Problem Set 3

Problem 1

Given that all stars have the same mass m, and $Nm = M$, and all stars have a mean velocity of V, the total kinetic energy becomes

$$\begin{aligned}
\sum_{i=1}^{N} \frac{1}{2} m_i v_i^2 &= \frac{1}{2} \sum_{i=1}^{N} m V^2 \\
&= \frac{1}{2} m V^2 \sum_{i=1}^{N} 1 \\
&= \frac{1}{2} m V^2 N,
\end{aligned}$$

and the potential energy is

$$\begin{aligned}
-\sum_{i=1}^{N}\sum_{j=1}^{N} \frac{G m_i m_j}{|r_i - r_j|} &= -\sum_{i=1}^{N}\sum_{j=1}^{N} \frac{G m^2}{R} \\
&= -\frac{G m^2}{R} N^2.
\end{aligned}$$

Using the virial theorem, we find that

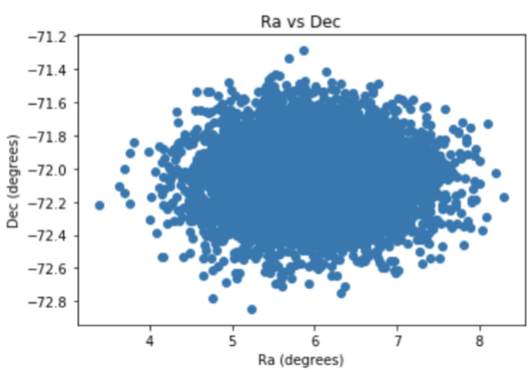
$$\begin{aligned}
K &= -\frac{U}{2} \\
\implies \frac{1}{2} N m V^2 &= \frac{G N^2 m^2}{2R} \\
\implies \qquad V^2 &= \frac{G N m}{R} \\
&= \frac{G M}{R}
\end{aligned}$$

Problem 2

(i)

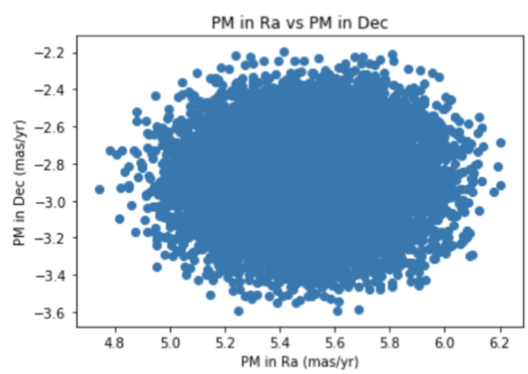
```
plt.scatter(ra, dec)
plt.title('Ra vs Dec')
plt.xlabel('Ra (degrees)')
plt.ylabel('Dec (degrees)')
```

```
Text(0, 0.5, 'Dec (degrees)')
```




```
plt.scatter(pmra, pmdec)
plt.title('PM in Ra vs PM in Dec')
plt.xlabel('PM in Ra (mas/yr)')
plt.ylabel('PM in Dec (mas/yr)')
```

```
Text(0, 0.5, 'PM in Dec (mas/yr)')
```



(ii)

```
print(f'''Values obtained from dataset
------------------------------------
Position of center:
Ra: {ra.mean():.3f}\tDec: {dec.mean():.3f}

Proper motion:
PM in Ra:   {pmra.mean():.3f}\t({(pmra.mean()*4.5*u.kpc).to(u.km/u.s, equivalencies=u.dimensionless_angles()):.3f})
PM in Dec: {pmdec.mean():.3f}\t({(pmdec.mean()*4.5*u.kpc).to(u.km/u.s, equivalencies=u.dimensionless_angles()):.3f})


Reference values from Vasiliev 2019
------------------------------------
Position of center:
Ra: 6.024 deg\tDec: -72.081 deg

Proper motions:
PM in Ra:   5.237 mas / yr\t({(5.237*u.mas/u.yr*4.5*u.kpc).to(u.km/u.s, equivalencies=u.dimensionless_angles()):.3f})
PM in Dec: -2.524 mas / yr\t({(-2.524*u.mas/u.yr*4.5*u.kpc).to(u.km/u.s, equivalencies=u.dimensionless_angles()):.3f})
''')
```

```
Values obtained from dataset
------------------------------------
Position of center:
Ra: 5.981 deg   Dec: -72.056 deg

Proper motion:
PM in Ra:   5.504 mas / yr      (117.407 km / s)
PM in Dec: -2.885 mas / yr      (-61.541 km / s)


Reference values from Vasiliev 2019
------------------------------------
Position of center:
Ra: 6.024 deg   Dec: -72.081 deg

Proper motions:
PM in Ra:   5.237 mas / yr      (111.716 km / s)
PM in Dec: -2.524 mas / yr      (-53.842 km / s)
```

(iii)

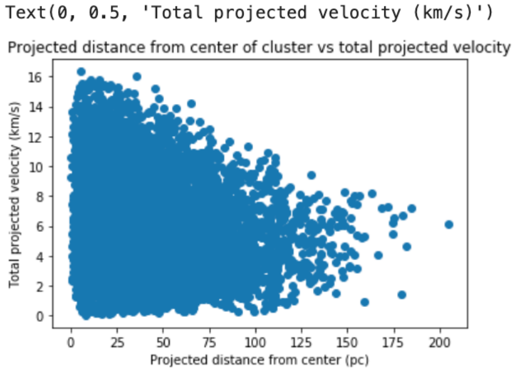
```
# coordinate transformation
ra2 = ra - ra.mean()
dec2 = dec - dec.mean()
pmra2 = pmra - pmra.mean()
pmdec2 = pmdec - pmdec.mean()
```

```
# distances to center. C&O Eq. 1.8
angdists = np.sqrt(np.square(ra2 * np.cos(dec2)) + np.square(dec2))
# convert angular distances to distances in pc
dists2center = (angdists * 4.5 * u.kpc).to(u.pc, equivalencies=u.dimensionless_angles())

# getting total projected velocity
totprojvels = np.sqrt(np.square(pmra2 * np.cos(dec2)) + np.square(pmdec2))
totprojvels *= 4.5 * u.kpc
totprojvels = totprojvels.to(u.km / u.s, equivalencies=u.dimensionless_angles())
```

```
plt.scatter(dists2center, totprojvels)
plt.title('Projected distance from center of cluster vs total projected velocity')
plt.xlabel('Projected distance from center (pc)')
plt.ylabel('Total projected velocity (km/s)')
```

```
Text(0, 0.5, 'Total projected velocity (km/s)')
```



(iv) Velocity is given by Hint 1:

$$V = \sqrt{3\sigma_{1D}^2},$$

where σ_{1D}^2 is the variance in the projected velocity. The radius is taken to be the maximum of the projected distances from the center of the cluster.

```
vsq = 3 * np.var(totprojvels)
vsq = vsq.to(u.m **2 / u.s ** 2)
r = np.max(abs(dists2center)).to(u.m)
mass = vsq * r / const.G
mass
```

2.6059035×10^{36} kg

(v)

```
# calculating luminosity in terms of solar luminosity
lum = const.L_sun * 100 ** ((4.83 + 9.42) / 5)
# getting mass to light ratio with the sun as a baseline
mlratio = (mass / const.M_sun) / (lum / const.L_sun)
mlratio
```

2.6148839

The mass-to-light ratio is somewhat low and it seems plausible that 47 Tuc contains very little dark matter.

(vi) The code is the same and won't be shown here. Performing the same analysis, the results for Keanu are as follows:

- mass estimate: 2.23×10^{39} kg
- mass-to-light ratio: 99.5 $\Upsilon_{\odot}$

The high mass-to-light ratio of Keanu indicates that it is dominated by dark matter.

Problem 3

(i) Solving the lensing equation for θ gives

$$\begin{aligned}\beta &= \theta - \frac{\theta_E^2}{\theta^2}\theta \\ &= \theta - \frac{\theta_E^2}{\theta}.\end{aligned}$$

We multiply both sides of the equation by θ and solve the quadratic in θ:

$$\begin{aligned} & \theta^2 - \beta\theta - \theta_E^2 = 0 \\ \implies \qquad & \theta = \frac{\beta \pm \sqrt{\beta^2 - 4(-\theta_E^2)}}{2} \\ & \quad = \frac{\beta \pm \sqrt{\beta^2 + 4\theta_E^2}}{2}.\end{aligned}$$

(ii) The Einstein angle for the given parameters is:

```
def einstein(ds, dd, m):
    dds = ds - dd
    return np.sqrt((4 * const.G * m * dds) / (ds * dd * const.c ** 2)) * u.rad
```

```
thetae = einstein(10 * u.kpc, 5 * u.kpc, 1 * u.M_sun).to(u.rad)
thetae
```

4.3751187×10^{-9} rad

For each of the different β values (in the order given), using the smallest $|\theta|$, $|\theta|/\theta_E$ is:

```
def theta(beta, einstein):
    return (beta + np.sqrt(beta**2 + 4*einstein**2)) / 2, (beta - np.sqrt(beta**2 + 4*einstein**2)) / 2
```

```
xs = np.array([1, 10, 100, 1000])
for i in xs:
    y = np.min(np.abs(theta(i * thetae.value, thetae.value))) / thetae.value
    print(f'beta = {i}\t|theta|/theta_e in mas: {y*thetae.value:.3E}\tin terms of Einstein angle: {y:.3E}')
```

```
beta = 1        |theta|/theta_e in mas: 2.704E-09        in terms of Einstein angle: 6.180E-01
beta = 10       |theta|/theta_e in mas: 4.332E-10        in terms of Einstein angle: 9.902E-02
beta = 100      |theta|/theta_e in mas: 4.375E-11        in terms of Einstein angle: 9.999E-03
beta = 1000     |theta|/theta_e in mas: 4.375E-12        in terms of Einstein angle: 1.000E-03
```

As $\beta \to \infty$, θ becomes

$$\begin{aligned}\lim_{\beta\to\infty} \theta &= \lim_{\beta\to\infty} \frac{\beta \pm \sqrt{\beta^2 + 4\theta_E^2}}{2} \\ &= \lim_{\beta\to\infty} \frac{\beta \pm \sqrt{\beta^2}}{2} \\ &= \lim_{\beta\to\infty} \frac{\beta \pm \beta}{2} \\ \implies \theta &\to \beta, 0\end{aligned}$$

The smaller solution approaches 0.

(iii) Assuming that the radius of such a star is the same as that of the Sun, and using the small angle approximation, β, in terms of the previously computed Einstein angle, is[1]

```
t = (const.R_sun / (5 * u.kpc)).to(u.dimensionless_unscaled)
b = t - thetae.value / t
b
```

-970.26057

Converting this into mas:

```
(970.26 * thetae).to(u.mas)
```

875.59465 mas

(iv) Assume that there are roughly 10^{11} stars in the sky (the number of stars in the galaxy), and that they are uniformly distributed. Also assume that you can only see half of the sky (the ground is blocking the other half). The area of the sky is 4π steradians (sr), so the area of the visible portion is 2π sr. Then the "density" of stars in the sky is

$$\frac{5 \times 10^{10} \text{ stars}}{2\pi \text{ sr}} = 7.958 \times 10^9 \text{ stars/sr}.$$

The reciprocal of this gives the area of the sky taken up by each star:

$$\frac{1}{7.96 \times 10^9 \text{ stars/sr}} = 1.257 \times 10^{-10} \text{ sr/star}.$$

Ignoring that circles have a maximum packing density of 0.9069 (or ignoring overlaps if the circles are forced closer together), assume that the area taken up by each star is in the shape of a circle. Using this assumption, we can calculate the diameter of each star from the area:

$$r = \sqrt{\frac{A}{\pi}} = \sqrt{\frac{1.257 \times 10^{-10} \text{ sr}}{\pi}} = 1.121 \times 10^{-5} \text{ rad}$$
$$\implies d = 2r = 2.242 \times 10^{-5} \text{ rad} \simeq 4.624 \text{ mas}.$$

This is smaller than the result we found in the previous part, and this suggests that, for a typical lens-source system in the sky, there is no second image.

[1] Physically it doesn't really make sense for β to be negative, but this doesn't really matter in terms of calculating distances—if you try to plug the negative result into the formula to calculate θ (the one from part a) and take the solution with the smaller absolute value, you get the same result as plugging in +970.26.

(v) We take the limit of the magnification as $\beta \to \infty$:

$$\begin{aligned}
\lim_{\beta\to\infty} \mu &= \lim_{\beta\to\infty} \frac{1}{4}\left(\frac{\beta}{\sqrt{\beta^2+4\theta_E^2}} + \frac{\sqrt{\beta^2+\theta_E^2}}{\beta} \pm 2\right) \\
&= \lim_{\beta\to\infty} \frac{1}{4}\left(\frac{\beta}{\sqrt{\beta^2}} + \frac{\sqrt{\beta^2}}{\beta} \pm 2\right) \\
&= \lim_{\beta\to\infty} \frac{1}{4}\left(1+1\pm 2\right) \\
\implies \mu &\to 1, 0
\end{aligned}$$

The magnification of the second image approaches 0 (i.e. the image disappears).