Symmetric Monoidal ∞-Categories

- Note: we will call $(\infty, 1)$-categories ∞-categories. Sorry Tom, it's just convenient.
- Main references: [4], [5], [2], [1], [3], [6]. How does [6, Example 01UB] work?

1 Pseudofunctors, opfibrations, and symmetric monoidal categories

1.1 Pseudofunctors to Cat are Grothendieck opfibrations

- Grothendieck op-fibrations correspond to pseudofunctors to **Cat**.

The below gives an example to complement [2].

Example of a pseudofunctor to **Cat**: Let $\mathcal{C}$ be a category with pullbacks. Recall that for a map $f : C \to D$ in $\mathcal{C}$, we define a pullback functor

$$\begin{array}{c} f^* : \quad \mathcal{C}_{/D} \quad \to \quad \mathcal{C}_{/C}, \\ (h : X \to D) \mapsto (f^*h : P \to C), \end{array}$$

where we have formed a pullback

$$\begin{tikzcd} P \arrow[r, "h^*f"] \arrow[d, "f^*h"'] & X \arrow[d, "h"] \\ C \arrow[r, "f"'] & D \end{tikzcd}$$

in $\mathscr{C}$. For any map

$$\begin{tikzcd} X \arrow[rr, "\phi"] \arrow[dr, "h"'] & & X' \arrow[dl, "h'"] \\ & D & \end{tikzcd}$$

from h to h' in $\mathcal{C}_{/D}$, we define $f^*\phi$ to be the unique map making the diagram below commute.

$$\begin{tikzcd} X \arrow[rr, "\phi"] \arrow[dr, "h"'] & & X' \arrow[dl, "h'"] \\ & D & \\ & & \\ P \arrow[uuu, "h^*f"] \arrow[rr, dashed, "f^*\phi"] \arrow[dr, "f^*h"'] & & P' \arrow[uuu, "h'^*f"'] \arrow[dl, "f^*h'"] \\ & C \arrow[uuu, "f"] & \end{tikzcd}$$

Now, we may wish to define a functor

$$\begin{aligned} F : \mathcal{C}^{\mathrm{op}} &\to \mathbf{Cat}, \\ C &\mapsto \mathcal{C}_{/C}, \end{aligned}$$

which sends a map $f : C \to D$ in $\mathcal{C}$ to a pullback functor $f^* : \mathcal{C}_{/D} \to \mathcal{C}_{/C}$. However a problem arises when we check that F respects composition: suppose $f : C \to D$, $g : D \to E$ are maps in $\mathcal{C}$. Then

$$F(g \circ f)(h : X \to E) = (g \circ f)^* h : P \to C,$$

corresponding to the pullback

$$\begin{tikzcd} P \arrow[r, "h^*(g\circ f)"] \arrow[d, "(g\circ f)^* h"'] & X \arrow[d, "h"] \\ C \arrow[r, "g\circ f"'] & E \end{tikzcd}$$

in $\mathcal{C}$. On the other hand,

$$(F(g) \circ F(f))(h : X \to E) = f^*(g^* h) : P'' \to C,$$

which corresponds to the diagram below.

$$\begin{tikzcd} P'' \arrow[r, "(g^*h)^* f"] \arrow[d, "f^*(g^*h)"'] & P' \arrow[r, "h^* g"] \arrow[d, "g^* h"] & X \arrow[d, "h"] \\ C \arrow[r, "f"'] & D \arrow[r, "g"'] & E \end{tikzcd}$$

The outer square is indeed a pullback square, since the inner two squares are, so we have a unique isomorphism $P \cong P''$. However, we do not in general have equality. This is because pullbacks are only unique up to unique isomorphism, and in defining a pullback functor we made arbitrary (and not necessarily compatible) choices of P, P' and P''. Thus, we have not defined a functor $\mathcal{C}^{\mathrm{op}} \to \mathbf{Cat}$, rather, we have defined what is known as a *pseudofunctor*.

(definition of a pseudofunctor)

The above example is one way in which pseudofunctors into $\mathbf{Cat}$ naturally arise; another common example is the pseudofunctor

$$\begin{aligned} \mathbf{CRing} &\to \mathbf{Cat} \\ R &\mapsto R\text{-}\mathbf{Mod}, \end{aligned}$$

which sends a ring homomorphism $\phi : R \to S$ to the functor $- \otimes_R S : R\text{-}\mathbf{Mod} \to S\text{-}\mathbf{Mod}$ (extension of scalars). However, to give the data of a pseudofunctor $F : \mathcal{C} \to \mathbf{Cat}$, we must specify not only the functions $\mathrm{ob}(\mathcal{C}) \to \mathrm{ob}(\mathbf{Cat})$ and $\mathrm{Hom}_{\mathcal{C}}(X, Y) \to \mathrm{Hom}_{\mathbf{Cat}}(F(X), F(Y))$ for each $X, Y \in \mathcal{C}$, but also natural isomorphisms

$$F(\mathrm{id}_X) \cong \mathrm{id}_{F(X)}, \quad F(g \circ f) \cong F(g) \circ F(f).$$

That's a pain, let's use Grothendieck opfibrations instead.

The definitions below are from [1].

DEFINITION 1.1.1. Let $p : X \to \mathcal{C}$ be a functor, and let $f : c \to d$ be a morphism in $\mathcal{C}$. A morphism Φ in X lying over f is *p-cocartesian* if for any other morphism Ψ of X lying over f, there exists a morphism I of X such that $p(I) = \mathrm{id}_d$ and $\Psi = I \circ \Phi$.

Definition 1.1.2. Let $p : X \to \mathcal{C}$ be a functor. Then p is a *Grothendieck opfibration* if for any morphism of $\mathcal{C}$ and any lift of its source, there is a p-cocartesian morphism with that source lying over it.

And the reconstructed definitions that actually (hopefully) work are below:

Definition 1.1.3. Let $p : X \to \mathcal{C}$ be a functor, and let $f : c \to d$ be a morphism in $\mathcal{C}$. A morphism $\Phi : x \to x'$ in X lying over f is *p-cocartesian* if for any other morphism $\Psi : x \to x''$ in X, and for any morphism $g : d \to p(x'')$ in $\mathcal{C}$ satisfying $g \circ f = p(\Psi)$, there exists a unique morphism $I : x' \to x''$ such that $p(I) = g$ and $\Psi = I \circ \Phi$.

$$
\begin{tikzcd}
 & & x'' \arrow[ddd] & & \\
x \arrow[r, "\Phi"'] \arrow[rru, "\forall\Psi"] \arrow[d] & x' \arrow[ru, dashed, "\exists! I"'] \arrow[d] & & & X \arrow[d, "p"'] \\
c \arrow[r, "f"] \arrow[rrd, "p(\Psi)"'] & d \arrow[rd, "\forall g"] & & & \mathcal{C} \\
 & & p(x'') & &
\end{tikzcd}
$$

Definition 1.1.4. Let $p : X \to \mathcal{C}$ be a functor. Then p is a *Grothendieck opfibration* if for any morphism of $\mathcal{C}$ and any lift of its source, there is a p-cocartesian morphism with that source lying over it.

1.2 Symmetric monoidal categories are special pseudofunctors to Cat

- Now that we know how to move between pseudofunctors to **Cat** and opfibrations, let's write the data of a symmetric monoidal category as a pseudofunctor to **Cat**.

1.3 ...which are special opfibrations

- The above implies there's some category $\mathcal{D}$ such that opfibrations $\mathcal{D} \to \mathrm{Fin}_*$ are the same as symmetric monoidal categories. Let's see what $\mathcal{D}$ is.
- (Usual) definition of symmetric monoidal category and translation into the language of op-fibrations into Fin_*, running example of $\mathbf{Vect}_k$ with $\otimes$ or $\times$.
- Possibly mention swapping out $\mathbf{Fin}_*$ for Δ^{op} gives a monoidal category rather than a symmetric monoidal category (how do we get a braided monoidal category?)
- Correspondence of symmetric monoidal functors with morphisms of opfibrations.

The construction below is in [4].

Let $(\mathcal{C}, \otimes)$ be a symmetric monoidal category. We define a new category $\mathcal{C}^{\otimes}$, whose objects are finite (possibly empty) sequences of objects of $\mathcal{C}$, denoted by $[C_1, ..., C_n]$. A morphism

$$[C_1, ..., C_n] \to [C'_1, ..., C'_m]$$

consists of a subset $S \subseteq \{1, ..., n\}$, a map of finite sets $\alpha : S \to \{1, ..., m\}$, and a collection of morphisms $\{f_j : \bigotimes_{i \in \alpha^{-1}\{j\}} C_i \to C'_j\}_{1 \leq j \leq m}$ in $\mathcal{C}$.

For two morphisms $f:[C_1,...,C_n]\to[C'_1,...,C'_m]$ and $g:[C'_1,...,C'_m]\to[C''_1,...,C''_l]$, determining two subsets $S\subseteq\{1,...,n\}$ and $T\subseteq\{1,...,m\}$ and maps $\alpha:S\to\{1,...,m\}$, $\beta:T\to\{1,...,l\}$, the composition $g\circ f$ is given by the subset $U=\alpha^{-1}T\subseteq\{1,...,n\}$, the map $\beta\circ\alpha:U\to\{1,...,l\}$ and the maps

$$\left\{\bigotimes_{i\in(\beta\circ\alpha)^{-1}\{k\}} C_i \cong \bigotimes_{j\in\beta^{-1}\{k\}}\bigotimes_{i\in\alpha^{-1}\{j\}} C_i \to \bigotimes_{j\in\beta^{-1}\{k\}} C'_j \to C''_k\right\}_{1\le k\le l}.$$

For example, let

$$f:[C_1,C_2,C_3,C_4]\to[C'_1,C'_2,C'_3]$$

be a morphism in $\mathcal{C}^\otimes$ consisting of the subset $\{1,2,3\}\subseteq\{1,2,3,4\}$, the map

$$\begin{aligned}\alpha:\{1,2,3\}&\to\{1,2,3\},\\ 1&\mapsto 1,\\ 2&\mapsto 2,\\ 3&\mapsto 3,\end{aligned}$$

and morphisms

$$f_1:C_1\to C'_1,\quad f_2:C_2\otimes C_3\to C'_2,\quad f_3:\mathbf{1}\to C'_3,$$

and let

$$g:[C'_1,C'_2,C'_3]\to[C''_1,C''_2,C''_3]$$

be a morphism in $\mathcal{C}^\otimes$ consisting of the subset $\{1,2,3\}\subseteq\{1,2,3\}$, the map

$$\begin{aligned}\beta:\{1,2,3\}&\to\{1,2,3\},\\ 1,2,3&\mapsto 3,\end{aligned}$$

and morphisms

$$g_1:\mathbf{1}\to C''_1,\quad g_2:\mathbf{1}\to C''_2,\quad g_3:C'_1\otimes C'_2\otimes C'_3\to C''_3.$$

Then the composition $g\circ f$ consists of the subset $\alpha^{-1}\{1,2,3\}=\{1,2,3\}\subseteq\{1,2,3,4\}$, the map

$$\begin{aligned}\beta\circ\alpha:\{1,2,3\}&\to\{1,2,3\},\\ 1,2,3&\mapsto 3,\end{aligned}$$

and the morphisms

$$(g\circ f)_1=g_1,\quad (g\circ f)_2=g_2,\quad (g\circ f)_3=g_3\circ(f_1\otimes f_2\otimes f_3).$$

(really?)

(some intuition on this, tensor along the fibres, etc)

Claim: the forgetful functor

$$\begin{aligned}p:\quad \mathcal{C}^\otimes &\to \mathrm{Fin}_*,\\ [C_1,...,C_n]&\mapsto \langle n\rangle_*\end{aligned}$$

is an opfibration. (It almost tautologically is).

1.4 Generalisation to ∞-categories

- Translation of the above into ∞-categorical language.
- A functor $p : D \to C$ between ordinary categories is a Grothendieck opfibration if and only if the induced functor $N(p) : N(D) \to N(C)$ on nerves is a cocartesian fibration – I cannot for the life of me prove this, there's possibly a proof in Kerodon (note: Kerodon calls Grothendieck opfibrations "cocartesian fibrations of categories")...
- Some examples (nerve of an ordinary symmetric monoidal category, currently trying to find more examples – many people talk about **Sp**, but it seems like I'd need a lot of background to understand this).
- If an ∞-category has finite (co)products, there is a (co)cartesian monoidal structure on $\mathcal{C}$. And we would have hoped so, because it's definitely true for 1-categories!
- Algebra objects in monoidal (∞-)categories
- Possibly generalisation to ∞-operads, depending on how much the above comes to or if I find anything fun to do with symmetric monoidal ∞-categories.

DEFINITION 1.4.1 ([1], Def 2.1). A functor $p : X \to Y$ between simplicial sets is an *inner fibration* if for all $n \geq 2$, all $0 < k < n$, and any solid arrow commutative square

$$\begin{tikzcd} \Lambda^n_k \arrow[r] \arrow[d, hook] & X \arrow[d, "p"] \\ \Delta^n \arrow[r] \arrow[ur, dashed] & Y \end{tikzcd}$$

there exists a dotted lift.

The following proposition is stated without proof in Section 2.3 of [5].

PROPOSITION 1.4.2. Let $p : X \to Y$ be an inner fibration, and suppose that the diagram below is a pullback square in **sSet**.

$$\begin{tikzcd} X' \arrow[r, "f"] \arrow[d, "p'"'] \arrow[dr, phantom, "\lrcorner", very near start] & X \arrow[d, "p"] \\ Y' \arrow[r, "g"'] & Y \end{tikzcd}$$

Then p' is also an inner fibration.

PROOF. Consider the (commutative) solid arrow diagram below.

$$\begin{tikzcd} \Lambda^n_k \arrow[r, "\lambda"] \arrow[d, hook, "\iota"'] & X' \arrow[r, "f"] \arrow[d, "p'"] \arrow[dr, phantom, "\lrcorner", very near start] & X \arrow[d, "p"] \\ \Delta^n \arrow[r, "\delta"'] \arrow[urr, dashed, "\phi"] & Y' \arrow[r, "g"'] & Y \end{tikzcd}$$

Since p is a fibration, there exists a dotted lift ϕ of $g\delta$; that is, $p\phi = g\delta$ and $\phi\iota = f\lambda$. Further, since the right square is a pullback diagram, there exists a unique map $\phi' : \Delta^n \to X'$ making the diagram below commute.

$$\begin{tikzcd}
\Delta^n \arrow[drr, bend left, "\phi"] \arrow[dr, dashed, "\phi'"] \arrow[ddr, bend right, "\delta"'] & & \\
& X' \arrow[r, "f"] \arrow[d, "p'"'] \arrow[dr, phantom, "\lrcorner", very near start] & X \arrow[d, "p"] \\
& Y' \arrow[r, "g"'] & Y
\end{tikzcd}$$

It remains to show that the triangle below commutes.

$$\begin{tikzcd}
\Lambda^n_k \arrow[r, "\lambda"] \arrow[d, hook, "\iota"'] & X' \\
\Delta^n \arrow[ur, dashed, "\phi'"'] &
\end{tikzcd}$$

Again, using the universal property of pullbacks, we see that there exist unique dotted maps such that the diagrams below commute.

$$\begin{tikzcd}
\Lambda^n_k \arrow[drr, bend left, "f\lambda"] \arrow[dr, dashed] \arrow[ddr, bend right, "\delta\iota"'] & & \\
& X' \arrow[r, "f"] \arrow[d, "p'"'] \arrow[dr, phantom, "\lrcorner", very near start] & X \arrow[d, "p"] \\
& Y' \arrow[r, "g"'] & Y
\end{tikzcd}
\qquad
\begin{tikzcd}
\Lambda^n_k \arrow[drr, bend left, "f\phi'\iota"] \arrow[dr, dashed] \arrow[ddr, bend right, "\delta\iota"'] & & \\
& X' \arrow[r, "f"] \arrow[d, "p'"'] \arrow[dr, phantom, "\lrcorner", very near start] & X \arrow[d, "p"] \\
& Y' \arrow[r, "g"'] & Y
\end{tikzcd}$$

The maps λ and $\phi'\iota$ make the left and right diagrams commute respectively. Further, we note that $f\phi' = \phi$ (by the second diagram) and $\phi\iota = f\lambda$ (since p is an inner fibration), so $f\phi'\iota = f\lambda$. Therefore, the above two diagrams are identical. Thus, by the uniqueness property of pullbacks, $\lambda = \phi'\iota$.

$\square$

(Stupid note to self, very obvious but I forget it every now and again):

- If $X : \Delta^{\text{op}} \to \mathbf{Set}$ is a simplicial set, and $\Delta^0 : \Delta^{\text{op}} \to \mathbf{Set} := \text{Hom}(-, [0])$, then a map $F : X \to \Delta^0$ is a natural transformation $(F_n : X_n \to *)_{n \in \mathbb{N}_0}$. That is, such a natural transformation is a family of maps down to a point. In other words, there's only really one natural transformation, so we really *can* view Δ^0 as a point.
- If Y is a simplicial set, and $y \in Y_0$ is a vertex of Y, we can view $\{y\}$ as a copy of Δ^0. Why is this? We can view $\{y\}$ as the constant simplicial set, sending everything to y. Then a natural isomorphism $\Delta^0 \cong \{y\}$ is a collection of isomorphisms $(* \to *)$, of which there is exactly one. Why is it natural? Well, there's only one map from a one-point set to another one-point set, so the square always commutes.

Example 1.4.3 ([1], Ex 2.2). Let $p : X \to \Delta^0$ be the canonical map, and suppose we have the diagram below, such that the outer square commutes.

$$\begin{tikzcd}
\Lambda^n_k \arrow[r] \arrow[d, hook] & X \arrow[d, "p"] \\
\Delta^n \arrow[r] \arrow[ur, dashed] & \Delta^0
\end{tikzcd}$$

The lower triangle commutes automatically, so the statement that p is an inner fibration is equivalent to the statement that for all $n \geq 2$, all $0 < k < n$, and any map $\Lambda^n_k \to X$, there exists a dotted lift.

$$\begin{tikzcd} \Lambda^n_k \arrow[r] \arrow[d, hook] & X \\ \Delta^n \arrow[ur, dashed] & \end{tikzcd}$$

That is, X is an ∞-category.

Now, combining the above argument with Proposition 1.4.2, we see that for any inner fibration $p : X \to Y$, each fibre $X \times_Y \{y\}$ is an ∞-category.

Definition 1.4.4 ([1], Def 3.1). Let $p : X \to Y$ be an inner fibration. An edge $f : \Delta^1 \to X$ of X is *p-cocartesian* if for all $n \geq 2$, any extension

$$\begin{tikzcd} \Delta^{\{0,1\}} \arrow[r, "f"] \arrow[d, hook] & X \\ \Lambda^n_0 \arrow[ur, "F"'] & \end{tikzcd}$$

and any solid arrow commutative diagram

$$\begin{tikzcd} \Lambda^n_0 \arrow[r, "F"] \arrow[d, hook] & X \arrow[d, "p"] \\ \Delta^n \arrow[r] \arrow[ur, dashed] & Y \end{tikzcd}$$

a dotted lift exists.

Definition 1.4.5. Let $p : X \to Y$ be an inner fibration. Then p is a cocartesian fibration if for any edge $\phi : y \to y'$ in Y_1, and for every $x \in X_0$ lying over y, there exists a p-cocartesian edge $f : x \to x'$ of X lying over ϕ.

References

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