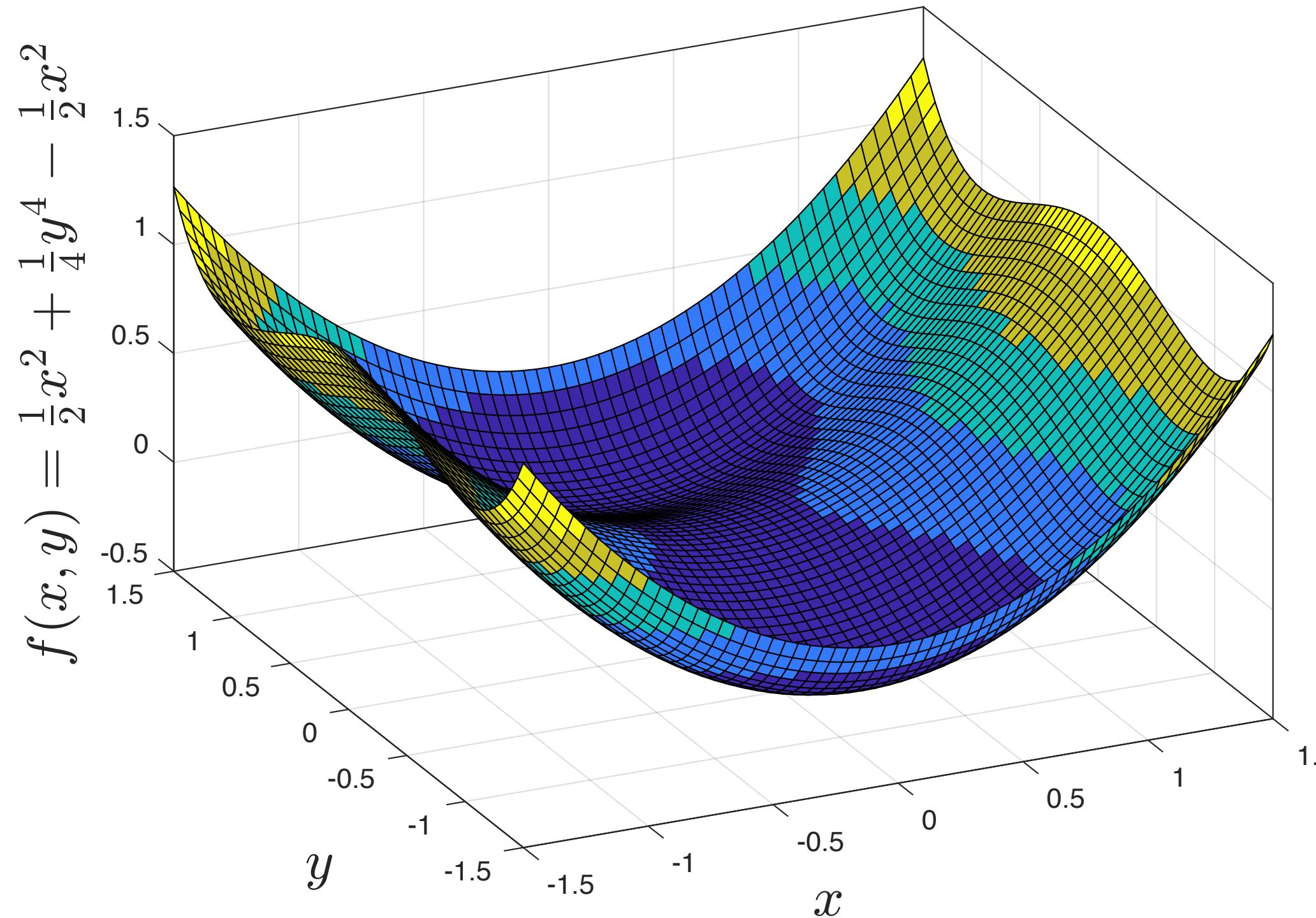
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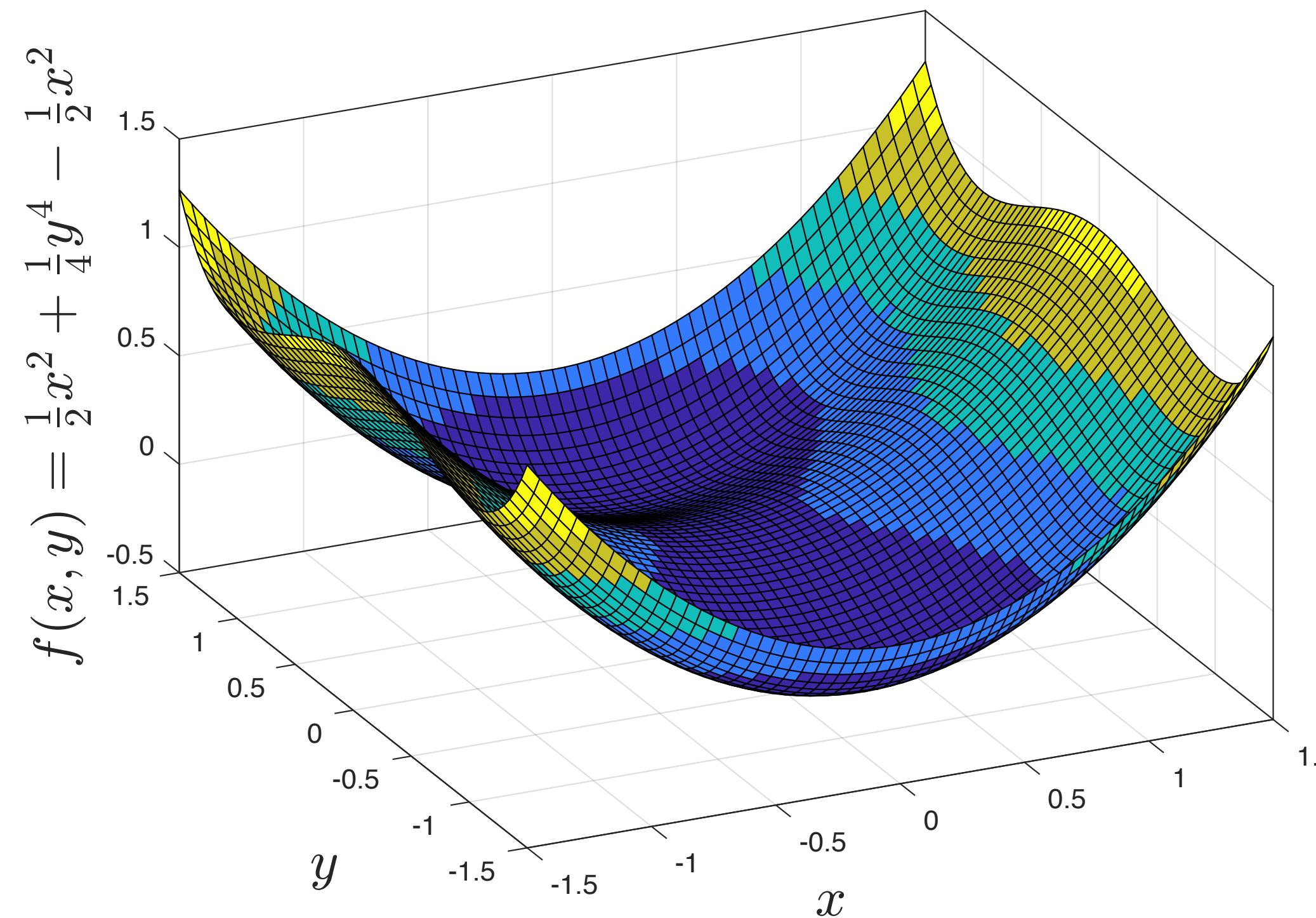
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- Critical points:
 $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$, $\begin{bmatrix} 0 \\ -1 \end{bmatrix}$, $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$

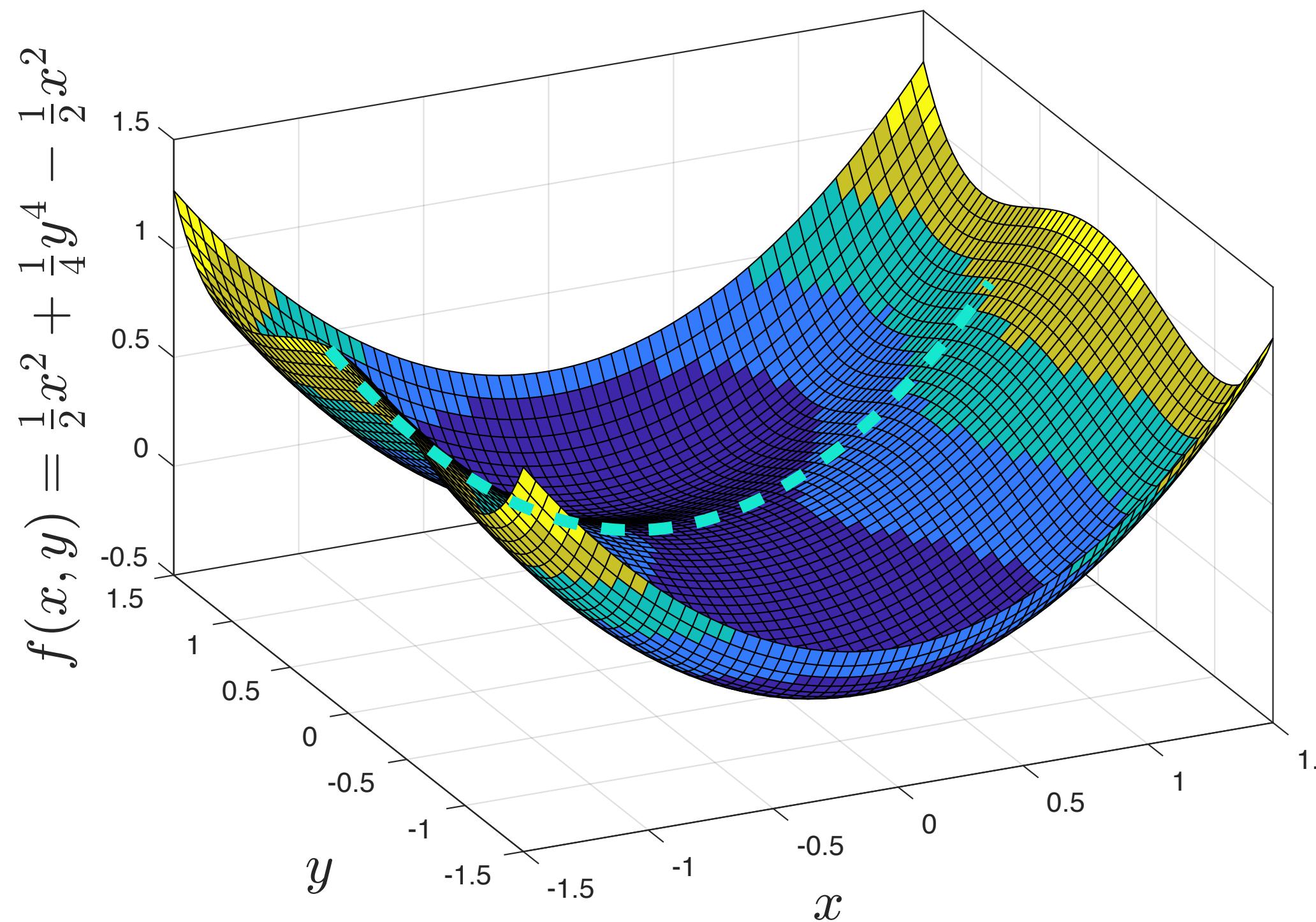
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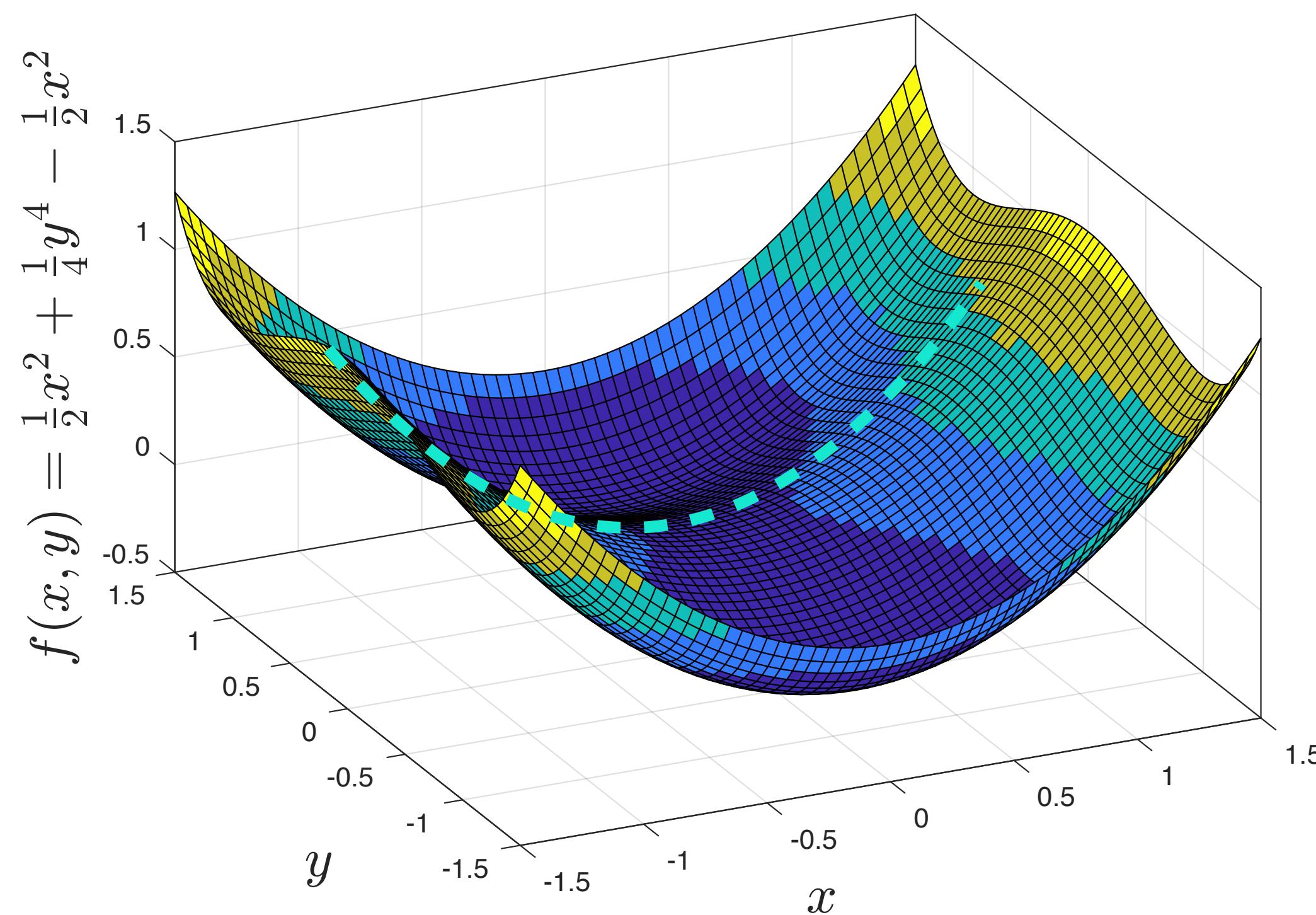
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 - Any initialization of the form $\begin{bmatrix} x \\ 0 \end{bmatrix}$ (cyan) leads to convergence to the saddle point
 - But, any other initialization converges to local minimizer!
(With random initialization, this happens with prob. 1)

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- The idea is that gradient descent satisfies such a theorem, and the set of saddle points (under the assumptions made by the theory so far) has measure zero!
- In practice: if you pick any random initial point, you are safe not to converge to a saddle point

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Whiteboard

- Similar results have been proven for: phase retrieval, matrix completion, dictionary recovery, semidefinite programming (SDPs), signal recovery from quadratic measurements, ..

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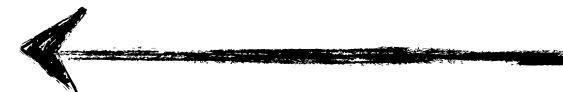
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 and given stationary point x and compare with global minima (to show potential equivalence)
 - For saddles, identify a (negative) upper bound for
$$\lambda_{\min} (\nabla^2 f(x))$$

Conclusion

- We discussed about **types of stationary points**, focus on **saddle points** and study some of their properties
- We introduced conditions that allow **escaping from saddle points**
- We studied (overview) matrix sensing as a test case, and how to prove “no spurious local minima” arguments