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STA257: Probability and Statistics I

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1 Probability and Counting

1.1 Introduction

Experiments: situations wherre outcome is random. eg. flipping a coin is an experiment.
Sample space: set of all possible outcomes, denoted S or Ω. The number of elements in the sample space (cardinality) is denoted $|\Omega|$.
Event: a subset of a sample space.
Outcome: a particular element of a sample space $s_1 \in \Omega$.

1.2 Set Theory

1.2.1 Definitions

- **union**: $A \cup B$. Elements in either A or B.
- **intersection**: $A \cap B$. Elements in both A and B.
- **complement** of A: A^c. Elements not in A.
- **empty set**: $\varnothing$. Set with no elements in it.
- A and B are **disjoint**: $A \cap B = \varnothing$. There are no elements in the intersection of A and B.

1.2.2 Laws of Set Theory

1. **commutativity**: $A \cup B = B \cup A$, $A \cap B = B \cap A$
2. **associativity**: $(A \cup B) \cup C = A \cup (B \cup C)$, $(A \cap B) \cap C = A \cap (B \cap B)$
3. **distributivity**: $(A \cup B) \cap C = (A \cap C) \cup (B \cap C)$, $(A \cap B) \cup C = (A \cup C) \cap (B \cup C)$

1.3 Probability Measures

Probability measure: a function which maps subsets of Ω, which can be defined on any space, to real numbers $\mathbb{R}$.

1.3.1 Axioms of Probability Measures

- $P(\Omega) = 1$
- $\forall A \in \Omega, P(A) \geq 0$
- if $A_1, A_2, \ldots A_n, \ldots$ are mutually disjoint, then $P(\bigcup_{i=1}^{\infty} A_i) = \sum_{i=1}^{\infty} P(A_i)$.

1.3.2 Properties of Probability Measures

- $\forall A \in \Omega, P(A^c) = 1 - P(A)$
  Proof:
  $$\begin{aligned} 1 &= P(\Omega) && \text{by Axiom 1} \\ &= P(A \cup A^c) && \text{by definition of complement} \\ &= P(A) + P(A^c) && \text{by Axiom 3 (since } A, A^c \text{ are disjoint)} \end{aligned}$$
  Rearrange this to see that $P(A^c) = 1 - P(A)$.
- $P(\varnothing) = 0$
  Proof:
  $$\begin{aligned} P(\Omega) &= P(\Omega \cup \varnothing) && \text{since } \Omega \cup \varnothing = \Omega \\ &= P(\Omega) + P(\varnothing) && \text{by Axiom 3 (since } \Omega, \varnothing \text{ are disjoint)} \end{aligned}$$
  So $P(\varnothing) = 0$.

- For $A, B \subseteq \Omega, A \subseteq B \implies P(A) \leq P(B)$

  Proof:

$$\begin{aligned} P(B) &= P(A \cup (B \cap A^c)) \\ &= P(A) + P(B \cap A^c) && \text{by Axiom 3 (since } A, A^c \text{ are disjoint)} \end{aligned}$$

  But note that $P(B \cap A^c) \geq 0$ by Axiom 2.

  Then $P(B) = P(A) + P(B \cap A^c) \geq P(A)$.

- For $A, B \subseteq \Omega, P(A \cup B) = P(A) + P(B) - P(A \cap B)$

  Proof:

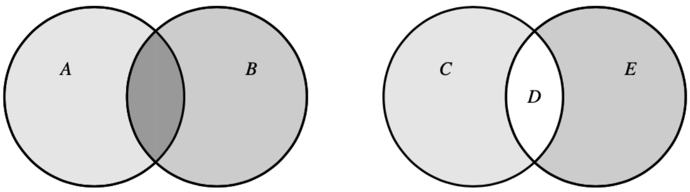
  Case 1: A, B are disjoint. Then $A \cap B = \varnothing \implies P(A \cap B) = 0$.

$$\begin{aligned} P(A \cup B) &= P(A) + P(B) && \text{by Axiom 3 (since } A, B \text{ are disjoint)} \\ &= P(A) + P(B) + P(A \cap B) && \text{since we can add 0 wherever we want} \end{aligned}$$

  Case 2: A, B not disjoint. Then $A \cap B \neq \varnothing$.

  Let $C = A \cap B^c$, $D = A \cap B$, $E = A^c \cap B$.

  Then C, D, E are disjoint, and $A = C \cup D$, $B = D \cup E$, and $A \cup B = C \cup D \cup E$.



$$\begin{aligned} P(A) + P(B) - P(A \cap B) &= P(C \cup D) + P(D \cup E) - P(D) && \text{by how we defined } C, D, E \\ &= P(C) + P(D) + P(D) + P(E) - P(D) && \text{by Axiom 3 and disjointness of } C, D, E \\ &= P(C) + P(D) + P(E) \\ &= P(C \cup D \cup E) && \text{by Axiom 3 and disjointness of } C, D, E \\ &= P(A \cup B) \end{aligned}$$

1.4 Counting

Multiplication principle: if there are m ways to do one thing, and n ways to do another thing, then there are mn ways to do both things.
Permutation: ordered arrangement of objects.

- Sampling **with replacement** means that duplicate item selection is allowed. (can pick the same object twice). For a set of size n and a **sample size** (number of items selected) r, there are n^r possible selections.
- Sampling **without replacement** means that each item is selected once at most. For a set of size n and a sample size r, there are $n(n-1)\ldots(n-r+1) = \frac{n!}{(n-r)!}$ possible selections. In particular, there are $n(n-1)\ldots(1) = n!$ ways to order n elements.

Combination: arrangement of objects *without regard to order*. Think about this as the ways to select objects without replacement, divided by the ways that those objects can be ordered. For a set of size n and a sample size r, we express the combination as follows:

$$\binom{n}{r} = \frac{n(n-1)\ldots(n-r+1)}{r!} = \frac{n!}{(n-r)!\,r!}$$

Binomial expansion:

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^k\, b^{n-k}$$

In particular, for $a = b = 1$,

$$(1+1)^n = 2^n = \sum_{k=0}^{n} \binom{n}{k} (1)^k (1)^{n-k} = \sum_{k=0}^{n} \binom{n}{k}$$

The number of ways to group n objects into r classes, with n_i objects in the ith classes, where $1 < i < r$, is given by the following formula:

$$\binom{n}{n_1 n_2 \dots n_r} = \frac{n!}{n_1!\, n_2! \dots n_r!}$$

Proof: There are $\binom{n}{n_1}$ ways to select the first class, $\binom{n-n_1}{n_2}$ ways to select the second class, and so on. Repeat this for all r classes, and then apply the multiplication rule. Then we have that

$$\begin{aligned}
\binom{n}{n_1}\binom{n-n_1}{n_2} \dots \binom{n-n_1-\dots-n_{r-1}}{n_r} &= \frac{n!}{n_1!\,(n-n_1)!} \frac{(n-n_1)!}{n_2!\,(n-n_1-n_2)!} \dots \frac{(n-n_1-\dots-n_{r-1})!}{n_r!\,0!} \\
&= \frac{n!}{n_1!\,\cancel{(n-n_1)!}} \frac{\cancel{(n-n_1)!}}{n_2!\,\cancel{(n-n_1-n_2)!}} \dots \frac{\cancel{(n-n_1-\dots-n_{r-1})!}}{n_r!\,0!} \\
&= \frac{n!}{n_1!\,n_2! \dots n_r!}
\end{aligned}$$

1.5 Conditional Probability and Independence

Conditional probability: probability that some event will occur given that another event has occured:

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

Two events are **independent** if knowing that one has occured gives no information about the likelihood of the other occuring. Events A, B are independent if and only if any of the following hold:

- $P(A|B) = P(A)$
- $P(B|A) = P(B)$
- $(A \cap B) = P(A)P(B)$

From the definition of conditional probability, we can derive **Bayes' Rule**, which describes the probability of an event given prior knowledge of conditions related to the event:

$$\begin{aligned}
& P(A|B)\,P(B) = P(A \cap B) = P(B \cap A) = P(B|A)\,P(A) \\
\implies & P(A|B)\,P(B) = P(B|A)\,P(A) \\
\implies & P(A|B) = \frac{P(B|A)\,P(A)}{P(B)}
\end{aligned}$$

1.6 Law of Total Probability

Let $B_1, B_2, \dots, B_n$ be sets satisfying the following conditions:

1. $B_1 \cup B_2 \cup \dots \cup B_n = \Omega$ (the B_i's are a **partition** of Ω)
2. $\forall i, j \in [1, n] \subseteq \mathbb{N}, i \neq j \implies B_i \cap B_j = \varnothing$ (pairwise disjoint)
3. $\forall i \in [1, n] \subseteq \mathbb{N}, P(B_i) > 0$ (strictly positive probability),

we can conclude that for any event A, we have that

$$P(A) = \sum_{i=1}^{n} P(A|B_i)\,P(B_i)$$

Proof: Since all B_i's are pairwise disjoint, all $A \cap B_i$'s must also be pairwise disjoint. Then

$$
\begin{aligned}
P(A) &= P(A \cap \Omega) \\
&= P\left(A \cap \left(\bigcup_{i=1}^{n} B_i\right)\right) && \text{by condition 1} \\
&= P\left(\bigcup_{i=1}^{n} (A \cap B_i)\right) \\
&= \sum_{i=1}^{n} P(A \cap B_i) && \text{by Axiom 3 of probability measures} \\
&= \sum_{i=1}^{n} P(A|B_i)\, P(B_i) && \text{by definition of conditional probability}
\end{aligned}
$$

Under the same conditions, we can also conclude an alternate form of Bayes' Rule:

$$P(B_j|A) = \frac{P(A|B_j)\, P(B_j)}{\sum_{i=1}^{n} P(A|B_i)\, P(B_i)}$$

1.7 Equations

Axioms	$P(\Omega) = 1$
	$(A) \geq 0$
	for mutually disjoint sets, $P(\bigcup_{i=1}^{\infty} A_i) = \sum_{i=1}^{\infty} P(A_i)$
Properties	$P(A^c) = 1 - P(A)$
	$P(\varnothing) = 0, P(\Omega) = 1$
	$A \subseteq B \implies P(A) \leq P(B)$
	$P(A \cup B) = P(A) + P(B) - P(A \cap B)$
Permutations	with replacement: n^r
	without replacement: $\frac{n!}{(n-r)!}$
Combinations	$\binom{n}{r} = \frac{n!}{(n-r)!\, r!}$
Binomial Expansion	$(a+b)^n = \sum_{i=1}^{n} \binom{n}{k} a^k\, b^{n-k}$
Grouping n objects into r classes	$\binom{n}{n_1 n_2 \ldots n_r} = \frac{n!}{n_1!\, n_2! \ldots n_r!}$
Conditional probability	$P(A\|B) = \frac{P(A \cap B)}{P(B)}$
Bayes' Rule	$P(A\|B) = \frac{P(B\|A)\, P(A)}{P(B)}$
	$P(B_j\|A) = \frac{P(A\|B_j)\, P(B_j)}{\sum_{i=1}^{n} P(A\|B_i)\, P(B_i)}$
Law of Total Probability	$P(A) = \sum_{i=1}^{n} P(A\|B_i)\, P(B_i)$

2 Random Variables

2.1 Discrete Random Variables

2.1.1 Bernoulli

2.1.2 Binomial

2.1.3 Geometric

2.1.4 Negative Binomial

2.1.5 Hypergeometric

2.1.6 Poisson

2.2 Continuous Random Variables

2.2.1 Uniform

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