Lecture 6.
Review of Linear Optimization
ECEN 5283 Computer Vision

Dr. Guoliang Fan
School of Electrical and Computer Engineering
Oklahoma State University

Goals

- To review geometric camera modeling.

- To review a simple linear optimization technique that will be used for camera calibration.

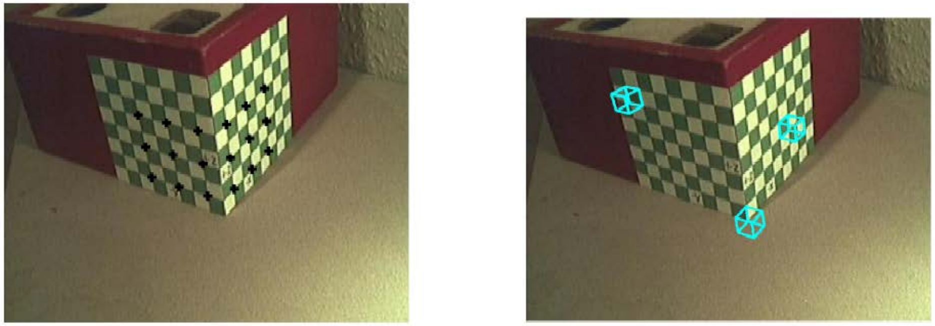
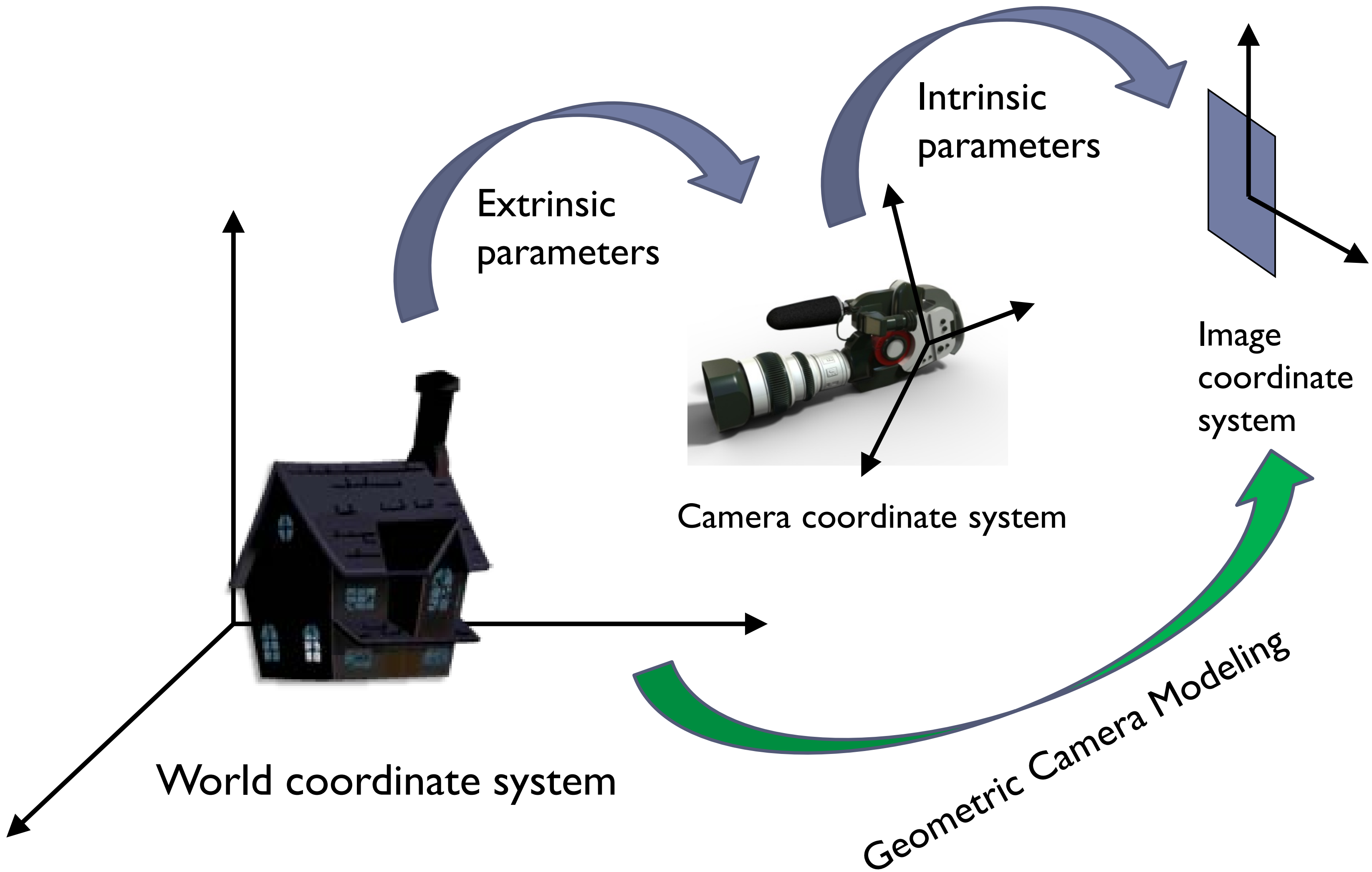


Figure 1: An image of a calibration rig, left with calibration points, right with cubes to check the calibration.

Geometric Camera Modeling: Intrinsic and Extrinsic Parameters



Camera Projection Matrix

- A projection matrix is written explicitly as a function of both intrinsic and extrinsic parameters as follows
  - Five intrinsic parameters $\alpha,\ \beta,\ u_0,\ v_0, \theta$
  - Six extrinsic ones (three angles and three coordinates of t).

$$\mathbf{p} = \frac{1}{z} M\mathbf{P} \quad \text{where } M = K\begin{pmatrix} R & t \end{pmatrix} \quad K = \begin{pmatrix} \alpha & -\alpha\cot\theta & u_0 \\ 0 & \dfrac{\beta}{\sin\theta} & v_0 \\ 0 & 0 & 1 \end{pmatrix} \quad R = \begin{pmatrix} \mathbf{r}_1^T \\ \mathbf{r}_2^T \\ \mathbf{r}_3^T \end{pmatrix} \text{ and } t = \begin{pmatrix} t_x \\ t_y \\ t_z \end{pmatrix}$$

$$M = K\begin{pmatrix} R & t \end{pmatrix} = \begin{pmatrix} KR & Kt \end{pmatrix} = \begin{pmatrix} \alpha\mathbf{r}_1^T - \alpha\cot\theta\mathbf{r}_2^T + u_0\mathbf{r}_3^T & \alpha t_x - \alpha\cot\theta t_y + u_0 t_z \\ \dfrac{\beta}{\sin\theta}\mathbf{r}_2^T + v_0\mathbf{r}_3^T & \dfrac{\beta}{\sin\theta}t_y + v_0 t_z \\ \mathbf{r}_3^T & t_z \end{pmatrix}_{3\times 4}$$

Homogeneous Systems and Eigenvalue Problems

- Let us now consider a linear equation system which is a homogeneous equation in **x** as

$$\begin{cases} u_{11}x_1 + u_{12}x_2 + \cdots + u_{1q}x_q = 0 \\ u_{21}x_1 + u_{22}x_2 + \cdots + u_{2q}x_q = 0 \\ \quad \cdots\cdots \\ u_{p1}x_1 + u_{p2}x_2 + \cdots + u_{pq}x_q = 0 \end{cases} \Leftrightarrow \mathbf{Ux} = 0$$

p conditions
q variables

$$\mathbf{U} = \begin{pmatrix} u_{11} & u_{12} & .. & u_{1q} \\ u_{21} & u_{22} & ... & u_{2q} \\ : & : & : & : \\ u_{p1} & u_{p2} & ... & u_{pq} \end{pmatrix}$$

- In this context, minimizing $E = |\boldsymbol{U}\mathbf{x}|^2$ only makes sense when some additional constraint, such as $|\mathbf{x}|^2 = 1$.

Homogeneous Systems and Eigenvalue Problems (Cont'd)

- We can re-write the error as

$$E(\mathbf{x}) = |\mathbf{U}\mathbf{x}|^2 = (\mathbf{U}\mathbf{x})^T \mathbf{U}\mathbf{x} = \mathbf{x}^T (\mathbf{U}^T \mathbf{U})\mathbf{x}$$

$$\mathbf{U} = \begin{pmatrix} u_{11} & u_{12} & .. & u_{1q} \\ u_{21} & u_{22} & ... & u_{2q} \\ : & : & : & : \\ u_{p1} & u_{p2} & ... & u_{pq} \end{pmatrix}$$

- The q x q matrix $\mathbf{U}^T\mathbf{U}$ is symmetric positive semi-definite (i.e., its eigenvalues are all non-negative.).
- $\mathbf{U}^T\mathbf{U}$ can be diagonalized in an orthonormal basis of eigenvectors associated with the eigenvalues as

$$\mathbf{U}^T \mathbf{U} = \mathbf{Q}\mathbf{\Lambda}\,\mathbf{Q}^T$$

$$\mathbf{Q} = (\mathbf{e}_1 \quad \mathbf{e}_2 \quad ... \quad \mathbf{e}_q)$$

$$\mathbf{\Lambda} = \begin{bmatrix} \lambda_1^2 & 0 & 0 & 0 \\ 0 & \lambda_2^2 & 0 & 0 \\ 0 & 0 & ... & 0 \\ 0 & 0 & 0 & \lambda_q^2 \end{bmatrix}$$

$$0 \le \lambda_1^2 \le \lambda_2^2 ... \le \lambda_q^2$$

$$\mathbf{e}_i^T (\mathbf{U}^T \mathbf{U})\mathbf{e}_i = \mathbf{e}_i^T (\mathbf{Q}\mathbf{\Lambda}\,\mathbf{Q}^T)\mathbf{e}_i$$

$$\mathbf{Q} = (\mathbf{e}_1 \quad \mathbf{e}_2 \quad ... \quad \mathbf{e}_q)$$

$$= (0 \quad ... \quad 1 \quad ... \quad 0)\mathbf{\Lambda}\begin{pmatrix} 0 \\ : \\ 1 \\ : \\ 0 \end{pmatrix}$$

$$= \lambda_i^2$$

Homogeneous Systems and Eigenvalue Problems (Cont'd)

- We can write any unit vector as

$$\mathbf{x} = \mu_1 \mathbf{e}_1 + \ldots + \mu_q \mathbf{e}_q \quad \text{with } \mu_1^{\ 2} + \mu_2^{\ 2} + \ldots + \mu_q^{\ 2} = 1.$$

$$E(\mathbf{x}) = \mathbf{x}^T \left(\mathbf{U}^T \mathbf{U} \right) \mathbf{x} = (\mu_1 \mathbf{e}_1 + \ldots + \mu_q \mathbf{e}_q)^T (\mathbf{U}^T \mathbf{U})(\mu_1 \mathbf{e}_1 + \ldots + \mu_q \mathbf{e}_q)$$

$$= (\mu_1 \mathbf{e}_1 + \ldots + \mu_q \mathbf{e}_q)^T (\mathbf{Q} \boldsymbol{\Lambda} \, \mathbf{Q}^T)(\mu_1 \mathbf{e}_1 + \ldots + \mu_q \mathbf{e}_q)$$

$$= \lambda_1^2 \mu_1^2 + \lambda_2^2 \mu_2^2 + \ldots. + \lambda_q^2 \mu_q^2 \quad \left((\mu_i \mathbf{e}_i)^T (\mathbf{Q} \boldsymbol{\Lambda} \, \mathbf{Q}^T)(\mu_i \mathbf{e}_i) = \lambda_i^2 \mu_i^2 \right)$$

$$\geq \lambda_1^2 (\mu_1^2 + \cdots + \mu_q^2) = \lambda_1^2$$

- That means the lower bound of *E* is the smallest eigenvalue.

Solution

- It is shown that the lower bound of $E(\mathbf{x})$ is determined by the minimum eigenvalue, i.e.,

$$E(\mathbf{x}) = \left|\mathbf{U}\mathbf{x}\right|^2 = \mathbf{x}^T(\mathbf{U}^T\mathbf{U})\mathbf{x} \geq \lambda_1^2$$

and when $\mathbf{x}$ is chosen to be the eigenvector that is associated with the smallest eigenvalue, i.e.,

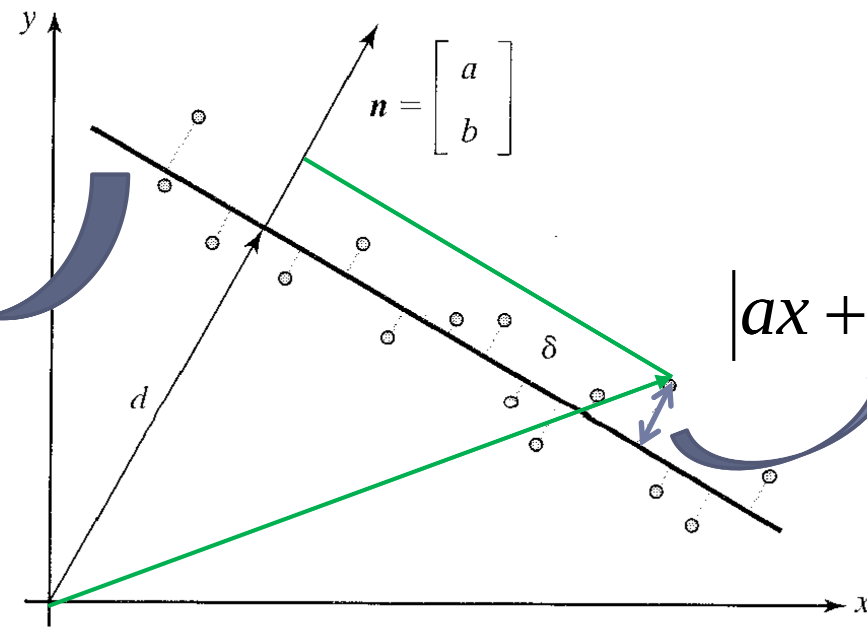
$$E(\mathbf{x} = \mathbf{e}_1) = \lambda_1^2$$

- The solution that minimizes E is the eigenvector $\mathbf{e}_1$ that is associated with the minimum eigenvalue.

Example: Fitting a line to points in a plane (Cont'd)

- Consider n points in a plane in a fixed coordinate system. What is the straight line best fit these points?

$$ax + by = d$$



$$|ax + by - d|$$

Figure 3.2 The line that best fits n points in the plane can be defined as the line δ that minimizes the mean-squared perpendicular distance to these points (i.e., in this diagram, the mean-squared length of the short parallel line segments joining δ to the points).

Example: Fitting a line to points in a plane (Cont'd)

- A line with unit normal $\mathbf{n} = (a, b)^T$, lying at distant *d* from the origin is $ax + by = d$.
- The perpendicular distance between a point with coordinate $(x, y)^T$, and this line is $|ax + by - d|$.
- We can therefore use the error measure as

$$E(a,b,d) = \sum_{i=1}^{n} (ax_i + by_i - d)^2$$

We may want to find *d* first.

- Thus, the line-fitting problem reduces to the minimization of E with respect to *a*, *b*, and *d* under the constraint

$$a^2 + b^2 = 1.$$

Example: Fitting a line to points in a plane (Cont'd)

- Differentiating E with respect to *d* shows that

$$E(a,b,d)=\sum_{i=1}^{n}(ax_i+by_i-d)^2$$

$$\frac{\partial E}{\partial d}=-2\sum_{i=1}^{n}(ax_i+by_i-d)=0$$

$$\sum_{i=1}^{n}(ax_i+by_i-d)=0 \Rightarrow \sum_{i=1}^{n}ax_i+by_i=nd$$

$$d=a\overline{x}+b\overline{y},\ \text{where } \overline{x}=\frac{1}{n}\sum_{i=1}^{n}x_i \ \text{ and } \overline{y}=\frac{1}{n}\sum_{i=1}^{n}y_i.$$

Example: Fitting a line to points in a plane (Cont'd)

- Substituting this expression for *d* in the definition of *E*

$$d = a\bar{x} + b\bar{y} \quad \rightarrow \quad E(a,b,d) = \sum_{i=1}^{n}(ax_i + by_i - d)^2 = \sum_{i=1}^{n}\left((ax_i + by_i) - (a\bar{x} + b\bar{y})\right)^2$$

$$E = \sum_{i=1}^{n}\left(a(x_i - \bar{x}) + b(y_i - \bar{y})\right)^2 = \left|\mathbf{Un}\right|^2$$

$$\text{where } \mathbf{U} = \begin{pmatrix} x_1 - \bar{x} & y_1 - \bar{y} \\ \cdots & \cdots \\ x_n - \bar{x} & y_n - \bar{y} \end{pmatrix}_{n\times 2} \text{ and } \mathbf{n} = \begin{pmatrix} a \\ b \end{pmatrix}.$$

Covariance matrix

- The optimal solution of **n** is the eigenvector of $\mathbf{U}^T\mathbf{U}$ that is associated with the smallest eigenvalue.

Example: Fitting a line to points in a plane (Cont'd)

- Can we do it in a simpler way?

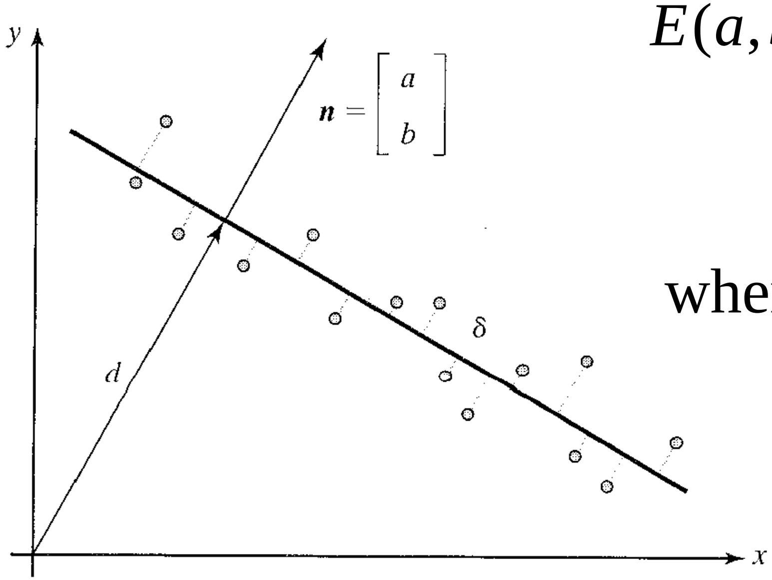


Figure 3.2 The line that best fits n points in the plane can be defined as the line δ that minimizes the mean-squared perpendicular distance to these points (i.e., in this diagram, the mean-squared length of the short parallel line segments joining δ to the points).

$$E(a,b,d) = \sum_{i=1}^{n} (ax_i + by_i - d)^2 = |\mathbf{Un}|^2$$

$$\text{where } \mathbf{U} = \begin{pmatrix} x_1 & y_1 & -1 \\ \dots & \dots & \dots \\ x_n & y_n & -1 \end{pmatrix}_{n\times 3} \text{ and } \mathbf{n} = \begin{pmatrix} a \\ b \\ d \end{pmatrix}.$$

The optimal solution of **n** is the eigenvector of $\mathbf{U}^T\mathbf{U}$ that is associated with the smallest eigenvalue.