AST 222 Problem Set 3

*Due on Fri. March. 6 at the **start** of class*

Please staple all of your work together

Student name: *Student number:*

Problem 1: (2 points)
For a set of bodies interacting with a $1/r^2$ force that is in dynamical equilibrium, the virial theorem states that K=-U/2, where K is the total kinetic energy of the system and U is the total potential energy. For a star cluster of N stars, the total kinetic energy is $\sum_{i=1}^{N} \frac{1}{2} m_i v_i^2$ and the total potential energy is $\sum_{i=1}^{N}\sum_{j=1}^{N} Gm_i m_j / \mid \vec{r_i} - \vec{r_j} \mid$. Here m, v and $\vec{r}$ are the mass, velocity and position vector of each star. If we assume all the stars have the same mass m (where Nm=M), a mean velocity of V, and a mean separation equal to the radius of the system R, prove that the virial theorem yields $V^2 = GM/R$ for star clusters.

Problem 2: (15 points)
As mentioned in Assignment 1, the `gaia` spacecraft is currently measuring the positions and velocities of billions of stars in the Milky Way. However `gaia` is limited by both is photometric and spatial resolution, meaning it is unable to measure faint (low-mass) stars or stars in high density regions (like the centres of star clusters). Hence for `gaia` observations of dense star clusters, the oldest of which are called globular clusters, only a fraction of the actual stellar population is detected. By assuming a cluster is in virial equilibrium, we can still use the spatial and kinematic information of the stars that are observed to determine the cluster's total mass. In this problem we will try and do this for the Galactic globular cluster 47 Tuc (aka NGC 104), which is one of the closest clusters to the Sun at a distance of 4.5 kpc. Unfortunately the line-of-sight distance and velocity of individual stars cannot be measured, so we are stuck using projected (two-dimensional) values. (Note: this assignment will ask you to read data in from a file. I personally recommend the `numpy` python software package, which is easy to install from your terminal by typing `pip install numpy`. The numpy function `loadtxt` is especially useful for reading data into arrays for analysis. You will also be asked to perform a similar analysis on a second dataset, so keep your analysis as modular as possible).

A catalogue of 47 Tuc stars that have been detected with `gaia` can be found on Quercus in the file 47Tuc.dat. The four columns represent each star's Right Ascension (RA) (degrees), Declination (Dec) (degrees), proper motion in Right Ascension (μ_{RA} mas/yr) and proper motion in Declination (μ_{Dec} mas/yr). To determine the total mass of 47 Tuc, read the data and do the following:

i) Check the data by making plots of the positions (Dec vs Ra in degrees) and proper motions (μ_{Dec} vs μ_{Ra} in km/s) of all the stars in the dataset. It should be clear that you are looking at a cluster of stars that have similar velocities. [2 pts]
ii) Determine the position (Ra_{centre}, Dec_{centre} in degrees) and velocity ($\mu_{RA,centre}$,$\mu_{Dec,centre}$ in **BOTH** mas/yr and km/s) of the cluster's centre. Compare your results to recent work done in Vasiliev 2019 (https://arxiv.org/abs/1807.09775 - see Table C1). [2 pts]
iii) Perform a coordinate transformation on the dataset so the cluster's centre is at the origin and the origin has a velocity of zero. Then convert each star's angular distance from the origin in degrees to a distance in parsecs. Make a plot of each stars total projected velocity (in km/s) versus its projected distance from the cluster's centre (in pc). [2 pts]
iv) Assuming the cluster is in virial equilibrium, use your result from Problem 1 to determine its total mass as implied by the observed stellar population. Justify your choices for V and R. [4 pts].
Hint 1:: Approximating the cluster as an isotropic system, the mean three dimensional velocity V is related to the 1D velocity dispersion via $V^2 = 3\sigma_{1D}^2$.
Hint 2: A globular cluster is spherically symmetric, which may come in handy when estimating its three dimensional radius.
v) Using the Hubble Space Telescope, 47 Tuc's total absolute visual magnitude was measured to be -9.42. Given this information, calculate 47 Tuc's mass-to-light ratio and discuss why you do or do not believe 47 Tuc contains dark matter. [2 pts]
vi) Repeat the same analysis on data from the fictional dwarf galaxy Keanu (data available on Quercus in the file keanu.dat, which has the same format as the 47 Tuc data). Keanu is at a distance of 50 kpc and has a total visual magnitude of -12.8. For submission, only your estimate of Keanu's total mass, mass-to-light-ratio, and a discussion on whether or not you believe Keanu contains dark matter is necessary. [3 pts]

Problem 3: (10 points)

We discussed the case of lensing by a point-mass in detail in class and saw that a background source lensed by a point-like lens always produces two observed images (unless when they are exactly lined up, in which case you get an Einstein ring). This means that every star in the sky is lensed by every star between us and the star and if there are N stars in the Milky Way, we should expect to see about $\approx N^2$ images of stars on the sky. That's crazy! Let's see why this does not happen in practice.
(i) Solve the lensing equation for a point-mass lens for the case where $\beta \neq 0$. Because the lensed image's position θ is along the line that connects the lens and the source position β, you can work in one dimension (that is, β and θ are numbers rather than two-dimensional vectors on the sky). That is, give θ as a function of β for a given point-mass. Express your result in terms of the Einstein angle θ_E (your solution should only contain β and θ_E).[2 pts]
(ii) Examine what happens to the solution with the smallest $|\theta|$ as $|\beta|$ increases. What is $|\theta|/\theta_E$ for $\beta = \theta_E, 10\theta_E, 100\theta_E, 1000\theta_E$? Compute θ_E from its definition for a lens star at 5 kpc with $M = 1$ $M_\odot$ and a source star at 10 kpc and use this to express your result for $|\theta|$ in terms of milli-arcseconds as well for this lens-source setup. What is the limit for $\beta \to \infty$?[2 pts]
(iii) In reality, stars are not point-sources, but they have a finite angular size given by their

physical radius divided by the distance. When a gravitationally-lensed image computed using the point-source model falls within this angular radius, we cannot trust the solution, because the point-source assumption is broken, and in reality no second image exists. For a $M = 1\,\mathrm{M}_\odot$ main-sequence star lens at 5 kpc and a source at 10 kpc, what is β for which the image with the smallest $|\theta|$ lies within the lens star? Express your result both in terms of θ_E and in milli-arcseconds.[2 pts]
(iv) Estimate the typical separation between two stars on the sky. (A simple estimate is enough, but you need to give your reasoning and calculations). How does this separation compare to the lens-source separation that you found in (c)? What does this imply about the existence for the second lensed image for typical lens-source systems on the sky?[2 pts]
(v) Another reason that the second image (that with the smallest $|\theta|$) is typically irrelevant is that its magnifiction is small. Compute the magnifiction of the second image for $\beta = \theta_E, 10\theta_E, 100\theta_E, 1000\theta_E$ for the lens-source setup that we have considered in other parts of this question. What is the limit of the magnifiction of the second image for $\beta \to \infty$[2 pts]?