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Numerical Integration of a Mass-Spring System

Introduction:

This experiment explores Hooke's law and harmonic oscillators. In this lab, the experiment is done in two parts to see the effects of a dampening component. The importance of this experiment is to reconfirm Hooke's law and the relationship between time and position (as well as velocity and energy) in a harmonic oscillator. This lab also introduces numerical integration: the experiment was simulated with two numerical methods (Forward Euler and Euler-Cromer), and the results were compared to the experimental data.

Method:

Materials:

1. A spring
2. A mass (~200g)
3. A flat disk (~17g)
4. A pole (with which to hang the spring off of)
5. A motion sensor and motion sensor program (MotionSensor.avi)

Experiment:

This experiment was done in two different sessions and therefore the assumption is that two different masses and two different springs were used for each session.

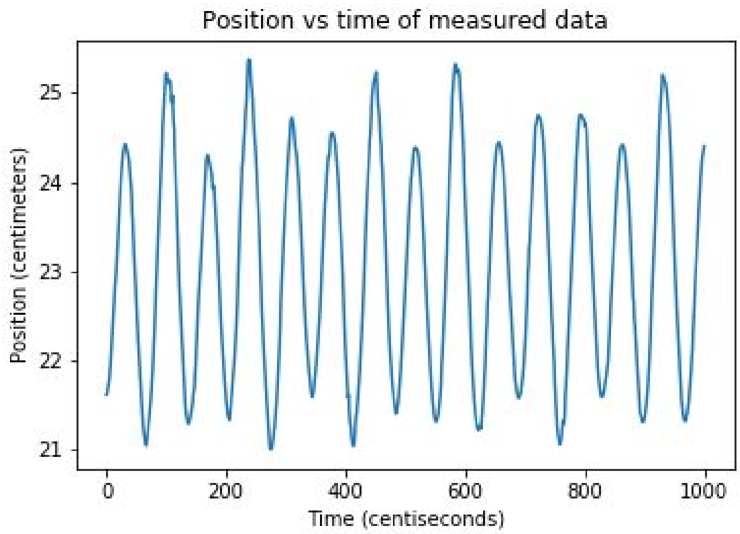
For the first session, the mass was measured to be 200.8 ± 0.4 g. First, hang the mass on the spring so it is about 20cm above the motion sensor at rest. Next, set the sample/seconds to 100. After setting the sample, move the mass from its resting state and release so it starts to bob. Collect the data for 10-20 seconds. Lastly, plot the data of distance vs time.

For the second session, the mass (with the disc) was measured at 217.6 ± 0.3 g. Set up the spring mass system as in session one, but with the disk attached to the bottom of the mass. After setting everything up as before, move the mass from its resting state and release so it starts to bob. Collect the data for about 2 minutes. Measure the data as before.

The frequency, spring constant and Reynolds number (for session 2 only) are calculated for both sessions separately.

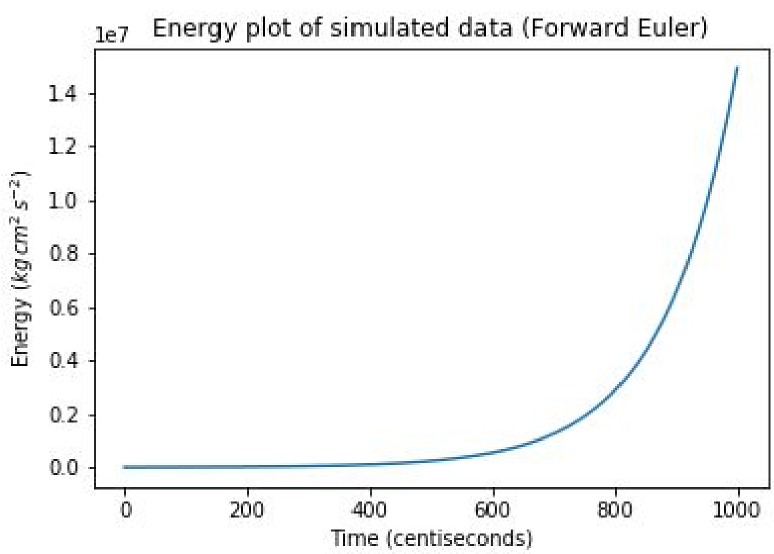
Results:
The results are shown in the figures below as there are too many raw data points to print seperately.

Analysis:
Figure 1



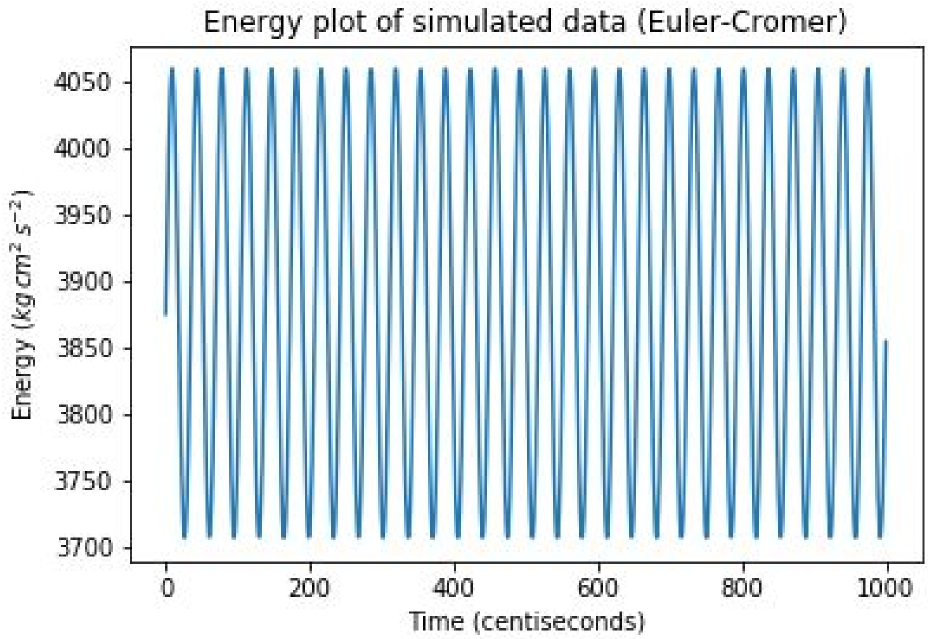
This plot graphs the relationship between the time and position of the mass on the spring during the first session for the data that we measured.

Figure 2



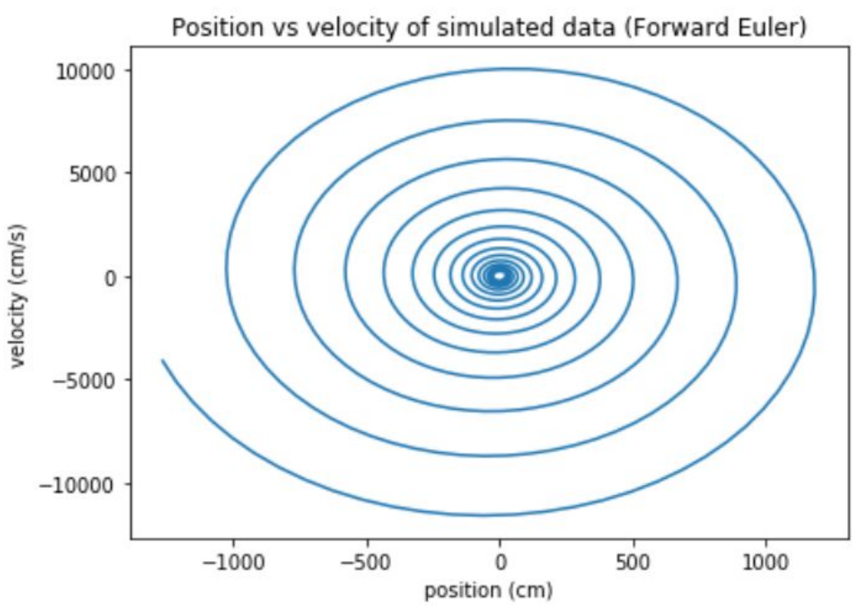
This shows the energy of the system over time for a system simulated with the Forward Euler method. Since the Forward Euler method is unstable, energy increases over time.

Figure 3



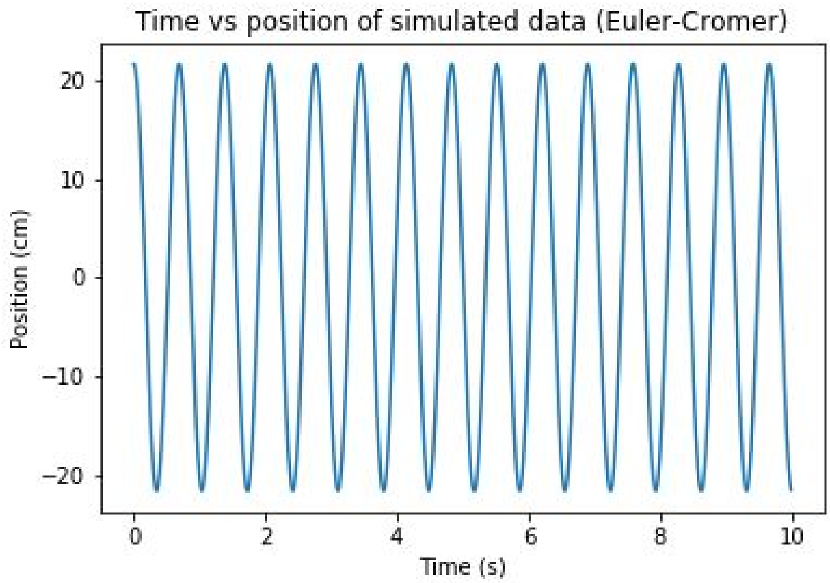
This shows energy over time for a system simulated with the Euler-Cromer method. Energy is conserved (in contrast to the Forward Euler method).

Figure 4



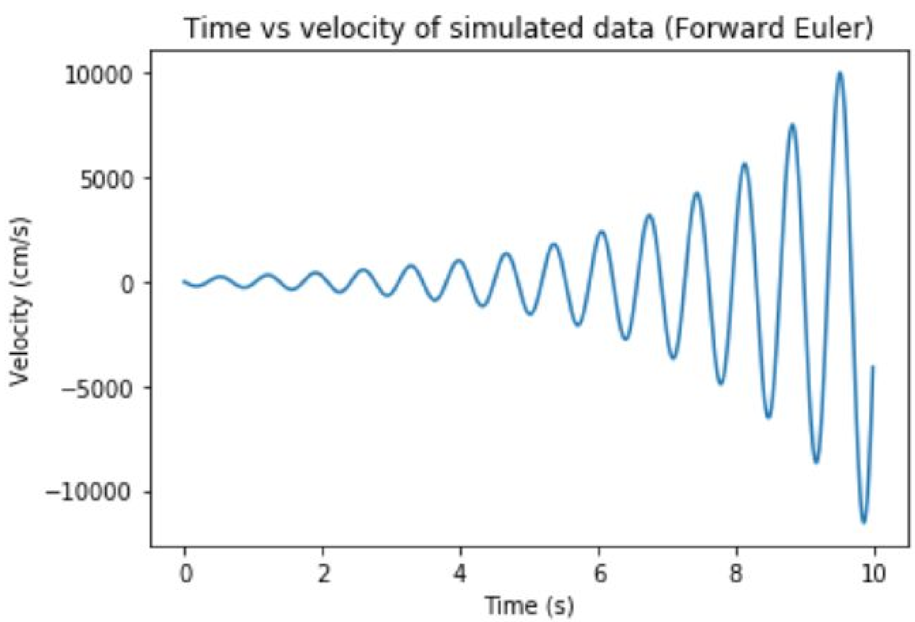
This plot graphs the relationship between the velocity and the position of the mass on the spring during the first session. Due to the instability of the Forward Euler method, energy increases.

Figure 5



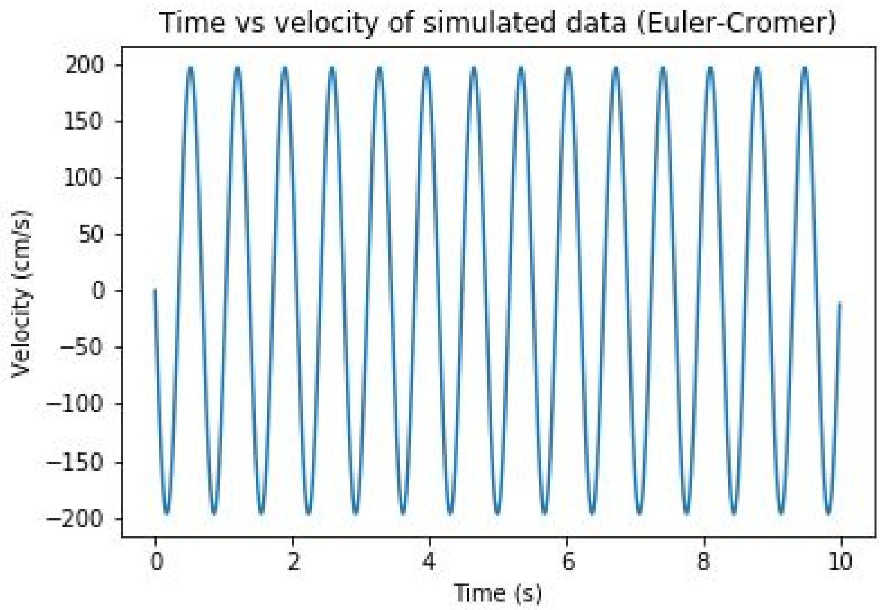
This plot shows the relationship between time and position for a system simulated with the Euler-Cromer method.

Figure 6



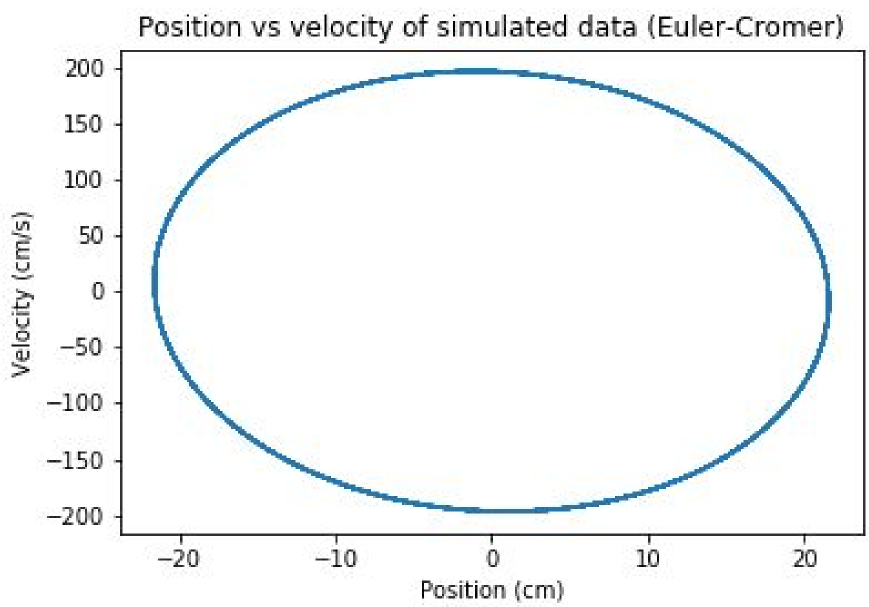
This plot graphs the relation between time and velocity for a system simulated with the Forward Euler method. Due to the instability of the method, the amplitude of oscillation increases.

Figure 7



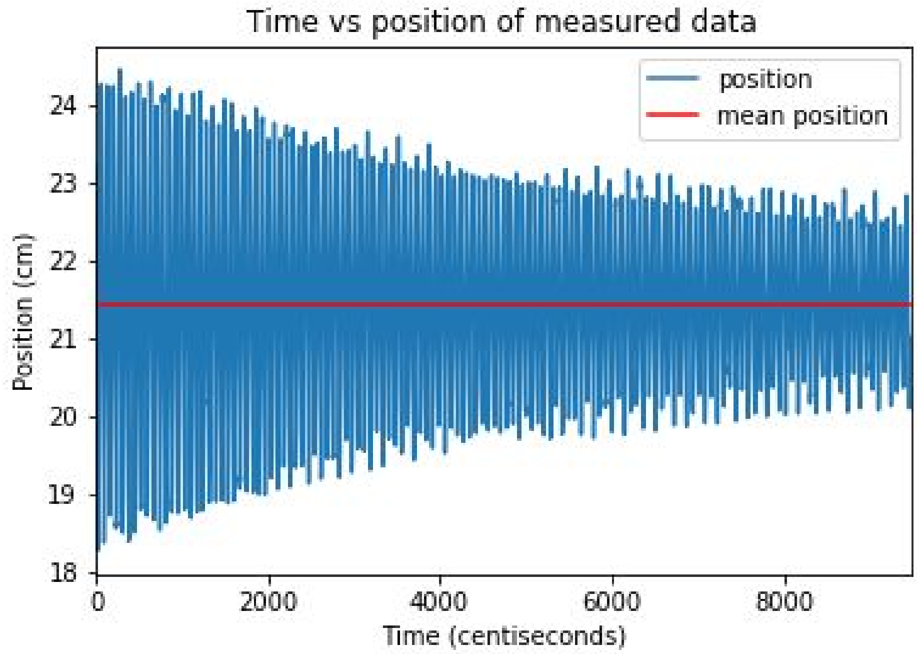
This plot graphs the data of a perfect system where energy would not escape it (other variables like mass stay the same as in the experiment). This is why the amplitude of velocity is constant.

Figure 8



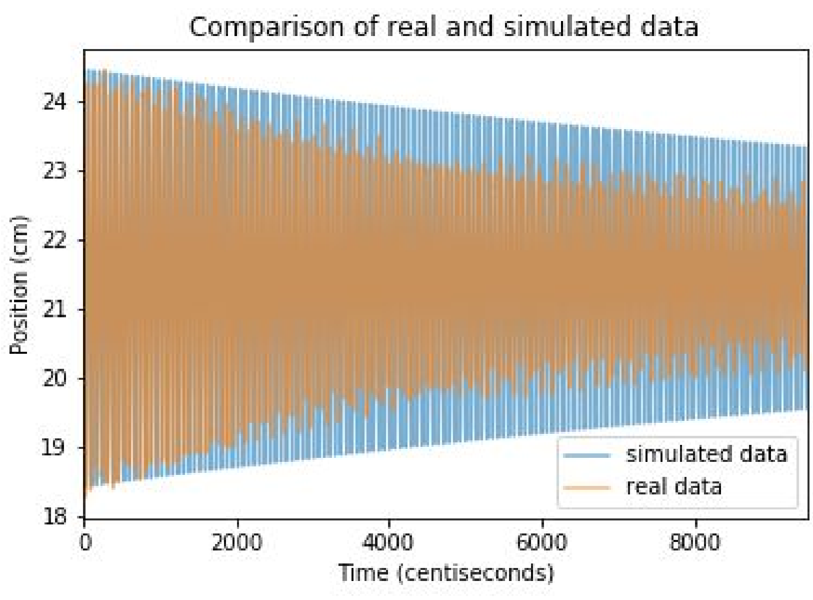
This is a phase plot of the system simulated with the Euler-Cromer method, without a damping term. Energy is constant, and the phase plot is an ellipse.

Figure 9



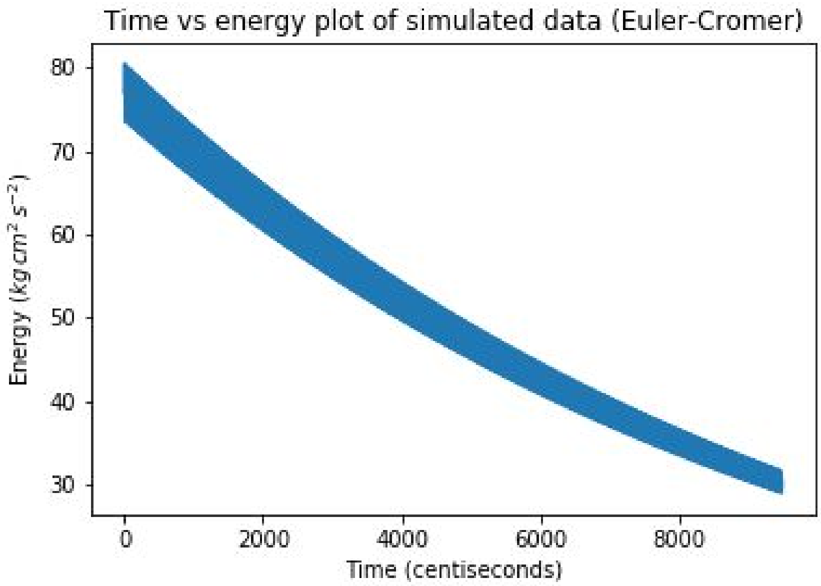
This is a plot of the position of the spring over time for Session 2. The red line across the middle indicates the mean position of the spring.

Figure 10



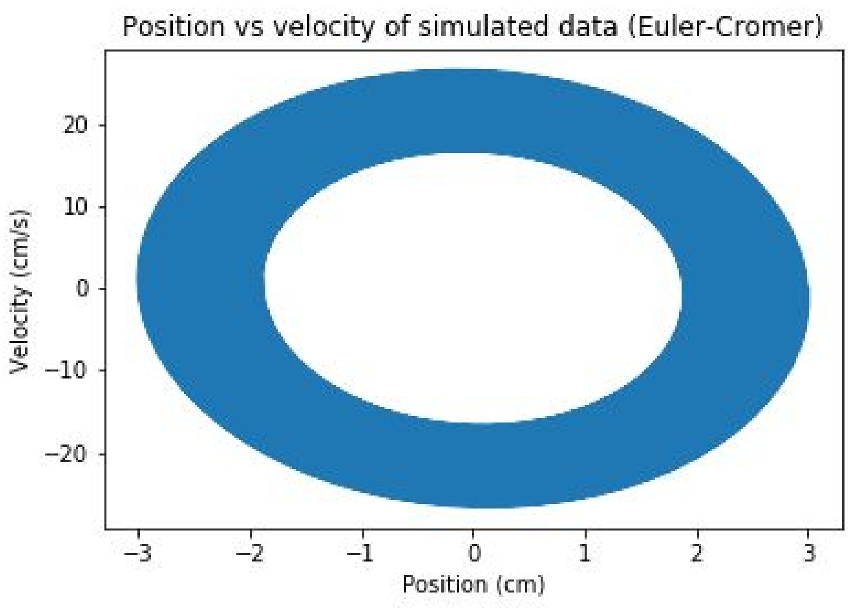
This plot shows the relationship between time and position for both the real observed data and the data that was simulated (with a damping term) using the Euler-Cromer method. It can be seen that the real data decays more rapidly than the simulated data.

Figure 11



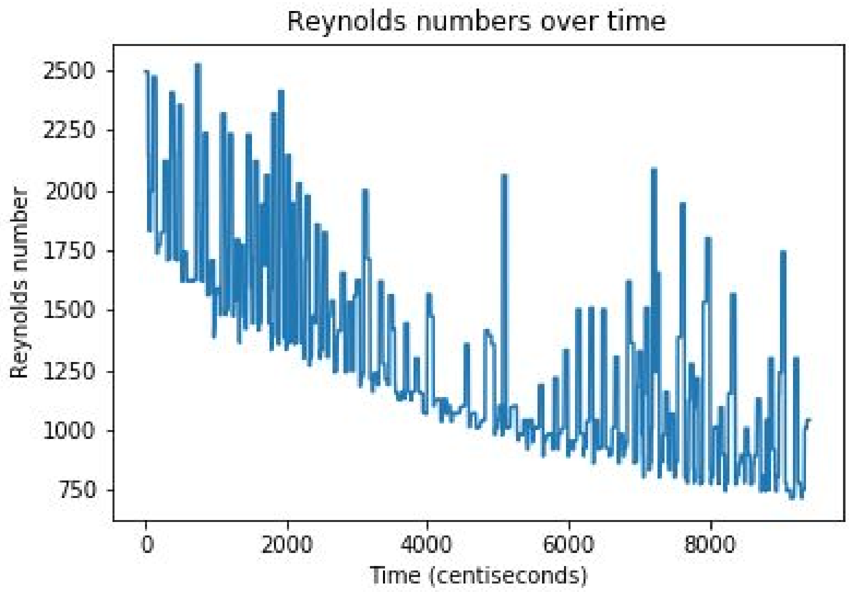
This plot shows the energy of a system when damping is taken into consideration. Energy decreases over time.

Figure 12



This is a phase plot of the system where damping decreases the energy over time.

Figure 13



This is a plot of the Reynolds number over time in a local section of data. The number at some time t is calculated by taking the maximum Reynolds number in the interval [t, t+0.5 seconds].

The sources of error can come from the friction of the spring and mass and the air resistance of the mass. The spring force may vary in different parts of the spring. The springs are also likely not ideal and therefore would vary the results. The Earth is always moving and therefore small vibrations may also affect the spring mass system.

Questions/Answers:

1. We know that $F = -ky$, $F = ma$ and $a = \frac{dv}{dt} = \frac{d^2y}{dt^2}$

   Therefore:

   $ma = -ky$

   $a = \frac{-ky}{m}$

   $\frac{d^2y}{dt^2} = \frac{-ky}{m}$

   $\frac{d^2y}{dt^2} + \frac{ky}{m} = 0$

2. We know that $a = \frac{dv}{dt} = \frac{d^2y}{dt^2}$, $p(t + \Delta t) = p(t) + \Delta t * F(q(t))$, $q(t + \Delta t) = q(t) + \Delta t * \frac{p(t)}{m}$, $\frac{dv}{dt} = -\Omega_0{}^2 y$ and $F = ma$

$$p(t+\Delta t) = p(t) + \Delta t * F(q(t))$$
$$= p(t+\Delta t) = p(t) + \Delta t\ * ma$$
$$= p(t+\Delta t) = p(t) + \Delta t\ * m * (-\Omega_0{}^2 y_i)$$
$$= \tfrac{p(t+\Delta t)}{m} = \tfrac{p(t)}{m} + \Delta t\ * (-\Omega_0{}^2 y_i)$$
$$(\tfrac{p(t)}{m} = v_i)$$
Therefore:
$$v_{i+1} = v_i - \Delta t * \Omega_0{}^2 y_i$$

$$q(t+\Delta t) = q(t) + \Delta t * \tfrac{p(t)}{m}$$
$$(\tfrac{p(t)}{m} = v_i)$$
$$(q(t) = y_i)$$
Therefore:
$$y_{i+1} = y_i + \Delta t * v$$

3. We know that $2\pi/T = \Omega$ and $\frac{dv}{dt} =- \Omega_0{}^2 y$
   $ma =- ky$
   $> -\Omega_0{}^2 y * m =- ky$
   $> \Omega_0{}^2 * m = k$
   Therefore:
   $k = 16.6\ \frac{kg}{s^2}$ (for the first session)
   $k = 17.04\ \frac{kg}{s^2}$ (for the second session)

4. In our plot (figure 6) we see that the amplitude of the oscillations increases over time. We had expected the amplitude to get smaller instead of bigger. This plot suggests that the energy is increasing over time.

5. The energy plot shows that the net energy is increasing. Figure 2 does explain why the previous graphs (figure 6 from question 4 and figure 4) appeared the way they did as the graphs both indicate the same result.

6. If energy is constant then the equation (Eq.10 from lab manual) becomes the equation of an ellipse. Additionally, if energy remains constant then the spring oscillations should remain constant, which indicates that the distance from the sensor would loop without deviations (in a perfect system) and so it would appear as an ellipse.

7. Taylor Expansion Formula:
   $f(x+h) = f(x) + hf'(x) + \frac{h^2}{d} f''(x) + \ldots$

   Our equation (eq.5a from the lab manual):
   $p(t+\Delta t) = p(t) + \Delta t * F(q(t))$

   Our equation (eq.5b from the lab manual):
   $q(t+\Delta t) = q(t) + \frac{\Delta t}{m} p(t)$

If we compare the two equations we can see that the third term of the Taylor Expansion is the first terms that are missing from the equations are:
$\frac{\Delta t^2}{2} * \frac{dF}{dt}$ and $\frac{\Delta t^2}{2m} * F(q(t))$ respectively.

8. The symplectic method is an explicit method which conserves energy. This conservation is apparent in both the energy and the phase plots. The energy plot (figure 3) has an amplitude which remains constant over time. The phase plot, (figure 8), as predicted before for a constant-energy system, is an ellipse.

9. The maximum Reynolds number for our data is 2524.6.

10. With the dampening effect, the energy plot (figure 11) shows that the energy of the system decreases over time. This is also shown in the phase plot (figure 12) , which has a (slow) spiral inward towards the center.

11. Figure 10 compares the experimental data to the simulated data. The simulated data demonstrates a decay which is slower than what we observed. This may be due to the fact that the Reynolds number and the decay rate change over time. However, for our simulation, we use a single decay rate of 0.00984 (the mean of three rates, each calculated from data spaced roughly 30 seconds apart).

Conclusion:

In this experiment, our goal was to explore harmonic oscillators and numerical integration methods. The experiment was done in two sessions, where the spring-mass system is on its own in one, and when a disc is added to the system (to add dampening) in the other. The same systems were then simulated in three ways: with an unstable method (Forward Euler), a stable method (Euler-Cromer), and with a stable method and a dampening term. Our findings show that with the unstable method, energy increases over time. With the Euler-Cromer method, energy is conserved. The addition of a dampening term causes the energy of the system to decrease over time. Our experimental data follows a similar pattern to that of the data simulated with the dampening term. However, the rates of decay do not perfectly match up.