

Problem 1

Using the distance modulus formula, this distance is (ignoring extinction)

$$d = 10^{(m-M+5)/5} = 10^{(25+19.5+5)/5} = 7.94 \text{ Gpc.}$$

Problem 2

Density for matter is given by $\rho_{m,0} = \Omega_{m,0}(1+z)^3$, and for radiation by $\rho_{r,0} = \Omega_{r,0}(1+z)^4$. Equate the two and solve for z :

$$\begin{aligned} \rho_{m,0} &= \rho_{r,0} \\ \Rightarrow \Omega_{m,0}(1+z)^3 &= \Omega_{r,0}(1+z)^4 \\ \Rightarrow z &= \frac{\Omega_{m,0}}{\Omega_{r,0}} - 1 = \frac{0.317}{10^{-4}} - 1 = 3169. \end{aligned}$$

Problem 3

The radius of the sphere of influence of the black hole is

$$r = \frac{GM}{\sigma^2} = \frac{6.67 \times 10^{-11} \text{ m}^3 \text{ kg}^{-1} \times 2 \times 10^9 \text{ M}_{\odot}}{4 \times 10^5 \text{ m s}^{-1}} = 1.66 \times 10^{18} \text{ m.}$$

Given that the distance to M87 is $53.5 \text{ Mly} = 5.06 \times 10^{23} \text{ m}$, the angular radius is

$$\theta \simeq \tan \theta = \frac{r}{d} = \frac{1.66 \times 10^{18} \text{ m}}{5.06 \times 10^{23} \text{ m}} = 3.28 \times 10^{-6} \text{ rad} = 0.67 \text{ arcsec.}$$

Whether this is observable without adaptive optics probably depends on the location of the ground-based telescope and the atmospheric conditions. The Internet¹ and the lecture slides from week 1 say that seeing of $0.5''$ can be achieved in locations such as Mauna Kea.

Problem 4

The parallax formula is

$$\begin{aligned} \theta &\simeq \tan \theta = \frac{r}{d} \\ \Rightarrow d &= \frac{r}{\theta}, \end{aligned}$$

where d is the distance we want, θ is the parallax angle, and r is the distance from Gaia to the Sun. Gaia is at the Earth-Sun L2 Lagrangian point, which has a radius of orbit of approximately 1.01 au. Plugging these values in, we find that d is

$$d = \frac{1.01 \text{ au}}{0.05 \text{ arcsec}} = 20.22 \text{ pc.}$$

At this distance,

¹Racine, R. 1989, Publications of the Astronomical Society of the Pacific, 101, 436.

Problem 5

```
import numpy as np
import matplotlib.pyplot as plt
%matplotlib inline
from astropy import units as u
from astropy import constants as const
```

```
dists, vels = np.loadtxt('ShenJeff.dat', unpack=True)
```

```
# fit a linear model to the data. the slope should be the hubble parameter H_0
slope, offset = np.polyfit(dists, vels, 1)
slope, offset
```

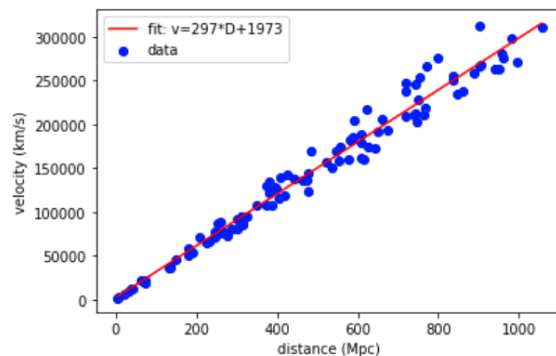
```
(296.7834527807584, 1972.9606705987662)
```

The present-day value of the Hubble parameter of this universe is $H_0 = 297 \text{ km/s/Mpc}$.

```
xvals = np.linspace(min(dists), max(dists))
```

```
# plot data and fit
plt.scatter(dists, vels, c='b', label='data')
plt.plot(xvals, offset + slope * xvals, c='r', label=f'fit: v={m:.0f}*D+{b:.0f}')
plt.xlabel('distance (Mpc)')
plt.ylabel('velocity (km/s)')
plt.legend()
```

```
<matplotlib.legend.Legend at 0x1239e8350>
```



```
# assuming constant expansion, the age of the universe is given by 1/H_0
h0 = m * (u.km / u.s) / u.Mpc
(1/h0).to(u.Gyr)
```

```
3.2946319 Gyr
```

The age of this universe is 3.3 Gyr.

```
# use the friedmann equations to find the critical density:
((3 * h0**2) / (8 * np.pi * const.G)).to(u.kg / u.m**3)
```

```
1.6544511 × 10-25  $\frac{\text{kg}}{\text{m}^3}$ 
```

The critical density of the universe is $\rho_c = 1.65 \times 10^{-25} \text{ kg/m}^3$. From week 10's lecture slides, we know that $\rho_{m,0} = 2.56 \times 10^{-27} \text{ kg/m}^3$ and $\rho_{b,0} = 4.17 \times 10^{-28} \text{ kg/m}^3$. This means that if the density of matter in this alternate universe is the same as the density of our universe, that universe has a density parameter $\Omega = \rho/\rho_c$ less than 1, so the universe is open.

Problem 6

(i) C&O Eq. 29.10:

$$\left[\left(\frac{1}{R} \frac{dR}{dt} \right)^2 - \frac{8}{3} \pi G \rho \right] R^2 = -kc^2.$$

Multiplying this by R , we get

$$\left(\frac{dR}{dt} \right)^2 R - \frac{8}{3} \pi G \rho R^3 = -kc^2 R$$

Taking the time derivative and applying the product and chain rules to the first term, we get

$$2 \frac{dR}{dt} \frac{d^2 R}{dt^2} R + \left(\frac{dR}{dt} \right)^3 - \frac{8}{3} \pi G \frac{d}{dt} (\rho R^3) = \frac{d}{dt} (-kc^2 R).$$

We use C&O Eq. 29.50 to replace the third term on the left side, then expand the derivative using the chain rule and cancel out a dR/dt term:

$$\begin{aligned} 2 \frac{dR}{dt} \frac{d^2 R}{dt^2} R + \left(\frac{dR}{dt} \right)^3 + \frac{8}{3} \pi G \frac{P}{c^2} \frac{d(R^3)}{dt} &= -kc^2 \frac{dR}{dt} \\ \Rightarrow 2 \frac{dR}{dt} \frac{d^2 R}{dt^2} R + \left(\frac{dR}{dt} \right)^3 + \frac{8}{3} \pi G \frac{P}{c^2} 3R^2 \frac{dR}{dt} &= -kc^2 \frac{dR}{dt} \\ \Rightarrow 2 \frac{d^2 R}{dt^2} R + \left(\frac{dR}{dt} \right)^2 + 8 \pi G \frac{P}{c^2} R^2 &= -kc^2. \end{aligned}$$

We can use Eq. 29.10 again to replace the $-kc^2$ term on the right side:

$$2 \frac{d^2 R}{dt^2} R + \left(\frac{dR}{dt} \right)^2 + 8 \pi G \frac{P}{c^2} R^2 = \left(\frac{dR}{dt} \right)^2 - \frac{8}{3} \pi G \rho R^2.$$

Cancelling terms and rearranging, we get

$$2 \frac{d^2 R}{dt^2} R = -8 \pi G R^2 \left(\frac{\rho}{3} + \frac{P}{c^2} \right).$$

Dividing both sides by $2R$ and then factoring out $1/3$ from the right side, we arrive at C&O Eq. 29.51:

$$\frac{d^2 R}{dt^2} = -\frac{4}{3} \pi G R \left(\rho + \frac{3P}{c^2} \right).$$

(ii) From C&O Eq. 29.8, we know that

$$HR = \frac{dR}{dt}, \tag{1}$$

and by (a rearrangement of) the definition of the density parameter Ω , we know that

$$H^2 = \frac{8 \pi G \rho}{3 \Omega}. \tag{2}$$

Using Eq. 1 and Eq. 2 above to replace the denominator of the expression for the deceleration parameter, and the acceleration equation to replace the numerator, we can rewrite the deceleration parameter as

$$\begin{aligned}
 q &= -\frac{R \frac{d^2 R}{dt^2}}{\left(\frac{dR}{dt}\right)^2} \\
 &= -\frac{R \frac{d^2 R}{dt^2}}{H^2 R^2} \\
 &= -\frac{\frac{d^2 R}{dt^2}}{H^2 R} \\
 &= -\frac{-\frac{4}{3}\pi G R(\rho + \frac{3P}{c^2})}{\frac{8\pi G \rho}{3\Omega} R} \\
 &= \frac{\rho + \frac{3P}{c^2}}{\frac{2\rho}{\Omega}} \\
 &= \frac{\frac{\rho c^2 + 3P}{c^2}}{\frac{2\rho}{\Omega}} \\
 &= \frac{(\rho c^2 + 3P)\Omega}{2\rho c^2}.
 \end{aligned}$$

Using the equation of state $P = \omega \rho c^2$ to replace the P in the above expression, we get

$$\begin{aligned}
 q &= \frac{(\rho c^2 + 3\omega \rho c^2)\Omega}{2\rho c^2} \\
 &= \frac{\Omega \rho c^2(1 + 3\omega)}{2\rho c^2} \\
 &= \frac{\Omega}{2}(1 + 3\omega).
 \end{aligned}$$

(iii) We begin with multiplying the acceleration by R^2 to get

$$\frac{d^2 R}{dt^2} R^2 = -\frac{4}{3}\pi G R^3 \left(\rho + \frac{3P}{c^2}\right).$$

Taking the time derivative and applying the product rule to the left side and the chain rule to both sides, we get

$$\frac{d^3 R}{dt^3} R^2 + \frac{d^2 R}{dt^2} 2R \frac{dR}{dt} = -\frac{4}{3}\pi G 3R^2 \frac{dR}{dt} \left(\rho + \frac{3P}{c^2}\right).$$

We can cancel out an R term, move the second term from the left to the right side, then multiply both sides of the equation by $R^2 / (dR/dt)^3$ to get the expression for jerk on the left:

$$j(t) = \frac{R^2 \frac{d^3 R}{dt^3}}{\left(\frac{dR}{dt}\right)^3} = -4\pi G R^3 \left(\frac{dR}{dt}\right)^{-2} \left(\rho + \frac{3P}{c^2} + 2 \frac{d^2 R}{dt^2} \frac{1}{4\pi G R}\right).$$

Using Eq. 1 from Q6(ii), we can introduce an H term into the right side:

$$j(t) = -\frac{4\pi GR}{H^2} \left(\rho + \frac{3P}{c^2} + \frac{\frac{d^2 R}{dt^2}}{2\pi GR} \right).$$

We can use the equation of state to replace the first two terms in the parentheses, and the acceleration equation to replace the numerator in the last term:

$$j(t) = -\frac{4\pi GR}{H^2} \left(\rho(1 + 3\omega) - \frac{\frac{4}{3}\pi GR(\rho + \frac{3P}{c^2})}{2\pi GR} \right).$$

After cancelling terms, we can again use the equation of state to rewrite the last term:

$$j(t) = -\frac{4\pi GR}{H^2} \left(\rho(1 + 3\omega) - \frac{2}{3}(\rho(1 + 3\omega)) \right) = -\frac{\frac{4}{3}\pi GR\rho(1 + 3\omega)}{H^2}.$$

Using Eq. 2 from Q6(ii), we replace the H^2 term and arrive at an expression for j in terms of Ω and ω :

$$j(t) = -\frac{\frac{4}{3}\pi GR\rho(1 + 3\omega)}{\frac{8\pi GR\rho}{3\Omega}} = -\frac{R\Omega}{2}(1 + 3\omega).$$

For matter, with $\omega = 0$, we get

$$j(t) = -\frac{R\Omega}{2}(1 + 0) = -\frac{R\Omega}{2},$$

and for dark energy, with $\omega = -1$, we get

$$j(t) = -\frac{R\Omega}{2}(1 - 3) = R\Omega.$$