Lecture 12
Laplacian of Gaussian (LoG)
ECEN 5283 Computer Vision

Dr. Guoliang Fan
School of Electrical and Computer Engineering
Oklahoma State University

Goals

- To review edge detection theory.
- To introduce the Laplacian-of-Gaussian (LoG) edge detector that uses the second-order spatial differentiation.

Edge Detection Theory

- Fundamentals of edge detection
  - Edge detection is essentially involving spatial differentiation (difference) that is sensitive to fast intensity changes
  - Differentiating a function is the same as emphasizing high-frequency components and deemphasizing low-frequency components.

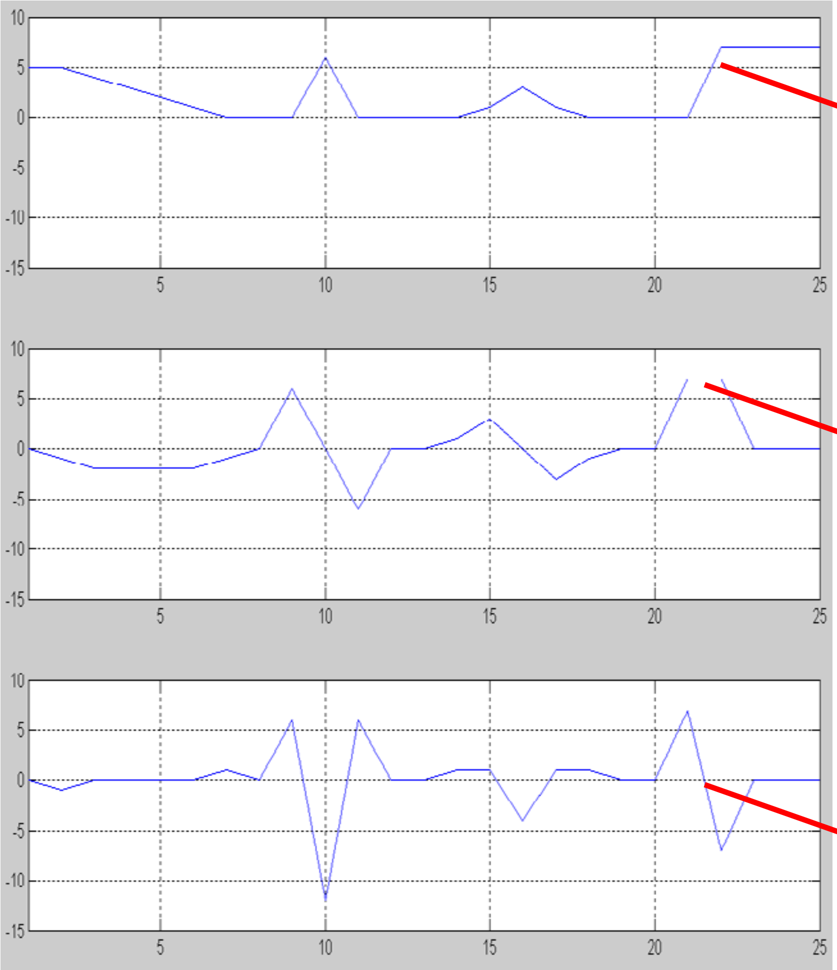
- Three issues of edge detection
  - Issue 1: How to model the noise in an image?
    - The Additive White Gaussian Noise (AWGN) model is often used.
  - Issue 2: How to reduce the noise effect in spatial differentiation?
    - Develop the difference of smoothing filters and use it for edge detection.
  - Issue 3: How to select the optimal smoothing filter?
    - The optimal smoothing filter is a Gaussian-like one.

Differentiation for Edge Detection

A 1-D signal

First-order derivative

Second-order derivative



Edge point

Local maximum

Zero-crossing

Laplacian-of-Gaussian (LoG) for Edge Detection (1)

- We define a 2-D second order derivative as the Laplacian function which is isotropic.

$$\frac{d^2 f}{dx^2} = f(x) - 2f(x-1) + f(x-2) \quad \Leftarrow \quad \frac{df}{dx} = f(x) - f(x-1)$$

$$(\nabla^2 f)(x, y) = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2}$$

$$\frac{\partial^2 f}{\partial y^2} = f(x, y+1) + f(x, y-1) - 2f(x, y)$$

$$\frac{\partial^2 f}{\partial x^2} = f(x+1, y) + f(x-1, y) - 2f(x, y)$$

- We could represent taking the Laplacian as convolving the image with some kernel as:

0	1	0
0	-2	0
0	1	0

$+$

0	0	0
1	-2	1
0	0	0

$=$

0	1	0
1	-4	1
0	1	0

$$\frac{\partial^2 f}{\partial y^2} = f(x, y+1) + f(x, y-1) - 2f(x, y) \qquad \frac{\partial^2 f}{\partial x^2} = f(x+1, y) + f(x-1, y) - 2f(x, y) \qquad (\nabla^2 f)(x, y) = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2}$$

Laplacian-of-Gaussian (LoG) for Edge Detection (2)

- If an image is pre-smoothed by a Gaussian filter, then we have the Laplacian-of-Gaussian (LoG) operation that is defined as

$$(K_{\nabla^2} ** (G_\sigma ** I))$$

$$= (K_{\nabla^2} ** G_\sigma) ** I$$

$$= \left(\frac{\partial^2 G_\sigma}{\partial x^2} + \frac{\partial^2 G_\sigma}{\partial y^2}\right) ** I.$$

$$= (\nabla^2 G_\sigma) ** I.$$

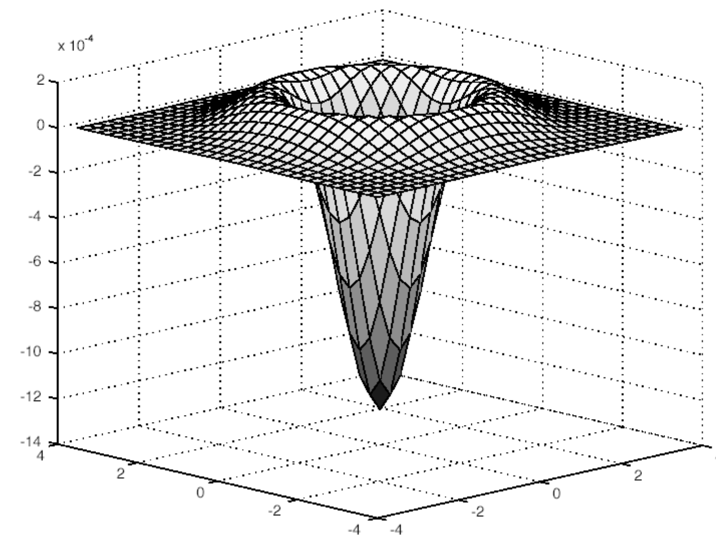
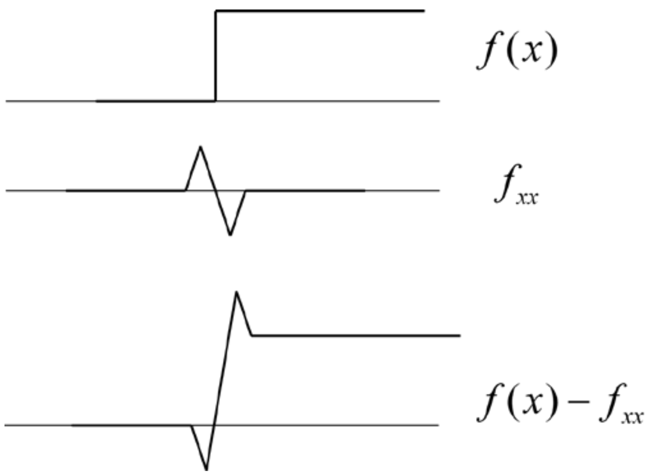


Figure 9.7. The Laplacian of Gaussian filter kernel, shown here for σ one pixel, can be thought of as subtracting the center pixel from a weighted average of the surround (hence the analogy with unsharp masking, described in the text). It is quite common to replace this kernel with the difference of two Gaussians, one with a small value of σ and the other with a large value of σ.

where $\nabla^2 G_\sigma(x, y) = \left(\frac{1}{2\pi\sigma^4}\right)\left[\frac{x^2 + y^2}{\sigma^2} - 2\right] e^{-\frac{x^2+y^2}{2\sigma^2}}$

Separable?

Laplacian-of-Gaussian (LoG) for Image Enhancement




LoG image ($\sigma = 1$)


Enhanced image ($\sigma = 1$)


Original image


LoG image ($\sigma = 0.25$)


Enhanced image ($\sigma = 0.25$)

Determining a Discrete LoG Kernel

$$\nabla^2 G_\sigma(x,y) = \left(\frac{1}{2\pi\sigma^4}\right)\left[\frac{x^2+y^2}{\sigma^2} - 2\right] e^{-\frac{x^2+y^2}{2\sigma^2}} \quad \text{(continuous LoG)}$$

$$K[i,j] = \left(\frac{1}{2\pi\sigma^4}\right)\left[\frac{(i-k-1)^2+(j-k-1)^2}{\sigma^2} - 2\right] e^{-\frac{(i-k-1)^2+(j-k-1)^2}{2\sigma^2}} \quad \text{(Discrete LoG)}$$

- The dimension of the kernel is (2k+1)$\times$(2k+1).
- The variance will determine the dimension of the kernel.
- π is usually ignored in computing the LoG coefficients.
- Some rounding maybe involved to make coefficient integers to speed up the 2D convolution (may not be necessary in Matlab).
- The average of all kernel coefficients must be zero (why?).

A LoG kernel example

$$K[i,j] = \left(\frac{1}{2\sigma^4}\right)\left[\frac{(i-k-1)^2+(j-k-1)^2}{\sigma^2}-2\right]e^{-\frac{(i-k-1)^2+(j-k-1)^2}{2\sigma^2}} \quad \text{(Discrete LoG)}$$

$$\delta = 0.5$$

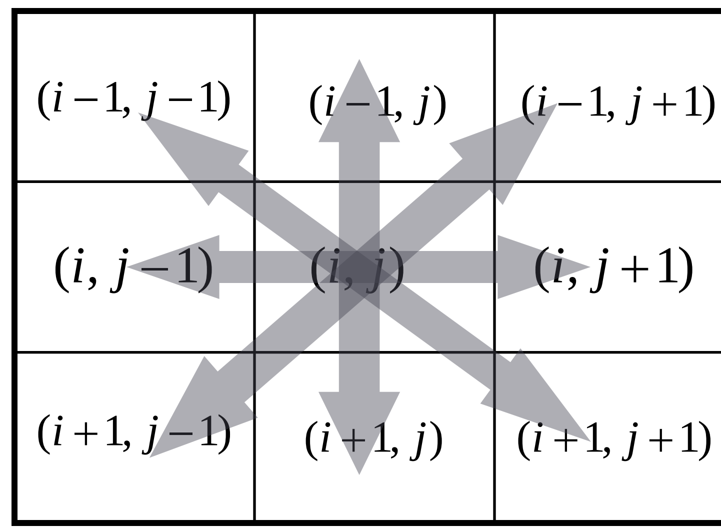
$$\mathbf{K} = \begin{bmatrix} 0.0000 & 0.0065 & 0.0376 & 0.0065 & 0.0000 \\ 0.0065 & 0.8792 & 2.1654 & 0.8792 & 0.0065 \\ 0.0376 & 2.1654 & -16.0 & 2.1654 & 0.0376 \\ 0.0065 & 0.8792 & 2.1654 & 0.8792 & 0.0065 \\ 0.0000 & 0.0065 & 0.0376 & 0.0065 & 0.0000 \end{bmatrix} \Rightarrow \mathbf{K}^* = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 2 & 1 & 0 \\ 1 & 2 & -16 & 2 & 1 \\ 0 & 1 & 2 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix}$$

The Steps of LoG Edge Detection

- Applying the LoG to the image.

$$\begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 2 & 1 & 0 \\ 1 & 2 & -16 & 2 & 1 \\ 0 & 1 & 2 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix}$$

- Detection of zero-crossings (ZCs) in the image (how?).
- Threshold the ZCs to keep only strong ones with significant difference between the positive and negative values.
- Remove isolated edge points caused by noise by counting the number of pixels in each connected component and removing smaller components.
  - (To be discussed in more details Lecture 14.)



512 256 128 64 32

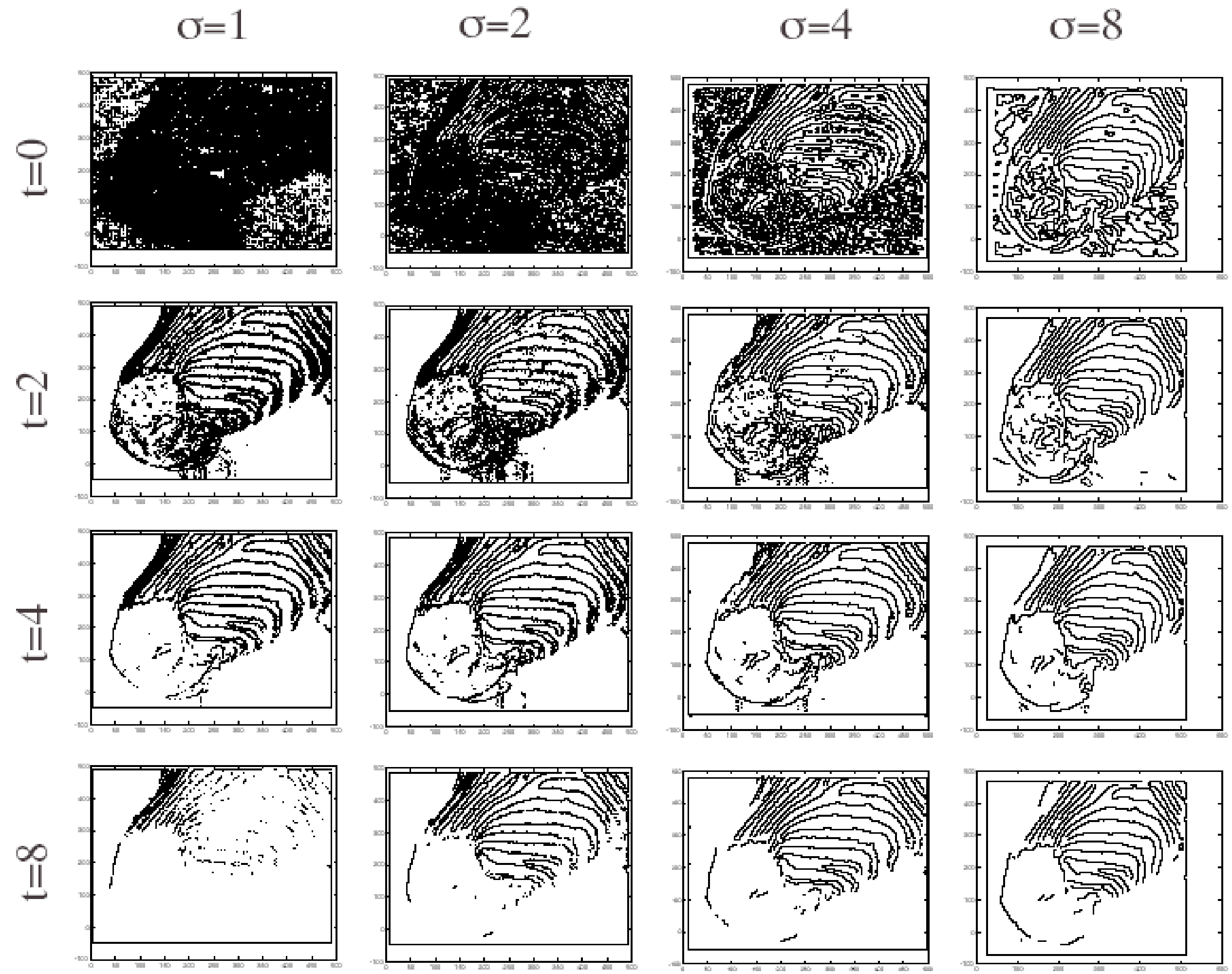


Figure 9.8. Zero crossings of the Laplacian of Gaussian for various scales and at various gradient magnitude thresholds. Each column shows a fixed scale, with t, the threshold on gradient magnitude increasing as one moves down (by a factor of two from image to image). Each row shows a fixed t, with scale increasing from σ one pixel to σ eight pixels, by factors of two. Notice that the fine scale, low threshold edges contain a quantity of detailed information that may or may not be useful (depending on one's interest in the hairs on the zebra's nose). As the scale increases, the detail is suppressed; as the threshold increases, small regions of edge drop out. No scale or threshold gives the outline of the zebra's head; all respond to its stripes, though as the scale increases, the narrow stripes on the top of the muzzle are no longer resolved.

Advantages/disadvantages of LoG

- **Advantages**
  - **Location:** ZCs are easier to find compared with peaks. Small variances have high precision while large ones are more robust.
  - **Robustness:** the second derivative is much less noise-sensitive when Gaussian smoothing is applied first.
- **Disadvantages**
  - The LoG filter is not oriented (isotropic), its response is composed of an average across an edge and one along the edge.
  - The ZCs are slightly displaced when the LoG is applied to objects having corners and thin line structures.