Problem 1

The spectrum would depend on the angle between the observer and the rotational axis of the AGN. Narrow emission lines come from lower-velocity material that is more distant from the center of the AGN, and are visible regardless of whether the angle is large or small. Broad emission lines are caused by Doppler broadening due to the higher velocities of material closer to the center. These are only visible if the angle is small, because at larger angles, the torus of dust/gas surrounding the center blocks the view of the center.

Problem 2

Radio-loud AGN have jets/lobes, while radio-quiet AGN do not. These jets are caused by relativistic particles accelerated out of the center of the AGN. The particles emit synchrotron radiation because of strong magnetic fields, and synchrotron emissions peak in the radio regime.

Problem 3

In free-fall, kinetic energy is given by

$$E_k = \frac{1}{2}mv^2,$$

where m is the mass of the parcel and v is the velocity of the parcel. Gravitational potential energy is given by

$$E_g = -\frac{GMm}{r},$$

where M and r are the mass and radius of the neutron star, respectively. In free-fall, energy is conserved, and

$$\frac{1}{2}mv^2 = \frac{GMm}{r}.$$

We also know that luminosity is energy emitted per unit time, which is the time derivative of the above. M and r are constant, so on the right, we take the time derivative of m:

$$L = \frac{1}{2}\dot{m}v^2 = \frac{GM\dot{m}}{r}.$$

We want an mc^2 term, so we multiply both the numerator and denominator by c^2:

$$L = \frac{GM\dot{m}}{r} \times \frac{c^2}{c^2} = \frac{GM}{rc^2} \times \dot{m}c^2.$$

We can call the first term (the coefficient on the $\dot{m}c^2$) the efficiency term η. For the parameters given in the question, we find that

$$\eta = \frac{GM}{rc^2} = \frac{6.67 \times 10^{-11}\ \mathrm{m^3\,kg^{-1}} \times 1.4 \times 1.989 \times 10^{30}\ \mathrm{kg}}{10^4\ \mathrm{m} \times (2.99 \times 10^8\ \mathrm{m\,s^{-1}})^2} = 0.207,$$

which is the fraction of the rest-mass energy energy released.

Problem 4

(i) The rest wavelength of the $H\alpha$ emission line in a vacuum is 6564.614 Å $= 0.656\ \mu m$[1]. The redshift of the source is

$$z = \frac{4\ \mu m - 0.656\ \mu m}{0.656\ \mu m} = 5.01.$$

(ii) We can relate the velocity dispersion to the width of the emission line using the following equation:

$$\frac{\Delta\lambda}{\lambda} = \frac{\Delta v}{c}.$$

Assume that the width $\Delta\lambda$ is measured from the bottom left to the bottom right of the peak (not the FWHM). Rearranging for Δv and plugging in the appropriate values, we find that Δv is:

$$\Delta v = \frac{\Delta\lambda}{\lambda} \times c = \frac{0.025\ \mu m}{0.656\ \mu m} \times 2.99 \times 10^8\ \mathrm{m\,s^{-1}} \simeq 11425\ \mathrm{km\,s^{-1}}.$$

Δv corresponds to the difference between the maximum and minimum velocities, and assuming that velocity dispersion is $\sigma = \frac{1}{2}\Delta v$, it seems that this does not correspond to a velocity dispersion of 2000 $\mathrm{km\,s^{-1}}$.

Problem 5

The mass of Sgr A* is approximately $4 \times 10^6\ \mathrm{M_\odot}$[2]. The Schwarzchild radius is

$$R_S = \frac{2GM}{c^2} = \frac{2 \times (6.67 \times 10^{-11}\ \mathrm{m^3\,kg^{-1}\,s^{-2}}) \times (4 \times 10^6\ \mathrm{M_\odot}) \times (1.989 \times 10^{30}\ \mathrm{kg\,M_\odot^{-1}})}{(2.99 \times 10^8\ \mathrm{m\,s^{-1}})^2} = 1.18 \times 10^{10}\ \mathrm{m}.$$

The distance between the Sun and the center of the Milky Way is 8 kpc $= 2.47 \times 10^{20}$ m. We can use this, along with the small angle approximation, to convert the Schwarzchild radius of Sgr A* to an angular size (which is the resolution required):

$$\theta \simeq \tan\theta = \frac{R_s}{d} = \frac{1.18 \times 10^{10}\ \mathrm{m}}{2.47 \times 10^{20}\ \mathrm{m}} = 4.78 \times 10^{-11}\ \mathrm{rad} \simeq 0.01\ \mathrm{mas}.$$

If VLBI is used and the effective diameter of the "telescope" is the diameter of the Earth, then the wavelength that would have to be used is

$$\begin{aligned} \theta &= \frac{1.22\lambda}{2R_E} \\ \implies \lambda &= \frac{\theta \times 2R_E}{1.22} \\ &= \frac{4.78 \times 10^{-11}\ \mathrm{rad} \times 2 \times 6.378 \times 10^6\ \mathrm{m}}{1.22} \\ &= 0.0005\ \mathrm{m}. \end{aligned}$$

[1] https://classic.sdss.org/dr3/products/spectra/vacwavelength.html

[2] https://arxiv.org/pdf/1607.05726.pdf

Problem 6

C&O Equation 26.2:

$$t_c = \frac{2\pi v_M r_i^2}{CGM}$$

where t_c is the time required for the cluster to spiral in to the center of the Milky Way, v_M is the orbital speed ($\sim$ 220 km s^{-1}), r_i is the initial distance of the cluster from the center, and M is the mass of the cluster. Plugging in the appropriate values and performing the necessary unit conversions, we get

$$t_c = \frac{2\pi \times 220 \text{ km s}^{-1} \times 5 \text{ kpc}}{75 \times 6.67 \times 10^{-11} \text{ m}^3 \text{ kg}^{-1} \times 5 \times 10^6 \text{ M}_\odot} = 20.95 \text{ Gyr}.$$

This means that dynamical friction is not relevant for Milky Way globular clusters, as the time required to sink to the center is greater than the age of the Universe.

Problem 7

(i) We can do some algebra and find $g(t)$:

$$Z(t) = -y \ln\left(\frac{g(t)}{M_b}\right) = Z_x \tag{1}$$

$$\implies \quad -\frac{Z_x}{y} = \ln\left(\frac{g(t)}{M_b}\right) \tag{2}$$

$$\implies e^{-Z_x/y} = \frac{g(t)}{M_b} \tag{3}$$

$$\implies \quad g(t) = e^{-Z_x/y} M_b. \tag{4}$$

Using this, we can write $s(t)$ in terms of M_b and Z_x:

$$s(t) = M_b - g(t) \tag{5}$$

$$= M_b - e^{-Z_x/y} M_b \tag{6}$$

$$= M_b(1 - e^{-Z_x/y}). \tag{7}$$

(ii) The fraction of low metallicity stars is given by

$$\frac{s_{Z<Z_\odot/3}}{s_{Z<Z_\odot}} = \frac{M(1 - e^{-Z_\odot/3y})}{M(1 - e^{-Z_\odot/y})}. \tag{8}$$

From Eq. 3 above, we know that $e^{-Z_\odot/y} = g/M_b$. Similarly, we can rewrite the $e^{-Z_\odot/3y}$ term by starting from Eq. 2:

$$-\frac{Z_x}{y} = \ln\left(\frac{g(t)}{M_b}\right) \tag{9}$$

$$\implies \quad -\frac{Z_x}{3y} = \frac{1}{3}\ln\left(\frac{g(t)}{M_b}\right) = \ln\left(\left(\frac{g(t)}{M_b}\right)^{1/3}\right) \tag{10}$$

$$\implies e^{-Z_\odot/3y} = \left(\frac{g(t)}{M_b}\right)^{1/3}. \tag{11}$$

Then we can rewrite Eq. 8 as

$$\frac{1-(g/M_b)^{1/3}}{1-(g/M_b)}. \tag{12}$$

For $g/M_b = 0.1$, we have

$$\frac{1-0.1^{1/3}}{1-0.1} \simeq 0.5954. \tag{13}$$

(iii) We integrate the given equation to find the entire supply of baryons lost from time $t' = 0$ to time $t' = t$ (call this $M_L(t)$). Note that η and α are constants:

$$M_L(t) = \int_0^t \frac{dM(t')}{dt'}dt' \tag{14}$$

$$= -\eta\alpha \int_0^t \Psi(t')dt' \tag{15}$$

$$= -\eta\alpha(S(t) - S(0)). \tag{16}$$

Since we assume that at the birth of the galaxy, all baryonic matter is in the form of gas, $S(0) = 0$. Furthermore, $s = \alpha S$, so

$$M_L(t) = -\eta s(t). \tag{17}$$

Then $M_b(t)$, the mass at time t, is the sum of the initial mass $M_b(0)$ and the total change in mass up until time t:

$$M_b(t) = M_b(0) + M_L(t) = M_b(0) - \eta s(t). \tag{18}$$

(iv) Using the result from (iii), PS 3.40 becomes

$$\frac{dg}{dt} + \frac{ds}{dt} = \frac{M_b}{dt} = -\eta\alpha\Psi. \tag{19}$$

By definition, we have the relation

$$\frac{ds}{dt} = \alpha\frac{dS}{dt} = \alpha\Psi. \tag{20}$$

Using this, we find that

$$\frac{dg}{dt} = -\eta\alpha\Psi - \frac{ds}{dt} \tag{21}$$

$$= -\eta\frac{ds}{dt} - \frac{ds}{dt} \tag{22}$$

$$= \frac{ds}{dt}(-\eta - 1). \tag{23}$$

Dividing both sides by ds/dt, we get

$$\frac{dg}{ds} = -\eta - 1. \tag{24}$$

We also adjust the the equations at the bottom left of PS p. 143 to account for the expelled gas. The rate of metal mass lost is given by

$$Z\frac{dM_L}{dt} = Z\frac{d}{dt}(-\eta s) = -Z\eta\alpha\Psi, \tag{25}$$

where M_L is given in Eq. 17. Including this extra term, we find that

$$\begin{aligned}
\frac{d(gZ)}{dt} &= \Psi(RZ+q) - Z\Psi - Z\eta\alpha\Psi && (26)\\
\implies \frac{d(gZ)}{dS} &= RZ + q - Z - Z\eta\alpha && (27)\\
&= Z(R - 1 - \eta\alpha) + q && (28)\\
&= Z(-\alpha - \eta\alpha) + q && (29)\\
&= Z\alpha(-\eta - 1) + q && (30)\\
\implies \frac{d(gZ)}{ds} &= Z(-\eta - 1) + \frac{q}{\alpha} && (31)
\end{aligned}$$

Using the product rule and the Eq. 24, we can also express the left hand side as

$$\frac{d(gZ)}{ds} = \frac{dg}{ds}Z + \frac{dZ}{ds}g = (-\eta - 1)Z + \frac{dZ}{ds}g. \tag{32}$$

Equating Eq. 31 and Eq. 32, we find that

$$\begin{aligned}
(-\eta - 1)Z + \frac{dZ}{ds}g &= Z(-\eta - 1) + \frac{q}{\alpha} && (33)\\
\implies \qquad \frac{dZ}{ds}g &= \frac{q}{\alpha}. && (34)
\end{aligned}$$

Again using Eq. 24, we can rewrite the left hand side:

$$\frac{dZ}{ds}g = \frac{dZ}{dg}g\frac{dg}{ds} = \frac{dZ}{dg}g(-\eta - 1) = \frac{dZ}{d\ln g}(-\eta - 1). \tag{35}$$

Substituting this into Eq. 34, we get

$$\begin{aligned}
\frac{dZ}{d\ln g}(-\eta - 1) &= \frac{q}{\alpha} && (36)\\
\implies \qquad \frac{dZ}{d\ln g} &= \frac{q/\alpha}{-\eta - 1}. && (37)
\end{aligned}$$

Redefining y as $y = \frac{q/\alpha}{\eta+1}$, we arrive at the same result as PS Eq. 3.42:

$$\frac{dZ}{d\ln g} = -y. \tag{38}$$

(v) Since $g(t)$ is constant, we know that $dg/ds = 0$. So, PS Eq. 3.41 becomes

$$\frac{d(gZ)}{ds} = \cancelto{0}{\frac{dg}{ds}}Z + \frac{dZ}{ds}g = \frac{dZ}{ds}g = y - Z. \tag{39}$$

Multiplying by dt/dt, we can rewrite Eq. 39 as

$$\frac{dZ}{ds}g = \frac{dZ}{dt}g\frac{dt}{ds} = y - Z \tag{40}$$

$$\implies \frac{dZ}{dt} = \frac{y - Z}{g}\frac{ds}{dt}. \tag{41}$$

We cancel out dt from both sides and move the $y - Z$ term to the left side. We get a separable differential equation:

$$\frac{1}{y - Z}dZ = \frac{1}{g}ds. \tag{42}$$

We can integrate this from 0 to t, noting that g is constant with respect to s:

$$\ln{(y - Z)}(-1)\Big|_{Z=0}^{Z=t} = \frac{s}{g}\Big|_{s=0}^{s=t}, \tag{43}$$

with the -1 on the left coming from the chain rule. We can move it over to the right hand side, then evaluate the left hand side using $Z(0) = 0$:

$$\ln{(y - Z(t))} - \ln{(y - Z(0))} = \ln{(y - Z(t))} - \ln y. \tag{44}$$

Using $s(0) = 0$, the right hand side (with the extra negative sign) evalutes to

$$-\frac{s(t)}{g} + \frac{s(0)}{g} = -\frac{s(t)}{g}. \tag{45}$$

We equate Eq. 44 and Eq. 45, then take the exponent on both sides. On the left side, we have

$$e^{\ln{(y - Z(t))} - \ln y} = e^{\ln{(y-Z(t))}}e^{-\ln y} = \frac{e^{\ln{(y-Z(t))}}}{e^{\ln y}} = \frac{y - Z(t)}{y}. \tag{46}$$

Equating this to the right side and rearranging, we find that

$$\frac{y - Z(t)}{y} = e^{-s(t)/g} \tag{47}$$

$$\implies \qquad y - Z(t) = ye^{-s(t)/g} \tag{48}$$

$$\implies \qquad Z(t) = y - ye^{-s(t)/g} = y(1 - e^{-s(t)/g}). \tag{49}$$

(vi) Since g is constant, we can directly rearrange Eq. 49 for $s(t)$:

$$Z(t) = y(1 - e^{-s(t)/g}) = Z_x \tag{50}$$

$$\implies \frac{Z_x}{y} = 1 - e^{-s(t)/g} \tag{51}$$

$$\implies e^{-s(t)/g} = 1 - \frac{Z_x}{y} \tag{52}$$

$$\implies -\frac{s(t)}{g} = \ln{(1 - \frac{Z_x}{y})} \tag{53}$$

$$\implies s(t) = -g\ln{(1 - \frac{Z_x}{y})}, \tag{54}$$

which gives the mass of stars with metallicity less than some metallity Z_x. We want to find y (the 10^{-2} given in the slides doesn't work—we end up taking the logarithm of a negative number). Assuming a present-day gas fraction of $g/M_b = 0.1$ (from part (ii)), we know that $s/M_b = 0.9$, which means that $s/g = 9$. From Eq. 51, we solve for y, leaving it in terms of Z_x:

$$y = \frac{Z_x}{1 - e^{-s(t)/g}} = \frac{1}{1 - e^{-9}} Z_x. \tag{55}$$

Now we use Eq. 54 and find the ratio of stars with $Z_\odot$ and with $Z_\odot/3$:

$$\frac{-g \ln\left(1 - \frac{Z_\odot}{3y}\right)}{-g \ln\left(1 - \frac{Z_\odot}{y}\right)} = \frac{\ln\left(1 - \frac{Z_\odot}{\frac{3Z_\odot}{1-e^{-9}}}\right)}{\ln\left(1 - \frac{Z_\odot}{\frac{Z_\odot}{1-e^{-9}}}\right)} \tag{56}$$

$$= \frac{\ln\left(1 - \frac{1-e^{-9}}{3}\right)}{\ln\left(1 - (1 - e^{-9})\right)} \tag{57}$$

$$= 0.045. \tag{58}$$

The fraction of metal-poor stars predicted by the accreting box model is much closer to observed values.

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