

COMP 414/514: Optimization – Algorithms, Complexity and Approximations

Lecture 1

Overview

$$\begin{array}{ll} \min & f(x) \\ x & \\ \text{s.t.} & x \in \mathcal{C} \end{array}$$

Overview

$$\min_x$$
$$\text{s.t.}$$
$$f(x)$$
$$x \in C$$

- Different objective classes
- Different strategies within each problem
- Different approaches based on computational capabilities
- Different approaches based on constraints

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s.t.

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$$x \in C$$

- Different objective classes
- Different strategies within each problem
- Different approaches based on computational capabilities
- Different approaches based on constraints

And, always having in mind applications in machine learning,
AI and signal processing

Motivation

(no fancy images included)

Provable efficiency

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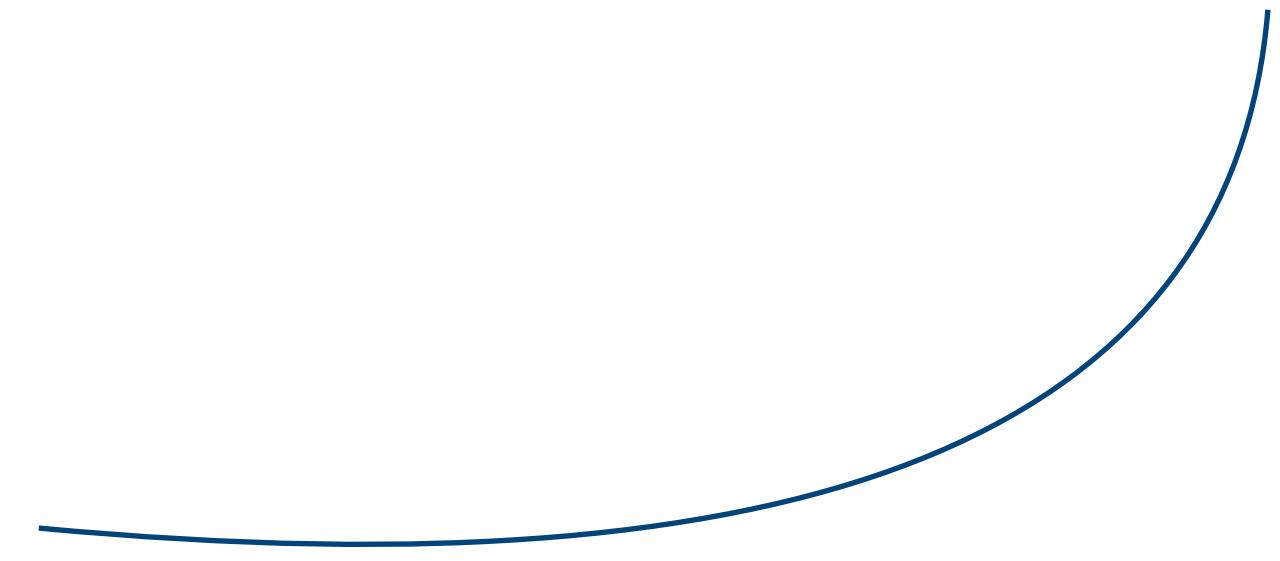
Lots of data

Motivation

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Provable efficiency Harder problems

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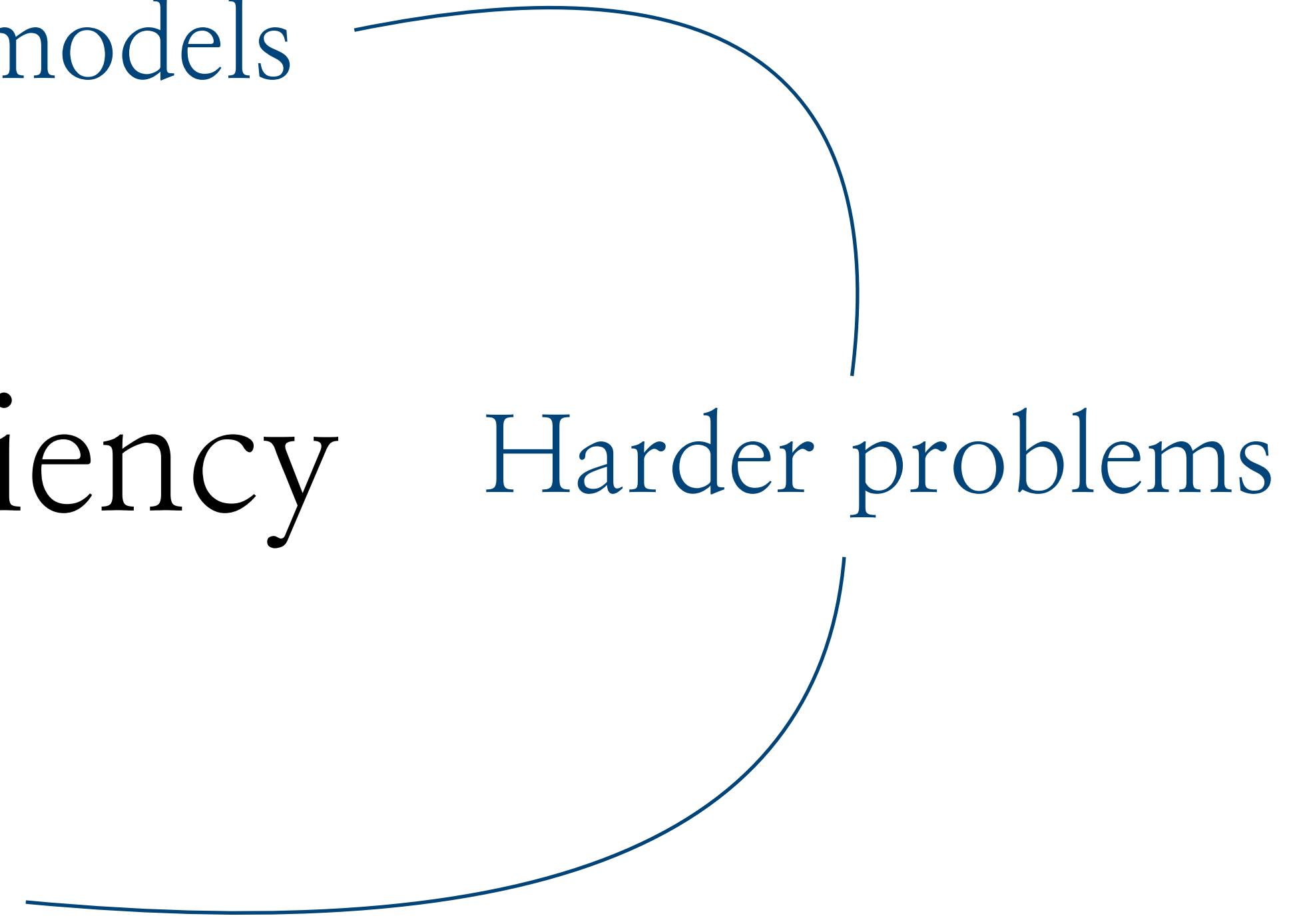
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More complicated models

Provable efficiency

Lots of data

Harder problems



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Better results

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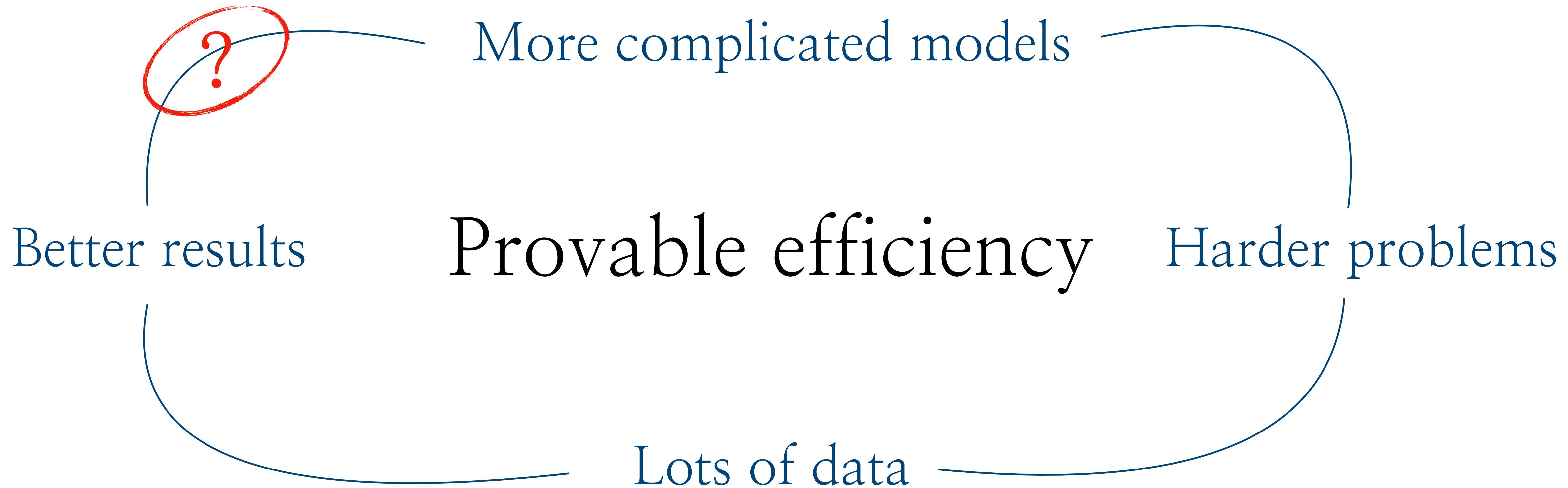
Provable efficiency

Harder problems

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Motivation

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Motivation

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Provable efficiency

“What shall we do?”

Motivation

(no fancy images included)

Provable efficiency

“What shall we do?”

Set up algo nicely

Use prior knowledge

Converge faster

Exploit resources

Topics

- Continuous optimization (in general)
 - See syllabus
 - Both theory and practice

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- Recent applications that drive research
- When no theory applies, some intuition

Examples

- Least squares / linear regression

(No, we will not re-define it)

$$\min_x f(x)$$

$$\text{s.t. } x \in \mathcal{C}$$

Examples

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$$\begin{array}{ll} \min_x & f(x) \\ \text{s.t.} & x \in C \end{array} \Rightarrow \min_x \frac{1}{n} \sum_{i=1}^n (y_i - a_i^\top x)^2$$

Examples

- Quantum state tomography from limited samples

$$\min_X f(X)$$

$$\text{s.t. } X \in \mathcal{C}$$

Examples

- Quantum state tomography from limited samples

$$\begin{array}{ll} \min_X f(X) & \Rightarrow \\ \text{s.t. } X \in \mathcal{C} & \begin{array}{ll} \min_X \sum_{i=1}^n (y_i - \text{Tr}(A_i^\top X))^2 \\ \text{s.t. } \text{Tr}(X) \leq 1 \\ X \succeq 0 \end{array} \end{array}$$

Examples

- Fleet management

$$\min_X f(X)$$

$$\text{s.t.} \quad X \in \mathcal{C}$$

Examples

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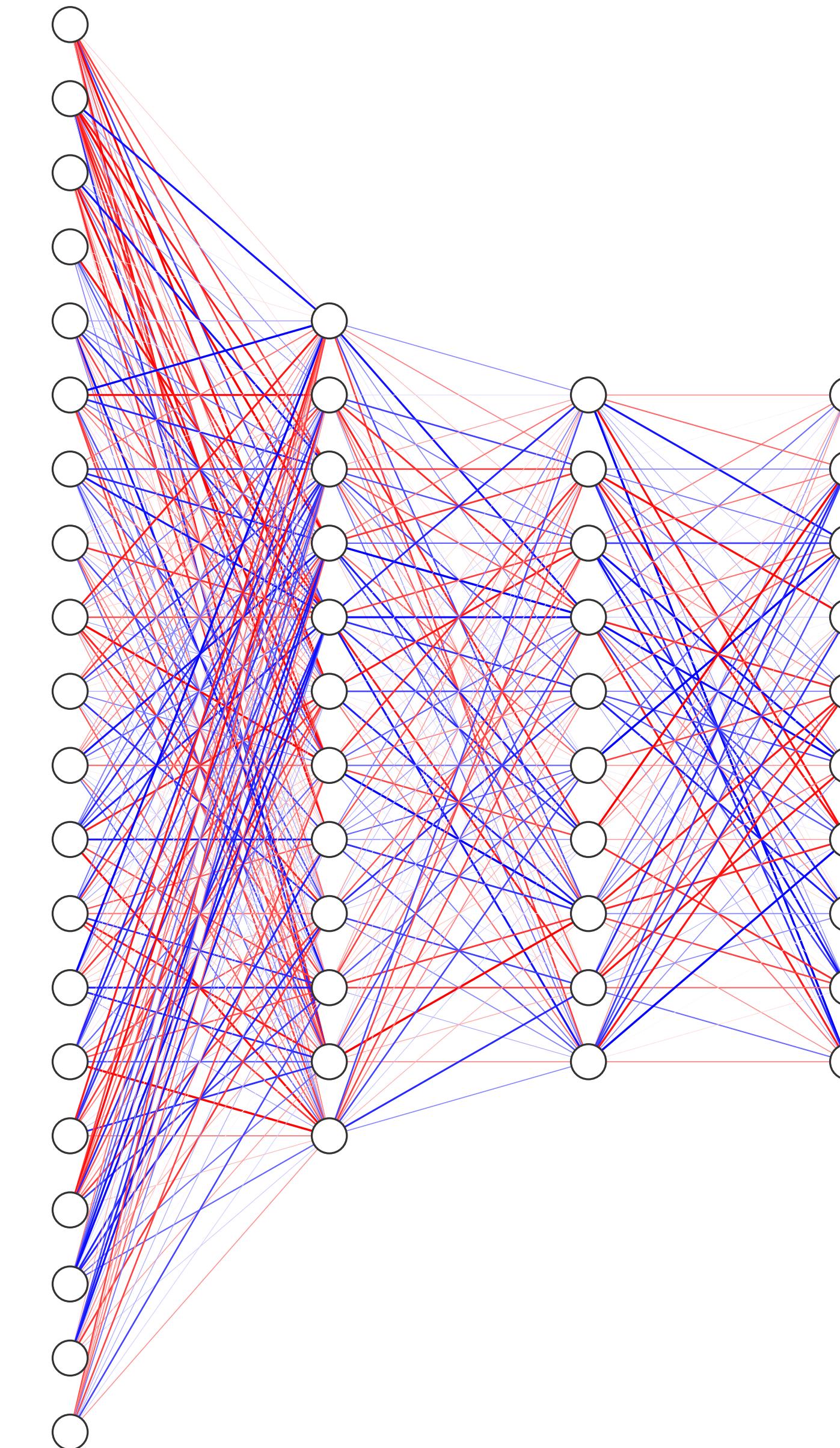
$$\min_X f(X) \quad \Rightarrow$$

$$\text{s.t.} \quad X \in \mathcal{C}$$

$$\begin{aligned} & \min_{x \in \{0,1\}^m, y \in \{0,1\}} f(y) = \sum_{i \in \mathcal{V}} \sum_{k=1}^p d_i (1-q) q^{k-1} y_{ik} \\ & \text{s.t.} \\ & \quad \sum_{j \in \mathcal{W}_i} x_j \geq \sum_{k=1}^p y_{ik}, i \in \mathcal{V} \\ & \quad \sum_{j \in \mathcal{W}} x_j = p \\ & \quad x_j \leq p_j \end{aligned}$$

Examples

- Neural networks



Input Layer $\in \mathbb{R}^{20}$ Hidden Layer $\in \mathbb{R}^{12}$ Hidden Layer $\in \mathbb{R}^{10}$ Output Layer $\in \mathbb{R}^{10}$

Examples

- Neural networks

$$\begin{array}{ll} \min_X & f(X) \\ \text{s.t.} & X \in \mathcal{C} \end{array} \qquad \Rightarrow \qquad$$

$$\min_{W_i} \quad f(W_1, W_2) := \frac{1}{n} \sum_{i=1}^n \mathcal{L}(\hat{y}_i, y_i).$$

where

$$\hat{y}_i = \text{softmax}(\sigma(W_2 \cdot \sigma(W_1 \cdot x_i)))$$

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- Bayesian algorithms

Any questions?

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- Master or even undergraduates that want to start research
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- Just auditing is fine by me

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(Feedback on GoogleDoc)

Course format

- Lectures (slides) + whiteboard + in-class code running

(Some lectures have presentations, others will be handwritten)

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- Your workload:
 - Graduate – HWs, final project or final exam
 - Undergraduate – HWs, final exam

(Additional workload: scribing)

Regarding assignments

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- Some questions are harder than others
- Some questions might not feel intuitive
("I'm a computer scientist! Why should I care about optimization?")
- Try to do the best you can
(There will be a reweighing at the end of the course, only if necessary)

Goals + outcomes

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- Read and review recent papers

My goals

- Not to judge you on small details in HWs
(But judge whether you have thought about solving the questions)

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- Not to judge you on small details in HWs
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- Spark your interest in research
where math and practice are combined together

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- A quiz was usually provided for self-assessment, but I decided to make it an additional HW

Grading policy

- 45% HWs
- 50% project/final exam
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- 5%: scribing notes (bonus)

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Usually there is scaling in final grades.
For me, a good grade is given based
on the overall performance of the
students: I value self-motivation,
being proactive and enthusiasm.

Participation + scribing

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(Moderation is key)

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- Deliverable in LaTEX
(template available online)

HWs

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Reviews (when applicable)

- Select papers from a pile of .pdfs that will be provided
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(Reviews will be related to the topics currently taught)
- Single page reviews, similar to NIPS/ICML standards:
(but not random as it usually is now)
 - Comment on novelty, clarity, importance
 - Main comments + your overall score

Presentations (for final projects)

(not certain yet)

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- Grading: slides quality, clarity of main ideas

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Example 1

Example 2

Example 3

Presentation example

Final Project

(Course website)

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Please come find me the earliest to discuss projects

Final Project

(Course website)

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You should start reading papers soon, so that around mid-way
you have a good project proposal

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- HWs: will be sent to you via Canvas every week.
(please do not distribute)

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- Every week I will try to upload an updated chapter; however I would appreciate any help with scribing throughout the semester

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- In case I don't have the time to cover fully a session, I will decide whether you will read it yourself, or I will teach it the next time.

Any questions?

Setting up the background

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(Associative)

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$$\alpha(x + y) = \alpha x + \alpha y, \quad x, y \in \mathbb{R}^p \quad (\text{Distributive})$$

Vectors

- Span of a set of vectors:

$$\text{span} \{x_1, x_2, \dots, x_k\} = \{\alpha_1 x_1 + \alpha_2 x_2 + \dots + \alpha_k x_k \mid \alpha_i \in \mathbb{R}, i = [1, k]\}$$

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- Inner product:

$$x^\top y = \langle x, y \rangle = \sum_{i=1}^p x_i \cdot y_i$$

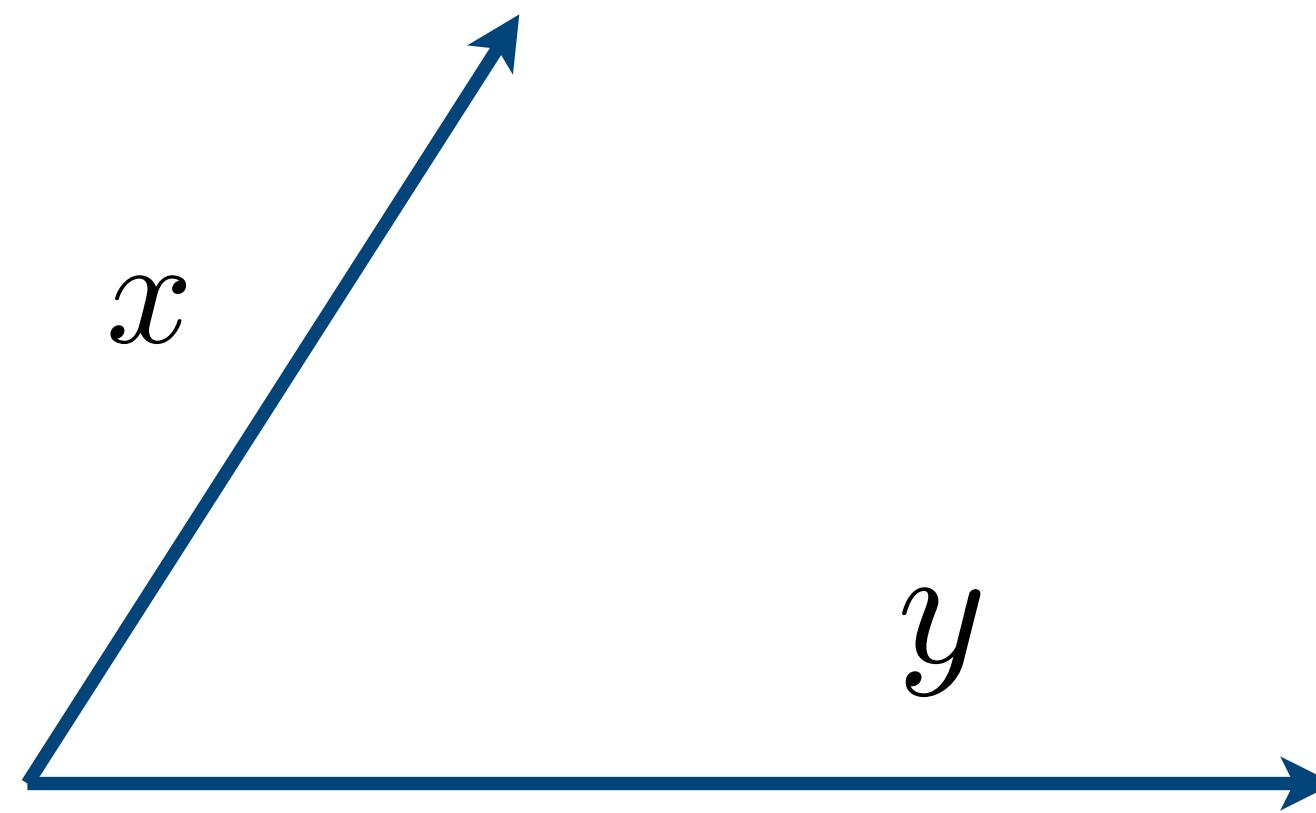
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- Inner production interpretation:

$$\langle x, y \rangle = \|x\| \cdot \|y\| \cdot \cos \theta$$

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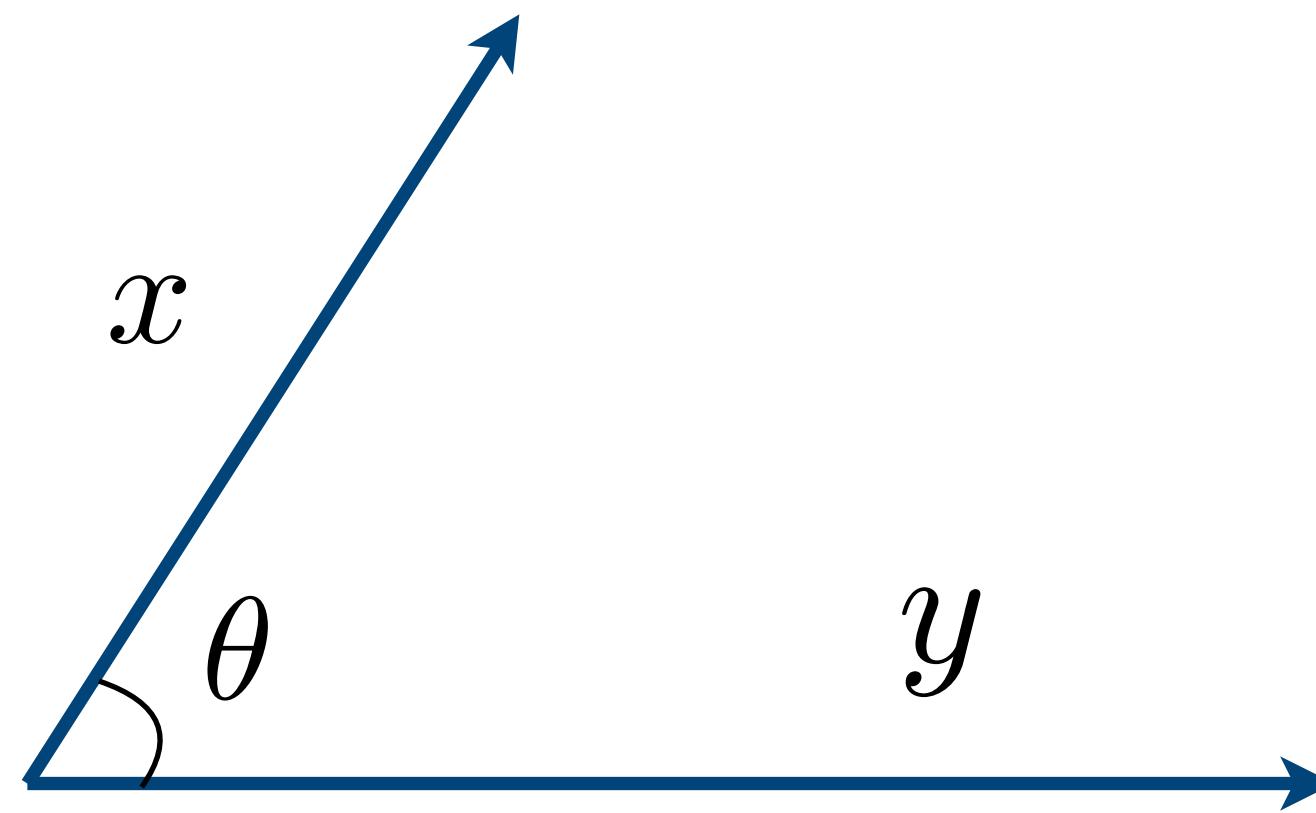
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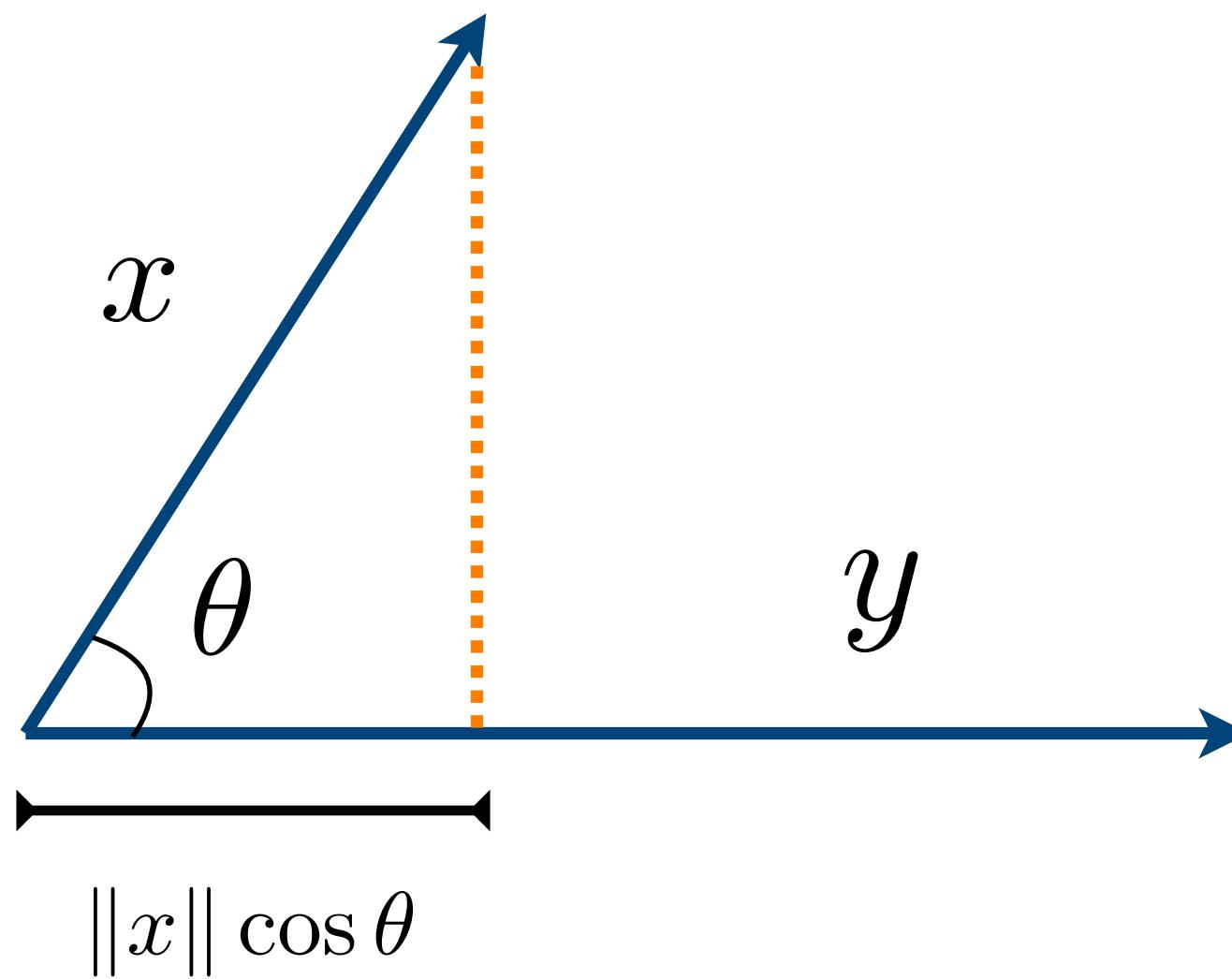
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Vectors

- Norms = notion of distance in multiple dimensions

$$\|x\| \geq 0, \forall x \in \mathbb{R}^p$$

$$\|x\| = 0, \text{ iff } x = 0$$

Properties:

$$\|\alpha x\| = |\alpha| \|x\|, \forall \alpha \in \mathbb{R}$$

$$\|x + y\| \leq \|x\| + \|y\|$$

$$|x^\top y| \leq \|x\| \|y\|$$

(Triangle inequality)

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$$\|x + y\| \leq \|x\| + \|y\| \quad (\text{Triangle inequality})$$

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- Famous wanna-be norms: $\|x\|_0 = \text{card}(x)$

Matrices

- Matrix in m, n dimensions: $A \in \mathbb{R}^{m \times n}$

$$A = \begin{bmatrix} A_{11} & A_{12} & \cdots & A_{1n} \\ A_{21} & A_{22} & \cdots & A_{2n} \\ \vdots & & \ddots & \vdots \\ A_{m1} & A_{m2} & \cdots & A_{mn} \end{bmatrix}$$

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- Names: Square, tall, fat, zero, identity, diagonal
- Properties:

$$A + B = B + A, \quad \forall A, B \in \mathbb{R}^{m \times n}$$

$$(A + B) + C = A + (B + C), \quad \forall A, B, C \in \mathbb{R}^{m \times n}$$

$$A + 0 = 0 + A, \quad \forall A \in \mathbb{R}^{m \times n}$$

$$(A + B)^\top = A^\top + B^\top, \quad \forall A, B \in \mathbb{R}^{m \times n}$$

Matrices

- Matrix multiplication: $C = AB$ where $C \in \mathbb{R}^{m \times p}$, $A \in \mathbb{R}^{m \times n}$, and $B \in \mathbb{R}^{n \times p}$

$$AB = \begin{bmatrix} A_{11} & A_{12} & \cdots & A_{1n} \\ A_{21} & A_{22} & \cdots & A_{2n} \\ \vdots & \ddots & & \vdots \\ A_{m1} & A_{m2} & \cdots & A_{mn} \end{bmatrix} \cdot \begin{bmatrix} B_{11} & B_{12} & \cdots & B_{1p} \\ B_{21} & B_{22} & \cdots & B_{2p} \\ \vdots & & \ddots & \vdots \\ B_{n1} & B_{n2} & \cdots & B_{np} \end{bmatrix}$$

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- Special cases: vector inner product, matrix–vector mult., outer product
- Properties:

$$(AB)C = A(BC), \quad \forall A, B, C$$

$$\alpha(AB) = (\alpha A)B, \quad \forall A, B$$

$$A(B + C) = AB + AC, \quad \forall A, B, C$$

$$(AB)^\top = B^\top A^\top, \quad \forall A, B$$

$$AB \neq BA$$

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- Nullspace of a matrix: $\{x \mid Ax = 0\}$
- Positive semi-definite matrices: $A \succeq 0$
 1. $A \in \mathbb{R}^{n \times n}$
 2. A is symmetric
 3. $x^\top Ax \geq 0, \forall x \in \mathbb{R}^n, x \neq 0$

Matrices

- Matrix singular value decomposition: $A \in \mathbb{R}^{m \times n}$

$$A = U\Sigma V^\top = \sum_{i=1}^r \sigma_i u_i v_i^\top, \quad U \in \mathbb{R}^{m \times r}, \Sigma \in \mathbb{R}^{r \times r}, V \in \mathbb{R}^{n \times r} \quad r \leq \{m, n\}$$

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- Left and right singular vectors are orthogonal: $U^\top U = I$ and $V^\top V = I$

Matrices

- Norms:

$$\|A\|_F = \sqrt{\sum_{ij} A_{ij}^2}$$

(Frobenius norm)

$$\|A\|_* = \sum_i^r \sigma_i$$

(Nuclear norm)

$$\|A\|_2 = \max_i \sigma_i$$

(Spectral norm)

..there are more norms to worry about (e.g., operator norms)
but we will skip them here..

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 - There exists a square matrix, A^{-1} , such that $A^{-1}A = AA^{-1} = I$

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- We have set up background and notation w.r.t. linear algebra
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Next lecture

- Brief introduction to convex optimization and related topics

Demo