Problem 1

The Oort constants A and B are

$$A = -\frac{1}{2}\left[\frac{d\Theta}{dR}\bigg|_{R_0} - \frac{\Theta_0}{R_0}\right]$$
$$B = -\frac{1}{2}\left[\frac{d\Theta}{dR}\bigg|_{R_0} + \frac{\Theta_0}{R_0}\right]$$

Substituting in $\Theta(R) = \Theta_0(R/R_0)^{-0.5}$, A becomes

$$\begin{aligned}
A &= -\frac{1}{2}\left[\frac{d}{dR}\left(\Theta_0 - \sqrt{\frac{R_0}{R}}\right)\bigg|_{R_0} - \frac{\Theta_0}{R_0}\right] \\
&= -\frac{1}{2}\left[\Theta_0\sqrt{R_0}\frac{d}{dR}\left(R^{-0.5}\right)\bigg|_{R_0} - \frac{\Theta_0}{R_0}\right] \\
&= -\frac{1}{2}\left[\Theta_0\sqrt{R_0}\left(-\frac{1}{2}R^{-1.5}\right)\bigg|_{R_0} - \frac{\Theta_0}{R_0}\right] \\
&= -\frac{1}{2}\left[\Theta_0\sqrt{R_0}\left(-\frac{1}{2}R_0^{-1.5}\right) - \frac{\Theta_0}{R_0}\right] \\
&= -\frac{1}{2}\left[-\frac{1}{2}\frac{\Theta_0}{R_0} - \frac{\Theta_0}{R_0}\right] \\
&= -\frac{1}{2}\left[-\frac{3}{2}\frac{\Theta_0}{R_0}\right] \\
&= \frac{3}{4}\frac{\Theta_0}{R_0}.
\end{aligned}$$

Following a similar procedure (with some steps omitted below) for B, we get

$$\begin{aligned}
B &= -\frac{1}{2}\left[\frac{d}{dR}\left(\Theta_0 - \sqrt{\frac{R_0}{R}}\right)\bigg|_{R_0} - \frac{\Theta_0}{R_0}\right] \\
&= -\frac{1}{2}\left[\Theta_0\sqrt{R_0}\left(-\frac{1}{2}R_0^{-1.5}\right) + \frac{\Theta_0}{R_0}\right] \\
&= -\frac{1}{2}\left[-\frac{1}{2}\frac{\Theta_0}{R_0} + \frac{\Theta_0}{R_0}\right] \\
&= -\frac{1}{2}\left[\frac{1}{2}\frac{\Theta_0}{R_0}\right] \\
&= -\frac{1}{4}\frac{\Theta_0}{R_0}.
\end{aligned}$$

Taking $\Theta_0 = 220\ \mathrm{km\,s^{-1}}$ and $R_0 = 8$ kpc, we get the values

$$A = 20.625\ \mathrm{km\,s^{-1}\,kpc^{-1}} \qquad \text{and} \qquad B = -6.875\ \mathrm{km\,s^{-1}\,kpc^{-1}}.$$

Observed values are

$$A = 15.3\ \mathrm{km\,s^{-1}\,kpc^{-1}} \qquad \text{and} \qquad B = -11.9\ \mathrm{km\,s^{-1}\,kpc^{-1}}.$$

This shows that Keplerian rotation does not perfectly describe the rotation of the Milky Way near the Sun.

Problem 2

(a) The vertical position of the Sun is given by the equation

$$z(t) = A_Z \sin(vt + \varphi), \tag{1}$$

and the vertical velocity is given by its derivative with respect to time:

$$\begin{aligned} z'(t) &= A_Z \cos(vt + \varphi)v \\ \implies \frac{z'(t)}{v} &= A_Z \cos(vt + \varphi). \end{aligned} \tag{2}$$

Adding the squares of Eq. 1 and Eq. 2 give the following:

$$z(t)^2 + \left(\frac{z'(t)}{v}\right)^2 = A_Z^2 \sin^2(x) + A_Z^2 \cos^2(x) = A_Z^2(\sin^2(x) + \cos^2(x)) = A_Z^2.$$

Using this to solve for A_Z, we get

$$A_Z = \sqrt{z(t)^2 + \left(\frac{z'(t)}{v}\right)^2}.$$

We know that at some $t = t_0$, we have $z(t_0) = 20$ pc $= 6.171 \times 10^{14}$ km and $z'(t_0) = 7.25\ \mathrm{km\,s^{-1}}$. We also know that the vertical oscillation frequency v is given by $v = \frac{2\pi}{p}$, where the period p is equal to 85 Myr $= 2.68 \times 10^{15}$ s. Then, at $t = t_0$, A_Z is given by

$$A_Z = \sqrt{(6.171 \times 10^{14}\ \mathrm{km})^2 + \left(\frac{7.25\ \mathrm{km\,s^{-1}}}{\frac{2\pi}{2.68\times10^{15}\ \mathrm{s}}}\right)^2} = 3.15 \times 10^{15}\ \mathrm{km} \simeq 102.2\ \mathrm{pc}.$$

(b) The Sun's position oscillates radially with some amplitude A_R, given by

$$A_R = \frac{9.16\ \mathrm{kpc} - 7.92\ \mathrm{kpc}}{2} = 0.62\ \mathrm{kpc},$$

about some average radial position R_A, given by

$$R_A = \frac{9.16\ \mathrm{kpc} + 7.92\ \mathrm{kpc}}{2} = 8.54\ \mathrm{kpc}.$$

The radial position is given by

$$r(t) = A_R \sin(\kappa t),$$

and the velocity by its time derivative

$$r'(t) = A_R \cos(\kappa t)\kappa.$$

We know that at the current time $t = t_0$, the following is true:

$$r(t_0) = R_0 = A_R \sin(\kappa t_0).$$

Rearranging to solve for t_0, we find that

$$\begin{aligned}
R_0 &= A_R \sin(\kappa t_0) \\
\implies \frac{R_0}{A_R} &= \sin(\kappa t_0) \\
\implies \kappa t_0 &= \arcsin\left(\frac{R_0}{A_R}\right) \\
\implies t_0 &= \frac{\arcsin\left(\frac{R_0}{A_R}\right)}{\kappa}.
\end{aligned}$$

I didn't do it here, but it is probably better to skip the following step and to just compute $r'(t_0)$ from here. This "avoids" the problem of having a negative time, and because the κ term cancels out, some (annoying) conversions and rounding steps can be avoided. Doing this gives $r'(t_0) = 11.0\ \mathrm{km\,s^{-1}}$.

In this case, we take $R_0 = 8.0$ kpc relative to the midpoint of the radial oscillation, $R_A = 8.54$ kpc. Thus, in the equation above, we use $R_0 = 8.0\ \mathrm{kpc} - 8.54\ \mathrm{kpc} = -0.54$ kpc. Using this value, and $\kappa = 36\ \mathrm{km\,s^{-1}\,kpc^{-1}}$, we find that the current time t_0 is

$$\begin{aligned}
t_0 &= \frac{\arcsin\left(\frac{R_0}{A_R}\right)}{\kappa} \\
&= \frac{\arcsin\left(\frac{-0.54\ \mathrm{kpc}}{0.62\ \mathrm{kpc}}\right)}{36\ \mathrm{km\,s^{-1}\,kpc^{-1}}} \\
&= -0.0294\ \mathrm{kpc\,s^{-1}\,km^{-1}} \times \frac{3.068 \times 10^{16}\ \mathrm{km}}{\mathrm{kpc}} \\
&= -9.01 \times 10^{14}\ \mathrm{s}.
\end{aligned}$$

We can ignore the fact that the time is negative because this is simply an intermediate value that will be put into cos, which, as an even function, satisfies $\cos(x) = \cos(-x)$. We can convert the units of the epicycle frequency κ into seconds: $\kappa = 36\ \mathrm{km\,s^{-1}\,kpc^{-1}} = 1.167 \times 10^{-15}\ \mathrm{s^{-1}}$. We can use this and the current time to calculate the current velocity in the radial direction:

$$\begin{aligned}
r'(t_0) &= A_R \cos(\kappa t_0)\kappa \\
&= 0.62\ \mathrm{kpc} \times \cos\Big(1.167 \times 10^{-15}\ \mathrm{s^{-1}} \times -9.01 \times 10^{14}\ \mathrm{s}\Big) \times 36\ \mathrm{km\,s^{-1}\,kpc^{-1}} \\
&= 11.1\ \mathrm{km\,s^{-1}}.
\end{aligned}$$

Problem 4

Some useful information:

- $G = 6.67 \times 10^{-11}\ \mathrm{m^3\,kg^{-1}\,s^{-2}}$
- $1\ \mathrm{pc} = 3.086 \times 10^{16}\ \mathrm{m}$
- $1\ \mathrm{M_\odot} = 1.989 \times 10^{30}\ \mathrm{kg}$

(a) We equate gravitational acceleration with centripetal acceleration and solve for circular velocity:

$$\begin{aligned} \frac{GM_{BH}m}{R^2} &= \frac{mv_c^2}{R} \\ \implies \qquad v_c^2 &= \frac{GM_{BH}}{R} \\ \implies \qquad v_c &= \sqrt{\frac{GM_{BH}}{R}}, \end{aligned}$$

where $M_{BH} = 4 \times 10^6\ \mathrm{M_\odot}$ is the mass of the black hole in solar masses, and R is the distance from the black hole in parsecs. We can convert these to SI units, which will yield a velocity in meters per second. Then, we can divide the velocity by 1000 to get a velocity in kilometers per second:

$$v_c(R)\ [\mathrm{km\,s^{-1}}] = \sqrt{\frac{G\ [\mathrm{m^3\,kg^{-1}\,s^{-2}}] \times M_{BH}\ [\mathrm{M_\odot}] \times 1.989 \times 10^{30}\ [\mathrm{kg\,M_\odot^{-1}}]}{R\ [\mathrm{pc}] \times 3.086 \times 10^{16}\ [\mathrm{m\,pc^{-1}}]}} \times \frac{1\ [\mathrm{km}]}{1000\ [\mathrm{m}]} \simeq \frac{131}{\sqrt{R\ [\mathrm{pc}]}}$$

(b) The density at some radius is given by $\rho(r) = \rho_0(\frac{r}{1\ \mathrm{pc}})^{-1.9}$. We want to find ρ_0. The mass enclosed in a sphere with a radius of R pc is

$$\begin{aligned} M(< R\ \mathrm{pc}) &= \int_0^{2\pi}\int_0^{\pi}\int_0^{r} \rho_0(\frac{r}{1\ \mathrm{pc}})^{-1.9}\, r^2 \sin\theta\, dr\, d\theta\, d\phi \\ &= 4\pi\rho_0 \int_0^R r^{0.1}\, dr \\ &= 4\pi\rho_0 \left(\frac{r^{1.1}}{1.1}\right)\Big|_0^R \\ &= \frac{4\pi\rho_0}{1.1} R^{1.1}. \end{aligned}$$

The mass enclosed has units of $M_\odot$ since ρ_0 has units of $M_\odot\ pc^{-3}$, and the volume integral gives units of pc^3.

At $R = 1$, we know that the mass enclosed is same as the mass of the black hole, $4 \times 10^6\ \mathrm{M_\odot}$. We can use this information to solve for ρ_0:

$$\begin{aligned} M(< 1\ \mathrm{pc}) &= \frac{4\pi\rho_0}{1.1}(1)^{1.1} = \frac{4\pi\rho_0}{1.1} = 4 \times 10^6\ \mathrm{M_\odot} \\ \implies \qquad \rho_0 &= 4 \times 10^6 \times\ \mathrm{M_\odot} \times \frac{1.1}{4\pi}\ \mathrm{pc^{-3}} \simeq 3.5 \times 10^5\ \mathrm{M_\odot\,pc^{-3}}. \end{aligned}$$

(c) This is similar to (a), but instead of using the mass of the black hole, we use the enclosed mass of the star cluster:

$$v_c = \sqrt{\frac{GM_{enc}}{R}},$$

where M_{enc} is given by the following (omitting units in the intermediate steps, but masses are in solar masses and distances are in parsecs):

$$\begin{aligned}
M_{enc}(R) &= \int_0^{2\pi}\int_0^{\pi}\int_0^{r} \rho_0 r^{-\alpha}\, r^2 \sin\theta \, dr\, d\theta\, d\phi \\
&= 4\pi\rho_0 \int_0^R r^{2-\alpha}\, dr \\
&= 4\pi\rho_0 \frac{1}{3-\alpha}(r^{3-\alpha})\Big|_0^R \\
&= \frac{4\pi\rho_0}{3-\alpha} R^{3-\alpha}\ [\mathrm{M}_\odot].
\end{aligned}$$

So the velocity is given by

$$v_c = \sqrt{\frac{G}{R}\frac{4\pi\rho_0 R^{3-\alpha}}{3-\alpha}}.$$

Again, we need to ensure that the units are consistent, and we can do this by multiplying by some conversion factors:

$$v_c = \sqrt{\frac{G}{R\times 3.086\times 10^{16}}\frac{4\pi\rho_0 R^{3-\alpha}}{3-\alpha}\times 1.989\times 10^{30}}\times \frac{1}{1000},$$

which gives circular velocity in kilometers per second.

(d) As before, circular velocity is given by

$$v_c = \sqrt{\frac{GM}{R}}.$$

Here, M is the total mass enclosed, and v_c is the velocity due to the entire enclosed mass. We can separate this mass term into the mass of the black hole, M_{BH}, and the enclosed mass of the stellar cluster, M_{SC}:

$$v_c = \sqrt{\frac{G}{R}(M_{BH} + M_{SC})}.$$

Then, we can rewrite the component masses in terms of the velocities due to those masses:

$$\begin{aligned}
v_{BH} &= \sqrt{\frac{GM_{BH}}{R}} \qquad & v_{SC} &= \sqrt{\frac{GM_{SC}}{R}} \\
\implies M_{BH} &= \frac{R}{G}v_{BH}^2 \qquad & M_{SC} &= \frac{R}{G}v_{SC}^2
\end{aligned}$$

Substituting this into the previous expression, we find that

$$\begin{aligned} v_c &= \sqrt{\frac{G}{R}(M_{BH} + M_{SC})} \\ &= \sqrt{\frac{G}{R}\left(\frac{R}{G}v_{BH}^2 + \frac{R}{G}v_{SC}^2\right)} \\ &= \sqrt{v_{BH}^2 + v_{SC}^2}. \end{aligned}$$

(e)