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STA257: Probability and Statistics I

University of Toronto — Fall 2019

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1 Probability and Counting

1.1 Introduction

Experiments: situations wherre outcome is random. eg. flipping a coin is an experiment.
Sample space: set of all possible outcomes, denoted S or Ω. The number of elements in the sample space (cardinality) is denoted $|\Omega|$.
Event: a subset of a sample space.
Outcome: a particular element of a sample space $s_1 \in \Omega$.

1.2 Set Theory

1.2.1 Definitions

- **union**: $A \cup B$. Elements in either A or B.
- **intersection**: $A \cap B$. Elements in both A and B.
- **complement** of A: A^c. Elements not in A.
- **empty set**: $\varnothing$. Set with no elements in it.
- A and B are **disjoint**: $A \cap B = \varnothing$. There are no elements in the intersection of A and B.

1.2.2 Laws of Set Theory

1. **commutativity**: $A \cup B = B \cup A$, $A \cap B = B \cap A$
2. **associativity**: $(A \cup B) \cup C = A \cup (B \cup C)$, $(A \cap B) \cap C = A \cap (B \cap B)$
3. **distributivity**: $(A \cup B) \cap C = (A \cap C) \cup (B \cap C)$, $(A \cap B) \cup C = (A \cup C) \cap (B \cup C)$

1.3 Probability Measures

Probability measure: a function which maps subsets of Ω, which can be defined on any space, to real numbers $\mathbb{R}$.

1.3.1 Axioms of Probability Measures

- $P(\Omega) = 1$
- $\forall A \in \Omega, P(A) \geq 0$
- if $A_1, A_2, \dots A_n, \dots$ are mutually disjoint, then $P(\bigcup_{i=1}^{\infty} A_i) = \sum_{i=1}^{\infty} P(A_i)$.

1.3.2 Properties of Probability Measures

- $\forall A \in \Omega, P(A^c) = 1 - P(A)$

  Proof:

  $$\begin{aligned} 1 &= P(\Omega) && \text{by Axiom 1} \\ &= P(A \cup A^c) && \text{by definition of complement} \\ &= P(A) + P(A^c) && \text{by Axiom 3 (since } A, A^c \text{ are disjoint)} \end{aligned}$$

  Rearrange this to see that $P(A^c) = 1 - P(A)$.

- $P(\varnothing) = 0$

  Proof:

  $$\begin{aligned} P(\Omega) &= P(\Omega \cup \varnothing) && \text{since } \Omega \cup \varnothing = \Omega \\ &= P(\Omega) + P(\varnothing) && \text{by Axiom 3 (since } \Omega, \varnothing \text{ are disjoint)} \end{aligned}$$

  So $P(\varnothing) = 0$.

- For $A, B \subseteq \Omega, A \subseteq B \implies P(A) \leq P(B)$

  Proof:

$$\begin{aligned} P(B) &= P(A \cup (B \cap A^c)) \\ &= P(A) + P(B \cap A^c) && \text{by Axiom 3 (since } A, A^c \text{ are disjoint)} \end{aligned}$$

  But note that $P(B \cap A^c) \geq 0$ by Axiom 2.

  Then $P(B) = P(A) + P(B \cap A^c) \geq P(A)$.

- For $A, B \subseteq \Omega, P(A \cup B) = P(A) + P(B) - P(A \cap B)$

  Proof:

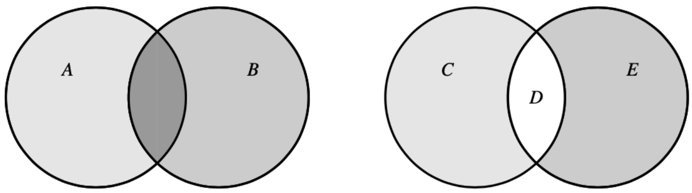
  Case 1: A, B are disjoint. Then $A \cap B = \varnothing \implies P(A \cap B) = 0$.

$$\begin{aligned} P(A \cup B) &= P(A) + P(B) && \text{by Axiom 3 (since } A, B \text{ are disjoint)} \\ &= P(A) + P(B) + P(A \cap B) && \text{since we can add 0 wherever we want} \end{aligned}$$

  Case 2: A, B not disjoint. Then $A \cap B \neq \varnothing$.

  Let $C = A \cap B^c$, $D = A \cap B$, $E = A^c \cap B$.

  Then C, D, E are disjoint, and $A = C \cup D$, $B = D \cup E$, and $A \cup B = C \cup D \cup E$.



$$\begin{aligned} P(A) + P(B) - P(A \cap B) &= P(C \cup D) + P(D \cup E) - P(D) && \text{by how we defined } C, D, E \\ &= P(C) + P(D) + P(D) + P(E) - P(D) && \text{by Axiom 3 and disjointness of } C, D, E \\ &= P(C) + P(D) + P(E) \\ &= P(C \cup D \cup E) && \text{by Axiom 3 and disjointness of } C, D, E \\ &= P(A \cup B) \end{aligned}$$

1.4 Permutations and Combinations

1.5 Conditional Probability

1.6 Independence, Law of Total Probability

2 Random Variables

2.1 Discrete Random Variables

2.1.1 Bernoulli

2.1.2 Binomial

2.1.3 Geometric

2.1.4 Negative Binomial

2.1.5 Hypergeometric

2.1.6 Poisson

2.2 Continuous Random Variables

2.2.1 Uniform

2.2.2 Exponential

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2.2.4 Beta

2.2.5 Uniform

2.2.6 Standard Normal

2.2.7 General Normal

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3.1.1 LOTUS

3.1.2 Inequalities

3.2 Moment Generating Functions

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4.1 Joint and Marginal Distributions

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4.1.2 Continuous

4.2 Independence in Joint Distributions

4.3 Conditional Distributions

4.3.1 Discrete

4.3.2 Continuous

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