Lecture 3.
Review of Euclidean Geometry
ECEN 5283 Computer Vision

Dr. Guoliang Fan
School of Electrical and Computer Engineering
Oklahoma State University

Goals

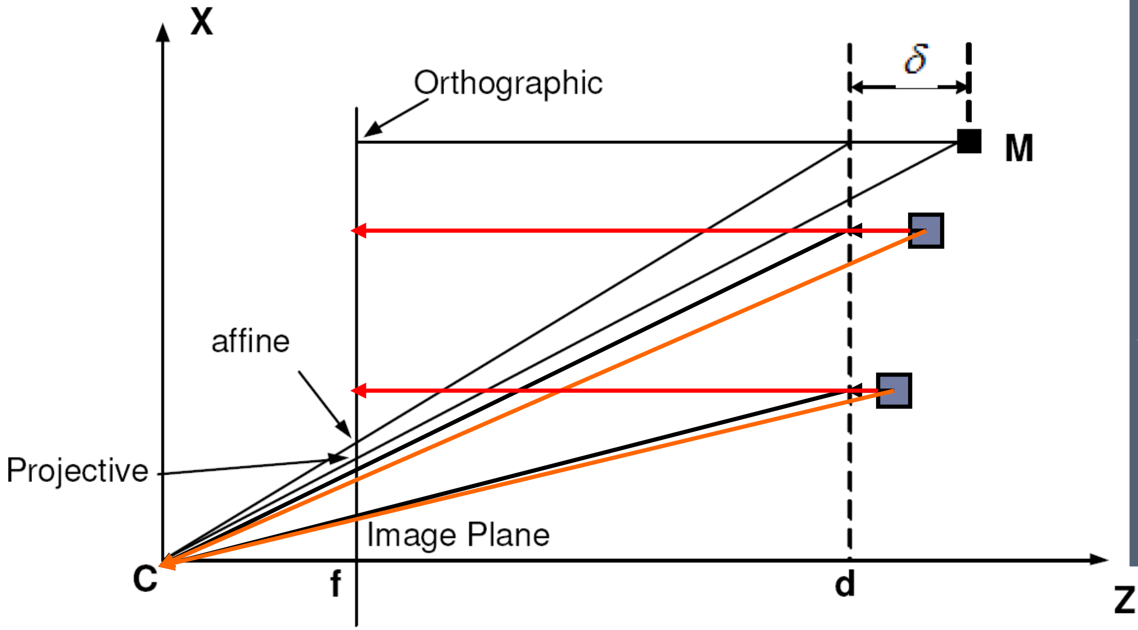
- To review three camera models.

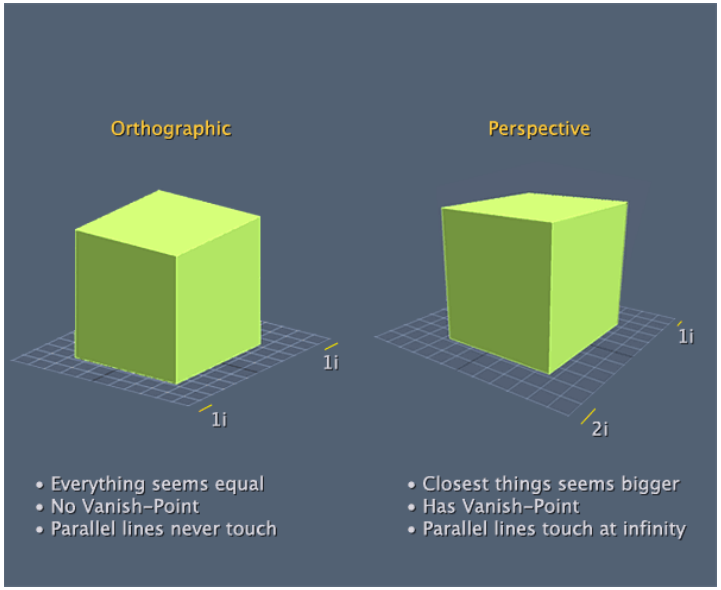
- To review Euclidean geometry that forms the foundation for geometric camera modeling.
  - Coordinate systems
  - Geometric definition of a plane
  - Homogeneous coordinate
  - Coordinate system changes

Euclid (Greek: Εὐκλείδης — Eukleidēs), fl. 300 BC, also known as Euclid of Alexandria, "The Father of Geometry" was a Greek mathematician.

Review of Camera Models

- Perspective projection is a standard camera model
- Affine projection is a simplified linear camera model
- Orthographic projection is an idealized camera model





Elements of Analytical Euclidean Geometry

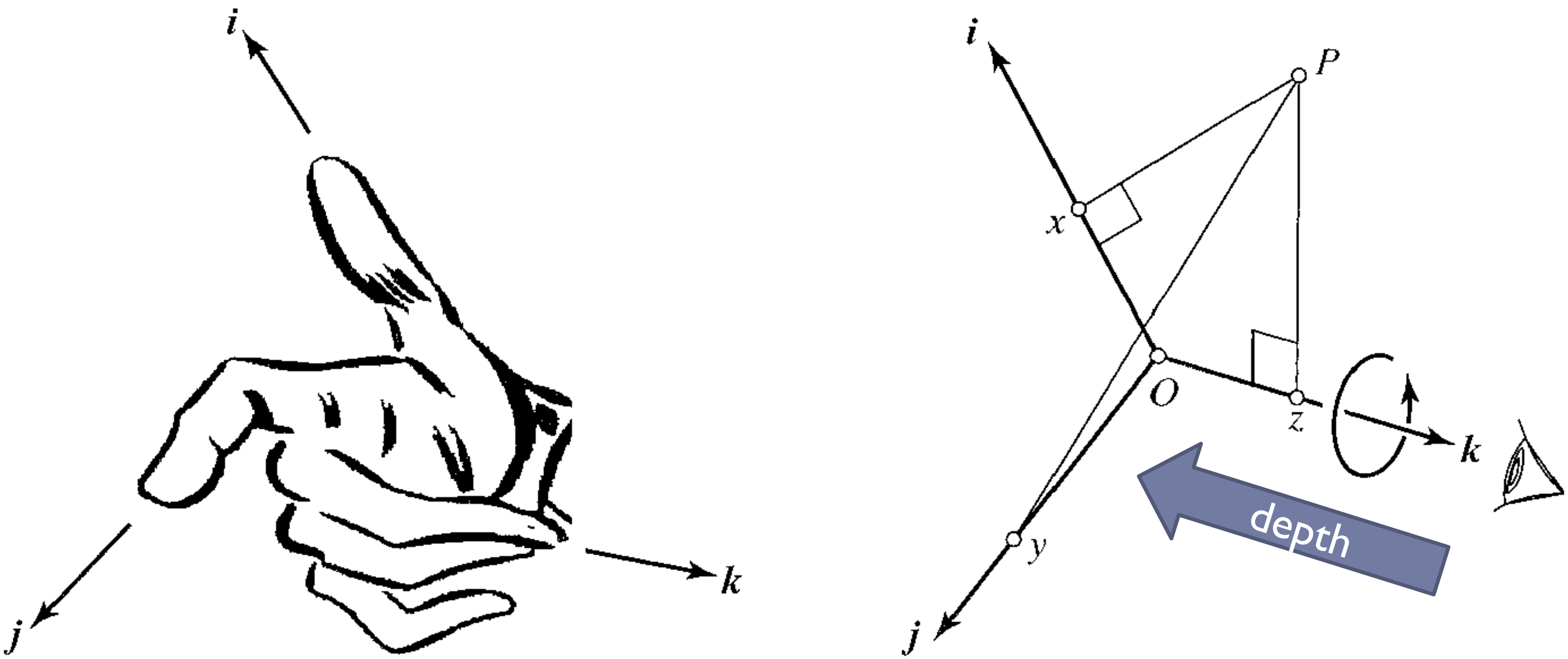
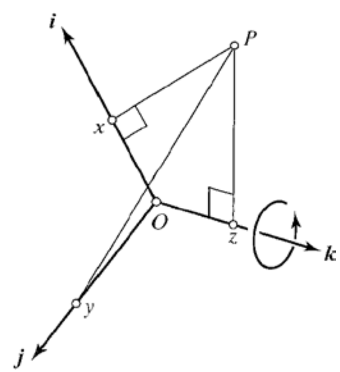


Figure 2.1 A right-handed coordinate system and the coordinates x, y, and z of a point P.

Coordinate Systems

- Orthonormal coordinate frame (F) is composed of an origin **O** in the physical 3-D Euclidean space E^3 and three basis vectors, ***i***, ***j***, ***k***, orthogonal to each other.
- The coordinates (x, y, z) of a point P in this frame is the (signed) length of the orthogonal projections of the vector $\overrightarrow{OP}$.

$$\begin{cases} x = \overrightarrow{OP} \cdot \mathbf{i} \\ y = \overrightarrow{OP} \cdot \mathbf{j} \\ z = \overrightarrow{OP} \cdot \mathbf{k} \end{cases} \Leftrightarrow \overrightarrow{OP} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}.$$



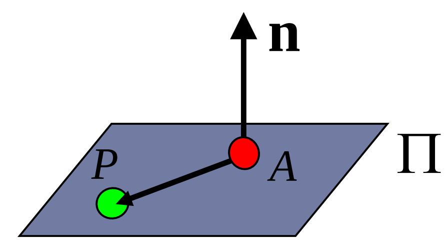
- The coordinate vector of the point P is defined in (F).

$$\mathbf{P} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \boldsymbol{R}^3$$

Geometric Definition of the Equation of a Plane

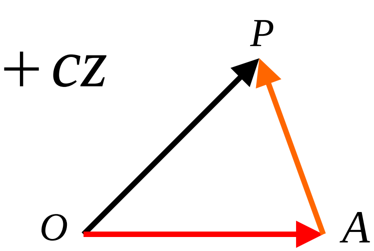
- Let's consider the plane Π, an arbitrary point A in Π, and a unit vector **n** perpendicular to the plane. The points *P* lying in Π are characterized by

$$\overrightarrow{AP} \cdot \mathbf{n} = 0$$



- In a coordinate system (*F*), where the coordinates of *P* are *x* , *y*, *z*, and the coordinates of **n** are *a*, *b*, *c*.

$$\text{Given } \overrightarrow{AP} = \overrightarrow{OP} - \overrightarrow{OA} \qquad \overrightarrow{OP} \cdot \mathbf{n} = ax + by + cz$$



$$\overrightarrow{AP} \cdot \mathbf{n} = 0 \Leftrightarrow \overrightarrow{OP} \cdot \mathbf{n} - \overrightarrow{OA} \cdot \mathbf{n} = 0$$

$$\rightarrow \boldsymbol{ax} + \boldsymbol{by} + \boldsymbol{cz} - \boldsymbol{d} = 0, \quad \text{where } d = \overrightarrow{OA} \cdot \mathbf{n} \text{ (what is } d\text{?)}$$

Example of Geometric Definition of the Equation of a Plane

$$ax + by + cz - d = 0$$

(Geometric definition of a plane)

$d = \overrightarrow{OA} \cdot \mathbf{n}$ (what is d?)

The (signed) distance from the origin to the plane.

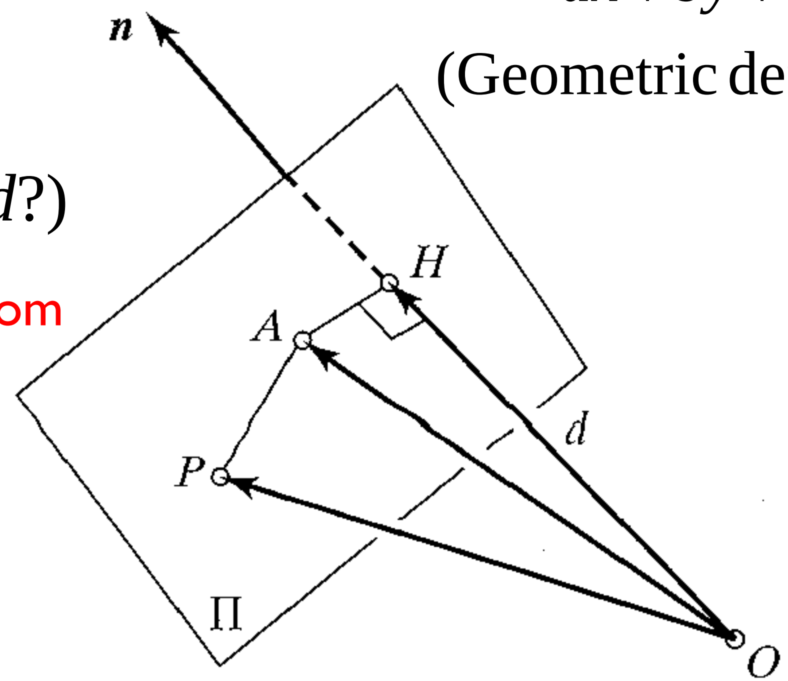


Figure 2.2 The geometric definition of the equation of a plane. The distance d between the origin and plane is reached at the point H where the normal vector passing through the origin pierces the plane.

Homogeneous Coordinate

- It is useful to use to *homogeneous coordinate* represent points, vectors, and planes.

$$ax + by + cz - d = 0 \rightarrow (a, b, c, -d)\begin{pmatrix} x \\ y \\ z \\ 1 \end{pmatrix} = 0$$

$$\text{or } \Pi \cdot P = 0 \quad \text{where } \Pi = \begin{pmatrix} a \\ b \\ c \\ -d \end{pmatrix} \text{ and } P = \begin{pmatrix} x \\ y \\ z \\ 1 \end{pmatrix}.$$

Homogeneous coordinate vector

Homogenous Coordinate Example

- Let us consider a sphere *S* radius *R* centered at the origin. A necessary and sufficient condition for the point P with coordinates, *x*, *y*, *z* to belong to *S* is that

$$x^2 + y^2 + z^2 = R^2$$

which is equivalent to

$$(x, y, z, 1)\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -R^2 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \\ 1 \end{pmatrix} = 0$$

Coordinate System Changes

- When several different coordinate systems are considered at the same time, we denote the coordinate vector of the point *P* in the frame *F* as

$$ {}^{F}P = {}^{F}\overrightarrow{OP} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \Leftrightarrow \overrightarrow{OP} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}. $$

- Let us consider two coordinate systems (two frames)

$$ (A) = (O_A, \mathbf{i}_A, \mathbf{j}_A, \mathbf{k}_A) $$

$$ (B) = (O_B, \mathbf{i}_B, \mathbf{j}_B, \mathbf{k}_B) $$

  - **Question**: How to express ${}^{B}P$ as a function of ${}^{A}P$.

Coordinate System Changes: Pure Translation

- Case I: $O_A \neq O_B, \mathbf{i}_A = \mathbf{i}_B, \mathbf{j}_A = \mathbf{j}_B, \mathbf{k}_A = \mathbf{k}_B.$

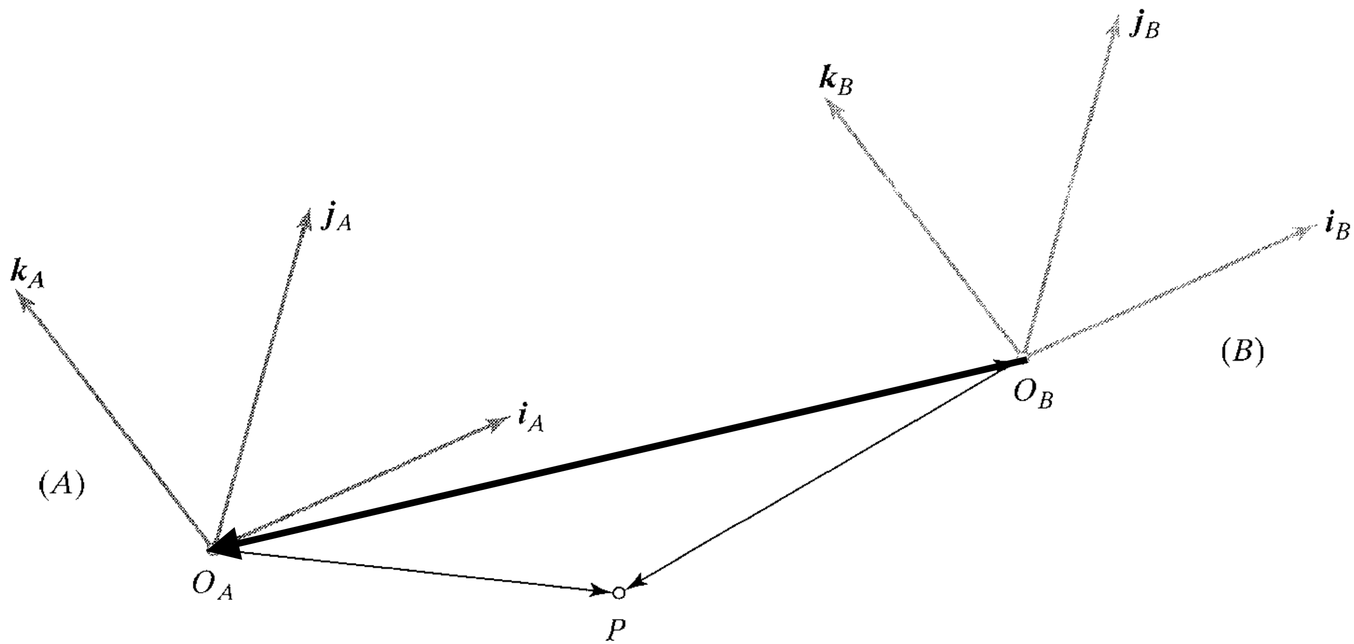


Figure 2.3 Change of coordinates between two frames: pure translation.

$$\overrightarrow{O_B P} = \overrightarrow{O_B O_A} + \overrightarrow{O_A P} \rightarrow {}^{B}P = {}^{B}O_A + {}^{A}P$$

Coordinate System Changes: Pure Rotation

- Case II: $O_A = O_B = O, \mathbf{i}_A \neq \mathbf{i}_B, \mathbf{j}_A \neq \mathbf{j}_B, \mathbf{k}_A \neq \mathbf{k}_B.$

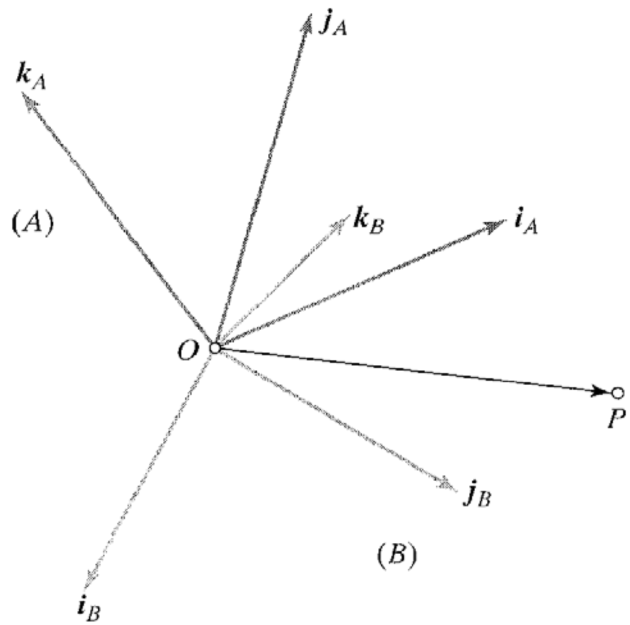


Figure 2.4 Change of coordinates between two frames: pure rotation.

$$ {}^B P = {}^B_A R \, {}^A P $$

Rotation matrix

$$ {}^B_A R = \begin{pmatrix} {}^B\mathbf{i}_A & {}^B\mathbf{j}_A & {}^B\mathbf{k}_A \end{pmatrix} $$

$$ {}^B_A \mathcal{R} \stackrel{\text{def}}{=} \begin{pmatrix} \boxed{\begin{matrix} i_A \cdot i_B \\ i_A \cdot j_B \\ i_A \cdot k_B \end{matrix}} & \boxed{\begin{matrix} j_A \cdot i_B \\ j_A \cdot j_B \\ j_A \cdot k_B \end{matrix}} & \boxed{\begin{matrix} k_A \cdot i_B \\ k_A \cdot j_B \\ k_A \cdot k_B \end{matrix}} \end{pmatrix}. $$

$$ {}^B_A R = {}^A_B R^T = \left({}^A_B R \right)^{-1} $$

Unitary matrix

$$ \mathbf{U}^T = \mathbf{U}^{-1} \qquad \det(\mathbf{U}) = 1 $$

Coordinate System Changes: Pure Rotation (Cont'd)

- Case III: $O_A = O_B = O, \mathbf{k}_A = \mathbf{k}_B = \mathbf{k}.$

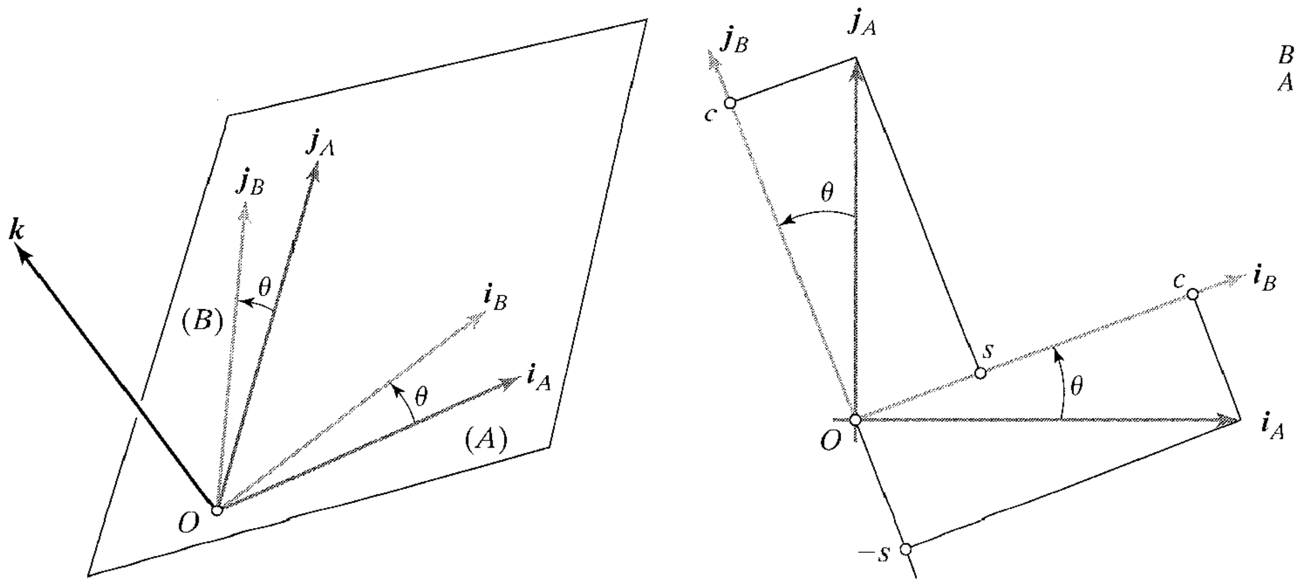


Figure 2.5 Two coordinate frames separated by a rotation of angle θ about their common $\boldsymbol{k}$ basis vector. As shown in the right of the figure, $\boldsymbol{i}_A = c\boldsymbol{i}_B - s\boldsymbol{j}_B$ and $\boldsymbol{j}_A = s\boldsymbol{i}_B + c\boldsymbol{j}_B$, where $c = \cos\theta$ and $s = \sin\theta$.

$$
{}^B_A\mathcal{R} \stackrel{\text{def}}{=} \begin{pmatrix} \boldsymbol{i}_A \cdot \boldsymbol{i}_B & \boldsymbol{j}_A \cdot \boldsymbol{i}_B & \boldsymbol{k}_A \cdot \boldsymbol{i}_B \\ \boldsymbol{i}_A \cdot \boldsymbol{j}_B & \boldsymbol{j}_A \cdot \boldsymbol{j}_B & \boldsymbol{k}_A \cdot \boldsymbol{j}_B \\ \boldsymbol{i}_A \cdot \boldsymbol{k}_B & \boldsymbol{j}_A \cdot \boldsymbol{k}_B & \boldsymbol{k}_A \cdot \boldsymbol{k}_B \end{pmatrix}.
$$

$$
\downarrow
$$

$$
{}^B_AR = \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}
$$

$$
{}^BP = {}^B_AR\,{}^AP
$$

Coordinate System Changes: Rigid Transform

- Case IV: $O_A \neq O_B, \mathbf{i}_A \neq \mathbf{i}_B, \mathbf{j}_A \neq \mathbf{j}_B, \mathbf{k}_A \neq \mathbf{k}_B.$
  - When the origins and basis vectors of the two coordinate systems are different, we say the frames are separated by a *general rigid transform*

$^{B}O_{A}$

$$^{B}P = {}^{C}_{A}R\,{}^{A}P + {}^{B}O_{C} = {}^{B}_{A}R\,{}^{A}P + {}^{B}O_{A}$$

Rotation first then shift

$O_C = O_A, \mathbf{i}_C = \mathbf{i}_B, \mathbf{j}_C = \mathbf{j}_B, \mathbf{k}_C = \mathbf{k}_B$

(intermediate frame for the first transform)

$$^{B}P = {}^{B}_{D}R({}^{A}P + {}^{D}O_{A}) = {}^{B}_{A}R\,{}^{A}P + {}^{B}_{D}R\,{}^{D}O_{A}$$

Shift first then ration

$O_D = O_B, \mathbf{i}_D = \mathbf{i}_A, \mathbf{j}_D = \mathbf{j}_A, \mathbf{k}_D = \mathbf{k}_A$

(intermediate frame for the first transform)

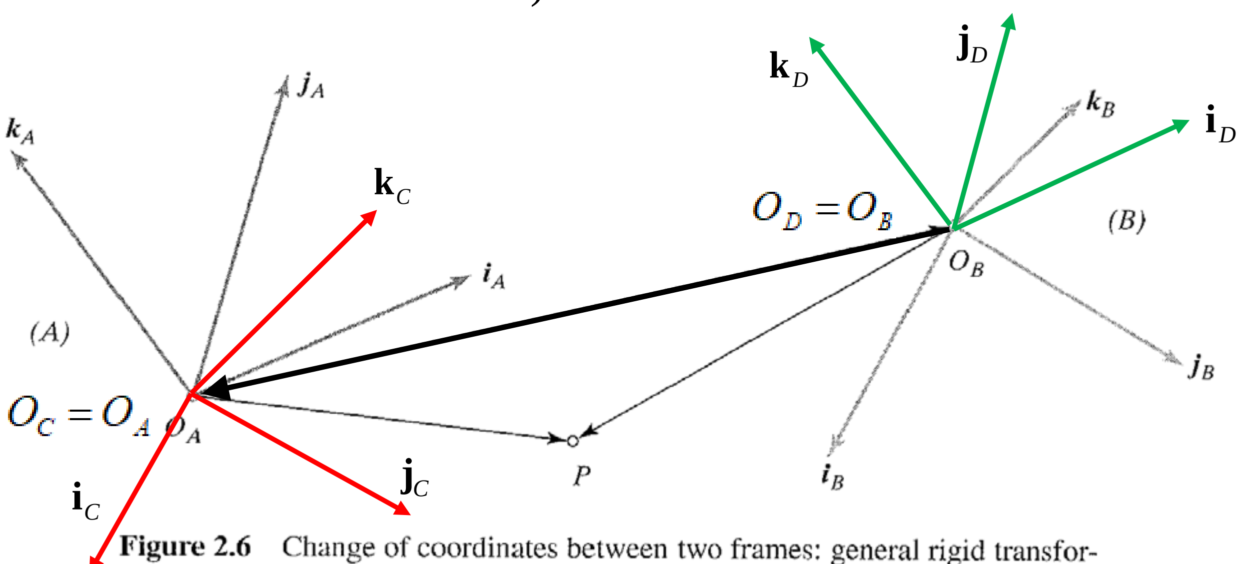


Figure 2.6 Change of coordinates between two frames: general rigid transformation.

Coordinate System Changes: Rigid Transform

- Rigid transformation using homogeneous coordinates

$$ {}^{B}P = {}^{B}_{A}R\,{}^{A}P + {}^{B}O_{A} $$

$$ \begin{pmatrix} {}^{B}P \\ 1 \end{pmatrix} = \begin{pmatrix} {}^{B}_{A}R & {}^{B}O_{A} \\ \mathbf{0}^{T} & 1 \end{pmatrix} \begin{pmatrix} {}^{A}P \\ 1 \end{pmatrix} $$

$$ \begin{pmatrix} {}^{B}P \\ 1 \end{pmatrix} = {}^{B}_{A}T \begin{pmatrix} {}^{A}P \\ 1 \end{pmatrix}, \quad \text{where } {}^{B}_{A}T = \begin{pmatrix} {}^{B}_{A}R & {}^{B}O_{A} \\ \mathbf{0}^{T} & 1 \end{pmatrix} $$