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Problem Set 5

Problem 1

Using the distance modulus formula, this distance is (ignoring extinction)

$$d = 10^{(m-M+5)/5} = 10^{(25+19.5+5)/5} = 7.94 \text{ Gpc}.$$

Problem 2

Density for matter is given by $\rho_{m,0} = \Omega_{m,0}(1+z)^3$, and for radiation by $\rho_{r,0} = \Omega_{r,0}(1+z)^4$. Equate the two and solve for z:

$$\begin{aligned} \rho_{m,0} &= \rho_{r,0} \\ \implies \Omega_{m,0}(1+z)^3 &= \Omega_{r,0}(1+z)^4 \\ \implies \qquad z &= \frac{\Omega_{m,0}}{\Omega_{r,0}} - 1 = \frac{0.317}{10^{-4}} - 1 = 3169. \end{aligned}$$

Problem 3

The radius of the sphere of influence of the black hole is

$$r = \frac{GM}{\sigma^2} = \frac{6.67 \times 10^{-11}\ \mathrm{m^3\,kg^{-1}} \times 2 \times 10^9\ \mathrm{M_\odot}}{4 \times 10^5\ \mathrm{m\,s^{-1}}} = 1.66 \times 10^{18}\ \mathrm{m}.$$

Given that the distance to M87 is 53.5 Mly $= 5.06 \times 10^{23}$ m, the angular radius is

$$\theta \simeq \tan\theta = \frac{r}{d} = \frac{1.66 \times 10^{18}\ \mathrm{m}}{5.06 \times 10^{23}\ \mathrm{m}} = 3.28 \times 10^{-6}\ \mathrm{rad} = 0.67\ \mathrm{arcsec}.$$

Whether this is observable without adaptive optics probably depends on the location of the ground-based telescope and the atmospheric conditions. The Internet[1] and the lecture slides from week 1 say that seeing of 0.5″ can be achieved in locations such as Mauna Kea.

Problem 4

The parallax formula is

$$\begin{aligned} \theta \simeq \tan\theta &= \frac{r}{d} \\ \implies \qquad d &= \frac{r}{\theta}, \end{aligned}$$

where d is the distance we want, θ is the parallax angle, and r is the parallax baseline. Gaia is at the Earth-Sun L2 Lagrangian point, which has a radius of orbit of approximately 1.01 au. Plugging these values in, we find that d is

$$d = \frac{1.01\ \mathrm{au}}{0.05\ \mathrm{arcsec}} = 20.22\ \mathrm{pc}.$$

Taking 8% of this, we find that distances of 1.6 pc are achievable with errors of 8%. (seems rather small?)

[1] Racine, R. 1989, Publications of the Astronomical Society of the Pacific, 101, 436.

Problem 5

```python
import numpy as np
import matplotlib.pyplot as plt
%matplotlib inline
from astropy import units as u
from astropy import constants as const
```

```python
dists, vels = np.loadtxt('ShenJeff.dat', unpack=True)
```

```python
# fit a linear model to the data. the slope should be the hubble parameter H_0
slope, offset = np.polyfit(dists, vels, 1)
slope, offset
```

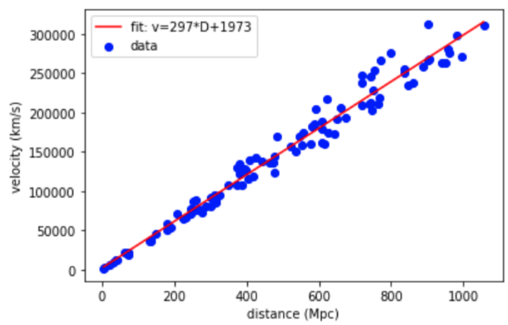
```
(296.7834527807584, 1972.9606705987662)
```

The present-day value of the Hubble parameter of this universe is $H_0 = 297$ km/s/Mpc.

```python
xvals = np.linspace(min(dists), max(dists))
```

```python
# plot data and fit
plt.scatter(dists, vels, c='b', label='data')
plt.plot(xvals, offset + slope * xvals, c='r', label=f'fit: v={m:.0f}*D+{b:.0f}'
plt.xlabel('distance (Mpc)')
plt.ylabel('velocity (km/s)')
plt.legend()
```

```
<matplotlib.legend.Legend at 0x1239e8350>
```




```python
# assuming constant expansion, the age of the universe is given by 1/H_0
h0 = m * (u.km / u.s) / u.Mpc
(1/h0).to(u.Gyr)
```

3.2946319 Gyr

The age of this universe is 3.3 Gyr.

```python
# use the friedmann equations to find the critical density:
((3 * h0**2) / (8 * np.pi * const.G)).to(u.kg / u.m**3)
```

$1.6544511 \times 10^{-25} \ \frac{\mathrm{kg}}{\mathrm{m}^3}$

The critical density of the universe is $\rho_c = 1.65 \times 10^{-25}$ kg/m^3. From week 10's lecture slides, we know that $\rho_{m,0} = 2.56 \times 10^{-27}$ kg/m^3 and $\rho_{b,0} = 4.17 \times 10^{-28}$ kg/m^3. This means that if the density of matter in this alternate universe is the same as the density of our universe, that universe has a density parameter $\Omega = \rho/\rho_c$ less than 1, so the universe is open.

I didn't force the fit through the origin. The difference is very small either way (the slope if forced through the origin is something like 299).

Problem 6

(i) C&O Eq. 29.10:

$$\left[\left(\frac{1}{R}\frac{dR}{dt}\right)^2 - \frac{8}{3}\pi G\rho\right] R^2 = -kc^2.$$

Multiplying this by R, we get

$$\left(\frac{dR}{dt}\right)^2 R - \frac{8}{3}\pi G\rho R^3 = -kc^2 R$$

Taking the time derivative and applying the product and chain rules to the first term, we get

$$2\frac{dR}{dt}\frac{d^2R}{dt}R + \left(\frac{dR}{dt}\right)^3 - \frac{8}{3}\pi G\frac{d}{dt}(\rho R^3) = \frac{d}{dt}(-kc^2R).$$

We use C&O Eq. 29.50 to replace the third term on the left side, then expand the derivative using the chain rule and cancel out a dR/dt term:

$$\begin{aligned}
&2\frac{dR}{dt}\frac{d^2R}{dt^2}R + \left(\frac{dR}{dt}\right)^3 + \frac{8}{3}\pi G\frac{P}{c^2}\frac{d(R^3)}{dt} = -kc^2\frac{dR}{dt}\\
\implies &2\frac{dR}{dt}\frac{d^2R}{dt^2}R + \left(\frac{dR}{dt}\right)^3 + \frac{8}{3}\pi G\frac{P}{c^2}3R^2\frac{dR}{dt} = -kc^2\frac{dR}{dt}\\
\implies &2\frac{d^2R}{dt^2}R + \left(\frac{dR}{dt}\right)^2 + 8\pi G\frac{P}{c^2}R^2 = -kc^2.
\end{aligned}$$

We can use Eq. 29.10 again to replace the $-kc^2$ term on the right side:

$$2\frac{d^2R}{dt^2}R + \left(\frac{dR}{dt}\right)^2 + 8\pi G\frac{P}{c^2}R^2 = \left(\frac{dR}{dt}\right)^2 - \frac{8}{3}\pi G\rho R^2.$$

Cancelling terms and rearranging, we get

$$2\frac{d^2R}{dt^2}R = -8\pi GR^2\left(\frac{\rho}{3} + \frac{P}{c^2}\right).$$

Dividing both sides by $2R$ and then factoring out $1/3$ from the right side, we arrive at C&O Eq. 29.51:

$$\frac{d^2R}{dt^2} = -\frac{4}{3}\pi GR\left(\rho + \frac{3P}{c^2}\right).$$

(ii) From C&O Eq. 29.8, we know that

$$HR = \frac{dR}{dt}, \tag{1}$$

and by (a rearrangement of) the definition of the density parameter Ω, we know that

$$H^2 = \frac{8\pi G\rho}{3\Omega}. \tag{2}$$

Using Eq. 1 and Eq. 2 above to replace the denominator of the expression for the deceleration parameter, and the acceleration equation to replace the numerator, we can rewrite the deceleration parameter as

$$\begin{aligned}
q &= -\frac{R\frac{d^2R}{dt^2}}{(\frac{dR}{dt})^2} \\
&= -\frac{R\frac{d^2R}{dt^2}}{H^2R^2} \\
&= -\frac{\frac{d^2R}{dt^2}}{H^2R} \\
&= -\frac{-\frac{4}{3}\pi GR(\rho + \frac{3P}{c^2})}{\frac{8\pi G\rho}{3\Omega}R} \\
&= \frac{\rho + \frac{3P}{c^2}}{\frac{2\rho}{\Omega}} \\
&= \frac{\frac{\rho c^2+3P}{c^2}}{\frac{2\rho}{\Omega}} \\
&= \frac{(\rho c^2 + 3P)\Omega}{2\rho c^2}.
\end{aligned}$$

Using the equation of state $P = \omega\rho c^2$ to replace the P in the above expression, we get

$$\begin{aligned}
q &= \frac{(\rho c^2 + 3\omega\rho c^2)\Omega}{2\rho c^2} \\
&= \frac{\Omega\rho c^2(1+3\omega)}{2\rho c^2} \\
&= \frac{\Omega}{2}(1+3\omega).
\end{aligned}$$

(iii) We begin with multiplying the acceleration by R^2 to get

$$\frac{d^2R}{dt^2}R^2 = -\frac{4}{3}\pi GR^3(\rho + \frac{3P}{c^2}).$$

Taking the time derivative and applying the product rule to the left side and the chain rule to both sides, we get

$$\frac{d^3R}{dt^3}R^2 + \frac{d^2R}{dt^2}2R\frac{dR}{dt} = -\frac{4}{3}\pi G3R^2\frac{dR}{dt}(\rho + \frac{3P}{c^2}).$$

We can cancel out an R term, move the second term from the left to the right side, then multiply both sides of the equation by $R^2/(dR/dt)^3$ to get the expression for jerk on the left:

$$j(t) = \frac{R^2\frac{d^3R}{dt^3}}{(\frac{dR}{dt})^3} = -4\pi GR^3\left(\frac{dR}{dt}\right)^{-2}\left(\rho + \frac{3P}{c^2} + 2\frac{d^2R}{dt^2}\frac{1}{4\pi GR}\right).$$

Using Eq. 1 from Q6(ii), we can introduce an H term into the right side:

$$j(t) = -\frac{4\pi GR}{H^2}\left(\rho + \frac{3P}{c^2} + \frac{\frac{d^2R}{dt^2}}{2\pi GR}\right).$$

We can use the equation of state to replace the first two terms in the parentheses, and the acceleration equation to replace the numerator in the last term:

$$j(t) = -\frac{4\pi GR}{H^2}\left(\rho(1+3\omega) - \frac{\frac{4}{3}\pi GR(\rho + \frac{3P}{c^2})}{2\pi GR}\right).$$

After cancelling terms, we can again use the equation of state to rewrite the last term:

$$j(t) = -\frac{4\pi GR}{H^2}\left(\rho(1+3\omega) - \frac{2}{3}(\rho(1+3\omega))\right) = -\frac{\frac{4}{3}\pi GR\rho(1+3\omega)}{H^2}.$$

Using Eq. 2 from Q6(ii), we replace the H^2 term and arrive at an expression for j in terms of Ω and ω:

$$j(t) = -\frac{\frac{4}{3}\pi GR\rho(1+3\omega)}{\frac{8\pi GR\rho}{3\Omega}} = -\frac{R\Omega}{2}(1+3\omega).$$

For matter, with $\omega = 0$, we get

$$j(t) = -\frac{R\Omega}{2}(1+0) = -\frac{R\Omega}{2},$$

and for dark energy, with $\omega = -1$, we get

$$j(t) = -\frac{R\Omega}{2}(1-3) = R\Omega.$$

:)