

# Symmetric Monoidal $\infty$ -Categories

- Note: we will call  $(\infty, 1)$ -categories  $\infty$ -categories. Sorry Tom, it's just convenient.
- Main references: [4], [5], [2], [1], [3], [6]. How does [6, Example 01UB] work?

## 1 Pseudofunctors, opfibrations, and symmetric monoidal categories

### 1.1 Pseudofunctors to **Cat** are Grothendieck opfibrations

- Grothendieck op-fibrations correspond to pseudofunctors to **Cat**.

The below gives an example to complement [2].

Example of a pseudofunctor to **Cat**: Let  $\mathcal{C}$  be a category with pullbacks. Recall that for a map  $f : C \rightarrow D$  in  $\mathcal{C}$ , we define a pullback functor

$$\begin{aligned} f^* : \mathcal{C}_{/D} &\rightarrow \mathcal{C}_{/C}, \\ (h : X \rightarrow D) &\mapsto (f^*h : P \rightarrow C), \end{aligned}$$

where we have formed a pullback

$$\begin{array}{ccc} P & \xrightarrow{h^*f} & X \\ f^*h \downarrow & & \downarrow h \\ C & \xrightarrow{f} & D \end{array}$$

in  $\mathcal{C}$ . For any map

$$\begin{array}{ccc} X & \xrightarrow{\phi} & X' \\ & \searrow h & \swarrow h' \\ & D & \end{array}$$

from  $h$  to  $h'$  in  $\mathcal{C}_{/D}$ , we define  $f^*\phi$  to be the unique map making the diagram below commute.

$$\begin{array}{ccccc} X & \xrightarrow{\phi} & X' & & \\ & \searrow h & \swarrow h' & & \\ & D & & & \\ & \uparrow f & & & \\ P & \xrightarrow{f^*\phi} & P' & & \\ & \searrow f^*h & \swarrow f^*h' & & \\ & C & & & \end{array}$$

Now, we may wish to define a functor

$$F : \mathcal{C}^{\text{op}} \rightarrow \mathbf{Cat},$$

$$C \mapsto \mathcal{C}_{/C},$$

which sends a map  $f : C \rightarrow D$  in  $\mathcal{C}$  to a pullback functor  $f^* : \mathcal{C}_{/D} \rightarrow \mathcal{C}_{/C}$ . However a problem arises when we check that  $F$  respects composition: suppose  $f : C \rightarrow D$ ,  $g : D \rightarrow E$  are maps in  $\mathcal{C}$ . Then

$$F(g \circ f)(h : X \rightarrow E) = (g \circ f)^*h : P \rightarrow C,$$

corresponding to the pullback

$$\begin{array}{ccc} P & \xrightarrow{h^*(g \circ f)} & X \\ (g \circ f)^*h \downarrow & & \downarrow h \\ C & \xrightarrow{g \circ f} & E \end{array}$$

in  $\mathcal{C}$ . On the other hand,

$$(F(g) \circ F(f))(h : X \rightarrow E) = f^*(g^*h) : P'' \rightarrow C,$$

which corresponds to the diagram below.

$$\begin{array}{ccccc} P'' & \xrightarrow{(g^*h)^*f} & P' & \xrightarrow{h^*g} & X \\ f^*(g^*h) \downarrow & & \downarrow g^*h & & \downarrow h \\ C & \xrightarrow{f} & D & \xrightarrow{g} & E \end{array}$$

The outer square is indeed a pullback square, since the inner two squares are, so we have a unique isomorphism  $P \cong P''$ . However, we do not in general have equality. This is because pullbacks are only unique up to unique isomorphism, and in defining a pullback functor we made arbitrary (and not necessarily compatible) choices of  $P, P'$  and  $P''$ . Thus, we have not defined a functor  $\mathcal{C}^{\text{op}} \rightarrow \mathbf{Cat}$ , rather, we have defined what is known as a *pseudofunctor*.

(definition of a pseudofunctor)

The above example is one way in which pseudofunctors into  $\mathbf{Cat}$  naturally arise; another common example is the pseudofunctor

$$\mathbf{CRing} \rightarrow \mathbf{Cat}$$

$$R \mapsto R\text{-}\mathbf{Mod},$$

which sends a ring homomorphism  $\phi : R \rightarrow S$  to the functor  $- \otimes_R S : R\text{-}\mathbf{Mod} \rightarrow S\text{-}\mathbf{Mod}$  (extension of scalars). However, to give the data of a pseudofunctor  $F : \mathcal{C} \rightarrow \mathbf{Cat}$ , we must specify not only the functions  $\text{ob}(\mathcal{C}) \rightarrow \text{ob}(\mathbf{Cat})$  and  $\text{Hom}_{\mathcal{C}}(X, Y) \rightarrow \text{Hom}_{\mathbf{Cat}}(F(X), F(Y))$  for each  $X, Y \in \mathcal{C}$ , but also natural isomorphisms

$$F(\text{id}_X) \cong \text{id}_{F(X)}, \quad F(g \circ f) \cong F(g) \circ F(f).$$

That's a pain, let's use Grothendieck opfibrations instead.

The definitions below are from [1].

**DEFINITION 1.1.1.** Let  $p : X \rightarrow \mathcal{C}$  be a functor, and let  $f : c \rightarrow d$  be a morphism in  $\mathcal{C}$ . A morphism  $\Phi$  in  $X$  lying over  $f$  is *p-cocartesian* if for any other morphism  $\Psi$  of  $X$  lying over  $f$ , there exists a morphism  $I$  of  $X$  such that  $p(I) = \text{id}_d$  and  $\Psi = I \circ \Phi$ .

**DEFINITION 1.1.2.** Let  $p : X \rightarrow \mathcal{C}$  be a functor. Then  $p$  is a *Grothendieck opfibration* if for any morphism of  $\mathcal{C}$  and any lift of its source, there is a *p-cocartesian* morphism with that source lying over it.

## 1.2 Symmetric monoidal categories are special pseudofunctors to **Cat**

- Now that we know how to move between pseudofunctors to **Cat** and opfibrations, let's write the data of a symmetric monoidal category as a pseudofunctor to **Cat**.

## 1.3 ...which are special opfibrations

- The above implies there's some category  $\mathcal{D}$  such that opfibrations  $\mathcal{D} \rightarrow \mathbf{Fin}_*$  are the same as symmetric monoidal categories. Let's see what  $\mathcal{D}$  is.
- (Usual) definition of symmetric monoidal category and translation into the language of op-fibrations into  $\mathbf{Fin}_*$ , running example of **Vect**<sub>k</sub> with  $\otimes$  or  $\times$ .
- Possibly mention swapping out **Fin**<sub>\*</sub> for  $\Delta^{\text{op}}$  gives a monoidal category rather than a symmetric monoidal category (how do we get a braided monoidal category?)
- Correspondence of symmetric monoidal functors with morphisms of opfibrations.

The construction below is in [4].

Let  $(\mathcal{C}, \otimes)$  be a symmetric monoidal category. We define a new category  $\mathcal{C}^{\otimes}$ , whose objects are finite (possibly empty) sequences of objects of  $\mathcal{C}$ , denoted by  $[C_1, \dots, C_n]$ . A morphism

$$[C_1, \dots, C_n] \rightarrow [C'_1, \dots, C'_m]$$

consists of a subset  $S \subseteq \{1, \dots, n\}$ , a map of finite sets  $\alpha : S \rightarrow \{1, \dots, m\}$ , and a collection of morphisms  $\{f_j : \bigotimes_{i \in \alpha^{-1}\{j\}} C_i \rightarrow C'_j\}_{1 \leq j \leq m}$  in  $\mathcal{C}$ .

For two morphisms  $f : [C_1, \dots, C_n] \rightarrow [C'_1, \dots, C'_m]$  and  $g : [C'_1, \dots, C'_m] \rightarrow [C''_1, \dots, C''_l]$ , determining two subsets  $S \subseteq \{1, \dots, n\}$  and  $T \subseteq \{1, \dots, m\}$  and maps  $\alpha : S \rightarrow \{1, \dots, m\}$ ,  $\beta : T \rightarrow \{1, \dots, l\}$ , the composition  $g \circ f$  is given by the subset  $U = \alpha^{-1}T \subseteq \{1, \dots, n\}$ , the map  $\beta \circ \alpha : U \rightarrow \{1, \dots, l\}$  and the maps

$$\left\{ \bigotimes_{i \in (\beta \circ \alpha)^{-1}\{k\}} C_i \cong \bigotimes_{j \in \beta^{-1}\{k\}} \bigotimes_{i \in \alpha^{-1}\{j\}} C_i \rightarrow \bigotimes_{j \in \beta^{-1}\{k\}} C'_j \rightarrow C''_k \right\}_{1 \leq k \leq l}.$$

For example, let

$$f : [C_1, C_2, C_3, C_4] \rightarrow [C'_1, C'_2, C'_3]$$

be a morphism in  $\mathcal{C}^{\otimes}$  consisting of the subset  $\{1, 2, 3\} \subseteq \{1, 2, 3, 4\}$ , the map

$$\begin{aligned} \alpha : \{1, 2, 3\} &\rightarrow \{1, 2, 3\}, \\ 1 &\mapsto 1, \\ 2 &\mapsto 2, \\ 3 &\mapsto 3, \end{aligned}$$

and morphisms

$$f_1 : C_1 \rightarrow C'_1, \quad f_2 : C_2 \otimes C_3 \rightarrow C'_2, \quad f_3 : \mathbf{1} \rightarrow C'_3,$$

and let

$$g : [C'_1, C'_2, C'_3] \rightarrow [C''_1, C''_2, C''_3]$$

be a morphism in  $\mathcal{C}^\otimes$  consisting of the subset  $\{1, 2, 3\} \subseteq \{1, 2, 3\}$ , the map

$$\begin{aligned}\beta : \{1, 2, 3\} &\rightarrow \{1, 2, 3\}, \\ 1, 2, 3 &\mapsto 3,\end{aligned}$$

and morphisms

$$g_1 : \mathbf{1} \rightarrow C_1'', \quad g_2 : \mathbf{1} \rightarrow C_2'', \quad g_3 : C_1' \otimes C_2' \otimes C_3' \rightarrow C_3''.$$

Then the composition  $g \circ f$  consists of the subset  $\alpha^{-1}\{1, 2, 3\} = \{1, 2, 3\} \subseteq \{1, 2, 3, 4\}$ , the map

$$\begin{aligned}\beta \circ \alpha : \{1, 2, 3\} &\rightarrow \{1, 2, 3\}, \\ 1, 2, 3 &\mapsto 3,\end{aligned}$$

and the morphisms

$$(g \circ f)_1 = g_1, \quad (g \circ f)_2 = g_2, \quad (g \circ f)_3 = g_3 \circ (f_1 \otimes f_2 \otimes f_3).$$

(really?)

(some intuition on this, tensor along the fibres, etc)

Claim: the forgetful functor

$$\begin{aligned}p : \mathcal{C}^\otimes &\rightarrow \mathbf{Fin}_*, \\ [C_1, \dots, C_n] &\mapsto \langle n \rangle_*\end{aligned}$$

is an opfibration. (It almost tautologically is).

## 1.4 Generalisation to $\infty$ -categories

- Translation of the above into  $\infty$ -categorical language.
- A functor  $p : D \rightarrow C$  between ordinary categories is a Grothendieck opfibration if and only if the induced functor  $N(p) : N(D) \rightarrow N(C)$  on nerves is a cocartesian fibration – I cannot for the life of me prove this, there's possibly a proof in Kerodon (note: Kerodon calls Grothendieck opfibrations "cocartesian fibrations of categories")...
- Some examples (nerve of an ordinary symmetric monoidal category, currently trying to find more examples – many people talk about  $\mathbf{Sp}$ , but it seems like I'd need a lot of background to understand this).
- If an  $\infty$ -category has finite (co)products, there is a (co)cartesian monoidal structure on  $\mathcal{C}$ . And we would have hoped so, because it's definitely true for 1-categories!
- Algebra objects in monoidal ( $\infty$ -)categories
- Possibly generalisation to  $\infty$ -operads, depending on how much the above comes to or if I find anything fun to do with symmetric monoidal  $\infty$ -categories.

DEFINITION 1.4.1 ([1], Def 2.1). A functor  $p : X \rightarrow Y$  between simplicial sets is an *inner fibration* if for all  $n \geq 2$ , all  $0 < k < n$ , and any solid arrow commutative square

$$\begin{array}{ccc} \Lambda_k^n & \longrightarrow & X \\ \downarrow & \nearrow & \downarrow p \\ \Delta^n & \longrightarrow & Y \end{array}$$

there exists a dotted lift.

The following proposition is stated without proof in Section 2.3 of [5].

**PROPOSITION 1.4.2.** Let  $p : X \rightarrow Y$  be an inner fibration, and suppose that the diagram below is a pullback square in  $\mathbf{sSet}$ .

$$\begin{array}{ccc} X' & \xrightarrow{f} & X \\ p' \downarrow \lrcorner & & \downarrow p \\ Y' & \xrightarrow{g} & Y \end{array}$$

Then  $p'$  is also an inner fibration.

**PROOF.** Consider the (commutative) solid arrow diagram below.

$$\begin{array}{ccccc} \Lambda_k^n & \xrightarrow{\lambda} & X' & \xrightarrow{f} & X \\ \iota \downarrow & & \downarrow \lrcorner & \nearrow \phi & \downarrow p \\ \Delta^n & \xrightarrow{\delta} & Y' & \xrightarrow{g} & Y \end{array}$$

Since  $p$  is a fibration, there exists a dotted lift  $\phi$  of  $g\delta$ ; that is,  $p\phi = g\delta$  and  $\phi\iota = f\lambda$ . Further, since the right square is a pullback diagram, there exists a unique map  $\phi' : \Delta^n \rightarrow X'$  making the diagram below commute.

$$\begin{array}{ccccc} \Delta^n & & & & \\ & \searrow \phi' & & \nearrow \phi & \\ & & X' & \xrightarrow{f} & X \\ & \searrow \delta & \downarrow \lrcorner & & \downarrow p \\ & & Y' & \xrightarrow{g} & Y \end{array}$$

It remains to show that the triangle below commutes.

$$\begin{array}{ccc} \Lambda_k^n & \xrightarrow{\lambda} & X' \\ \iota \downarrow & \nearrow \phi' & \\ \Delta^n & & \end{array}$$

Again, using the universal property of pullbacks, we see that there exist unique dotted maps such that the diagrams below commute.

$$\begin{array}{ccc} \Lambda_k^n & \xrightarrow{f\lambda} & X \\ \delta \searrow & \nearrow \phi' & \downarrow p \\ & X' & \xrightarrow{f} \\ & \downarrow \lrcorner & \\ & Y' & \xrightarrow{g} Y \end{array} \quad \begin{array}{ccc} \Lambda_k^n & \xrightarrow{f\phi'\iota} & X \\ \delta \searrow & \nearrow \phi' & \downarrow p \\ & X' & \xrightarrow{f} \\ & \downarrow \lrcorner & \\ & Y' & \xrightarrow{g} Y \end{array}$$

The maps  $\lambda$  and  $\phi'\iota$  make the left and right diagrams commute respectively. Further, we note that  $f\phi' = \phi$  (by the second diagram) and  $\phi\iota = f\lambda$  (since  $p$  is an inner fibration), so  $f\phi'\iota = f\lambda$ . Therefore, the above two diagrams are identical. Thus, by the uniqueness property of pullbacks,  $\lambda = \phi'\iota$ .

□

(Stupid note to self, very obvious but I forget it every now and again):

- If  $X : \Delta^{\text{op}} \rightarrow \mathbf{Set}$  is a simplicial set, and  $\Delta^0 : \Delta^{\text{op}} \rightarrow \mathbf{Set} := \text{Hom}(-, [0])$ , then a map  $F : X \rightarrow \Delta^0$  is a natural transformation  $(F_n : X_n \rightarrow *)_{n \in \mathbb{N}_0}$ . That is, such a natural transformation is a family of maps down to a point. In other words, there's only really one natural transformation, so we really *can* view  $\Delta^0$  as a point.
- If  $Y$  is a simplicial set, and  $y \in Y_0$  is a vertex of  $Y$ , we can view  $\{y\}$  as a copy of  $\Delta^0$ . Why is this? We can view  $\{y\}$  as the constant simplicial set, sending everything to  $y$ . Then a natural isomorphism  $\Delta^0 \cong \{y\}$  is a collection of isomorphisms  $(* \rightarrow *)$ , of which there is exactly one. Why is it natural? Well, there's only one map from a one-point set to another one-point set, so the square always commutes.

EXAMPLE 1.4.3 ([1], Ex 2.2). Let  $p : X \rightarrow \Delta^0$  be the canonical map, and suppose we have the diagram below, such that the outer square commutes.

$$\begin{array}{ccc} \Lambda_k^n & \longrightarrow & X \\ \downarrow & \nearrow & \downarrow p \\ \Delta^n & \longrightarrow & \Delta^0 \end{array}$$

The lower triangle commutes automatically, so the statement that  $p$  is an inner fibration is equivalent to the statement that for all  $n \geq 2$ , all  $0 < k < n$ , and any map  $\Lambda_k^n \rightarrow X$ , there exists a dotted lift.

$$\begin{array}{ccc} \Lambda_k^n & \longrightarrow & X \\ \downarrow & \nearrow & \\ \Delta^n & & \end{array}$$

That is,  $X$  is an  $\infty$ -category.

Now, combining the above argument with [Proposition 1.4.2](#), we see that for any inner fibration  $p : X \rightarrow Y$ , each fibre  $X \times_Y \{y\}$  is an  $\infty$ -category.

DEFINITION 1.4.4 ([1], Def 3.1). Let  $p : X \rightarrow Y$  be an inner fibration. An edge  $f : \Delta^1 \rightarrow X$  of  $X$  is *p-cocartesian* if for all  $n \geq 2$ , any extension

$$\begin{array}{ccc} \Delta^{\{0,1\}} & \xrightarrow{f} & X \\ \downarrow & \nearrow F & \\ \Lambda_0^n & & \end{array}$$

and any solid arrow commutative diagram

$$\begin{array}{ccc} \Lambda_0^n & \xrightarrow{F} & X \\ \downarrow & \nearrow & \downarrow p \\ \Delta^n & \longrightarrow & Y \end{array}$$

a dotted lift exists.

DEFINITION 1.4.5. Let  $p : X \rightarrow Y$  be an inner fibration. Then  $p$  is a *cocartesian fibration* if for any edge  $\phi : y \rightarrow y'$  in  $Y_1$ , and for every  $x \in X_0$  lying over  $y$ , there exists a  $p$ -cocartesian edge  $f : x \rightarrow x'$  of  $X$  lying over  $\phi$ .

## References

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