Symmetric Monoidal ∞-Categories

Ali Ramsey

1 Introduction

- Note: we will call $(\infty, 1)$-categories ∞-categories.
- We start with an example of a pseudofunctor to motivate Grothendieck opfibrations. We explain the relationship between pseudofunctors into **Cat** and Grothendieck opfibrations and how to pass between the two.
- We define a symmetric monoidal category $(\mathcal{C}, \otimes)$ in the usual way, then note that it can be written as a pseudofunctor to **Cat**, and thus as a Grothendieck opfibration. We construct $\mathcal{C}^{\otimes}$ using the process outlined earlier, and compare it to Lurie's construction.
- We define symmetric monoidal functors in the usual way, and then construct the correct definitions in terms of morphisms of opfibrations.
- We introduce inner fibrations and prove that they are stable under pullbacks and that the fibres are ∞-categories. We introduce (co)cartesian fibrations and prove that the nerve of a functor is a (co)cartesian fibration if and only if the original functor was an (op)fibration.
- We finally define symmetric monoidal ∞-categories, and functors between them. We give the trivial examples: the nerve of a symmetric monoidal category, and the symmetric monoidal (co)cartesian structure on an ∞-category with finite products.
- We give an interesting example: the derived category.
- References: [1], [2], [3], [4], [5], [6], [7], [8], [9], [10]. Referencing Kerodon test: [9, Example 01UB].

2 Symmetric monoidal 1-categories

2.1 The Grothendieck construction

(Something about motivation and the pain of higher coherences. Below we give an example to complement [3].)

Let $\mathcal{C}$ be a category with pullbacks. Recall that for a map $f : C \to D$ in $\mathcal{C}$, we may define a pullback functor

$$\begin{aligned} f^* : \quad \mathcal{C}_{/D} \quad &\to \quad \mathcal{C}_{/C}, \\ (h : X \to D) &\mapsto (f^*h : P \to C), \end{aligned}$$

where we have formed a pullback

$$\begin{tikzcd} P \arrow[r, "h^*f"] \arrow[d, "f^*h"'] \arrow[dr, phantom, "\lrcorner", very near start] & X \arrow[d, "h"] \\ C \arrow[r, "f"'] & D \end{tikzcd}$$

in $\mathcal{C}$. For any map

$$\begin{tikzcd}
X \arrow[rr, "\phi"] \arrow[dr, "h"'] & & X' \arrow[dl, "h'"] \\
& D &
\end{tikzcd}$$

from h to h' in $\mathcal{C}_{/D}$, we define $f^*\phi$ to be the unique map making the diagram below commute.

$$\begin{tikzcd}
X \arrow[rr, "\phi"] \arrow[dr, "h"'] & & X' \arrow[dl, "h'"] \\
& D & \\
P \arrow[uu, "h^*f"] \arrow[rr, dashed, "f^*\phi"] \arrow[dr, "f^*h"'] & & P' \arrow[uu, "h'^*f"'] \arrow[dl, "f^*h'"] \\
& C \arrow[uu, "f"'] &
\end{tikzcd}$$

Now, we may wish to define a functor

$$\begin{aligned} F : \mathcal{C}^{\mathrm{op}} &\to \mathbf{Cat}, \\ C &\mapsto \mathcal{C}_{/C}, \end{aligned}$$

which sends a map $f : C \to D$ in $\mathcal{C}$ to a pullback functor $f^* : \mathcal{C}_{/D} \to \mathcal{C}_{/C}$. However a problem arises when we check that F respects composition: suppose $f : C \to D$, $g : D \to E$ are maps in $\mathcal{C}$. Then

$$F(g \circ f)(h : X \to E) = (g \circ f)^*h : P \to C,$$

corresponding to the pullback

$$\begin{tikzcd}
P \arrow[r, "h^*(g\circ f)"] \arrow[d, "(g\circ f)^*h"'] \arrow[dr, phantom, "\lrcorner", very near start] & X \arrow[d, "h"] \\
C \arrow[r, "g\circ f"'] & E
\end{tikzcd}$$

in $\mathcal{C}$. On the other hand,

$$(F(g) \circ F(f))(h : X \to E) = f^*(g^*h) : P'' \to C,$$

which corresponds to the diagram below.

$$\begin{tikzcd}
P'' \arrow[r, "(g^*h)^*f"] \arrow[d, "f^*(g^*h)"'] \arrow[dr, phantom, "\lrcorner", very near start] & P' \arrow[r, "h^*g"] \arrow[d, "g^*h"'] \arrow[dr, phantom, "\lrcorner", very near start] & X \arrow[d, "h"] \\
C \arrow[r, "f"'] & D \arrow[r, "g"'] & E
\end{tikzcd}$$

The outer square is indeed a pullback square, since the inner two squares are, so we have a unique isomorphism $P \cong P''$. However, we do not in general have equality. This is because pullbacks are only unique up to unique isomorphism, and in defining a pullback functor we made arbitrary (and not necessarily compatible) choices of P, P' and P''. Thus, we have not defined a functor $\mathcal{C}^{\mathrm{op}} \to \mathbf{Cat}$, rather, we have defined what is known as a *pseudofunctor*; that is, a weak functor between 2-categories.

The above example is one way in which pseudofunctors into **Cat** naturally arise; another common example is the pseudofunctor

$$\begin{aligned} \mathbf{CRing} &\to \mathbf{Cat} \\ R &\mapsto R\text{-}\mathbf{Mod}, \end{aligned}$$

which sends a ring homomorphism $\phi : R \to S$ to the functor $- \otimes_R S : R\text{-}\mathbf{Mod} \to S\text{-}\mathbf{Mod}$ (extension of scalars). However, to give the data of a pseudofunctor $F : \mathcal{C} \to \mathbf{Cat}$, we must specify not only the functions $\mathrm{ob}(\mathcal{C}) \to \mathrm{ob}(\mathbf{Cat})$ and $\mathrm{Hom}_{\mathcal{C}}(X, Y) \to \mathrm{Hom}_{\mathbf{Cat}}(F(X), F(Y))$ for each $X, Y \in \mathcal{C}$, but also natural isomorphisms

$$F(\mathrm{id}_X) \cong \mathrm{id}_{F(X)}, \quad F(g \circ f) \cong F(g) \circ F(f).$$

(This problem only becomes worse as we consider functors between higher categories... Let's use Grothendieck opfibrations instead.)

Definition 2.1.1. Let $p : X \to \mathcal{C}$ be a functor, let $f : c \to c'$ be a morphism in $\mathcal{C}$, and let $\phi : x \to x'$ be a morphism in X lying over f.

We say that ϕ is *p-cartesian* if for any other morphism $\psi : x'' \to x'$ in X, and for any morphism $g : p(x'') \to c$ in $\mathcal{C}$ satisfying $f \circ g = p(\psi)$, there exists a unique morphism $\gamma : x'' \to x$ such that $p(\gamma) = g$ and $\psi = \gamma \circ \phi$.

Dually, ϕ is *p-cocartesian* if for any other morphism $\psi : x \to x''$ in X, and for any morphism $g : c' \to p(x'')$ in $\mathcal{C}$ satisfying $g \circ f = p(\psi)$, there exists a unique morphism $\gamma : x' \to x''$ such that $p(\gamma) = g$ and $\psi = \gamma \circ \phi$.

The left diagram below corresponds to a p-cartesian morphism, and the right diagram corresponds to p-cocartesian morphism.

$$\begin{tikzcd}
 & & x'' \arrow[dll, dashed, "\exists!\gamma"'] \arrow[dl, "\forall\psi"] \arrow[ddd] \\
x \arrow[r, "\phi"'] \arrow[d] & x' \arrow[d] & \\
c \arrow[r, "f"] & c' & \\
 & & p(x'') \arrow[ull, "\forall g"] \arrow[ul, "p(\psi)"']
\end{tikzcd}
\qquad
\begin{tikzcd}
 & & x'' \arrow[ddd] \\
x \arrow[urr, "\forall\psi"] \arrow[r, "\phi"'] \arrow[d] & x' \arrow[ur, dashed, "\exists!\gamma"'] \arrow[d] & \\
c \arrow[r, "f"] \arrow[drr, "p(\psi)"'] & c' \arrow[dr, "\forall g"] & \\
 & & p(x'')
\end{tikzcd}
\qquad
\begin{tikzcd}
X \arrow[d, "p"'] \\
\mathcal{C}
\end{tikzcd}$$

Definition 2.1.2. Let $p : X \to \mathcal{C}$ be a functor. Then p is a *Grothendieck fibration* if for any morphism of $\mathcal{C}$ and any lift of its target, there is a p-cartesian morphism with that target lying over it. Dually, p is a *Grothendieck opfibration* if for any morphism of $\mathcal{C}$ and any lift of its source, there is a p-cocartesian morphism with that source lying over it.

We will usually refer to Grothendieck (op)fibrations as just (op)fibrations for brevity. Note that in [3] and [8] these are referred to as *(co)cartesian fibrations*; we reserve this term for the ∞-category analogue.

Remark 2.1.3. A functor $p : X \to \mathcal{C}$ is an opfibration if and only if $p^{\mathrm{op}} : X^{\mathrm{op}} \to \mathcal{C}^{\mathrm{op}}$ is a fibration. We give the definition of an opfibration explicitly, since we will be working with these more often than fibrations.

THEOREM 2.1.4 ([10], Thm 2.4). There is an equivalence of 2-categories

$$\mathbf{Psd}[\mathcal{C}^{\mathrm{op}}, \mathbf{Cat}] \simeq \mathbf{Fib}(\mathcal{C}),$$

where $\mathbf{Psd}[\mathcal{C}^{\mathrm{op}}, \mathbf{Cat}]$ is the 2-category of pseudofunctors $\mathcal{C}^{\mathrm{op}} \to \mathbf{Cat}$, and $\mathbf{Fib}(\mathcal{C}) \hookrightarrow \mathbf{Cat}_{/\mathcal{C}}$ is the 2-category of fibrations into $\mathcal{C}$.

REMARK 2.1.5. Combining Remark 2.1.3 with Theorem 2.1.4 gives us a chain of equivalences

$$\mathbf{Psd}[\mathcal{C}, \mathbf{Cat}] \xrightarrow{\mathrm{op}\,\circ} \mathbf{Psd}[\mathcal{C}, \mathbf{Cat}] \simeq \mathbf{Psd}[(\mathcal{C}^{\mathrm{op}})^{\mathrm{op}}, \mathbf{Cat}] \simeq \mathbf{Fib}(\mathcal{C}^{\mathrm{op}}) \simeq \mathbf{opFib}(\mathcal{C}).$$

We will not prove the theorem above (see [10] or [5] for more details) but we will describe how to pass between $\mathbf{Psd}[\mathcal{C}, \mathbf{Cat}]$ and $\mathbf{opFib}(\mathcal{C})$ by dualising[1] the constructions in these sources.

Let $F : \mathcal{C} \to \mathbf{Cat}$ be a pseudofunctor. Define the category X as follows: the objects of X are pairs (c, x), with $c \in \mathcal{C}$, $x \in F(c)$. A map $(c, x) \to (d, y)$ is a pair (f, u), where $f : c \to d$ is a morphism in $\mathcal{C}$, and $u : (Ff)(x) \to y$ is a morphism in $F(d)$. For an object $(c, x) \in X$, the identity morphism is given by

$$(\mathrm{id}_c, F^0_x : F(\mathrm{id}_c)(x) \to x),$$

where F^0 is the natural isomorphism $F(\mathrm{id}_c) \cong \mathrm{id}_{F(c)}$. Further, given two maps $(f, u) : (c, x) \to (d, y)$ and $(g, v) : (d, y) \to (e, z)$, their composition $(g, v) \circ (f, u)$ is given by $g \circ f$, together with the map

$$(F(g \circ f))(x) \xrightarrow{(F^2_{g,f})_x} (Fg \circ Ff)(x) \xrightarrow{(Fg)(u)} (Fg)(y) \xrightarrow{v} z,$$

where $F^2_{g,f}$ is the natural isomorphism $F(g \circ f) \cong Fg \circ Ff$. One can show that the forgetful functor $X \to \mathcal{C}$ is an opfibration over $\mathcal{C}$.

Now, let $p : X \to \mathcal{C}$ be an opfibration over $\mathcal{C}$. Define a pseudofunctor

$$\begin{aligned} \mathcal{C} &\to \mathbf{Cat} \\ c &\mapsto p^{-1}\{c\}. \end{aligned}$$

For a map $f : c \to d$ in $\mathcal{C}$, we define a functor

$$f_* : p^{-1}\{c\} \to p^{-1}\{d\},$$

which sends $x \in p^{-1}\{c\}$ to the target x' of a cocartesian edge $\phi_x : x \to x'$ lying over f. Now, let $f^*(y) = y'$, with $\phi_y : y \to y'$, and let $g : x \to y$ be a morphism in $p^{-1}\{c\}$ (that is, a morphism in X lying over id_c). Then, since $\phi_y : y \to y'$ is a cocartesian edge, there is a unique lift $\overline{g} : x' \to y'$ of id_d making the square on the left commute, as shown in the diagram on the right.

[1] We use the equivalences in Remark 2.1.5; in particular we postcompose with the equivalence $\mathrm{op} : \mathbf{Cat} \to \mathbf{Cat}$. This is to ensure that we can define the pseudofunctor $F : \mathcal{C} \to \mathbf{Cat}$ to send $c \in \mathcal{C}$ to $p^{-1}\{c\}$ rather than $(p^{\mathrm{op}})^{-1}\{c\} \simeq (p^{-1}\{c\})^{\mathrm{op}}$.

$$
\begin{tikzcd}
x \arrow[r, "\phi_x"] \arrow[d, dashed, "g"'] & x' \arrow[d, "\exists!\overline{g}"] \\
y \arrow[r, "\phi_y"'] & y'
\end{tikzcd}
\qquad
\begin{tikzcd}
 & & y' \arrow[ddd, mapsto] \\
x \arrow[r, "\phi_x"'] \arrow[urr, "\phi_y \circ g"] \arrow[d, mapsto] & x' \arrow[ur, dashed, "\exists!\overline{g}"'] \arrow[d, mapsto] & \\
c \arrow[r, "f"] \arrow[drr, "f"'] & d \arrow[dr, "\mathrm{id}_d"] & \\
 & & d
\end{tikzcd}
$$

We thus define $f_* g = \overline{g}$ in $p^{-1}\{d\}$.

It can be shown that these data assemble into a pseudofunctor $\mathcal{C} \to \mathbf{Cat}$.

2.2 Symmetric monoidal categories and functors

Definition 2.2.1. A *symmetric monoidal category* is a category $\mathcal{C}$ equipped with a bifunctor

$$\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C},$$

an object $\mathbf{1} \in \mathcal{C}$, and natural isomorphisms

$$\alpha : \otimes \circ (\otimes \times \mathrm{id}) \xrightarrow{\sim} \otimes \circ (\mathrm{id} \times \otimes), \quad l : \otimes \circ (\mathbf{1} \times \mathrm{id}) \xrightarrow{\sim} \mathrm{id}, \quad r : \otimes \circ (\mathrm{id} \times \mathbf{1}) \xrightarrow{\sim} \mathrm{id},$$

$$\tau : \otimes \xrightarrow{\sim} \sigma \circ \otimes,$$

where $\sigma : \mathcal{C} \times \mathcal{C}$ swaps the order of the factors. These isomorphisms are subject to the condition that $\tau^2 = \mathrm{id}$, and that the diagrams below commute.

$$
\begin{tikzcd}
 & ((U \otimes V) \otimes W) \otimes X \arrow[dl, "\alpha_{U \otimes V, W, X}"'] \arrow[dr, "\alpha_{U,V,W} \otimes \mathrm{id}_X"] & \\
(U \otimes V) \otimes (W \otimes X) \arrow[d, "\alpha_{U,V,W \otimes X}"'] & & (U \otimes (V \otimes W)) \otimes X \arrow[d, "\alpha_{U, V \otimes W, X}"] \\
U \otimes (V \otimes (W \otimes X)) & & U \otimes ((V \otimes W) \otimes X) \arrow[ll, "\mathrm{id} \otimes \alpha_{V,W,X}"]
\end{tikzcd}
$$

$$
\begin{tikzcd}
(V \otimes \mathbf{1}) \otimes W \arrow[rr, "\alpha_{V,\mathbf{1},W}"] \arrow[dr, "r_V \otimes \mathrm{id}_W"'] & & V \otimes (\mathbf{1} \otimes W) \arrow[dl, "\mathrm{id}_V \otimes l_W"] \\
 & V \otimes W &
\end{tikzcd}
\qquad
\begin{tikzcd}
V \otimes \mathbf{1} \arrow[rr, "\tau_{V,\mathbf{1}}"] \arrow[dr, "r_V"'] & & \mathbf{1} \otimes V \arrow[dl, "l_V"] \\
 & V &
\end{tikzcd}
$$

$$
\begin{tikzcd}
& (U \otimes V) \otimes W \arrow[dl, "\alpha_{U,V,W}"'] \arrow[dr, "\tau_{U,V} \otimes \mathrm{id}_W"] & \\
U \otimes (V \otimes W) \arrow[d, "\tau_{U,V \otimes W}"'] & & (V \otimes U) \otimes W \arrow[d, "\alpha_{V,U,W}"] \\
(V \otimes W) \otimes U \arrow[dr, "\alpha_{V,W,U}"'] & & V \otimes (U \otimes W) \arrow[dl, "\mathrm{id}_V \otimes \tau_{U,W}"] \\
& V \otimes (W \otimes U) &
\end{tikzcd}
$$

Now that we know how to move between pseudofunctors to **Cat** and opfibrations, let's write the data of a symmetric monoidal category as a pseudofunctor to **Cat**.

Let $(\mathcal{C}, \otimes)$ be a symmetric monoidal category. Define a pseudofunctor

$$
\begin{aligned}
F : \mathrm{Fin}_* &\to \mathbf{Cat} \\
\langle n \rangle &\mapsto \mathcal{C}^{\times n}
\end{aligned}
$$

Let $f : \langle n \rangle \to \langle m \rangle$ be a morphism in Fin_*. This induces a morphism

$$f^* : (C_1, ..., C_n) \mapsto (C'_1, ..., C'_m),$$

where

$$C'_i = \bigotimes_{j \in f^{-1}\{i\}} C_j.$$

- Possibly mention swapping out $\mathbf{Fin}_*$ for Δ^{op} gives a monoidal category rather than a symmetric monoidal category. (How do we get a braided monoidal category? Apparently there is no base 1-category we can look at opfibrations into, because the correct formulation is with E_2, which has higher homotopy groups on the mapping spaces.)
- Correspondence of symmetric monoidal functors with morphisms of opfibrations.

Using the construction above, we can obtain a category $\mathcal{C}^{\otimes}$ and an opfibration $p : \mathcal{C}^{\otimes} \to \mathbf{Fin}_*$ corresponding to the pseudofunctor encoding $(\mathcal{C}, \otimes)$. Unravelling the definitions gives exactly the construction below, in [7].

Let $(\mathcal{C}, \otimes)$ be a symmetric monoidal category. We define a new category $\mathcal{C}^{\otimes}$, whose objects are finite (possibly empty) sequences of objects of $\mathcal{C}$, denoted by $[C_1, ..., C_n]$. A morphism

$$[C_1, ..., C_n] \to [C'_1, ..., C'_m]$$

consists of a subset $S \subseteq \{1, ..., n\}$, a map of finite sets $\alpha : S \to \{1, ..., m\}$, and a collection of morphisms $\{f_j : \bigotimes_{i \in \alpha^{-1}\{j\}} C_i \to C'_j\}_{1 \leq j \leq m}$ in $\mathcal{C}$.

For two morphisms $f : [C_1, ..., C_n] \to [C'_1, ..., C'_m]$ and $g : [C'_1, ..., C'_m] \to [C''_1, ..., C''_l]$, determining two subsets $S \subseteq \{1, ..., n\}$ and $T \subseteq \{1, ..., m\}$ and maps $\alpha : S \to \{1, ..., m\}$, $\beta : T \to \{1, ..., l\}$, the composition $g \circ f$ is given by the subset $U = \alpha^{-1}T \subseteq \{1, ..., n\}$, the map $\beta \circ \alpha : U \to \{1, ..., l\}$ and the maps

$$\left\{ \bigotimes_{i \in (\beta \circ \alpha)^{-1}\{k\}} C_i \cong \bigotimes_{j \in \beta^{-1}\{k\}} \bigotimes_{i \in \alpha^{-1}\{j\}} C_i \to \bigotimes_{j \in \beta^{-1}\{k\}} C'_j \to C''_k \right\}_{1 \leq k \leq l}.$$

For example, let

$$f : [C_1, C_2, C_3, C_4] \to [C_1', C_2', C_3']$$

be a morphism in $\mathcal{C}^{\otimes}$ consisting of the subset $\{1, 2, 3\} \subseteq \{1, 2, 3, 4\}$, the map

$$\begin{aligned} \alpha : \{1, 2, 3\} &\to \{1, 2, 3\}, \\ 1 &\mapsto 1, \\ 2 &\mapsto 2, \\ 3 &\mapsto 3, \end{aligned}$$

and morphisms

$$f_1 : C_1 \to C_1', \quad f_2 : C_2 \otimes C_3 \to C_2', \quad f_3 : \mathbf{1} \to C_3',$$

and let

$$g : [C_1', C_2', C_3'] \to [C_1'', C_2'', C_3'']$$

be a morphism in $\mathcal{C}^{\otimes}$ consisting of the subset $\{1, 2, 3\} \subseteq \{1, 2, 3\}$, the map

$$\begin{aligned} \beta : \{1, 2, 3\} &\to \{1, 2, 3\}, \\ 1, 2, 3 &\mapsto 3, \end{aligned}$$

and morphisms

$$g_1 : \mathbf{1} \to C_1'', \quad g_2 : \mathbf{1} \to C_2'', \quad g_3 : C_1' \otimes C_2' \otimes C_3' \to C_3''.$$

Then the composition $g \circ f$ consists of the subset $\alpha^{-1}\{1, 2, 3\} = \{1, 2, 3\} \subseteq \{1, 2, 3, 4\}$, the map

$$\begin{aligned} \beta \circ \alpha : \{1, 2, 3\} &\to \{1, 2, 3\}, \\ 1, 2, 3 &\mapsto 3, \end{aligned}$$

and the morphisms

$$(g \circ f)_1 = g_1, \quad (g \circ f)_2 = g_2, \quad (g \circ f)_3 = g_3 \circ (f_1 \otimes f_2 \otimes f_3).$$

(really?)

(some intuition on this, tensor along the fibres, etc)

Claim: the forgetful functor

$$\begin{aligned} p : \quad \mathcal{C}^{\otimes} \quad &\to \quad \mathrm{Fin}_*, \\ [C_1, ..., C_n] &\mapsto \quad \langle n \rangle_* \end{aligned}$$

is an opfibration. (It almost tautologically is).

Proposition 2.2.2 ([4], Prop 4.26). If $(\mathcal{C}, \otimes)$ is a symmetric monoidal category, then the forgetful functor $p : \mathcal{C}^{\otimes} \to \mathbf{Fin}_*$ given above is an opfibration. Moreover, p satisfies the *Segal condition*; that is, the Segal maps

$$(\rho_!^1, ..., \rho_!^n) : \mathcal{C}^{\otimes}_{\langle n \rangle} \to \mathcal{C}^{\times n}$$

are equivalences. Conversely any Grothendieck opfibration $p : \mathcal{C} \to \mathbf{Fin}_*$ satisfying the Segal condition gives rise to a symmetric monoidal structure on $\mathcal{C}_{\langle 1 \rangle}$.

Summary: we noticed that pseudofunctors $\mathcal{D} \to \mathbf{Cat}$ are the same thing as opfibrations into $\mathcal{D}$. We also noticed that symmetric monoidal categories could be written as special pseudofunctors

$\mathbf{Fin}_* \to \mathbf{Cat}$, which means they are special opfibrations into $\mathbf{Fin}_*$. We looked at the corresponding construction of $\mathcal{C}^{\otimes}$. So: symmetric monoidal categories, special opfibrations into $\mathbf{Fin}_*$ and special pseudofunctors $\mathbf{Fin}_* \to \mathbf{Cat}$ are all the same thing, we just hide the coherences in the opfibrations.

We've said that symmetric monoidal categories are special opfibrations to $\mathbf{Fin}_*$. Now, what are symmetric monoidal functors?

The definition below is basically definition 3.3 of [3].

Definition 2.2.3. Let $p : X \to \mathcal{C}$ and $q : Y \to \mathcal{C}$ be two Grothendieck (op)fibrations. A functor $F : X \to Y$ is a *morphism of (op)fibrations* from p to q if the diagram below commutes,

$$\begin{tikzcd} X \arrow[rr, "F"] \arrow[dr, "p"'] & & Y \arrow[dl, "q"] \\ & \mathcal{C} & \end{tikzcd}$$

and F sends p-(co)cartesian morphisms to q-(co)cartesian morphisms.

3 Generalisation to ∞-categories

- If an ∞-category has finite (co)products, there is a (co)cartesian monoidal structure on $\mathcal{C}$. And we would have hoped so, because it's definitely true for 1-categories!
- Might be cool to try to look at E_k algebras, to resolve the earlier mystery of how to write braided monoidal categories.

We first need an ∞-categorical analogue of Grothendieck opfibrations. We start by requiring that our functor is what's known as an *inner fibration*; there is no 1–categorical analogue of this, since all functors between 1-categories are automatically inner fibrations under the nerve functor (see Example 3.0.2). Think of it as a 'minimum niceness condition' – we want the fibres to be ∞-categories in much the same way as we want the fibres of ordinary functors to be categories themselves.

Definition 3.0.1 ([1], Def 2.1). A functor $p : X \to Y$ between simplicial sets is an *inner fibration* if for all $n \geq 2$, all $0 < k < n$, and any solid arrow commutative square

$$\begin{tikzcd} \Lambda^n_k \arrow[r] \arrow[d, hook] & X \arrow[d, "p"] \\ \Delta^n \arrow[r] \arrow[ur, dashed] & Y \end{tikzcd}$$

there exists a dotted lift.

Example 3.0.2. Let $\mathcal{C}, \mathcal{D}$ be categories, and $p : \mathcal{C} \to \mathcal{D}$ be a functor between them. Then $N(p) : N\mathcal{C} \to N\mathcal{D}$ is an inner fibration.

The following proposition is stated without proof in Section 2.3 of [8].

Proposition 3.0.3. Let $p : X \to Y$ be an inner fibration, and suppose that the diagram below is a pullback square in $\mathbf{sSet}$.

$$\begin{tikzcd} X' \arrow[r, "f"] \arrow[d, "p'"'] \arrow[dr, phantom, "\lrcorner", very near start] & X \arrow[d, "p"] \\ Y' \arrow[r, "g"'] & Y \end{tikzcd}$$

Then p' is also an inner fibration.

PROOF. Consider the (commutative) solid arrow diagram below.

$$\begin{tikzcd}
\Lambda^n_k \arrow[r, "\lambda"] \arrow[d, hook, "\iota"'] & X' \arrow[r, "f"] \arrow[d, "p'"] \arrow[dr, phantom, "\lrcorner", very near start] & X \arrow[d, "p"] \\
\Delta^n \arrow[r, "\delta"'] \arrow[urr, dashed, "\phi"] & Y' \arrow[r, "g"'] & Y
\end{tikzcd}$$

Since p is a fibration, there exists a dotted lift ϕ of $g\delta$; that is, $p\phi = g\delta$ and $\phi\iota = f\lambda$. Further, since the right square is a pullback diagram, there exists a unique map $\phi' : \Delta^n \to X'$ making the diagram below commute.

$$\begin{tikzcd}
\Delta^n \arrow[dr, dashed, "\phi'"] \arrow[drr, bend left, "\phi"] \arrow[ddr, bend right, "\delta"'] & & \\
& X' \arrow[r, "f"] \arrow[d, "p'"'] \arrow[dr, phantom, "\lrcorner", very near start] & X \arrow[d, "p"] \\
& Y' \arrow[r, "g"'] & Y
\end{tikzcd}$$

It remains to show that the triangle below commutes.

$$\begin{tikzcd}
\Lambda^n_k \arrow[r, "\lambda"] \arrow[d, hook, "\iota"'] & X' \\
\Delta^n \arrow[ur, dashed, "\phi'"'] &
\end{tikzcd}$$

Again, using the universal property of pullbacks, we see that there exist unique dotted maps such that the diagrams below commute.

$$\begin{tikzcd}
\Lambda^n_k \arrow[dr, dashed] \arrow[drr, bend left, "f\lambda"] \arrow[ddr, bend right, "\delta\iota"'] & & \\
& X' \arrow[r, "f"] \arrow[d, "p'"'] \arrow[dr, phantom, "\lrcorner", very near start] & X \arrow[d, "p"] \\
& Y' \arrow[r, "g"'] & Y
\end{tikzcd}
\qquad
\begin{tikzcd}
\Lambda^n_k \arrow[dr, dashed] \arrow[drr, bend left, "f\phi'\iota"] \arrow[ddr, bend right, "\delta\iota"'] & & \\
& X' \arrow[r, "f"] \arrow[d, "p'"'] \arrow[dr, phantom, "\lrcorner", very near start] & X \arrow[d, "p"] \\
& Y' \arrow[r, "g"'] & Y
\end{tikzcd}$$

The maps λ and $\phi'\iota$ make the left and right diagrams commute respectively. Further, we note that $f\phi' = \phi$ (by the second diagram) and $\phi\iota = f\lambda$ (since p is an inner fibration), so $f\phi'\iota = f\lambda$. Therefore, the above two diagrams are identical. Thus, by the uniqueness property of pullbacks, $\lambda = \phi'\iota$. $\square$

(Stupid note to self, very obvious but I forget it every now and again):

- If $X : \Delta^{\mathrm{op}} \to \mathbf{Set}$ is a simplicial set, and $\Delta^0 : \Delta^{\mathrm{op}} \to \mathbf{Set} := \mathrm{Hom}(-, [0])$, then a map $F : X \to \Delta^0$ is a natural transformation $(F_n : X_n \to *)_{n\in\mathbb{N}_0}$. That is, such a natural transformation is a family of maps down to a point. In other words, there's only really one natural transformation, so we really *can* view Δ^0 as a point.
- If Y is a simplicial set, and $y \in Y_0$ is a vertex of Y, we can view $\{y\}$ as a copy of Δ^0. Why is this? We can view $\{y\}$ as the constant simplicial set, sending everything to y. Then a natural isomorphism $\Delta^0 \cong \{y\}$ is a collection of isomorphisms $(* \to *)$, of which there is exactly one. Why is it natural? Well, there's only one map from a one-point set to another one-point set, so the square always commutes.

Example 3.0.4 ([1], Ex 2.2). Let $p : X \to \Delta^0$ be the canonical map, and suppose we have the diagram below, such that the outer square commutes.

$$\begin{tikzcd} \Lambda^n_k \arrow[r] \arrow[d, hook] & X \arrow[d, "p"] \\ \Delta^n \arrow[r] \arrow[ur, dashed] & \Delta^0 \end{tikzcd}$$

The lower triangle commutes automatically, so the statement that p is an inner fibration is equivalent to the statement that for all $n \geq 2$, all $0 < k < n$, and any map $\Lambda^n_k \to X$, there exists a dotted lift.

$$\begin{tikzcd} \Lambda^n_k \arrow[r] \arrow[d, hook] & X \\ \Delta^n \arrow[ur, dashed] & \end{tikzcd}$$

That is, X is an ∞-category.

Now, combining the above argument with Proposition 3.0.3, we see that for any inner fibration $p : X \to Y$, each fibre $X \times_Y \{y\}$ is an ∞-category.

Definition 3.0.5 ([1], Def 3.1). Let $p : X \to Y$ be an inner fibration. An edge $f : \Delta^1 \to X$ of X is *p-cocartesian* if for all $n \geq 2$, any extension

$$\begin{tikzcd} \Delta^{\{0,1\}} \arrow[r, "f"] \arrow[d, hook] & X \\ \Lambda^n_0 \arrow[ur, "F"'] & \end{tikzcd}$$

and any solid arrow commutative diagram

$$\begin{tikzcd} \Lambda^n_0 \arrow[r, "F"] \arrow[d, hook] & X \arrow[d, "p"] \\ \Delta^n \arrow[r] \arrow[ur, dashed] & Y \end{tikzcd}$$

a dotted lift exists.

Definition 3.0.6. Let $p : X \to Y$ be an inner fibration. Then p is a cocartesian fibration if for any edge $\phi : y \to y'$ in Y_1, and for every $x \in X_0$ lying over y, there exists a p-cocartesian edge $f : x \to x'$ of X lying over ϕ.

The following proposition tells us that the above definition is a reasonable generalisation of Definition 2.1.2. It is also stated without proof in [8], which did not do wonders for my ego.

Proposition 3.0.7 ([8], Rmk 2.4.2.2). Let $\mathcal{C}$, $\mathcal{D}$ be categories, and let $p : \mathcal{C} \to \mathcal{D}$ be a functor between them. Then p is a Grothendieck opfibration if and only if the induced map $N(p) : N\mathcal{C} \to N\mathcal{D}$ is a cocartesian fibration of simplicial sets.

Proof. Let $f : d \to d'$ be a morphism of $\mathcal{D}$, and let c lie over d.

Suppose p is a Grothendieck opfibration, let $F : \Lambda^n_0 \to N\mathcal{C}$ be an extension of f, and let

$$\begin{tikzcd} \Lambda^n_0 \arrow[r, "F"] \arrow[d, hook] & N\mathcal{C} \arrow[d, "N(p)"] \\ \Delta^n \arrow[r] & N\mathcal{D} \end{tikzcd}$$

be a commutative diagram. If $n = 2$, it follows immediately from the fact that p is an opfibration that a dotted lift exists. Further, if $n > 3$, there is nothing to check, since an n-simplex in a category commutes if and only if all of its triangles commute, which is guaranteed for any extension $F : \Lambda^n_0 \to N\mathcal{C}$. We thus prove the proposition for $n = 3$.

Suppose we have an extension $F : \Lambda^3_0 \to N\mathcal{C}$ of f; that is, a tetrahedron

$$\begin{tikzcd}
 & c \arrow[rr, "\chi"] \arrow[dl, "\phi"'] \arrow[dr, "\psi"] & & c''' \\
c' \arrow[rr, "\gamma"'] \arrow[urrr, "\gamma'"] & & c'' \arrow[ur, "\gamma''"'] &
\end{tikzcd}$$

such that all faces containing the vertex c commute. Let

$$\begin{tikzcd}
 & d \arrow[rr, "p(\chi)"] \arrow[dl, "f"'] \arrow[dr, "p(\psi)"] & & p(c''') \\
d' \arrow[rr, "p(\gamma)"'] \arrow[urrr, "p(\gamma')"] & & p(c'') \arrow[ur, "p(\gamma'')"'] &
\end{tikzcd}$$

be a commutative tetrahedron in $\mathcal{D}$. We claim that the tetrahedron in $\mathcal{C}$ commutes. First, note that $\gamma'' \circ \gamma$ is a lift of $p(\gamma')$, since $p(\gamma') = p(\gamma'') \circ p(\gamma) = p(\gamma'' \circ \gamma)$. Further,

$$\begin{aligned}
(\gamma \circ \gamma'') \circ \phi &= \gamma'' \circ \psi \\
&= \chi.
\end{aligned}$$

Thus, by the uniqueness in the universal property of ϕ, we have that $\gamma' = \gamma'' \circ \gamma$, as required.

Now, suppose $N(p)$ is a cocartesian fibration. Then there exists a lift $\phi : c \to c'$ of f, and, in particular, for any diagram

$$\begin{tikzcd}
 & c \arrow[dl, "\phi"'] \arrow[dr, "\psi"] & \\
c' & & c''
\end{tikzcd}$$

in $\mathcal{C}$, and any commutative diagram

$$\begin{tikzcd}
 & d \arrow[dl, "f"'] \arrow[dr, "p(\psi)"] & \\
d' \arrow[rr, "g"'] & & p(c'')
\end{tikzcd}$$

in $\mathcal{D}$, there exists a map $\gamma : c' \to c''$ such that γ lies over g and $\gamma \circ \phi = \psi$. It remains to show that γ is unique.

Suppose that there were two maps $\gamma_1, \gamma_2 : c' \to c''$ lying over g and satisfying $\gamma_1 \circ \phi = \gamma_2 \circ \phi = \psi$. Then we would have a tetrahedron

$$\begin{tikzcd}
 & c \arrow[rr, "\psi"] \arrow[dl, "\phi"'] \arrow[dr, "\psi"] & & c'' \\
c' \arrow[rr, "\gamma_1"'] \arrow[urrr, "\gamma_2"] & & c'' \arrow[ur, "\mathrm{id}"'] &
\end{tikzcd}$$

where all faces containing the vertex c commute. The image of this tetrahedron under p commutes in $\mathcal{D}$, so the original tetrahedron must commute in $\mathcal{C}$; that is, $\gamma_1 = \gamma_2$. □

Definition 3.0.8 ([8], Def 2.0.0.7). A *symmetric monoidal* ∞*-category* is a cocartesian fibration of simplicial sets $p : X^{\otimes} \to N(\mathbf{Fin}_*)$ such that for each $n \geq 0$, the maps

$$\{\rho^i : \langle n \rangle \to \langle 1 \rangle\}_{1 \leq i \leq n}$$

induce functors $\rho^i_! : X^{\otimes}_{\langle n \rangle} \to X^{\otimes}_{\langle 1 \rangle}$ which determine an equivalence $X^{\otimes}_{\langle n \rangle} \simeq \left(X^{\otimes}_{\langle 1 \rangle}\right)^n$.

Example 3.0.9. Let $(\mathcal{C}, \otimes)$ be a symmetric monoidal category. Then $p : N(\mathcal{C}^{\otimes}) \to N(\mathbf{Fin}_*)$ is a symmetric monoidal ∞-category.

4 A nontrivial example

Throughout this section, $\mathcal{A}$ is an abelian category, and $\mathcal{A}_{\text{proj}}$ is the full subcategory of $\mathcal{A}$ spanned by the projective objects.

Definition 4.0.1 ([7], Def 1.2.3.1). A *chain complex* with values in $\mathcal{A}$ is a composable sequence of morphisms

$$\cdots \to A_2 \xrightarrow{d(2)} A_1 \xrightarrow{d(1)} A_0 \xrightarrow{d(0)} A_{-1} \to \cdots$$

in $\mathcal{A}$ such that $d(n-1) \circ d(n) = 0$ for all $n \in \mathbb{Z}$. The collection of chain complexes with values in $\mathcal{A}$ is an additive category, $\mathrm{Ch}(\mathcal{A})$.

Definition 4.0.2 ([7], Not 1.3.2.6). $\mathrm{Ch}^-(\mathcal{A})$ is the full subcategory of $\mathrm{Ch}(\mathcal{A})$ spanned by those chain complexes M_* such that $M_n \simeq 0$ for $n << 0$.

Definition 4.0.3 ([7], Def 1.3.2.7). Suppose $\mathcal{A}$ has enough projective objects. We let $\mathcal{D}^-(\mathcal{A})$ denote the ∞-category $N_{\text{dg}}(\mathrm{Ch}^-(\mathcal{A}_{\text{proj}}))$. We refer to $\mathcal{D}^-(\mathcal{A})$ as the *derived* ∞*-category of* $\mathcal{A}$.

5 Miscellaneous stupid notes

5.1 Observations

Let $S \in \mathbf{Set}$. We define the constant simplicial set

$$\overline{S} : \Delta^{\text{op}} \longrightarrow \mathbf{Set}$$

$$\begin{array}{ccc} [n] & \longmapsto & S \\ {\scriptstyle f}\downarrow & & \uparrow{\scriptstyle \mathrm{id}} \\ [m] & \longmapsto & S \end{array}$$

It's a Kan complex. Why? Well, when you consider S as a discrete category, and take the nerve of it, you get $\overline{S}$. You can then either just see than it's a Kan complex (fill the horns with identities) or use the fact that the nerve of a groupoid is a Kan complex. It's surely in [8] or [9] somewhere.

Here are the original definitions of the Grothendieck equivalence (before I dualised them, just in case I did it wrong).

Let $F : \mathcal{C}^{\mathrm{op}} \to \mathbf{Cat}$ be a pseudofunctor. Define the category X as follows: the objects of X are pairs (c, x), with $c \in \mathcal{C}$, $x \in F(c)$. A map $(c, x) \to (d, y)$ is a pair (f, u), where $f : c \to d$ is a morphism in $\mathcal{C}$, and $u : x \to (Ff)(y)$ is a morphism in $F(c)$. For an object $(c, x) \in X$, the identity morphism is given by

$$(\mathrm{id}_c, F^0_x : x \to F(\mathrm{id}_c)(x)),$$

where F^0 is the natural isomorphism $\mathrm{id}_{F(c)} \cong F(\mathrm{id}_c)$. Further, given two maps $(f, u) : (c, x) \to (d, y)$ and $(g, v) : (d, y) \to (e, z)$, their composition $(g, v) \circ (f, u)$ is given by $g \circ f$, together with the map

$$x \xrightarrow{u} (Ff)(y) \xrightarrow{(Ff)(v)} (Ff \circ Fg)(z) \xrightarrow{\left(F^2_{f,g}\right)_z} (F(g \circ f))(z),$$

where $F^2_{f,g}$ is the natural isomorphism $Ff \circ Fg \cong F(g \circ f)$ (recall that the domain of F is $\mathcal{C}^{\mathrm{op}}$, while f, g are morphisms in $\mathcal{C}$). One can show that the forgetful functor $X \to \mathcal{C}$ is a fibration over $\mathcal{C}$.

Now, let $p : X \to \mathcal{C}$ be a fibration over $\mathcal{C}$. Define a pseudofunctor

$$\begin{aligned} \mathcal{C}^{\mathrm{op}} &\to \mathbf{Cat} \\ c &\mapsto p^{-1}\{c\}. \end{aligned}$$

For a map $f : c \to d$ in $\mathcal{C}$, we define a functor

$$f^* : p^{-1}\{d\} \to p^{-1}\{c\},$$

which sends $x \in p^{-1}\{d\}$ to a the source x' of a cartesian edge $\phi_x : x' \to x$ lying over f. Now, let $f^*(y) = y'$, with $\phi_y : y' \to y$, and let $g : x \to y$ be a morphism in $p^{-1}\{d\}$ (that is, a morphism in X lying over id_c). Then, since $\phi_y : y' \to y$ is a cartesian edge, there is a unique lift $\overline{g} : x' \to y'$ of id_c making the square on the left commute, as shown in the diagram on the right.

$$\begin{tikzcd}
x' \arrow[r, "\phi_x"] \arrow[d, dashed, "\exists!\overline{g}"'] & x \arrow[d, "g"] \\
y' \arrow[r, "\phi_y"'] & y
\end{tikzcd}
\qquad
\begin{tikzcd}
 & & & x' \arrow[dlll, dashed, "\exists!\overline{g}"'] \arrow[dll, "g\circ\phi_x"] \arrow[ddd, mapsto] \\
y' \arrow[r, "\phi_y"'] \arrow[d, mapsto] & y \arrow[d, mapsto] & & \\
c \arrow[r, "f"] & d & & \\
 & & & c \arrow[ulll, "\mathrm{id}_c"] \arrow[ull, "f"']
\end{tikzcd}$$

We thus define $f^* g = \overline{g}$ in $p^{-1}\{c\}$.

It can be shown that these data assemble into a pseudofunctor $\mathcal{C} \to \mathbf{Cat}$.

Here is the dualisation of the forward direction, which I think was correct, but actually not what I wanted:

Let $F : \mathcal{C} \to \mathbf{Cat}$ be a pseudofunctor. Define the category X as follows: the objects of X are pairs (c, x), with $c \in \mathcal{C}$, $x \in F(c)$. A map $(c, x) \to (d, y)$ is a pair (f, u), where $f : c \to d$ is a morphism in $\mathcal{C}$, and $u : y \to (Ff)(x)$ is a morphism in $F(d)$. For an object $(c, x) \in X$, the identity morphism is given by

$$(\mathrm{id}_c, F^0_x : x \to F(\mathrm{id}_c)(x)),$$

where F^0 is the natural isomorphism $\mathrm{id}_{F(c)} \cong F(\mathrm{id}_c)$. Further, given two maps $(f, u) : (c, x) \to (d, y)$ and $(g, v) : (d, y) \to (e, z)$, their composition $(g, v) \circ (f, u)$ is given by $g \circ f$, together with the map

$$z \xrightarrow{v} (Fg)(y) \xrightarrow{(Fg)(u)} (Fg \circ Ff)(x) \xrightarrow{(F^2_{g,f})_x} (F(g \circ f))(x),$$

where $F^2_{g,f}$ is the natural isomorphism $Fg \circ Ff \cong F(g \circ f)$. One can show that the forgetful functor $X \to \mathcal{C}$ is an opfibration over $\mathcal{C}$.

5.2 Questions

Questions:

- ...what *is* $\mathbf{Grpd}_\infty$?

5.3 Equivalent definitions

$\mathbf{Grpd}_\infty$

An algebraic category

An equivalence of ∞-categories

References

[1] Clark Barwick and Jay Shah. 'Fibrations in ∞-Category Theory'. In: *2016 MATRIX Annals*. Springer Cham, 2018, pp. 17–42.

[2] Francis Borceux. *Handbook of Categorical Algebra*. Encyclopedia of Mathematics and its Applications. Cambridge University Press, 1994.

[3] Julius Frank. *Symmetric monoidal ∞-categories*. 2021. URL: https://homepages.abdn.ac.uk/markus.upmeier/pages/JuliusMay13.pdf.

[4] Moritz Groth. *A Short Course on ∞-Categories*. 2015. arXiv: 1007.2925 [math.AT].

[5] Niles Johnson and Donald Yau. *2-Dimensional Categories*. 2020. arXiv: 2002.06055 [math.CT].

[6] Fosco Loregian and Emily Riehl. 'Categorical Notions of Fibration'. In: *Expositiones Mathematicae* 38.4 (2020), 496–514. ISSN: 0723-0869. DOI: 10.1016/j.exmath.2019.02.004. URL: http://dx.doi.org/10.1016/j.exmath.2019.02.004.

[7] Jacob Lurie. *Higher Algebra*. 2017. URL: https://www.math.ias.edu/~lurie/papers/HA.pdf.

[8] Jacob Lurie. *Higher Topos Theory*. 2008. arXiv: math/0608040 [math.CT].

[9] Jacob Lurie. *Kerodon*. https://kerodon.net. 2024.

[10] Joe Moeller and Christina Vasilakopoulou. *Monoidal Grothendieck Construction*. 2021. arXiv: 1809.00727 [math.CT].