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Problem Set 2

Problem 1

The Oort constants A and B are

$$A = -\frac{1}{2}\left[\frac{d\Theta}{dR}\Big|_{R_0} - \frac{\Theta_0}{R_0}\right]$$
$$B = -\frac{1}{2}\left[\frac{d\Theta}{dR}\Big|_{R_0} + \frac{\Theta_0}{R_0}\right]$$

Substituting in $\Theta(R) = \Theta_0(R/R_0)^{-0.5}$, A becomes

$$\begin{aligned}
A &= -\frac{1}{2}\left[\frac{d}{dR}\left(\Theta_0 - \sqrt{\frac{R_0}{R}}\right)\Big|_{R_0} - \frac{\Theta_0}{R_0}\right] \\
&= -\frac{1}{2}\left[\Theta_0\sqrt{R_0}\frac{d}{dR}\left(R^{-0.5}\right)\Big|_{R_0} - \frac{\Theta_0}{R_0}\right] \\
&= -\frac{1}{2}\left[\Theta_0\sqrt{R_0}\left(-\frac{1}{2}R^{-1.5}\right)\Big|_{R_0} - \frac{\Theta_0}{R_0}\right] \\
&= -\frac{1}{2}\left[\Theta_0\sqrt{R_0}\left(-\frac{1}{2}R_0^{-1.5}\right) - \frac{\Theta_0}{R_0}\right] \\
&= -\frac{1}{2}\left[-\frac{1}{2}\frac{\Theta_0}{R_0} - \frac{\Theta_0}{R_0}\right] \\
&= -\frac{1}{2}\left[-\frac{3}{2}\frac{\Theta_0}{R_0}\right] \\
&= \frac{3}{4}\frac{\Theta_0}{R_0}.
\end{aligned}$$

Following a similar procedure (with some steps omitted below) for B, we get

$$\begin{aligned}
B &= -\frac{1}{2}\left[\frac{d}{dR}\left(\Theta_0 - \sqrt{\frac{R_0}{R}}\right)\Big|_{R_0} - \frac{\Theta_0}{R_0}\right] \\
&= -\frac{1}{2}\left[\Theta_0\sqrt{R_0}\left(-\frac{1}{2}R_0^{-1.5}\right) + \frac{\Theta_0}{R_0}\right] \\
&= -\frac{1}{2}\left[-\frac{1}{2}\frac{\Theta_0}{R_0} + \frac{\Theta_0}{R_0}\right] \\
&= -\frac{1}{2}\left[\frac{1}{2}\frac{\Theta_0}{R_0}\right] \\
&= -\frac{1}{4}\frac{\Theta_0}{R_0}.
\end{aligned}$$

Taking $\Theta_0 = 220\ \mathrm{km\,s^{-1}}$ and $R_0 = 8$ kpc, we get the values

$$A = 20.625\ \mathrm{km\,s^{-1}\,kpc^{-1}} \qquad \text{and} \qquad B = -6.875\ \mathrm{km\,s^{-1}\,kpc^{-1}}.$$

Observed values are

$$A = 15.3\ \mathrm{km\,s^{-1}\,kpc^{-1}} \qquad \text{and} \qquad B = -11.9\ \mathrm{km\,s^{-1}\,kpc^{-1}}.$$

something about how the values compare?

Problem 2

(a) The vertical position of the Sun is given by the equation

$$z(t) = A_Z \sin(vt + \varphi), \tag{1}$$

and the vertical velocity is given by its derivative with respect to time:

$$\begin{aligned} z'(t) &= A_Z \cos(vt + \varphi) v \\ \implies \frac{z'(t)}{v} &= A_Z \cos(vt + \varphi). \end{aligned} \tag{2}$$

Adding the squares of Eq. 1 and Eq. 2 give the following:

$$z(t)^2 + \left(\frac{z'(t)}{v}\right)^2 = A_Z^2 \sin^2(x) + A_Z^2 \cos^2(x) = A_Z^2(\sin^2(x) + \cos^2(x)) = A_Z^2.$$

Using this to solve for A_Z, we get

$$A_Z = \sqrt{z(t)^2 + \left(\frac{z'(t)}{v}\right)^2}.$$

We know that at some $t = t_0$, we have $z(t_0) = 20\ \mathrm{pc} = 6.171 \times 10^{14}\ \mathrm{km}$ and $z'(t_0) = 7.25\ \mathrm{km\,s^{-1}}$. We also know that the vertical oscillation frequency v is given by $v = \frac{2\pi}{p}$, where the period p is equal to $85\ \mathrm{Myr} = 2.68 \times 10^{15}\ \mathrm{s}$. Then, at $t = t_0$, A_Z is given by

$$A_Z = \sqrt{(6.171 \times 10^{14}\ \mathrm{km})^2 + \left(\frac{7.25\ \mathrm{km\,s^{-1}}}{\frac{2\pi}{2.68 \times 10^{15}\ \mathrm{s}}}\right)^2} = 3.15 \times 10^{15}\ \mathrm{km} \simeq 102.2\ \mathrm{pc}.$$

(b) The Sun's position oscillates radially with some amplitude A_R, given by

$$A_R = \frac{9.16\ \mathrm{kpc} - 7.92\ \mathrm{kpc}}{2} = 0.62\ \mathrm{kpc},$$

about some average radial position R_A, given by

$$R_A = \frac{9.16\ \mathrm{kpc} + 7.92\ \mathrm{kpc}}{2} = 8.54\ \mathrm{kpc}.$$

The radial position is given by

$$r(t) = A_R \sin(\kappa t),$$

and the velocity by its time derivative

$$r'(t) = A_R \cos(\kappa t)\kappa.$$

We know that at the current time $t = t_0$, the following is true:

$$r(t_0) = R_0 = A_R \sin(\kappa t_0).$$

Rearranging to solve for t_0, we find that

$$\begin{aligned} R_0 &= A_R \sin(\kappa t_0) \\ \implies \frac{R_0}{A_R} &= \sin(\kappa t_0) \\ \implies \kappa t_0 &= \arcsin\left(\frac{R_0}{A_R}\right) \\ \implies \quad t_0 &= \frac{\arcsin\left(\frac{R_0}{A_R}\right)}{\kappa}. \end{aligned}$$

In this case, we take $R_0 = 8.0$ kpc relative to the midpoint of the radial oscillation, $R_A = 8.54$ kpc. Thus, in the equation above, we use $R_0 = 8.0 \text{ kpc} - 8.54 \text{ kpc} = -0.54$ kpc. Using this value, and $\kappa = 36 \text{ km s}^{-1}\text{ kpc}^{-1}$, we find that the current time t_0 is

$$\begin{aligned} t_0 &= \frac{\arcsin\left(\frac{R_0}{A_R}\right)}{\kappa} \\ &= \frac{\arcsin\left(\frac{-0.54 \text{ kpc}}{0.62 \text{ kpc}}\right)}{36 \text{ km s}^{-1}\text{ kpc}^{-1}} \\ &= -0.0294 \text{ kpc s}^{-1}\text{ km}^{-1} \times \frac{3.068 \times 10^{16} \text{ km}}{\text{kpc}} \\ &= -9.01 \times 10^{14} \text{ s}. \end{aligned}$$

can we just use the cos property earlier so we dont get a negative time? take abs of the arcsin thing? otherwise put this as a footnote? combining this + the velocity equation would eliminate the intermediate negative time and the kappa would cancel out perfectly and avoid rounding errors. We can ignore the fact that the time is negative because this is simply an intermediate value that will be put into cos, which, as an even function, satisfies $\cos(x) = \cos(-x)$. We can convert the units of the epicycle frequency κ into seconds: $\kappa = 36\ \mathrm{km\,s^{-1}\,kpc^{-1}} = 1.167 \times 10^{-15}\ \mathrm{s^{-1}}$. We can use this and the current time to calculate the current velocity in the radial direction:

$$\begin{aligned} r'(t_0) &= A_R \cos(\kappa t_0)\kappa \\ &= 0.62\ \mathrm{kpc} \times \cos\left(1.167 \times 10^{-15}\ \mathrm{s^{-1}} \times -9.01 \times 10^{14}\ \mathrm{s}\right) \times 36\ \mathrm{km\,s^{-1}\,kpc^{-1}} \\ &= 11.1\ \mathrm{km\,s^{-1}}. \end{aligned}$$

Problem 4

(a)