

# Symmetric Monoidal $\infty$ -Categories

- Note: we will call  $(\infty, 1)$ -categories  $\infty$ -categories. Sorry Tom, it's just convenient.
- Main references: [4], [5], [2], [1], [3], [6]. How does [6, Example 01UB] work?

## 1 Pseudofunctors, opfibrations, and symmetric monoidal categories

### 1.1 Pseudofunctors to **Cat** are Grothendieck opfibrations

- Grothendieck op-fibrations correspond to pseudofunctors to **Cat**.

The below gives an example to complement [2].

Example of a pseudofunctor to **Cat**: Let  $\mathcal{C}$  be a category with pullbacks. Recall that for a map  $f : C \rightarrow D$  in  $\mathcal{C}$ , we define a pullback functor

$$\begin{aligned} f^* : \mathcal{C}_{/D} &\rightarrow \mathcal{C}_{/C}, \\ (h : X \rightarrow D) &\mapsto (f^*h : P \rightarrow C), \end{aligned}$$

where we have formed a pullback

$$\begin{array}{ccc} P & \xrightarrow{h^*f} & X \\ f^*h \downarrow & & \downarrow h \\ C & \xrightarrow{f} & D \end{array}$$

in  $\mathcal{C}$ . For any map

$$\begin{array}{ccc} X & \xrightarrow{\phi} & X' \\ & \searrow h & \swarrow h' \\ & D & \end{array}$$

from  $h$  to  $h'$  in  $\mathcal{C}_{/D}$ , we define  $f^*\phi$  to be the unique map making the diagram below commute.

$$\begin{array}{ccccc} X & \xrightarrow{\phi} & X' & & \\ & \searrow h & \swarrow h' & & \\ & D & & & \\ & \uparrow f & & & \\ P & \xrightarrow{f^*\phi} & P' & & \\ & \searrow f^*h & \swarrow f^*h' & & \\ & C & & & \end{array}$$

Now, we may wish to define a functor

$$\begin{aligned} F : \mathcal{C}^{\text{op}} &\rightarrow \mathbf{Cat}, \\ C &\mapsto \mathcal{C}_{/C}, \end{aligned}$$

which sends a map  $f : C \rightarrow D$  in  $\mathcal{C}$  to a pullback functor  $f^* : \mathcal{C}_{/D} \rightarrow \mathcal{C}_{/C}$ . However a problem arises when we check that  $F$  respects composition: suppose  $f : C \rightarrow D$ ,  $g : D \rightarrow E$  are maps in  $\mathcal{C}$ . Then

$$F(g \circ f)(h : X \rightarrow E) = (g \circ f)^* h : P \rightarrow C,$$

corresponding to the pullback

$$\begin{array}{ccc} P & \xrightarrow{h^*(g \circ f)} & X \\ (g \circ f)^* h \downarrow & & \downarrow h \\ C & \xrightarrow{g \circ f} & E \end{array}$$

in  $\mathcal{C}$ . On the other hand,

$$(F(g) \circ F(f))(h : X \rightarrow E) = f^*(g^* h) : P'' \rightarrow C,$$

which corresponds to the diagram below.

$$\begin{array}{ccccc} P'' & \xrightarrow{(g^* h)^* f} & P' & \xrightarrow{h^* g} & X \\ f^*(g^* h) \downarrow & & \downarrow g^* h & & \downarrow h \\ C & \xrightarrow{f} & D & \xrightarrow{g} & E \end{array}$$

The outer square is indeed a pullback square, since the inner two squares are, so we have a unique isomorphism  $P \cong P''$ . However, we do not in general have equality. This is because pullbacks are only unique up to unique isomorphism, and in defining a pullback functor we made arbitrary (and not necessarily compatible) choices of  $P, P'$  and  $P''$ . Thus, we have not defined a functor  $\mathcal{C}^{\text{op}} \rightarrow \mathbf{Cat}$ , rather, we have defined what is known as a *pseudofunctor*.

(definition of a pseudofunctor)

The above example is one way in which pseudofunctors into  $\mathbf{Cat}$  naturally arise; another common example is the pseudofunctor

$$\begin{aligned} \mathbf{CRing} &\rightarrow \mathbf{Cat} \\ R &\mapsto R\text{-Mod}, \end{aligned}$$

which sends a ring homomorphism  $\phi : R \rightarrow S$  to the functor  $- \otimes_R S : R\text{-Mod} \rightarrow S\text{-Mod}$  (extension of scalars). However, to give the data of a pseudofunctor  $F : \mathcal{C} \rightarrow \mathbf{Cat}$ , we must specify not only the functions  $\text{ob}(\mathcal{C}) \rightarrow \text{ob}(\mathbf{Cat})$  and  $\text{Hom}_{\mathcal{C}}(X, Y) \rightarrow \text{Hom}_{\mathbf{Cat}}(F(X), F(Y))$  for each  $X, Y \in \mathcal{C}$ , but also natural isomorphisms

$$F(\text{id}_X) \cong \text{id}_{F(X)}, \quad F(g \circ f) \cong F(g) \circ F(f).$$

That's a pain, let's use Grothendieck opfibrations instead.

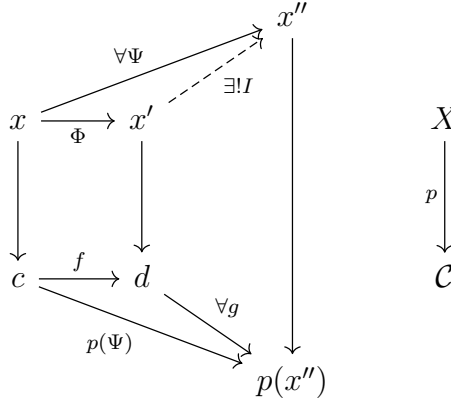
The definitions below are from [1].

**DEFINITION 1.1.1.** Let  $p : X \rightarrow \mathcal{C}$  be a functor, and let  $f : c \rightarrow d$  be a morphism in  $\mathcal{C}$ . A morphism  $\Phi$  in  $X$  lying over  $f$  is *p-cocartesian* if for any other morphism  $\Psi$  of  $X$  lying over  $f$ , there exists a morphism  $I$  of  $X$  such that  $p(I) = \text{id}_d$  and  $\Psi = I \circ \Phi$ .

DEFINITION 1.1.2. Let  $p : X \rightarrow \mathcal{C}$  be a functor. Then  $p$  is a *Grothendieck opfibration* if for any morphism of  $\mathcal{C}$  and any lift of its source, there is a  $p$ -cocartesian morphism with that source lying over it.

And the reconstructed definitions that actually (hopefully) work are below:

DEFINITION 1.1.3. Let  $p : X \rightarrow \mathcal{C}$  be a functor, and let  $f : c \rightarrow d$  be a morphism in  $\mathcal{C}$ . A morphism  $\Phi : x \rightarrow x'$  in  $X$  lying over  $f$  is  *$p$ -cocartesian* if for any other morphism  $\Psi : x \rightarrow x''$  in  $X$ , and for any morphism  $g : d \rightarrow p(x'')$  in  $\mathcal{C}$  satisfying  $g \circ f = p(\Psi)$ , there exists a unique morphism  $I : x' \rightarrow x''$  such that  $p(I) = g$  and  $\Psi = I \circ \Phi$ .



DEFINITION 1.1.4. Let  $p : X \rightarrow \mathcal{C}$  be a functor. Then  $p$  is a *Grothendieck opfibration* if for any morphism of  $\mathcal{C}$  and any lift of its source, there is a  $p$ -cocartesian morphism with that source lying over it.

## 1.2 Symmetric monoidal categories are special pseudofunctors to **Cat**

- Now that we know how to move between pseudofunctors to **Cat** and opfibrations, let's write the data of a symmetric monoidal category as a pseudofunctor to **Cat**.

## 1.3 ...which are special opfibrations

- The above implies there's some category  $\mathcal{D}$  such that opfibrations  $\mathcal{D} \rightarrow \mathbf{Fin}_*$  are the same as symmetric monoidal categories. Let's see what  $\mathcal{D}$  is.
- (Usual) definition of symmetric monoidal category and translation into the language of op-fibrations into  $\mathbf{Fin}_*$ , running example of  $\mathbf{Vect}_k$  with  $\otimes$  or  $\times$ .
- Possibly mention swapping out  $\mathbf{Fin}_*$  for  $\Delta^{\text{op}}$  gives a monoidal category rather than a symmetric monoidal category (how do we get a braided monoidal category?)
- Correspondence of symmetric monoidal functors with morphisms of opfibrations.

The construction below is in [4].

Let  $(\mathcal{C}, \otimes)$  be a symmetric monoidal category. We define a new category  $\mathcal{C}^{\otimes}$ , whose objects are finite (possibly empty) sequences of objects of  $\mathcal{C}$ , denoted by  $[C_1, \dots, C_n]$ . A morphism

$$[C_1, \dots, C_n] \rightarrow [C'_1, \dots, C'_m]$$

consists of a subset  $S \subseteq \{1, \dots, n\}$ , a map of finite sets  $\alpha : S \rightarrow \{1, \dots, m\}$ , and a collection of morphisms  $\{f_j : \bigotimes_{i \in \alpha^{-1}\{j\}} C_i \rightarrow C'_j\}_{1 \leq j \leq m}$  in  $\mathcal{C}$ .

For two morphisms  $f : [C_1, \dots, C_n] \rightarrow [C'_1, \dots, C'_m]$  and  $g : [C'_1, \dots, C'_m] \rightarrow [C''_1, \dots, C''_l]$ , determining two subsets  $S \subseteq \{1, \dots, n\}$  and  $T \subseteq \{1, \dots, m\}$  and maps  $\alpha : S \rightarrow \{1, \dots, m\}$ ,  $\beta : T \rightarrow \{1, \dots, l\}$ , the composition  $g \circ f$  is given by the subset  $U = \alpha^{-1}T \subseteq \{1, \dots, n\}$ , the map  $\beta \circ \alpha : U \rightarrow \{1, \dots, l\}$  and the maps

$$\left\{ \bigotimes_{i \in (\beta \circ \alpha)^{-1}\{k\}} C_i \cong \bigotimes_{j \in \beta^{-1}\{k\}} \bigotimes_{i \in \alpha^{-1}\{j\}} C_i \rightarrow \bigotimes_{j \in \beta^{-1}\{k\}} C'_j \rightarrow C''_k \right\}_{1 \leq k \leq l}.$$

For example, let

$$f : [C_1, C_2, C_3, C_4] \rightarrow [C'_1, C'_2, C'_3]$$

be a morphism in  $\mathcal{C}^\otimes$  consisting of the subset  $\{1, 2, 3\} \subseteq \{1, 2, 3, 4\}$ , the map

$$\begin{aligned} \alpha : \{1, 2, 3\} &\rightarrow \{1, 2, 3\}, \\ 1 &\mapsto 1, \\ 2 &\mapsto 2, \\ 3 &\mapsto 3, \end{aligned}$$

and morphisms

$$f_1 : C_1 \rightarrow C'_1, \quad f_2 : C_2 \otimes C_3 \rightarrow C'_2, \quad f_3 : \mathbf{1} \rightarrow C'_3,$$

and let

$$g : [C'_1, C'_2, C'_3] \rightarrow [C''_1, C''_2, C''_3]$$

be a morphism in  $\mathcal{C}^\otimes$  consisting of the subset  $\{1, 2, 3\} \subseteq \{1, 2, 3\}$ , the map

$$\begin{aligned} \beta : \{1, 2, 3\} &\rightarrow \{1, 2, 3\}, \\ 1, 2, 3 &\mapsto 3, \end{aligned}$$

and morphisms

$$g_1 : \mathbf{1} \rightarrow C''_1, \quad g_2 : \mathbf{1} \rightarrow C''_2, \quad g_3 : C'_1 \otimes C'_2 \otimes C'_3 \rightarrow C''_3.$$

Then the composition  $g \circ f$  consists of the subset  $\alpha^{-1}\{1, 2, 3\} = \{1, 2, 3\} \subseteq \{1, 2, 3, 4\}$ , the map

$$\begin{aligned} \beta \circ \alpha : \{1, 2, 3\} &\rightarrow \{1, 2, 3\}, \\ 1, 2, 3 &\mapsto 3, \end{aligned}$$

and the morphisms

$$(g \circ f)_1 = g_1, \quad (g \circ f)_2 = g_2, \quad (g \circ f)_3 = g_3 \circ (f_1 \otimes f_2 \otimes f_3).$$

(really?)

(some intuition on this, tensor along the fibres, etc)

Claim: the forgetful functor

$$\begin{aligned} p : \mathcal{C}^\otimes &\rightarrow \mathbf{Fin}_*, \\ [C_1, \dots, C_n] &\mapsto \langle n \rangle_* \end{aligned}$$

is an opfibration. (It almost tautologically is).

## 1.4 Generalisation to $\infty$ -categories

- Translation of the above into  $\infty$ -categorical language.
- A functor  $p : D \rightarrow C$  between ordinary categories is a Grothendieck opfibration if and only if the induced functor  $N(p) : N(D) \rightarrow N(C)$  on nerves is a cocartesian fibration – I cannot for the life of me prove this, there’s possibly a proof in Kerodon (note: Kerodon calls Grothendieck opfibrations “cocartesian fibrations of categories”)...
- Some examples (nerve of an ordinary symmetric monoidal category, currently trying to find more examples – many people talk about **Sp**, but it seems like I’d need a lot of background to understand this).
- If an  $\infty$ -category has finite (co)products, there is a (co)cartesian monoidal structure on  $\mathcal{C}$ . And we would have hoped so, because it’s definitely true for 1-categories!
- Algebra objects in monoidal ( $\infty$ -)categories
- Possibly generalisation to  $\infty$ -operads, depending on how much the above comes to or if I find anything fun to do with symmetric monoidal  $\infty$ -categories.

DEFINITION 1.4.1 ([1], Def 2.1). A functor  $p : X \rightarrow Y$  between simplicial sets is an *inner fibration* if for all  $n \geq 2$ , all  $0 < k < n$ , and any solid arrow commutative square

$$\begin{array}{ccc} \Lambda_k^n & \longrightarrow & X \\ \downarrow & \nearrow & \downarrow p \\ \Delta^n & \longrightarrow & Y \end{array}$$

there exists a dotted lift.

The following proposition is stated without proof in Section 2.3 of [5].

PROPOSITION 1.4.2. Let  $p : X \rightarrow Y$  be an inner fibration, and suppose that the diagram below is a pullback square in **sSet**.

$$\begin{array}{ccc} X' & \xrightarrow{f} & X \\ p' \downarrow \lrcorner & & \downarrow p \\ Y' & \xrightarrow{g} & Y \end{array}$$

Then  $p'$  is also an inner fibration.

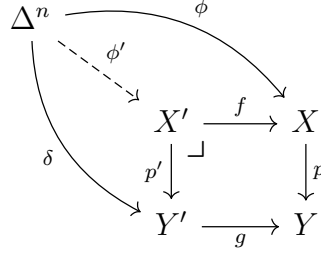
PROOF. Consider the (commutative) solid arrow diagram below.

$$\begin{array}{ccccc} \Lambda_k^n & \xrightarrow{\lambda} & X' & \xrightarrow{f} & X \\ \downarrow \iota & & \downarrow \lrcorner & \nearrow & \downarrow p \\ \Delta^n & \xrightarrow{\delta} & Y' & \xrightarrow{g} & Y \end{array}$$

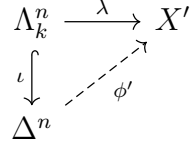
$\phi$  (dotted arrow from  $\Delta^n$  to  $X'$ )

Since  $p$  is a fibration, there exists a dotted lift  $\phi$  of  $g\delta$ ; that is,  $p\phi = g\delta$  and  $\phi\iota = f\lambda$ . Further, since the right square is a pullback diagram, there exists a unique map  $\phi' : \Delta^n \rightarrow X'$  making

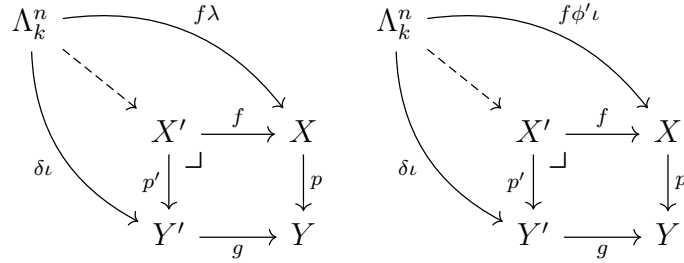
the diagram below commute.



It remains to show that the triangle below commutes.



Again, using the universal property of pullbacks, we see that there exist unique dotted maps such that the diagrams below commute.

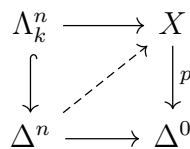


The maps  $\lambda$  and  $\phi'\iota$  make the left and right diagrams commute respectively. Further, we note that  $f\phi' = \phi$  (by the second diagram) and  $\phi\iota = f\lambda$  (since  $p$  is an inner fibration), so  $f\phi'\iota = f\lambda$ . Therefore, the above two diagrams are identical. Thus, by the uniqueness property of pullbacks,  $\lambda = \phi'\iota$ .  $\square$

(Stupid note to self, very obvious but I forget it every now and again):

- If  $X : \Delta^{\text{op}} \rightarrow \mathbf{Set}$  is a simplicial set, and  $\Delta^0 : \Delta^{\text{op}} \rightarrow \mathbf{Set} := \text{Hom}(-, [0])$ , then a map  $F : X \rightarrow \Delta^0$  is a natural transformation  $(F_n : X_n \rightarrow *)_{n \in \mathbb{N}_0}$ . That is, such a natural transformation is a family of maps down to a point. In other words, there's only really one natural transformation, so we really \*can\* view  $\Delta^0$  as a point.
- If  $Y$  is a simplicial set, and  $y \in Y_0$  is a vertex of  $Y$ , we can view  $\{y\}$  as a copy of  $\Delta^0$ . Why is this? We can view  $\{y\}$  as the constant simplicial set, sending everything to  $y$ . Then a natural isomorphism  $\Delta^0 \cong \{y\}$  is a collection of isomorphisms  $(* \rightarrow *)$ , of which there is exactly one. Why is it natural? Well, there's only one map from a one-point set to another one-point set, so the square always commutes.

EXAMPLE 1.4.3 ([1], Ex 2.2). Let  $p : X \rightarrow \Delta^0$  be the canonical map, and suppose we have the diagram below, such that the outer square commutes.



The lower triangle commutes automatically, so the statement that  $p$  is an inner fibration is equivalent to the statement that for all  $n \geq 2$ , all  $0 < k < n$ , and any map  $\Lambda_k^n \rightarrow X$ , there

exists a dotted lift.

$$\begin{array}{ccc} \Lambda_k^n & \longrightarrow & X \\ \downarrow & \nearrow \text{dotted} & \\ \Delta^n & & \end{array}$$

That is,  $X$  is an  $\infty$ -category.

Now, combining the above argument with [Proposition 1.4.2](#), we see that for any inner fibration  $p : X \rightarrow Y$ , each fibre  $X \times_Y \{y\}$  is an  $\infty$ -category.

DEFINITION 1.4.4 ([1], Def 3.1). Let  $p : X \rightarrow Y$  be an inner fibration. An edge  $f : \Delta^1 \rightarrow X$  of  $X$  is *p-cocartesian* if for all  $n \geq 2$ , any extension

$$\begin{array}{ccc} \Delta^{\{0,1\}} & \xrightarrow{f} & X \\ \downarrow & \nearrow F & \\ \Lambda_0^n & & \end{array}$$

and any solid arrow commutative diagram

$$\begin{array}{ccc} \Lambda_0^n & \xrightarrow{F} & X \\ \downarrow & \nearrow \text{dotted} & \downarrow p \\ \Delta^n & \longrightarrow & Y \end{array}$$

a dotted lift exists.

DEFINITION 1.4.5. Let  $p : X \rightarrow Y$  be an inner fibration. Then  $p$  is a cocartesian fibration if for any edge  $\phi : y \rightarrow y'$  in  $Y_1$ , and for every  $x \in X_0$  lying over  $y$ , there exists a *p-cocartesian* edge  $f : x \rightarrow x'$  of  $X$  lying over  $\phi$ .

## References

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