Lecture 5.
Camera Models: Extrinsic Parameters
ECEN 5283 Computer Vision

Dr. Guoliang Fan
School of Electrical and Computer Engineering
Oklahoma State University

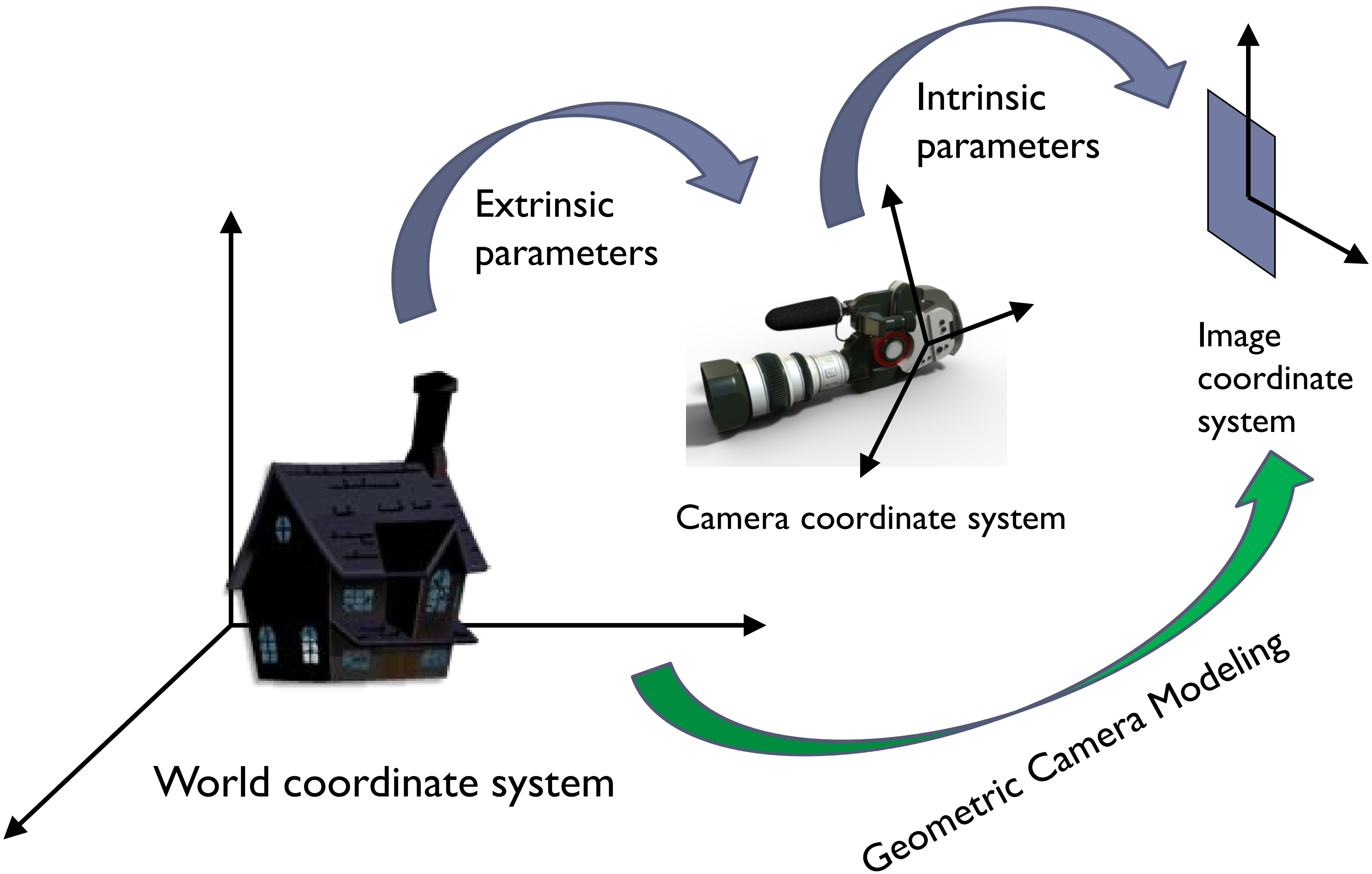
Goals

- To review the role of intrinsic parameters for geometric camera modeling.

- To study the role of extrinsic parameters for geometric camera modeling.

- To introduce the camera projection matrix that incorporates both intrinsic and extrinsic parameters.

Geometric Camera Modeling: Intrinsic and Extrinsic Parameters



Normalized Image Plane

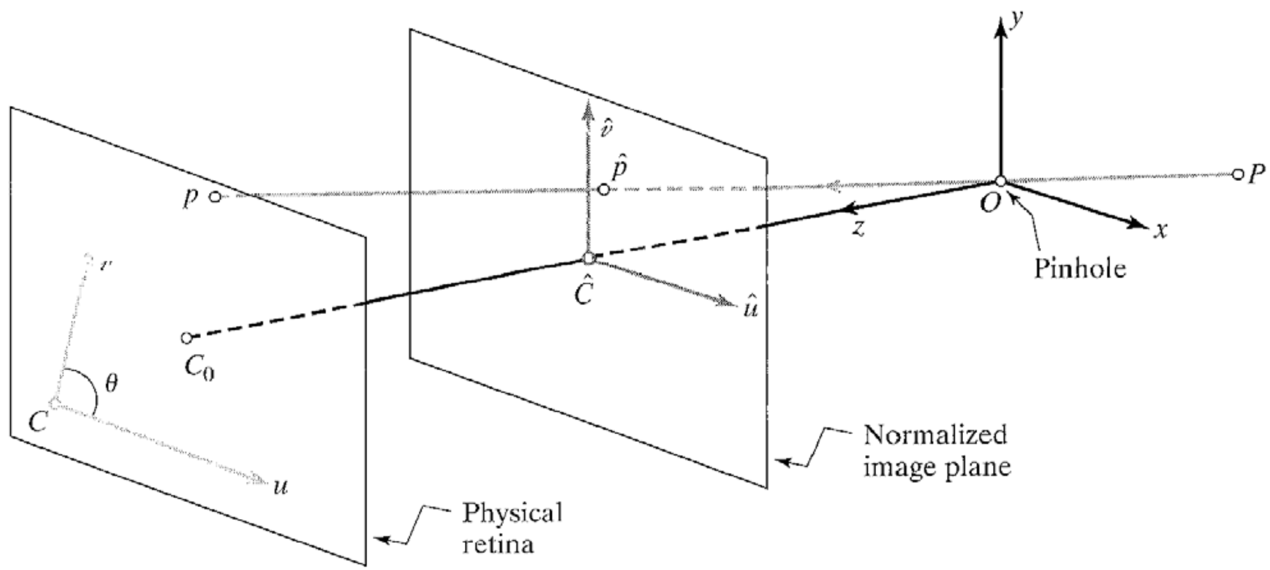


Figure 2.8 Physical and normalized image coordinate systems.

$$\begin{cases} \hat{u} = \dfrac{x}{z} \\ \hat{v} = \dfrac{y}{z} \end{cases} \Rightarrow \begin{bmatrix} \hat{u} \\ \hat{v} \\ 1 \end{bmatrix} = \frac{1}{z} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} \Leftrightarrow \hat{\mathbf{p}} = \frac{1}{z} (\mathbf{I} \quad 0) \mathbf{P}$$

Perspective projection

where $\begin{cases} \mathbf{P} = (x \quad y \quad z \quad 1)^T \\ \hat{\mathbf{p}} = (\hat{u} \quad \hat{v} \quad 1)^T \end{cases}$

$$\mathbf{I} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Physical Retina of the Camera

$C_0 = \{u_0, v_0\}$

(pixel coordinate of the image center)

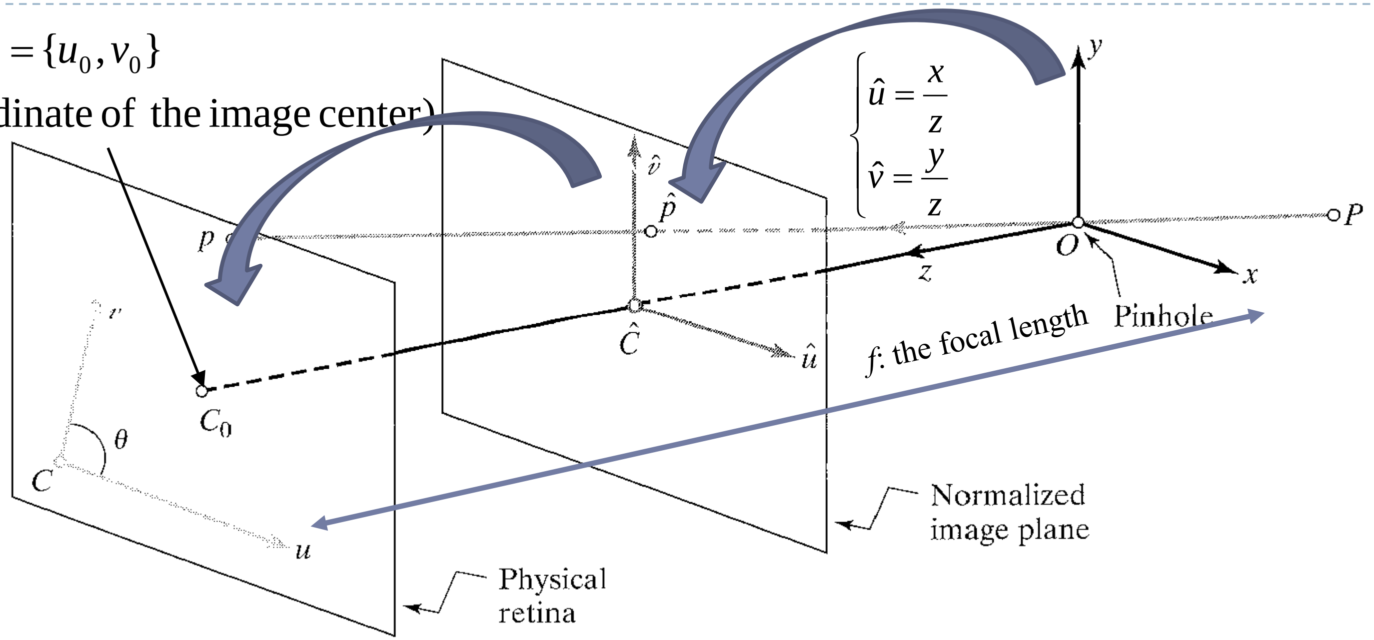


Figure 2.8 Physical and normalized image coordinate systems.

$$\begin{cases} u = kf\dfrac{x}{z} \\ v = lf\dfrac{y}{z} \end{cases}$$ where a pixel has dimension $\frac{1}{k} \times \frac{1}{l}$, f is the focal length. $\Rightarrow$ Pixel coordinates $$\begin{cases} u = \alpha\dfrac{x}{z} + u_0 \\ v = \beta\dfrac{y}{z} + v_0 \end{cases}$$

(assuming $\theta = \frac{\pi}{2}$)

$1/k$

$1/l$ Pixel

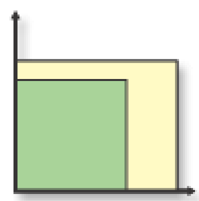
$$\alpha = kf \quad \text{and} \quad \beta = lf$$

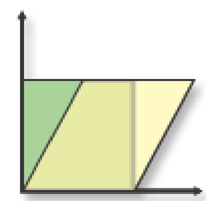
Affine Transformation Review

- An affine transformation can differentially scale the data, skew it, rotate it, and translate it.

$$\begin{pmatrix} u \\ v \\ 1 \end{pmatrix} = \begin{pmatrix} a & b & c \\ d & e & f \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \hat{u} \\ \hat{v} \\ 1 \end{pmatrix}$$

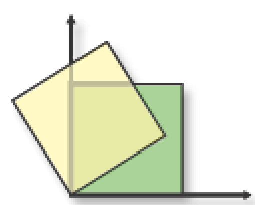
$$\begin{pmatrix} s_u & 0 & 0 \\ 0 & s_v & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

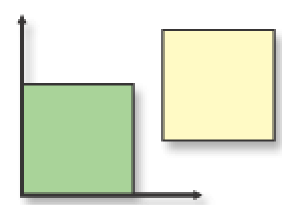

DIFFERENTIAL SCALING


SKEW

$$\begin{pmatrix} 1 & h_x & 0 \\ h_y & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$


ROTATION


TRANSLATION

$$\begin{pmatrix} 1 & 0 & u_0 \\ 0 & 1 & v_0 \\ 0 & 0 & 1 \end{pmatrix}$$

Normalized Image Plan and Physical Retina: Revisited

$C_0 = \{u_0, v_0\}$

(pixel coordinate of the image center)

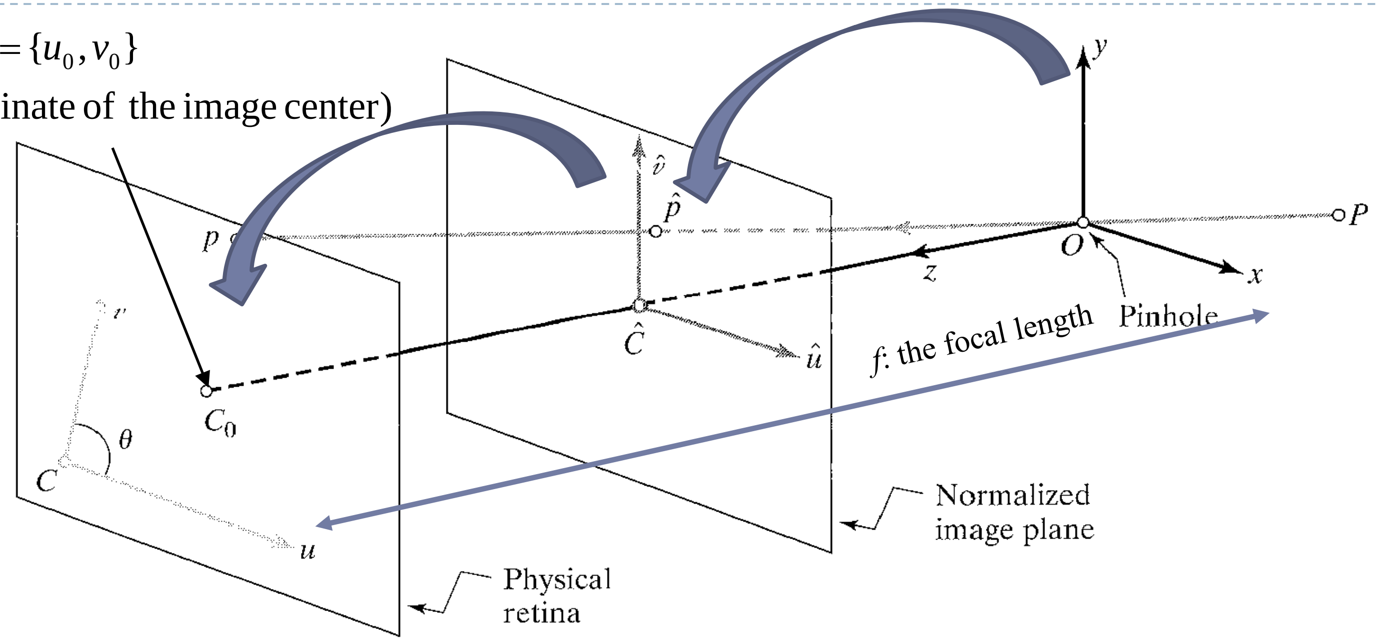


Figure 2.8 Physical and normalized image coordinate systems.

$$\begin{cases} \hat{u} = \dfrac{x}{z} \\ \hat{v} = \dfrac{y}{z} \end{cases} \xrightarrow{\text{Affine Transformation}} \begin{cases} u = \alpha\hat{u} - \alpha \cot\theta \hat{v} + u_0 \\ v = \dfrac{\beta}{\sin\theta}\hat{v} + v_0 \end{cases}$$

$$\left(\cot\theta = \frac{\cos\theta}{\sin\theta} \right)$$

$$\alpha = kf, \beta = lf$$

Planar Affine Transformation

$$\mathbf{p} = \begin{pmatrix} u \\ v \\ 1 \end{pmatrix} \Rightarrow \begin{cases} u = \alpha\hat{u} - \alpha\cot\theta\hat{v} + u_0 \\ v = \dfrac{\beta}{\sin\theta}\hat{v} + v_0 \end{cases}$$

$$\hat{\mathbf{p}} = \begin{pmatrix} \hat{u} \\ \hat{v} \\ 1 \end{pmatrix} \Rightarrow \begin{cases} \hat{u} = \dfrac{x}{z} \\ \hat{v} = \dfrac{y}{z} \end{cases}$$

(normalized image plane)

$$\hat{\mathbf{p}} = \begin{pmatrix} \hat{u} \\ \hat{v} \\ 1 \end{pmatrix} = \frac{1}{z}(\mathbf{I} \quad 0)\mathbf{P}$$

(3D to 2D perspective projection on the normalized image plane)

$$\mathbf{p} = K\hat{\mathbf{p}},$$

$$\text{where } K = \begin{pmatrix} \alpha & -\alpha\cot\theta & u_0 \\ 0 & \dfrac{\beta}{\sin\theta} & v_0 \\ 0 & 0 & 1 \end{pmatrix}.$$

$$\mathbf{p} = K\frac{1}{z}(\mathbf{I} \quad 0)\mathbf{P}$$

$$\begin{pmatrix} x \\ y \\ z \\ 1 \end{pmatrix}$$

Intrinsic Parameters

$$\mathbf{p} = \frac{1}{z}M\mathbf{P}, \quad \text{where } M = (K \quad \mathbf{0})$$

Extrinsic Parameters

- Let us consider the case where the camera frame (C) is distinct from the world frame (W). Noting that,

$$\begin{pmatrix} {}^{C}P \\ 1 \end{pmatrix} = \begin{pmatrix} {}^{C}_{W}R & {}^{C}O_{W} \\ \mathbf{0}^{T} & 1 \end{pmatrix} \begin{pmatrix} {}^{W}P \\ 1 \end{pmatrix},$$

$$\begin{pmatrix} {}^{C}P \\ 1 \end{pmatrix} = {}^{C}_{W}T \begin{pmatrix} {}^{W}P \\ 1 \end{pmatrix}, \quad \text{where } {}^{C}_{W}T = \begin{pmatrix} {}^{C}_{W}R & {}^{C}O_{W} \\ \mathbf{0}^{T} & 1 \end{pmatrix}$$

$$\mathbf{p} = \frac{1}{z} M' \mathbf{P}',$$

$M' = \begin{pmatrix} K & \mathbf{0} \end{pmatrix}$ (intrinsic parameters)

$\mathbf{P}'$ (3D homogeneous coordinate in the camera frame)

$$\mathbf{p} = \frac{1}{z} M' T \mathbf{P}$$

Projection Matrix: Definition

$$\mathbf{p} = \frac{1}{z} M'T\mathbf{P} \quad \text{where } M' = \begin{pmatrix} K & \mathbf{0} \end{pmatrix} \text{ and } T = \begin{pmatrix} \mathbf{R} & \mathbf{t} \\ \mathbf{0}^T & 1 \end{pmatrix}.$$

$$\downarrow$$

$$\mathbf{p} = \frac{1}{z} M\mathbf{P} \quad \text{where } M_{3\times 4} = M'T = \mathrm{K}_{3\times 3} \begin{pmatrix} R & \mathbf{t} \end{pmatrix}_{3\times 4}.$$

$R = {}^{C}_{W}R$ is a rotation matrix;

$\mathbf{t} = {}^{C}O_{W}$ is a translation vector;

$\mathbf{P} = \begin{pmatrix} W_x \\ W_y \\ W_z \\ 1 \end{pmatrix}$ denotes the homogeneous coordinate vector of P is the frame W.

Projection Matrix: Depth Constraint

- It is important to understand the the depth z is not independent of M and P.

$$\mathbf{p} = \frac{1}{z} M\mathbf{P} \quad \text{where } \mathbf{p} = \begin{pmatrix} u \\ v \\ 1 \end{pmatrix} \text{ and } M = \begin{pmatrix} \mathbf{m}_1^T \\ \mathbf{m}_2^T \\ \mathbf{m}_3^T \end{pmatrix}$$

What is this z?

$$\begin{pmatrix} u \\ v \\ 1 \end{pmatrix} = \frac{1}{z} \begin{pmatrix} \mathbf{m}_1^T \mathbf{P} \\ \mathbf{m}_2^T \mathbf{P} \\ \mathbf{m}_3^T \mathbf{P} \end{pmatrix} = \frac{1}{z} \begin{pmatrix} \mathbf{m}_1 \cdot \mathbf{P} \\ \mathbf{m}_2 \cdot \mathbf{P} \\ \mathbf{m}_3 \cdot \mathbf{P} \end{pmatrix} \Rightarrow \begin{cases} z = \mathbf{m}_3 \cdot \mathbf{P} \\ u = \dfrac{\mathbf{m}_1 \cdot \mathbf{P}}{\mathbf{m}_3 \cdot \mathbf{P}} \\ v = \dfrac{\mathbf{m}_2 \cdot \mathbf{P}}{\mathbf{m}_3 \cdot \mathbf{P}} \end{cases}$$

$$1 = \frac{\mathbf{m}_3^T \mathbf{P}}{z} \rightarrow \mathbf{m}_3^T \mathbf{P} = z$$

Projection Matrix: Parameters

- A projection matrix is written explicitly as a function of both intrinsic and extrinsic parameters as follows
  - Five intrinsic parameters $\alpha,\ \beta,\ u_0,\ v_0, \theta$
  - Six extrinsic ones (three angles and three coordinates of t).

$$\mathbf{p} = \frac{1}{z} M\mathbf{P} \quad \text{where } M = K\begin{pmatrix} R & \mathbf{t} \end{pmatrix} \quad K = \begin{pmatrix} \alpha & -\alpha\cot\theta & u_0 \\ 0 & \dfrac{\beta}{\sin\theta} & v_0 \\ 0 & 0 & 1 \end{pmatrix} \quad R = \begin{pmatrix} \mathbf{r}_1^T \\ \mathbf{r}_2^T \\ \mathbf{r}_3^T \end{pmatrix} \text{ and } \mathbf{t} = \begin{pmatrix} t_x \\ t_y \\ t_z \end{pmatrix}$$

$$M = K\begin{pmatrix} R & \mathbf{t} \end{pmatrix} = \begin{pmatrix} KR & K\mathbf{t} \end{pmatrix} = \begin{pmatrix} \alpha\mathbf{r}_1^T - \alpha\cot\theta\mathbf{r}_2^T + u_0\mathbf{r}_3^T & \alpha t_x - \alpha\cot\theta t_y + u_0 t_z \\ \dfrac{\beta}{\sin\theta}\mathbf{r}_2^T + v_0\mathbf{r}_3^T & \dfrac{\beta}{\sin\theta} t_y + v_0 t_z \\ \mathbf{r}_3^T & t_z \end{pmatrix}_{3\times4}$$