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STA257: Probability and Statistics I

University of Toronto — Fall 2019

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1 Probability and Counting

1.1 Introduction

Experiments: situations wherre outcome is random. eg. flipping a coin is an experiment.
Sample space: set of all possible outcomes, denoted S or Ω. The number of elements in the sample space (cardinality) is denoted $|\Omega|$.
Event: a subset of a sample space.
Outcome: a particular element of a sample space $s_1 \in \Omega$.

1.2 Set Theory

1.2.1 Definitions

- **union**: $A \cup B$. Elements in either A or B.
- **intersection**: $A \cap B$. Elements in both A and B.
- **complement** of A: A^c. Elements not in A.
- **empty set**: $\varnothing$. Set with no elements in it.
- A and B are **disjoint**: $A \cap B = \varnothing$. There are no elements in the intersection of A and B.

1.2.2 Laws of Set Theory

1. **commutativity**: $A \cup B = B \cup A$, $A \cap B = B \cap A$
2. **associativity**: $(A \cup B) \cup C = A \cup (B \cup C)$, $(A \cap B) \cap C = A \cap (B \cap B)$
3. **distributivity**: $(A \cup B) \cap C = (A \cap C) \cup (B \cap C)$, $(A \cap B) \cup C = (A \cup C) \cap (B \cup C)$

1.3 Probability Measures

Probability measure: a function which maps subsets of Ω, which can be defined on any space, to real numbers $\mathbb{R}$.

1.3.1 Axioms of Probability Measures

- $P(\Omega) = 1$
- $\forall A \in \Omega, P(A) \geq 0$
- if $A_1, A_2, \ldots A_n, \ldots$ are mutually disjoint, then $P(\bigcup_{i=1}^{\infty} A_i) = \sum_{i=1}^{\infty} P(A_i)$.

1.3.2 Properties of Probability Measures

- $\forall A \in \Omega, P(A^c) = 1 - P(A)$
  Proof:
  $$\begin{aligned} 1 &= P(\Omega) && \text{by Axiom 1} \\ &= P(A \cup A^c) && \text{by definition of complement} \\ &= P(A) + P(A^c) && \text{by Axiom 3 (since } A, A^c \text{ are disjoint)} \end{aligned}$$
  Rearrange this to see that $P(A^c) = 1 - P(A)$.
- $P(\varnothing) = 0$
  Proof:
  $$\begin{aligned} P(\Omega) &= P(\Omega \cup \varnothing) && \text{since } \Omega \cup \varnothing = \Omega \\ &= P(\Omega) + P(\varnothing) && \text{by Axiom 3 (since } \Omega, \varnothing \text{ are disjoint)} \end{aligned}$$
  So $P(\varnothing) = 0$.

- For $A, B \subseteq \Omega, A \subseteq B \implies P(A) \leq P(B)$

  Proof:

$$\begin{aligned} P(B) &= P(A \cup (B \cap A^c)) \\ &= P(A) + P(B \cap A^c) && \text{by Axiom 3 (since } A, A^c \text{ are disjoint)} \end{aligned}$$

  But note that $P(B \cap A^c) \geq 0$ by Axiom 2.

  Then $P(B) = P(A) + P(B \cap A^c) \geq P(A)$.

- For $A, B \subseteq \Omega, P(A \cup B) = P(A) + P(B) - P(A \cap B)$

  Proof:

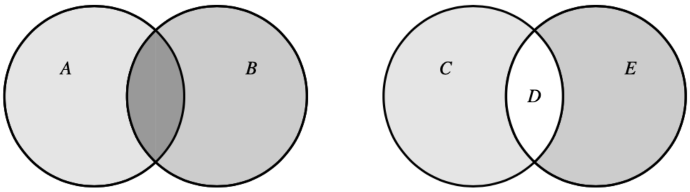
  Case 1: A, B are disjoint. Then $A \cap B = \varnothing \implies P(A \cap B) = 0$.

$$\begin{aligned} P(A \cup B) &= P(A) + P(B) && \text{by Axiom 3 (since } A, B \text{ are disjoint)} \\ &= P(A) + P(B) + P(A \cap B) && \text{since we can add 0 wherever we want} \end{aligned}$$

  Case 2: A, B not disjoint. Then $A \cap B \neq \varnothing$.

  Let $C = A \cap B^c$, $D = A \cap B$, $E = A^c \cap B$.

  Then C, D, E are disjoint, and $A = C \cup D$, $B = D \cup E$, and $A \cup B = C \cup D \cup E$.



$$\begin{aligned} P(A) + P(B) - P(A \cap B) &= P(C \cup D) + P(D \cup E) - P(D) && \text{by how we defined } C, D, E \\ &= P(C) + P(D) + P(D) + P(E) - P(D) && \text{by Axiom 3 and disjointness of } C, D, E \\ &= P(C) + P(D) + P(E) \\ &= P(C \cup D \cup E) && \text{by Axiom 3 and disjointness of } C, D, E \\ &= P(A \cup B) \end{aligned}$$

1.4 Counting

Multiplication principle: if there are m ways to do one thing, and n ways to do another thing, then there are mn ways to do both things.

Permutation: ordered arrangement of objects.

- Sampling **with replacement** means that duplicate item selection is allowed. (can pick the same object twice). For a set of size n and a **sample size** (number of items selected) r, there are n^r possible selections.
- Sampling **without replacement** means that each item is selected once at most. For a set of size n and a sample size r, there are $n(n-1)\ldots(n-r+1) = \frac{n!}{(n-r)!}$ possible selections. In particular, there are $n(n-1)\ldots(1) = n!$ ways to order n elements.

Combination: arrangement of objects *without regard to order*. Think about this as the ways to select objects without replacement, divided by the ways that those objects can be ordered. For a set of size n and a sample size r, we express the combination as follows:

$$\binom{n}{r} = \frac{n(n-1)\ldots(n-r+1)}{r!} = \frac{n!}{(n-r)!\,r!}$$

Binomial expansion:

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^k\, b^{n-k}$$

In particular, for $a = b = 1$,

$$(1+1)^n = 2^n = \sum_{k=0}^{n} \binom{n}{k} (1)^k (1)^{n-k} = \sum_{k=0}^{n} \binom{n}{k}$$

1.5 Conditional Probability

1.6 Independence, Law of Total Probability

1.7 Equations

2 Random Variables

2.1 Discrete Random Variables

2.1.1 Bernoulli

2.1.2 Binomial

2.1.3 Geometric

2.1.4 Negative Binomial

2.1.5 Hypergeometric

2.1.6 Poisson

2.2 Continuous Random Variables

2.2.1 Uniform

2.2.2 Exponential

2.2.3 Gamma

2.2.4 Beta

2.2.5 Uniform

2.2.6 Standard Normal

2.2.7 General Normal

2.3 Transformations of Random Variables

3 Expected Values

3.1 Mean and Variance

3.1.1 LOTUS

3.1.2 Inequalities

3.2 Moment Generating Functions

4 Joint Distributions

4.1 Joint and Marginal Distributions

4.1.1 Discrete

4.1.2 Continuous

4.2 Independence in Joint Distributions

4.3 Conditional Distributions

4.3.1 Discrete

4.3.2 Continuous

4.4 Functions of Joint Distributions

4.5 Order Statistics