Lecture 27
Sampling for Monte Carlo Methods
ECEN 5283 Computer Vision

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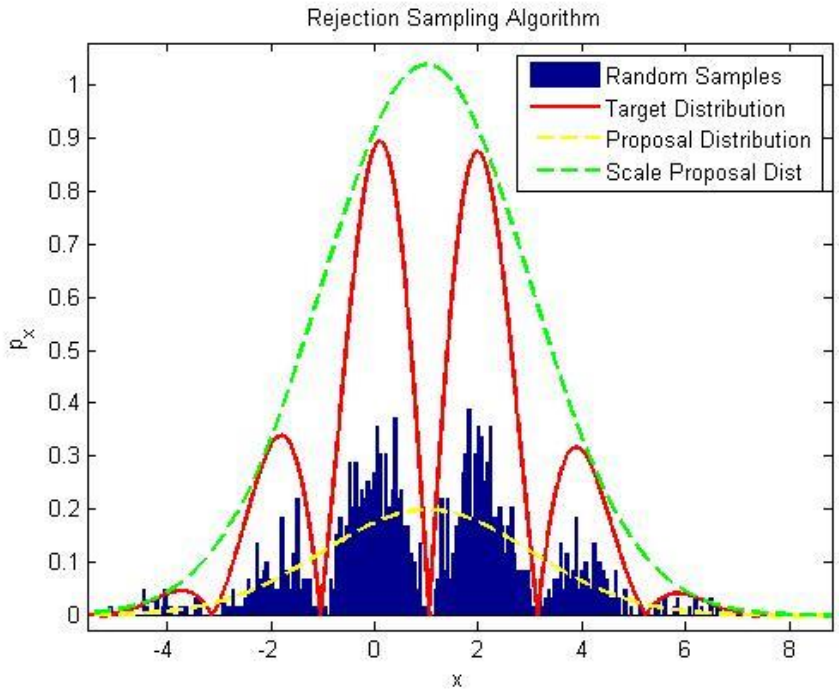
Goal

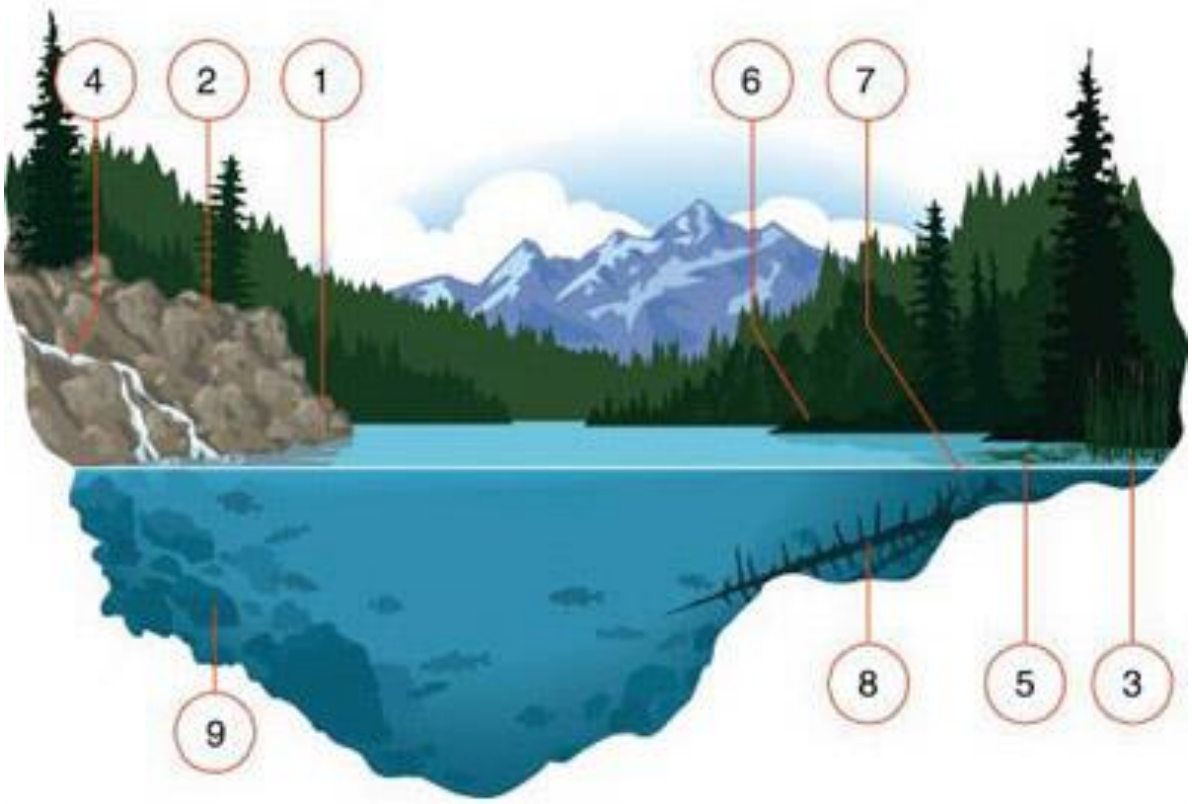
- To review the Monte Carlo methods.

- To discuss two direct sampling methods.
  - Important sampling
  - Rejection sampling

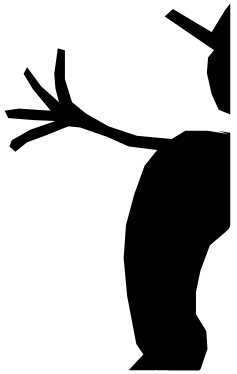


Monte Carlo Methods: Why?

- We want to develop a depth map for a big lake!
- The only thing we have is a boat and a sonar distance meter.

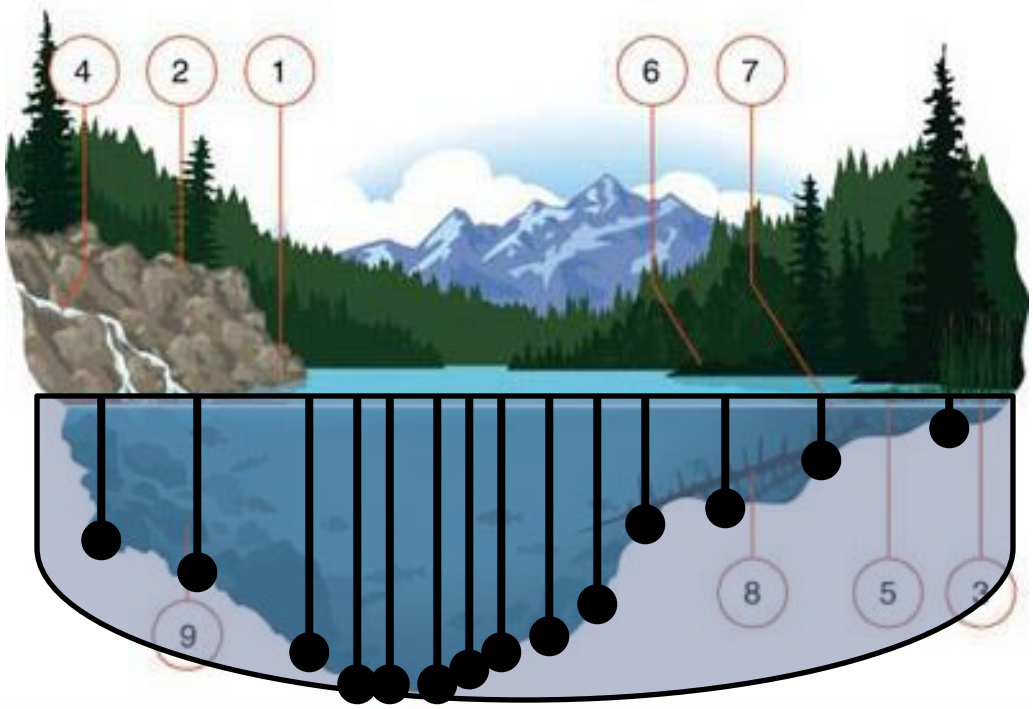


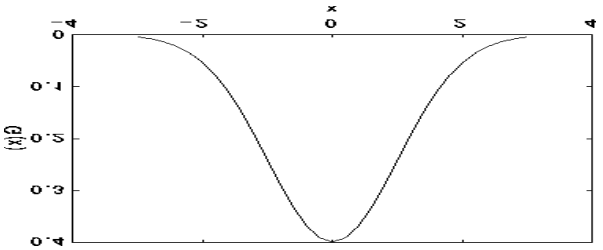
That is impossible!



Monte Carlo Methods: How?

- Let's find an ideal distribution where we can draw samples easily. Then we can evaluate the sample by going to the lake and measuring the depth…





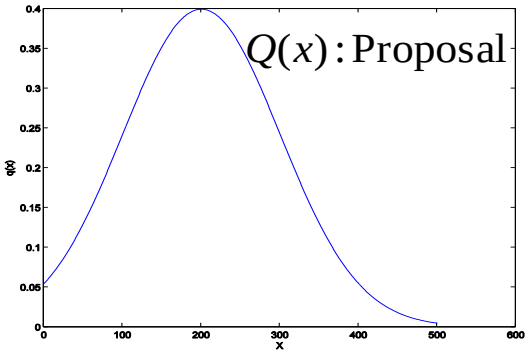
Objective and Approach

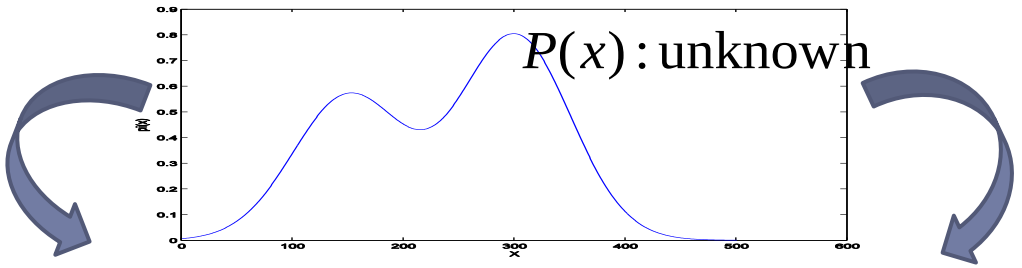
- We want to have a set of samples to represent an unknown distribution $P^*(x)$ (why do we care a distribution?).
  - What we don't know: How does $P^*(x)$ look like? That means we cannot directly sample from it.
  - What we know: Given a sample x_0, we can evaluate the density value of that sample, i.e., we know the value of $P^*(x_0)$.

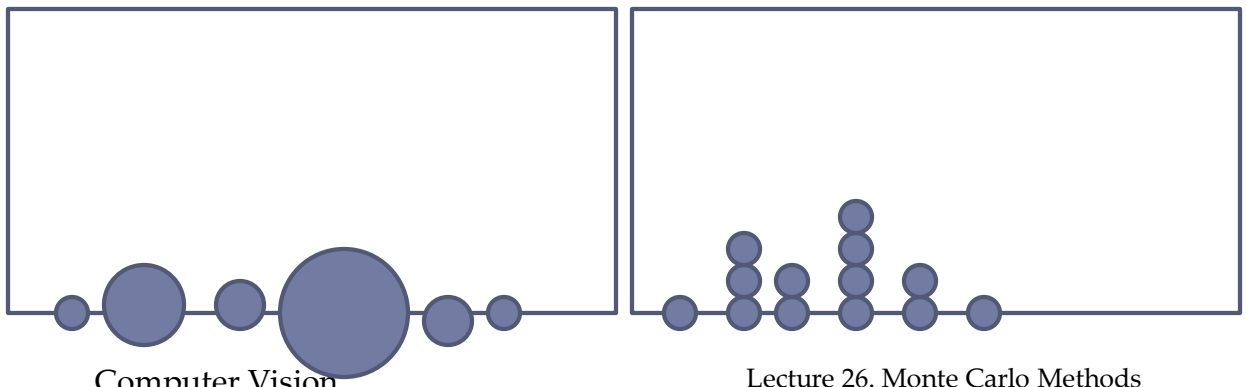
- It boils down to *how to efficiently generate a set of samples to approximate the underlying unknown distribution*.
  - A proposal is involved from which the samples can be drawn.
  - The samples will be evaluated, and then weighted, rejected or accepted.
    - Then we can have samples with different weights
    - Or we can have samples with the same weights after the accept/reject test

Two Major Issues of MC methods

- There are two major issues of implementing a MC method:
  - Find a proposal from which we can easily draw samples to approximate the unknown distribution.
  - Evaluate the samples to compute a weight for each sample or to keep some of good samples, so that the unknown distribution can be approximated by either weighted samples or samples of a same weight.







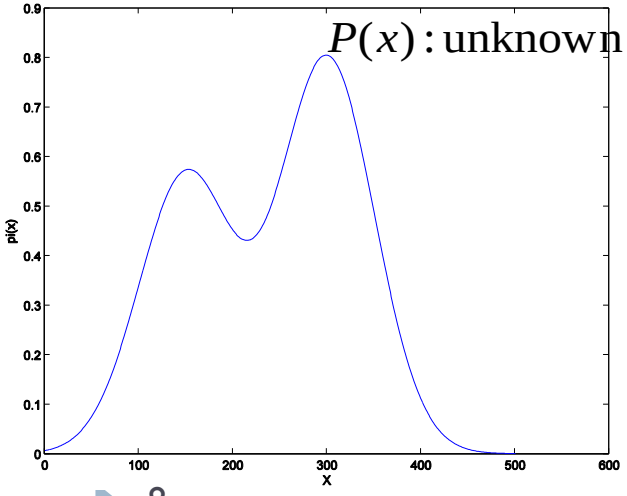
Sampling Methods

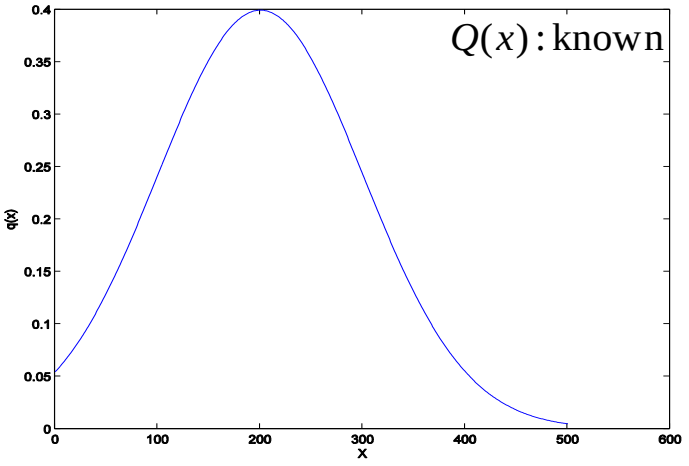
- Direct Sampling
  - Importance sampling
  - Rejection sampling
- Markov Chain Monte Carlo (MCMC) Sampling
  - Metropolis sampling
  - Gibbs sampling

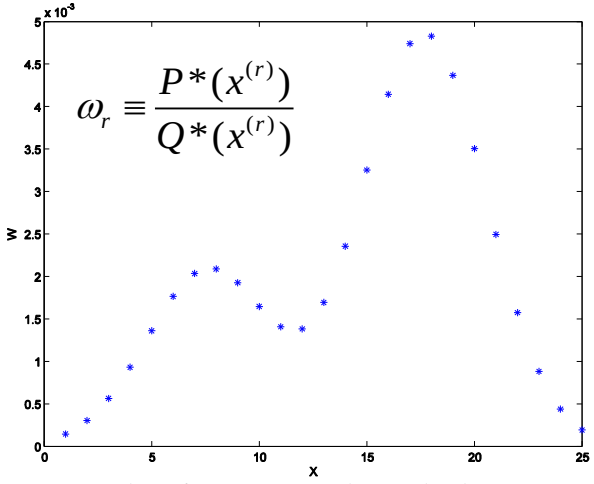
Importance Sampling

- **Importance sampling** uses weighted samples from a known distribution $Q(x)$ to approximate an unknown one $P(x)$.
  - Step 1: Generate samples from a proposal distribution $Q(x)$ rather than the distribution of interest.
  - Step 2: Weight each sample by computing the ratio between the values of unknown density $P(x)$ with that of the known density $Q(x)$.
  - Step 3: Use normalized samples to represent the unknown density.

$$P(x) \approx \omega_r \delta(x - x^{(r)})$$

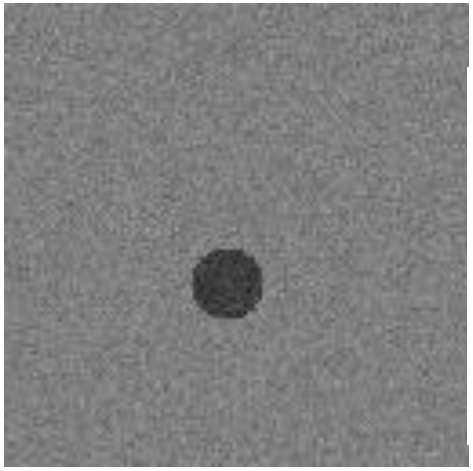


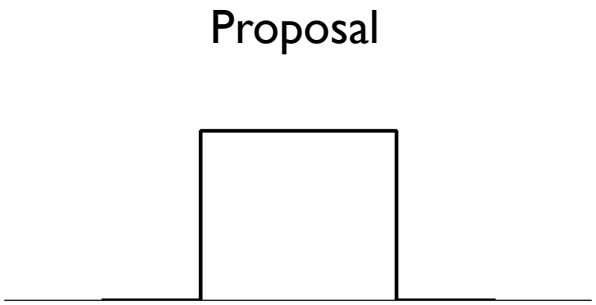


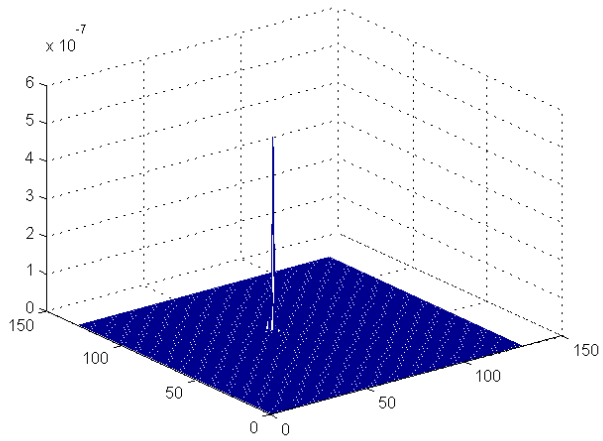


Importance Sampling Example

- For example, given a problem where we want to locate where is the object in an image, we need to answer the following two questions
  - What is the proposal function based on which we can draw samples as position hypotheses?
  - How to evaluate each sample drawn assuming a target template is available?

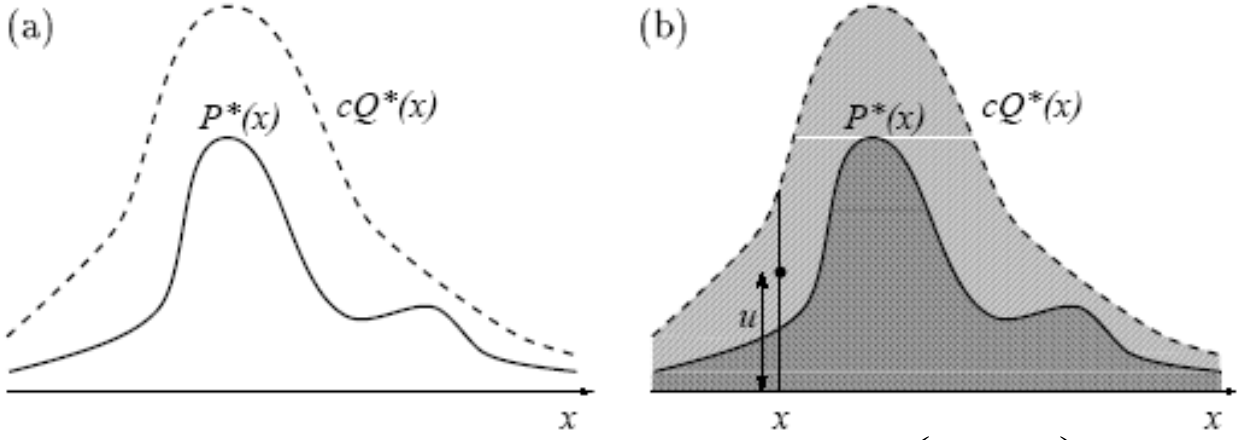






Rejection Sampling

- It is commonly called the acceptance-rejection method or "accept-reject algorithm". Try to reshape a known envelop distribution $Q(x)$ to look like the unknown one $P(x)$.

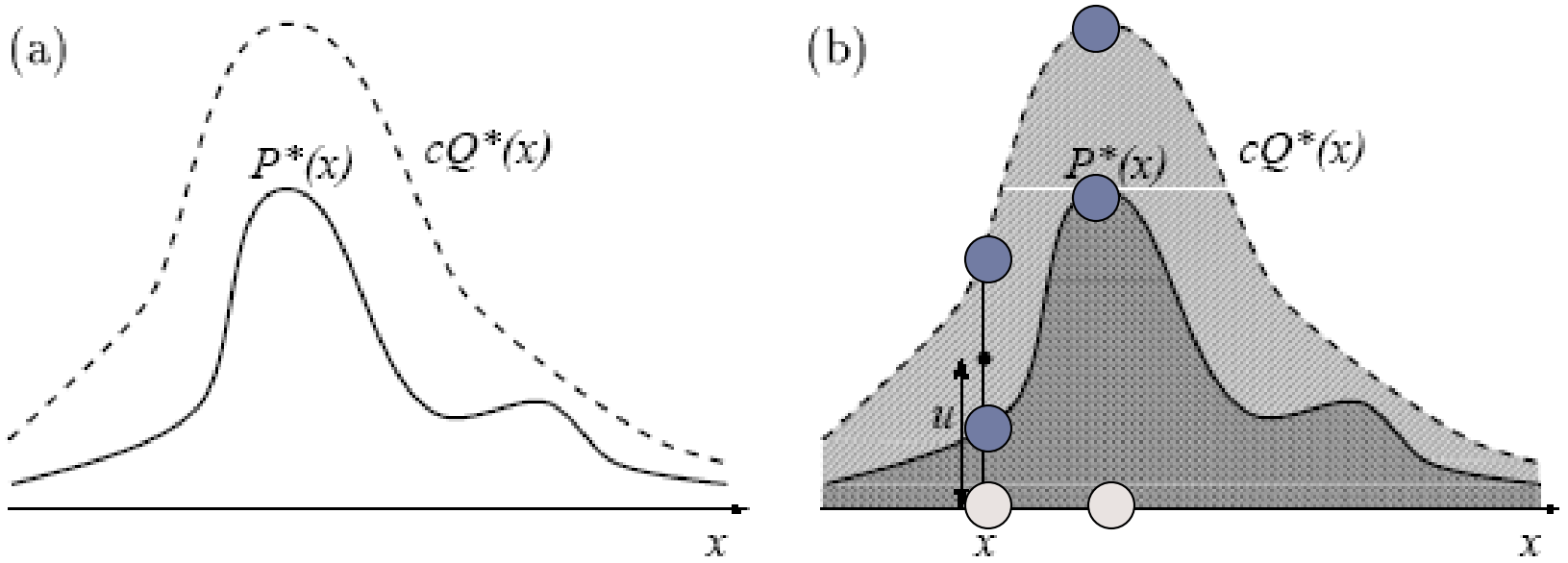


$$cQ(x) \geq P(x) \qquad \boldsymbol{c} > \max\left(\frac{\boldsymbol{P(x)}}{\boldsymbol{Q(x)}}\right)$$

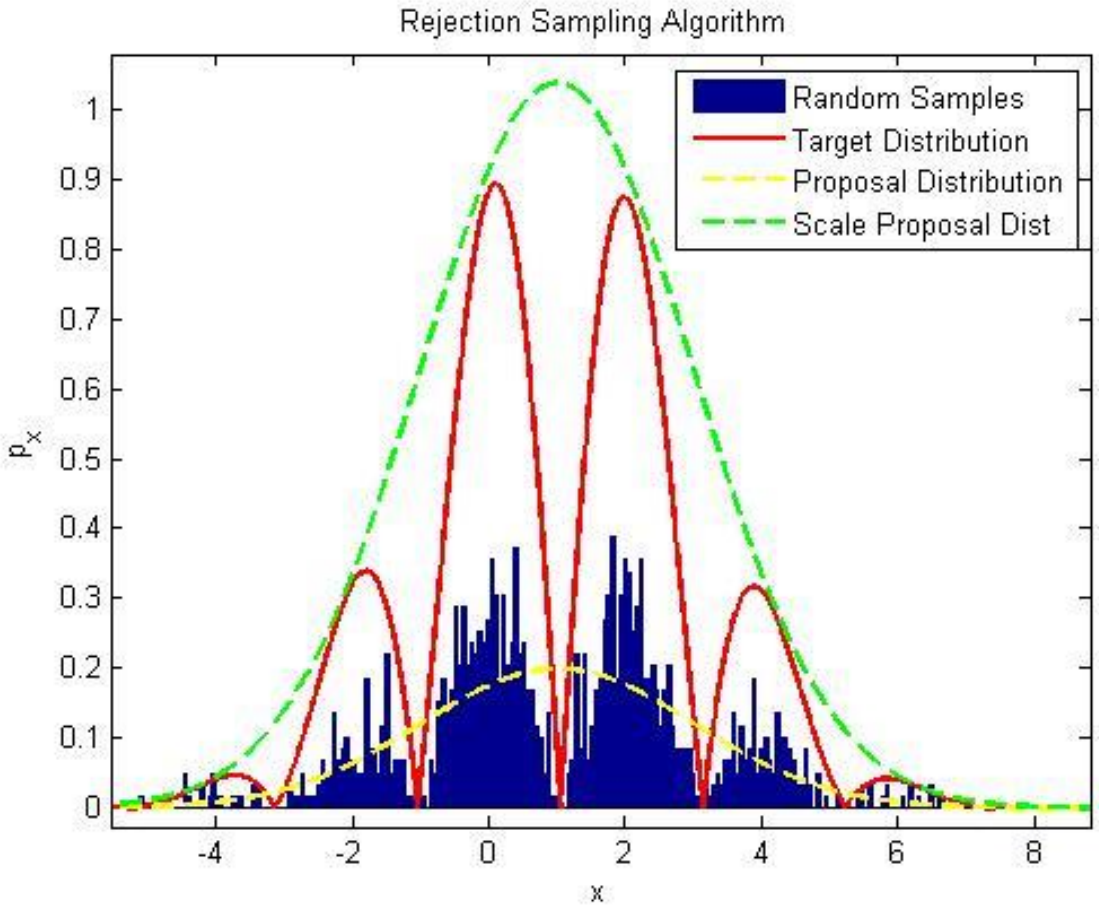
$$\text{Accept a sample with a probability of } \min\left(\frac{P(x)}{cQ(x)}, 1\right)$$

Rejection Sampling Algorithm

- Step 1: Generate a sample from Q(x)
- Step 2: Generate U from unit(0,1)
- Step 3: Accept the sample if $U < \frac{P(x)}{cQ(x)}$



Rejection Sampling Example



http://ct-cps.blogspot.com/2011/04/rejection-sampling-algorithm.html