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STA257: Probability and Statistics I

University of Toronto — Fall 2019

Jeff Shen

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1 Probability and Counting

1.1 Introduction

Experiments: situations wherre outcome is random. eg. flipping a coin is an experiment.
Sample space: set of all possible outcomes, denoted S or Ω. The number of elements in the sample space (cardinality) is denoted $|\Omega|$.
Event: a subset of a sample space.
Outcome: a particular element of a sample space $s_1 \in \Omega$.

1.2 Set Theory

1.2.1 Definitions

- **union**: $A \cup B$. Elements in either A or B.
- **intersection**: $A \cap B$. Elements in both A and B.
- **complement** of A: A^c. Elements not in A.
- **empty set**: $\varnothing$. Set with no elements in it.
- A and B are **disjoint**: $A \cap B = \varnothing$. There are no elements in the intersection of A and B.

1.2.2 Laws of Set Theory

1. **commutativity**: $A \cup B = B \cup A$, $A \cap B = B \cap A$
2. **associativity**: $(A \cup B) \cup C = A \cup (B \cup C)$, $(A \cap B) \cap C = A \cap (B \cap B)$
3. **distributivity**: $(A \cup B) \cap C = (A \cap C) \cup (B \cap C)$, $(A \cap B) \cup C = (A \cup C) \cap (B \cup C)$

1.3 Probability Measures

Probability measure: a function which maps subsets of Ω, which can be defined on any space, to real numbers $\mathbb{R}$.

1.3.1 Axioms of Probability Measures

- $P(\Omega) = 1$
- $\forall A \in \Omega, P(A) \geq 0$
- if $A_1, A_2, \ldots A_n, \ldots$ are mutually disjoint, then $P(\bigcup_{i=1}^{\infty} A_i) = \sum_{i=1}^{\infty} P(A_i)$.

1.3.2 Properties of Probability Measures

- $\forall A \in \Omega, P(A^c) = 1 - P(A)$

  Proof:

  $$\begin{aligned} 1 &= P(\Omega) && \text{by Axiom 1} \\ &= P(A \cup A^c) && \text{by definition of complement} \\ &= P(A) + P(A^c) && \text{by Axiom 3 (since } A, A^c \text{ are disjoint)} \end{aligned}$$

  Rearrange this to see that $P(A^c) = 1 - P(A)$.

- $P(\varnothing) = 0$

  Proof:

$$\begin{aligned} P(\Omega) &= P(\Omega \cup \varnothing) && \text{since } \Omega \cup \varnothing = \Omega \\ &= P(\Omega) + P(\varnothing) && \text{by Axiom 3 (since } \Omega, \varnothing \text{ are disjoint)} \end{aligned}$$

  So $P(\varnothing) = 0$.

- For $A, B \subseteq \Omega, A \subseteq B \implies P(A) \leq P(B)$

  Proof:

$$\begin{aligned} P(B) &= P(A \cup (B \cap A^c)) \\ &= P(A) + P(B \cap A^c) && \text{by Axiom 3 (since } A, A^c \text{ are disjoint)} \end{aligned}$$

  But note that $P(B \cap A^c) \geq 0$ by Axiom 2.

  Then $P(B) = P(A) + P(B \cap A^c) \geq P(A)$.

- For $A, B \subseteq \Omega, P(A \cup B) = P(A) + P(B) - P(A \cap B)$

  Proof:

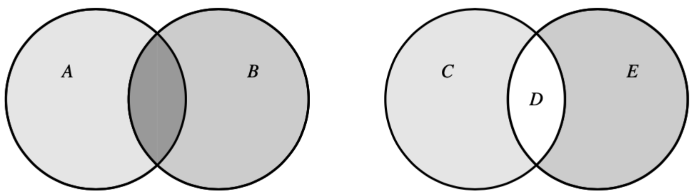
  Case 1: A, B are disjoint. Then $A \cap B = \varnothing \implies P(A \cap B) = 0$.

$$\begin{aligned} P(A \cup B) &= P(A) + P(B) && \text{by Axiom 3 (since } A, B \text{ are disjoint)} \\ &= P(A) + P(B) + P(A \cap B) && \text{since we can add 0 wherever we want} \end{aligned}$$

  Case 2: A, B not disjoint. Then $A \cap B \neq \varnothing$.

  Let $C = A \cap B^c$, $D = A \cap B$, $E = A^c \cap B$.

  Then C, D, E are disjoint, and $A = C \cup D$, $B = D \cup E$, and $A \cup B = C \cup D \cup E$.



$$\begin{aligned} P(A) + P(B) - P(A \cap B) &= P(C \cup D) + P(D \cup E) - P(D) && \text{by how we defined } C, D, E \\ &= P(C) + P(D) + P(D) + P(E) - P(D) && \text{by Axiom 3 and disjointness of } C, D, E \\ &= P(C) + P(D) + P(E) \\ &= P(C \cup D \cup E) && \text{by Axiom 3 and disjointness of } C, D, E \\ &= P(A \cup B) \end{aligned}$$

1.4 Counting

Multiplication principle: if there are m ways to do one thing, and n ways to do another thing, then there are mn ways to do both things.
Permutation: ordered arrangement of objects.

- Sampling **with replacement** means that duplicate item selection is allowed. (can pick the same object twice). For a set of size n and a **sample size** (number of items selected) r, there are n^r possible selections.
- Sampling **without replacement** means that each item is selected once at most. For a set of size n and a sample size r, there are $n(n-1)\dots(n-r+1) = \frac{n!}{(n-r)!}$ possible selections. In particular, there are $n(n-1)\dots(1) = n!$ ways to order n elements.

Combination: arrangement of objects *without regard to order*. Think about this as the ways to select objects without replacement, divided by the ways that those objects can be ordered. For a set of size n and a sample size r, we express the combination as follows:

$$\binom{n}{r} = \frac{n(n-1)\dots(n-r+1)}{r!} = \frac{n!}{(n-r)!\,r!}$$

Binomial expansion:

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^k\, b^{n-k}$$

In particular, for $a = b = 1$,

$$(1+1)^n = 2^n = \sum_{k=0}^{n} \binom{n}{k} (1)^k (1)^{n-k} = \sum_{k=0}^{n} \binom{n}{k}$$

The number of ways to group n objects into r classes, with n_i objects in the ith classes, where $1 < i < r$, is given by the following formula:

$$\binom{n}{n_1 n_2 \dots n_r} = \frac{n!}{n_1!\, n_2! \dots n_r!}$$

Proof: There are $\binom{n}{n_1}$ ways to select the first class, $\binom{n-n_1}{n_2}$ ways to select the second class, and so on. Repeat this for all r classes, and then apply the multiplication rule. Then we have that

$$\begin{aligned}
\binom{n}{n_1}\binom{n-n_1}{n_2}\dots\binom{n-n_1-\dots-n_{r-1}}{n_r} &= \frac{n!}{n_1!\,(n-n_1)!}\,\frac{(n-n_1)!}{n_2!\,(n-n_1-n_2)!}\dots\frac{(n-n_1-\dots-n_{r-1})!}{n_r!\,0!} \\
&= \frac{n!}{n_1!\,\cancel{(n-n_1)!}}\,\frac{\cancel{(n-n_1)!}}{n_2!\,\cancel{(n-n_1-n_2)!}}\dots\frac{\cancel{(n-n_1-\dots-n_{r-1})!}}{n_r!\,0!} \\
&= \frac{n!}{n_1!\,n_2!\dots n_r!}
\end{aligned}$$

1.5 Conditional Probability and Independence

Conditional probability: probability that some event will occur given that another event has occured:

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

Two events are **independent** if knowing that one has occured gives no information about the likelihood of the other occuring. Events A, B are independent if and only if any of the following hold:

- $P(A|B) = P(A)$
- $P(B|A) = P(B)$

- $(A \cap B) = P(A)P(B)$

From the definition of conditional probability, we can derive **Bayes' Rule**, which describes the probability of an event given prior knowledge of conditions related to the event:

$$\begin{aligned}
&P(A|B)\,P(B) = P(A \cap B) = P(B \cap A) = P(B|A)\,P(A) \\
\implies &P(A|B)\,P(B) = P(B|A)\,P(A) \\
\implies &P(A|B) = \frac{P(B|A)\,P(A)}{P(B)}
\end{aligned}$$

1.6 Law of Total Probability

Let $B_1, B_2, \ldots, B_n$ be sets satisfying the following conditions:

1. $B_1 \cup B_2 \cup \ldots \cup B_n = \Omega$ (the B_i's are a **partition** of Ω)
2. $\forall i, j \in [1, n] \subseteq \mathbb{N}, i \neq j \implies B_i \cap B_j = \varnothing$ (pairwise disjoint)
3. $\forall i \in [1, n] \subseteq \mathbb{N}, P(B_i) > 0$ (strictly positive probability),

we can conclude that for any event A, we have that

$$P(A) = \sum_{i=1}^{n} P(A|B_i)\,P(B_i)$$

Proof: Since all B_i's are pairwise disjoint, all $A \cap B_i$'s must also be pairwise disjoint. Then

$$\begin{aligned}
P(A) &= P(A \cap \Omega) \\
&= P\left(A \cap \left(\bigcup_{i=1}^{n} B_i\right)\right) && \text{by condition 1} \\
&= P\left(\bigcup_{i=1}^{n} (A \cap B_i)\right) \\
&= \sum_{i=1}^{n} P(A \cap B_i) && \text{by Axiom 3 of probability measures} \\
&= \sum_{i=1}^{n} P(A|B_i)\,P(B_i) && \text{by definition of conditional probability}
\end{aligned}$$

Under the same conditions, we can also conclude an alternate form of Bayes' Rule:

$$P(B_j|A) = \frac{P(A|B_j)\,P(B_j)}{\sum_{i=1}^{n} P(A|B_i)\,P(B_i)}$$

1.7 Equations

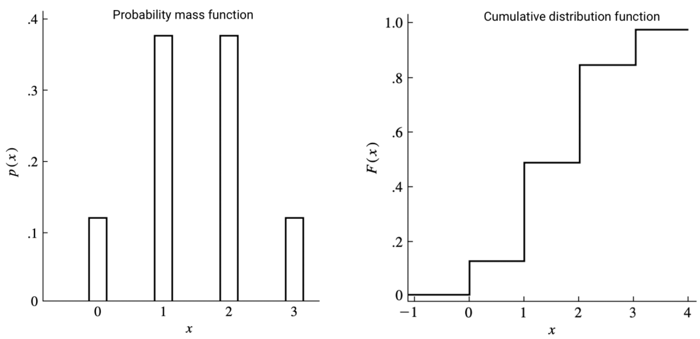
Axioms	$P(\Omega) = 1$
	$P(A) \geq 0$
	for mutually disjoint sets, $P(\bigcup_{i=1}^{\infty} A_i) = \sum_{i=1}^{\infty} P(A_i)$
Properties	$P(A^c) = 1 - P(A)$
	$P(\varnothing) = 0, P(\Omega) = 1$
	$A \subseteq B \implies P(A) \leq P(B)$
	$P(A \cup B) = P(A) + P(B) - P(A \cap B)$
Permutations	with replacement: n^r
	without replacement: $\frac{n!}{(n-r)!}$
Combinations	$\binom{n}{r} = \frac{n!}{(n-r)!\,r!}$
Binomial Expansion	$(a+b)^n = \sum_{i=1}^{n} \binom{n}{k} a^k\, b^{n-k}$
Grouping n objects into r classes	$\binom{n}{n_1 n_2 \ldots n_r} = \frac{n!}{n_1!\, n_2! \ldots n_r!}$
Conditional probability	$P(A \vert B) = \frac{P(A \cap B)}{P(B)}$
Bayes' Rule	$P(A \vert B) = \frac{P(B \vert A)\, P(A)}{P(B)}$
	$P(B_j \vert A) = \frac{P(A \vert B_j)\, P(B_j)}{\sum_{i=1}^{n} P(A \vert B_i)\, P(B_i)}$
Law of Total Probability	$P(A) = \sum_{i=1}^{n} P(A \vert B_i)\, P(B_i)$

2 Random Variables

2.1 Discrete Random Variables

Random variable: function from Ω to $\mathbb{R}$.
Discrete random variable: random variable which can only take a finite number of values, or a countably infinite number of values.
Probability mass function (PMF): describes probability properties of a random variable. For some discrete random variable X taking on values $x_1, x_2, \ldots$, the probability mass function is some function p such that $p(x_i) = P(X = x_i)$, and satisfies that $\sum_{i=1}^{\infty} p(x_i) = 1$.
Cumulative distribution function (CDF): defined as some function F such that $F(x) = P(X \leq x)$, where $-\infty < x < \infty$. It satisfies the following conditions:

- $\lim_{x \to -\infty} F(x) = 0$
- $\lim_{x \to \infty} F(x) = 1$
- is right-continuous (can have jumps, but must be continuous when approaching values from the right side). Continuity implies right-continuity.
- is non-decreasing.



For random varriables X, Y taking on values $x_1, x_2, \ldots$ and $y_1, y_2, \ldots$ respectively, X and Y are **independent** if $\forall i, j, P(X = x_i, Y = y_j) = P(X = x_i)P(Y = y_j)$.

2.1.1 Bernoulli

2.1.2 Binomial

2.1.3 Geometric

2.1.4 Negative Binomial

2.1.5 Hypergeometric

2.1.6 Poisson

2.2 Continuous Random Variables

2.2.1 Uniform

2.2.2 Exponential

2.2.3 Gamma

2.2.4 Beta

2.2.5 Uniform

2.2.6 Standard Normal

2.2.7 General Normal

2.3 Transformations of Random Variables

3 Expected Values

3.1 Mean and Variance

3.1.1 LOTUS

3.1.2 Inequalities

3.2 Moment Generating Functions

4 Joint Distributions

4.1 Joint and Marginal Distributions

4.1.1 Discrete

4.1.2 Continuous

4.2 Independence in Joint Distributions

4.3 Conditional Distributions

4.3.1 Discrete

4.3.2 Continuous

4.4 Functions of Joint Distributions

4.5 Order Statistics