

Symmetric Monoidal ∞ -Categories

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1 Introduction

- Note: we will call $(\infty, 1)$ -categories ∞ -categories.
- We start with an example of a pseudofunctor to motivate Grothendieck opfibrations. We explain the relationship between pseudofunctors into **Cat** and Grothendieck opfibrations and how to pass between the two.
- We define a symmetric monoidal category (\mathcal{C}, \otimes) in the usual way, then note that it can be written as a pseudofunctor to **Cat**, and thus as a Grothendieck opfibration. We construct \mathcal{C}^{\otimes} using the process outlined earlier, and compare it to Lurie's construction.
- We define symmetric monoidal functors in the usual way, and then construct the correct definitions in terms of morphisms of opfibrations.
- We introduce inner fibrations and prove that they are stable under pullbacks and that the fibres are ∞ -categories. We introduce (co)cartesian fibrations and prove that the nerve of a functor is a (co)cartesian fibration if and only if the original functor was an (op)fibration.
- We finally define symmetric monoidal ∞ -categories, and functors between them. We give the trivial examples: the nerve of a symmetric monoidal category, and the symmetric monoidal (co)cartesian structure on an ∞ -category with finite products.
- We give an interesting example: the derived category.
- References: [1], [2], [3], [4], [5], [6], [7], [8], [9], [10]. Referencing Kerodon test: [9, Example 01UB].

2 Symmetric monoidal 1-categories

2.1 The Grothendieck construction

(Something about motivation and the pain of higher coherences. Below we give an example to complement [3].)

Let \mathcal{C} be a category with pullbacks. Recall that for a map $f : C \rightarrow D$ in \mathcal{C} , we may define a pullback functor

$$\begin{aligned} f^* : \mathcal{C}_{/D} &\rightarrow \mathcal{C}_{/C}, \\ (h : X \rightarrow D) &\mapsto (f^*h : P \rightarrow C), \end{aligned}$$

where we have formed a pullback

$$\begin{array}{ccc} P & \xrightarrow{h^*f} & X \\ f^*h \downarrow \lrcorner & & \downarrow h \\ C & \xrightarrow{f} & D \end{array}$$

in \mathcal{C} . For any map

$$\begin{array}{ccc} X & \xrightarrow{\phi} & X' \\ & \searrow h & \swarrow h' \\ & D & \end{array}$$

from h to h' in $\mathcal{C}_{/D}$, we define $f^*\phi$ to be the unique map making the diagram below commute.

$$\begin{array}{ccccc} X & \xrightarrow{\phi} & X' & & \\ & \searrow h & \swarrow h' & & \\ & D & & & \\ \uparrow h^*f & & \uparrow h'^*f & & \\ P & \xrightarrow{f^*\phi} & P' & & \\ & \searrow f^*h & \swarrow f^*h' & & \\ & C & & & \end{array}$$

Now, we may wish to define a functor

$$\begin{aligned} F : \mathcal{C}^{\text{op}} &\rightarrow \mathbf{Cat}, \\ C &\mapsto \mathcal{C}_{/C}, \end{aligned}$$

which sends a map $f : C \rightarrow D$ in \mathcal{C} to a pullback functor $f^* : \mathcal{C}_{/D} \rightarrow \mathcal{C}_{/C}$. However a problem arises when we check that F respects composition: suppose $f : C \rightarrow D$, $g : D \rightarrow E$ are maps in \mathcal{C} . Then

$$F(g \circ f)(h : X \rightarrow E) = (g \circ f)^*h : P \rightarrow C,$$

corresponding to the pullback

$$\begin{array}{ccc} P & \xrightarrow{h^*(g \circ f)} & X \\ (g \circ f)^*h \downarrow \lrcorner & & \downarrow h \\ C & \xrightarrow{g \circ f} & E \end{array}$$

in \mathcal{C} . On the other hand,

$$(F(g) \circ F(f))(h : X \rightarrow E) = f^*(g^*h) : P'' \rightarrow C,$$

which corresponds to the diagram below.

$$\begin{array}{ccccc} P'' & \xrightarrow{(g^*h)^*f} & P' & \xrightarrow{h^*g} & X \\ f^*(g^*h) \downarrow \lrcorner & & g^*h \downarrow \lrcorner & & \downarrow h \\ C & \xrightarrow{f} & D & \xrightarrow{g} & E \end{array}$$

The outer square is indeed a pullback square, since the inner two squares are, so we have a unique isomorphism $P \cong P''$. However, we do not in general have equality. This is because pullbacks are only unique up to unique isomorphism, and in defining a pullback functor we made arbitrary (and not necessarily compatible) choices of P, P' and P'' . Thus, we have not defined a functor $\mathcal{C}^{\text{op}} \rightarrow \mathbf{Cat}$, rather, we have defined what is known as a *pseudofunctor*; that is, a weak functor between 2-categories.

The above example is one way in which pseudofunctors into \mathbf{Cat} naturally arise; another common example is the pseudofunctor

$$\begin{aligned} \mathbf{CRing} &\rightarrow \mathbf{Cat} \\ R &\mapsto R\text{-}\mathbf{Mod}, \end{aligned}$$

which sends a ring homomorphism $\phi : R \rightarrow S$ to the functor $- \otimes_R S : R\text{-}\mathbf{Mod} \rightarrow S\text{-}\mathbf{Mod}$ (extension of scalars). However, to give the data of a pseudofunctor $F : \mathcal{C} \rightarrow \mathbf{Cat}$, we must specify not only the functions $\mathrm{ob}(\mathcal{C}) \rightarrow \mathrm{ob}(\mathbf{Cat})$ and $\mathrm{Hom}_{\mathcal{C}}(X, Y) \rightarrow \mathrm{Hom}_{\mathbf{Cat}}(F(X), F(Y))$ for each $X, Y \in \mathcal{C}$, but also natural isomorphisms

$$F(\mathrm{id}_X) \cong \mathrm{id}_{F(X)}, \quad F(g \circ f) \cong F(g) \circ F(f).$$

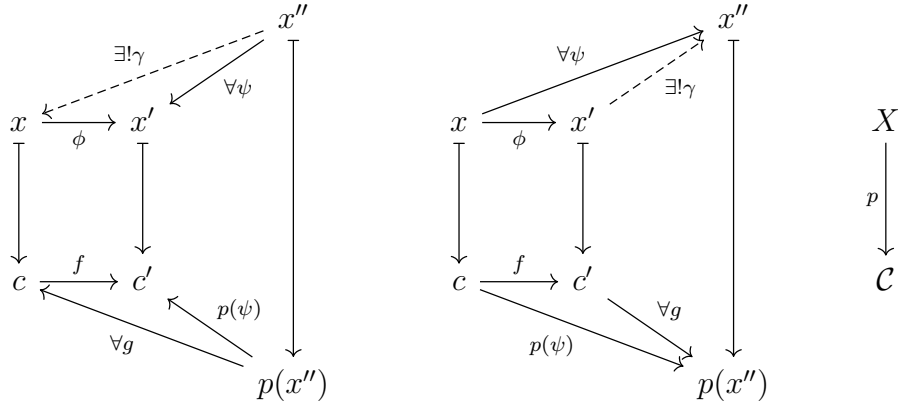
(This problem only becomes worse as we consider functors between higher categories... Let's use Grothendieck opfibrations instead.)

DEFINITION 2.1.1. Let $p : X \rightarrow \mathcal{C}$ be a functor, let $f : c \rightarrow c'$ be a morphism in \mathcal{C} , and let $\phi : x \rightarrow x'$ be a morphism in X lying over f .

We say that ϕ is *p-cartesian* if for any other morphism $\psi : x'' \rightarrow x'$ in X , and for any morphism $g : p(x'') \rightarrow c$ in \mathcal{C} satisfying $f \circ g = p(\psi)$, there exists a unique morphism $\gamma : x'' \rightarrow x$ such that $p(\gamma) = g$ and $\psi = \gamma \circ \phi$.

Dually, ϕ is *p-cocartesian* if for any other morphism $\psi : x \rightarrow x''$ in X , and for any morphism $g : c' \rightarrow p(x'')$ in \mathcal{C} satisfying $g \circ f = p(\psi)$, there exists a unique morphism $\gamma : x' \rightarrow x''$ such that $p(\gamma) = g$ and $\psi = \gamma \circ \phi$.

The left diagram below corresponds to a *p-cartesian* morphism, and the right diagram corresponds to a *p-cocartesian* morphism.



DEFINITION 2.1.2. Let $p : X \rightarrow \mathcal{C}$ be a functor. Then p is a *Grothendieck fibration* if for any morphism of \mathcal{C} and any lift of its target, there is a *p-cartesian* morphism with that target lying over it. Dually, p is a *Grothendieck opfibration* if for any morphism of \mathcal{C} and any lift of its source, there is a *p-cocartesian* morphism with that source lying over it.

We will usually refer to Grothendieck (op)fibrations as just (op)fibrations for brevity. Note that in [3] and [8] these are referred to as *(co)cartesian fibrations*; we reserve this term for the ∞ -category analogue.

REMARK 2.1.3. A functor $p : X \rightarrow \mathcal{C}$ is an opfibration if and only if $p^{\mathrm{op}} : X^{\mathrm{op}} \rightarrow \mathcal{C}^{\mathrm{op}}$ is a fibration. We give the definition of an opfibration explicitly, since we will be working with these more often than fibrations.

THEOREM 2.1.4 ([10], Thm 2.4). There is an equivalence of 2-categories

$$\mathbf{Psd}[\mathcal{C}^{\text{op}}, \mathbf{Cat}] \simeq \mathbf{Fib}(\mathcal{C}),$$

where $\mathbf{Psd}[\mathcal{C}^{\text{op}}, \mathbf{Cat}]$ is the 2-category of pseudofunctors $\mathcal{C}^{\text{op}} \rightarrow \mathbf{Cat}$, and $\mathbf{Fib}(\mathcal{C}) \hookrightarrow \mathbf{Cat}_{/\mathcal{C}}$ is the 2-category of fibrations into \mathcal{C} .

REMARK 2.1.5. Combining Remark 2.1.3 with Theorem 2.1.4 gives us a chain of equivalences

$$\mathbf{Psd}[\mathcal{C}, \mathbf{Cat}] \xrightarrow{\text{op} \circ} \mathbf{Psd}[\mathcal{C}, \mathbf{Cat}] \simeq \mathbf{Psd}[(\mathcal{C}^{\text{op}})^{\text{op}}, \mathbf{Cat}] \simeq \mathbf{Fib}(\mathcal{C}^{\text{op}}) \simeq \mathbf{opFib}(\mathcal{C}).$$

We will not prove the theorem above (see [10] or [5] for more details) but we will describe how to pass between $\mathbf{Psd}[\mathcal{C}, \mathbf{Cat}]$ and $\mathbf{opFib}(\mathcal{C})$ by dualising¹ the constructions in these sources.

Let $F : \mathcal{C} \rightarrow \mathbf{Cat}$ be a pseudofunctor. Define the category X as follows: the objects of X are pairs (c, x) , with $c \in \mathcal{C}$, $x \in F(c)$. A map $(c, x) \rightarrow (d, y)$ is a pair (f, u) , where $f : c \rightarrow d$ is a morphism in \mathcal{C} , and $u : (Ff)(x) \rightarrow y$ is a morphism in $F(d)$. For an object $(c, x) \in X$, the identity morphism is given by

$$(\text{id}_c, F_x^0 : F(\text{id}_c)(x) \rightarrow x),$$

where F^0 is the natural isomorphism $F(\text{id}_c) \cong \text{id}_{F(c)}$. Further, given two maps $(f, u) : (c, x) \rightarrow (d, y)$ and $(g, v) : (d, y) \rightarrow (e, z)$, their composition $(g, v) \circ (f, u)$ is given by $g \circ f$, together with the map

$$(F(g \circ f))(x) \xrightarrow{(F_{g,f}^2)_x} (Fg \circ Ff)(x) \xrightarrow{(Fg)(u)} (Fg)(y) \xrightarrow{v} z,$$

where $F_{g,f}^2$ is the natural isomorphism $F(g \circ f) \cong Fg \circ Ff$.

LEMMA 2.1.6. The forgetful functor $p : X \rightarrow \mathcal{C}$ is an opfibration over \mathcal{C} .

PROOF. Let $f : c \rightarrow d$ be a morphism in \mathcal{C} , and let x be a lift of its source. Consider the map $(f, \text{id}_{(Ff)(x)}) : (c, x) \rightarrow (d, (Ff)(x))$. Let $(g, \psi) : (c, x) \rightarrow (e, z)$ be a map in X , and let $h : d \rightarrow e$ be a map in \mathcal{C} such that $g = h \circ f$. We wish to find some map $\gamma : (Fh \circ Ff)(x) \rightarrow z$ such that the composition

$$F(h \circ f)(x) \xrightarrow{F_{h,f}^2} (Fh \circ Ff)(x) \xrightarrow{(Fh)(\text{id}_{(Ff)(x)})} (Fh \circ Ff)(x) \xrightarrow{\gamma} z$$

is equal to the map

$$(Fg)(x) \xrightarrow{\psi} z.$$

Since $h \circ f = g$, and $(Fh)(\text{id}_{(Ff)(x)}) = \text{id}_{(Fh \circ Ff)(x)}$, we see that $\psi = \gamma \circ F_{h,f}^2$. Since F^2 is a natural isomorphism, we have $\gamma = \psi \circ (F_{h,f}^2)^{-1}$, so γ exists and is unique. Therefore, for every morphism in \mathcal{C} and every lift of its source, there is a cocartesian morphism lying over it; that is, $p : X \rightarrow \mathcal{C}$ is an opfibration. \square

Now, let $p : X \rightarrow \mathcal{C}$ be an opfibration over \mathcal{C} . Define a pseudofunctor

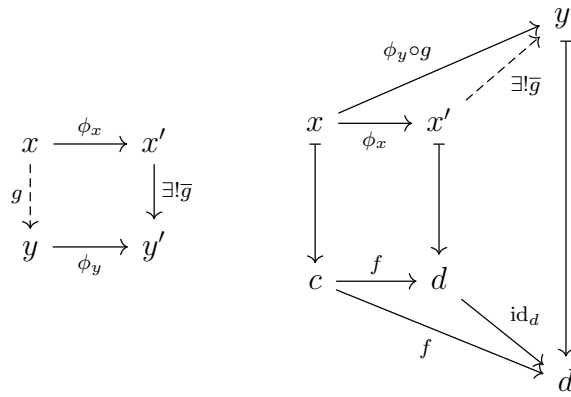
$$\begin{aligned} \mathcal{C} &\rightarrow \mathbf{Cat} \\ c &\mapsto p^{-1}\{c\}. \end{aligned}$$

¹We use the equivalences in Remark 2.1.5; in particular we postcompose with the equivalence $\text{op} : \mathbf{Cat} \rightarrow \mathbf{Cat}$. This is to ensure that we can define the pseudofunctor $F : \mathcal{C} \rightarrow \mathbf{Cat}$ to send $c \in \mathcal{C}$ to $p^{-1}\{c\}$ rather than $(p^{\text{op}})^{-1}\{c\} \simeq (p^{-1}\{c\})^{\text{op}}$.

For a map $f : c \rightarrow d$ in \mathcal{C} , we define a functor

$$f_* : p^{-1}\{c\} \rightarrow p^{-1}\{d\},$$

which sends $x \in p^{-1}\{c\}$ to the target x' of a cocartesian edge $\phi_x : x \rightarrow x'$ lying over f . Now, let $f^*(y) = y'$, with $\phi_y : y \rightarrow y'$, and let $g : x \rightarrow y$ be a morphism in $p^{-1}\{c\}$ (that is, a morphism in X lying over id_c). Then, since $\phi_y : y \rightarrow y'$ is a cocartesian edge, there is a unique lift $\bar{g} : x' \rightarrow y'$ of id_d making the square on the left commute, as shown in the diagram on the right.



We thus define $f_*g = \bar{g}$ in $p^{-1}\{d\}$. It can be shown that these data assemble into a pseudofunctor $\mathcal{C} \rightarrow \mathbf{Cat}$.

2.2 Symmetric monoidal categories and functors

DEFINITION 2.2.1. A *symmetric monoidal category* is a category \mathcal{C} equipped with a bifunctor

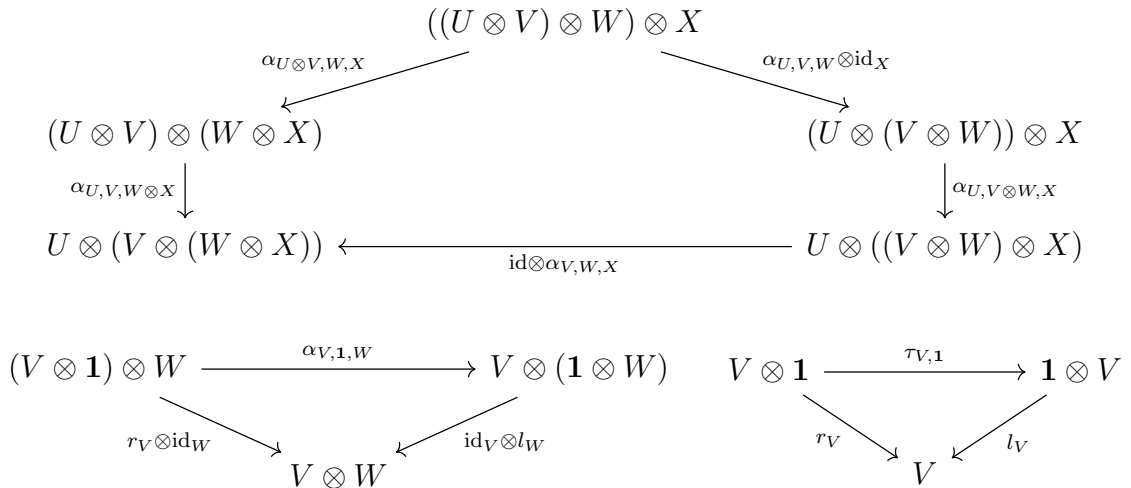
$$\otimes : \mathcal{C} \times \mathcal{C} \rightarrow \mathcal{C},$$

an object $\mathbf{1} \in \mathcal{C}$, and natural isomorphisms

$$\alpha : \otimes \circ (\otimes \times \text{id}) \xrightarrow{\sim} \otimes \circ (\text{id} \times \otimes), \quad l : \otimes \circ (\mathbf{1} \times \text{id}) \xrightarrow{\sim} \text{id}, \quad r : \otimes \circ (\text{id} \times \mathbf{1}) \xrightarrow{\sim} \text{id},$$

$$\tau : \otimes \xrightarrow{\sim} \sigma \circ \otimes,$$

where $\sigma : \mathcal{C} \times \mathcal{C}$ swaps the order of the factors. These isomorphisms are subject to the condition that $\tau^2 = \text{id}$, and that the diagrams below commute.



$$\begin{array}{ccc}
& (U \otimes V) \otimes W & \\
\alpha_{U,V,W} \swarrow & & \searrow \tau_{U,V} \otimes \text{id}_W \\
U \otimes (V \otimes W) & & (V \otimes U) \otimes W \\
\tau_{U,V} \otimes \text{id}_W \downarrow & & \downarrow \alpha_{V,U,W} \\
(V \otimes W) \otimes U & & V \otimes (U \otimes W) \\
\alpha_{V,W,U} \searrow & & \swarrow \text{id}_V \otimes \tau_{U,W} \\
& V \otimes (W \otimes U) &
\end{array}$$

Now that we know how to move between pseudofunctors to **Cat** and opfibrations, let's write the data of a symmetric monoidal category as a pseudofunctor to **Cat**.

Let (\mathcal{C}, \otimes) be a symmetric monoidal category. Define a pseudofunctor

$$\begin{aligned}
F : \mathbf{Fin}_* &\rightarrow \mathbf{Cat} \\
\langle n \rangle &\mapsto \mathcal{C}^{\times n}
\end{aligned}$$

Let $f : \langle n \rangle \rightarrow \langle m \rangle$ be a morphism in \mathbf{Fin}_* . This induces a morphism

$$f^* : (C_1, \dots, C_n) \mapsto (C'_1, \dots, C'_m),$$

where

$$C'_i = \bigotimes_{j \in f^{-1}\{i\}} C_j.$$

- Possibly mention swapping out **Fin**_{*} for Δ^{op} gives a monoidal category rather than a symmetric monoidal category. (How do we get a braided monoidal category? Apparently there is no base 1-category we can look at opfibrations into, because the correct formulation is with E_2 , which has higher homotopy groups on the mapping spaces.)
- Correspondence of symmetric monoidal functors with morphisms of opfibrations.

Using the construction above, we can obtain a category \mathcal{C}^{\otimes} and an opfibration $p : \mathcal{C}^{\otimes} \rightarrow \mathbf{Fin}_*$ corresponding to the pseudofunctor encoding (\mathcal{C}, \otimes) . Unravelling the definitions gives exactly the construction below, in [7].

Let (\mathcal{C}, \otimes) be a symmetric monoidal category. We define a new category \mathcal{C}^{\otimes} , whose objects are finite (possibly empty) sequences of objects of \mathcal{C} , denoted by $[C_1, \dots, C_n]$. A morphism

$$[C_1, \dots, C_n] \rightarrow [C'_1, \dots, C'_m]$$

consists of a subset $S \subseteq \{1, \dots, n\}$, a map of finite sets $\alpha : S \rightarrow \{1, \dots, m\}$, and a collection of morphisms $\{f_j : \bigotimes_{i \in \alpha^{-1}\{j\}} C_i \rightarrow C'_j\}_{1 \leq j \leq m}$ in \mathcal{C} .

For two morphisms $f : [C_1, \dots, C_n] \rightarrow [C'_1, \dots, C'_m]$ and $g : [C'_1, \dots, C'_m] \rightarrow [C''_1, \dots, C''_l]$, determining two subsets $S \subseteq \{1, \dots, n\}$ and $T \subseteq \{1, \dots, m\}$ and maps $\alpha : S \rightarrow \{1, \dots, m\}$, $\beta : T \rightarrow \{1, \dots, l\}$, the composition $g \circ f$ is given by the subset $U = \alpha^{-1}T \subseteq \{1, \dots, n\}$, the map $\beta \circ \alpha : U \rightarrow \{1, \dots, l\}$ and the maps

$$\left\{ \bigotimes_{i \in (\beta \circ \alpha)^{-1}\{k\}} C_i \cong \bigotimes_{j \in \beta^{-1}\{k\}} \bigotimes_{i \in \alpha^{-1}\{j\}} C_i \rightarrow \bigotimes_{j \in \beta^{-1}\{k\}} C'_j \rightarrow C''_k \right\}_{1 \leq k \leq l}.$$

For example, let

$$f : [C_1, C_2, C_3, C_4] \rightarrow [C'_1, C'_2, C'_3]$$

be a morphism in \mathcal{C}^\otimes consisting of the subset $\{1, 2, 3\} \subseteq \{1, 2, 3, 4\}$, the map

$$\begin{aligned} \alpha : \{1, 2, 3\} &\rightarrow \{1, 2, 3\}, \\ 1 &\mapsto 1, \\ 2 &\mapsto 2, \\ 3 &\mapsto 3, \end{aligned}$$

and morphisms

$$f_1 : C_1 \rightarrow C'_1, \quad f_2 : C_2 \otimes C_3 \rightarrow C'_2, \quad f_3 : \mathbf{1} \rightarrow C'_3,$$

and let

$$g : [C'_1, C'_2, C'_3] \rightarrow [C''_1, C''_2, C''_3]$$

be a morphism in \mathcal{C}^\otimes consisting of the subset $\{1, 2, 3\} \subseteq \{1, 2, 3\}$, the map

$$\begin{aligned} \beta : \{1, 2, 3\} &\rightarrow \{1, 2, 3\}, \\ 1, 2, 3 &\mapsto 3, \end{aligned}$$

and morphisms

$$g_1 : \mathbf{1} \rightarrow C''_1, \quad g_2 : \mathbf{1} \rightarrow C''_2, \quad g_3 : C'_1 \otimes C'_2 \otimes C'_3 \rightarrow C''_3.$$

Then the composition $g \circ f$ consists of the subset $\alpha^{-1}\{1, 2, 3\} = \{1, 2, 3\} \subseteq \{1, 2, 3, 4\}$, the map

$$\begin{aligned} \beta \circ \alpha : \{1, 2, 3\} &\rightarrow \{1, 2, 3\}, \\ 1, 2, 3 &\mapsto 3, \end{aligned}$$

and the morphisms

$$(g \circ f)_1 = g_1, \quad (g \circ f)_2 = g_2, \quad (g \circ f)_3 = g_3 \circ (f_1 \otimes f_2 \otimes f_3).$$

(really?)

(some intuition on this, tensor along the fibres, etc)

Claim: the forgetful functor

$$\begin{aligned} p : \mathcal{C}^\otimes &\rightarrow \mathbf{Fin}_*, \\ [C_1, \dots, C_n] &\mapsto \langle n \rangle_* \end{aligned}$$

is an opfibration. (It almost tautologically is).

PROPOSITION 2.2.2 ([4], Prop 4.26). If (\mathcal{C}, \otimes) is a symmetric monoidal category, then the forgetful functor $p : \mathcal{C}^\otimes \rightarrow \mathbf{Fin}_*$ given above is an opfibration. Moreover, p satisfies the *Segal condition*; that is, the Segal maps

$$(\rho_!^1, \dots, \rho_!^n) : \mathcal{C}_{\langle n \rangle}^\otimes \rightarrow \mathcal{C}^{\times n}$$

are equivalences. Conversely any Grothendieck opfibration $p : \mathcal{C} \rightarrow \mathbf{Fin}_*$ satisfying the Segal condition gives rise to a symmetric monoidal structure on $\mathcal{C}_{\langle 1 \rangle}$.

Summary: we noticed that pseudofunctors $\mathcal{D} \rightarrow \mathbf{Cat}$ are the same thing as opfibrations into \mathcal{D} . We also noticed that symmetric monoidal categories could be written as special pseudofunctors

$\mathbf{Fin}_* \rightarrow \mathbf{Cat}$, which means they are special opfibrations into \mathbf{Fin}_* . We looked at the corresponding construction of \mathcal{C}^\otimes . So: symmetric monoidal categories, special opfibrations into \mathbf{Fin}_* and special pseudofunctors $\mathbf{Fin}_* \rightarrow \mathbf{Cat}$ are all the same thing, we just hide the coherences in the opfibrations.

We've said that symmetric monoidal categories are special opfibrations to \mathbf{Fin}_* . Now, what are symmetric monoidal functors?

The definition below is basically definition 3.3 of [3].

DEFINITION 2.2.3. Let $p : X \rightarrow \mathcal{C}$ and $q : Y \rightarrow \mathcal{C}$ be two Grothendieck (op)fibrations. A functor $F : X \rightarrow Y$ is a *morphism of (op)fibrations* from p to q if the diagram below commutes,

$$\begin{array}{ccc} X & \xrightarrow{F} & Y \\ & \searrow p & \swarrow q \\ & \mathcal{C} & \end{array}$$

and F sends p -(co)cartesian morphisms to q -(co)cartesian morphisms.

3 Generalisation to ∞ -categories

- If an ∞ -category has finite (co)products, there is a (co)cartesian monoidal structure on \mathcal{C} . And we would have hoped so, because it's definitely true for 1-categories!
- Might be cool to try to look at E_k algebras, to resolve the earlier mystery of how to write braided monoidal categories.

We first need an ∞ -categorical analogue of Grothendieck opfibrations. We start by requiring that our functor is what's known as an *inner fibration*; there is no 1-categorical analogue of this, since all functors between 1-categories are automatically inner fibrations under the nerve functor (see Example 3.0.2). Think of it as a 'minimum niceness condition' – we want the fibres to be ∞ -categories in much the same way as we want the fibres of ordinary functors to be categories themselves.

DEFINITION 3.0.1 ([1], Def 2.1). A functor $p : X \rightarrow Y$ between simplicial sets is an *inner fibration* if for all $n \geq 2$, all $0 < k < n$, and any solid arrow commutative square

$$\begin{array}{ccc} \Lambda_k^n & \longrightarrow & X \\ \downarrow & \nearrow \text{dotted} & \downarrow p \\ \Delta^n & \longrightarrow & Y \end{array}$$

there exists a dotted lift.

EXAMPLE 3.0.2. Let \mathcal{C}, \mathcal{D} be categories, and $p : \mathcal{C} \rightarrow \mathcal{D}$ be a functor between them. Then $N(p) : N\mathcal{C} \rightarrow N\mathcal{D}$ is an inner fibration.

The following proposition is stated without proof in Section 2.3 of [8].

PROPOSITION 3.0.3. Let $p : X \rightarrow Y$ be an inner fibration, and suppose that the diagram below is a pullback square in \mathbf{sSet} .

$$\begin{array}{ccc} X' & \xrightarrow{f} & X \\ p' \downarrow \lrcorner & & \downarrow p \\ Y' & \xrightarrow{g} & Y \end{array}$$

Then p' is also an inner fibration.

PROOF. Consider the (commutative) solid arrow diagram below.

$$\begin{array}{ccccc}
 \Lambda_k^n & \xrightarrow{\lambda} & X' & \xrightarrow{f} & X \\
 \downarrow \iota & & \downarrow p' & \dashrightarrow & \downarrow p \\
 \Delta^n & \xrightarrow{\delta} & Y' & \xrightarrow{g} & Y
 \end{array}$$

Since p is a fibration, there exists a dotted lift ϕ of $g\delta$; that is, $p\phi = g\delta$ and $\phi\iota = f\lambda$. Further, since the right square is a pullback diagram, there exists a unique map $\phi' : \Delta^n \rightarrow X'$ making the diagram below commute.

$$\begin{array}{ccccc}
 \Delta^n & & & \xrightarrow{\phi} & X \\
 \downarrow \delta & \dashrightarrow \phi' & X' & \xrightarrow{f} & \downarrow p \\
 & & \downarrow p' & \dashrightarrow & Y
 \end{array}$$

It remains to show that the triangle below commutes.

$$\begin{array}{ccc}
 \Lambda_k^n & \xrightarrow{\lambda} & X' \\
 \downarrow \iota & \dashrightarrow \phi' & \\
 \Delta^n & &
 \end{array}$$

Again, using the universal property of pullbacks, we see that there exist unique dotted maps such that the diagrams below commute.

$$\begin{array}{ccc}
 \Lambda_k^n & \xrightarrow{f\lambda} & X \\
 \downarrow \delta\iota & \dashrightarrow & \downarrow p \\
 & & Y
 \end{array}
 \quad
 \begin{array}{ccc}
 \Lambda_k^n & \xrightarrow{f\phi'\iota} & X \\
 \downarrow \delta\iota & \dashrightarrow & \downarrow p \\
 & & Y
 \end{array}$$

The maps λ and $\phi'\iota$ make the left and right diagrams commute respectively. Further, we note that $f\phi' = \phi$ (by the second diagram) and $\phi\iota = f\lambda$ (since p is an inner fibration), so $f\phi'\iota = f\lambda$. Therefore, the above two diagrams are identical. Thus, by the uniqueness property of pullbacks, $\lambda = \phi'\iota$. \square

(Stupid note to self, very obvious but I forget it every now and again):

- If $X : \Delta^{\text{op}} \rightarrow \mathbf{Set}$ is a simplicial set, and $\Delta^0 : \Delta^{\text{op}} \rightarrow \mathbf{Set} := \text{Hom}(-, [0])$, then a map $F : X \rightarrow \Delta^0$ is a natural transformation $(F_n : X_n \rightarrow *)_{n \in \mathbb{N}_0}$. That is, such a natural transformation is a family of maps down to a point. In other words, there's only really one natural transformation, so we really *can* view Δ^0 as a point.
- If Y is a simplicial set, and $y \in Y_0$ is a vertex of Y , we can view $\{y\}$ as a copy of Δ^0 . Why is this? We can view $\{y\}$ as the constant simplicial set, sending everything to y . Then a natural isomorphism $\Delta^0 \cong \{y\}$ is a collection of isomorphisms $(* \rightarrow *)$, of which there is exactly one. Why is it natural? Well, there's only one map from a one-point set to another one-point set, so the square always commutes.

EXAMPLE 3.0.4 ([1], Ex 2.2). Let $p : X \rightarrow \Delta^0$ be the canonical map, and suppose we have the diagram below, such that the outer square commutes.

$$\begin{array}{ccc} \Lambda_k^n & \longrightarrow & X \\ \downarrow & \nearrow & \downarrow p \\ \Delta^n & \longrightarrow & \Delta^0 \end{array}$$

The lower triangle commutes automatically, so the statement that p is an inner fibration is equivalent to the statement that for all $n \geq 2$, all $0 < k < n$, and any map $\Lambda_k^n \rightarrow X$, there exists a dotted lift.

$$\begin{array}{ccc} \Lambda_k^n & \longrightarrow & X \\ \downarrow & \nearrow & \\ \Delta^n & & \end{array}$$

That is, X is an ∞ -category.

Now, combining the above argument with Proposition 3.0.3, we see that for any inner fibration $p : X \rightarrow Y$, each fibre $X \times_Y \{y\}$ is an ∞ -category.

DEFINITION 3.0.5 ([1], Def 3.1). Let $p : X \rightarrow Y$ be an inner fibration. An edge $f : \Delta^1 \rightarrow X$ of X is *p-cocartesian* if for all $n \geq 2$, any extension

$$\begin{array}{ccc} \Delta^{\{0,1\}} & \xrightarrow{f} & X \\ \downarrow & \nearrow F & \\ \Lambda_0^n & & \end{array}$$

and any solid arrow commutative diagram

$$\begin{array}{ccc} \Lambda_0^n & \xrightarrow{F} & X \\ \downarrow & \nearrow & \downarrow p \\ \Delta^n & \longrightarrow & Y \end{array}$$

a dotted lift exists.

DEFINITION 3.0.6. Let $p : X \rightarrow Y$ be an inner fibration. Then p is a cocartesian fibration if for any edge $\phi : y \rightarrow y'$ in Y_1 , and for every $x \in X_0$ lying over y , there exists a p -cocartesian edge $f : x \rightarrow x'$ of X lying over ϕ .

The following proposition tells us that the above definition is a reasonable generalisation of Definition 2.1.2. It is also stated without proof in [8], which did not do wonders for my ego.

PROPOSITION 3.0.7 ([8], Rmk 2.4.2.2). Let \mathcal{C}, \mathcal{D} be categories, and let $p : \mathcal{C} \rightarrow \mathcal{D}$ be a functor between them. Then p is a Grothendieck opfibration if and only if the induced map $N(p) : N\mathcal{C} \rightarrow N\mathcal{D}$ is a cocartesian fibration of simplicial sets.

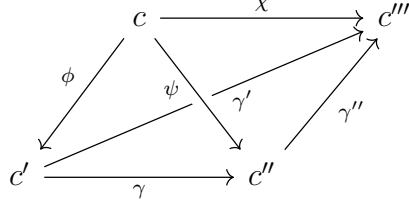
PROOF. Let $f : d \rightarrow d'$ be a morphism of \mathcal{D} , and let c lie over d .

Suppose p is a Grothendieck opfibration, let $F : \Lambda_0^n \rightarrow N\mathcal{C}$ be an extension of f , and let

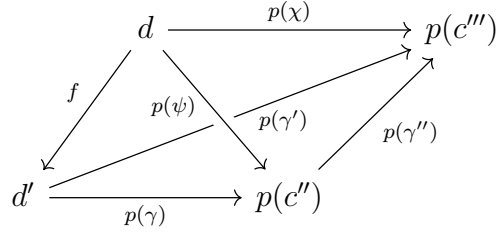
$$\begin{array}{ccc} \Lambda_0^n & \xrightarrow{F} & N\mathcal{C} \\ \downarrow & & \downarrow N(p) \\ \Delta^n & \longrightarrow & N\mathcal{D} \end{array}$$

be a commutative diagram. If $n = 2$, it follows immediately from the fact that p is an opfibration that a dotted lift exists. Further, if $n > 3$, there is nothing to check, since an n -simplex in a category commutes if and only if all of its triangles commute, which is guaranteed for any extension $F : \Lambda_0^n \rightarrow N\mathcal{C}$. We thus prove the proposition for $n = 3$.

Suppose we have an extension $F : \Lambda_0^3 \rightarrow N\mathcal{C}$ of f ; that is, a tetrahedron



such that all faces containing the vertex c commute. Let

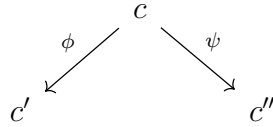


be a commutative tetrahedron in \mathcal{D} . We claim that the tetrahedron in \mathcal{C} commutes. First, note that $\gamma'' \circ \gamma$ is a lift of $p(\gamma')$, since $p(\gamma') = p(\gamma'') \circ p(\gamma) = p(\gamma'' \circ \gamma)$. Further,

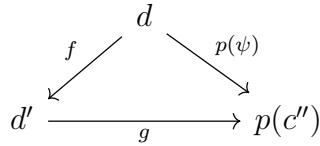
$$\begin{aligned} (\gamma \circ \gamma'') \circ \phi &= \gamma'' \circ \psi \\ &= \chi. \end{aligned}$$

Thus, by the uniqueness in the universal property of ϕ , we have that $\gamma' = \gamma'' \circ \gamma$, as required.

Now, suppose $N(p)$ is a cocartesian fibration. Then there exists a lift $\phi : c \rightarrow c'$ of f , and, in particular, for any diagram

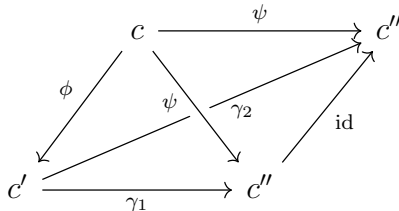


in \mathcal{C} , and any commutative diagram



in \mathcal{D} , there exists a map $\gamma : c' \rightarrow c''$ such that γ lies over g and $\gamma \circ \phi = \psi$. It remains to show that γ is unique.

Suppose that there were two maps $\gamma_1, \gamma_2 : c' \rightarrow c''$ lying over g and satisfying $\gamma_1 \circ \phi = \gamma_2 \circ \phi = \psi$. Then we would have a tetrahedron



where all faces containing the vertex c commute. The image of this tetrahedron under p commutes in \mathcal{D} , so the original tetrahedron must commute in \mathcal{C} ; that is, $\gamma_1 = \gamma_2$. \square

DEFINITION 3.0.8 ([8], Def 2.0.0.7). A *symmetric monoidal ∞ -category* is a cocartesian fibration of simplicial sets $p : X^\otimes \rightarrow N(\mathbf{Fin}_*)$ such that for each $n \geq 0$, the maps

$$\{\rho^i : \langle n \rangle \rightarrow \langle 1 \rangle\}_{1 \leq i \leq n}$$

induce functors $\rho_i^! : X_{\langle n \rangle}^\otimes \rightarrow X_{\langle 1 \rangle}^\otimes$ which determine an equivalence $X_{\langle n \rangle}^\otimes \simeq \left(X_{\langle 1 \rangle}^\otimes\right)^n$.

EXAMPLE 3.0.9. Let (\mathcal{C}, \otimes) be a symmetric monoidal category. Then $p : N(\mathcal{C}^\otimes) \rightarrow N(\mathbf{Fin}_*)$ is a symmetric monoidal ∞ -category.

4 A nontrivial example

Throughout this section, \mathcal{A} is an abelian category, and $\mathcal{A}_{\text{proj}}$ is the full subcategory of \mathcal{A} spanned by the projective objects.

DEFINITION 4.0.1 ([7], Def 1.2.3.1). A *chain complex* with values in \mathcal{A} is a composable sequence of morphisms

$$\cdots \rightarrow A_2 \xrightarrow{d(2)} A_1 \xrightarrow{d(1)} A_0 \xrightarrow{d(0)} A_{-1} \rightarrow \cdots$$

in \mathcal{A} such that $d(n-1) \circ d(n) = 0$ for all $n \in \mathbb{Z}$. The collection of chain complexes with values in \mathcal{A} is an additive category, $\text{Ch}(\mathcal{A})$.

DEFINITION 4.0.2 ([7], Not 1.3.2.6). $\text{Ch}^-(\mathcal{A})$ is the full subcategory of $\text{Ch}(\mathcal{A})$ spanned by those chain complexes M_* such that $M_n \simeq 0$ for $n \ll 0$.

DEFINITION 4.0.3 ([7], Def 1.3.2.7). Suppose \mathcal{A} has enough projective objects. We let $\mathcal{D}^-(\mathcal{A})$ denote the ∞ -category $N_{\text{dg}}(\text{Ch}^-(\mathcal{A}_{\text{proj}}))$. We refer to $\mathcal{D}^-(\mathcal{A})$ as the *derived ∞ -category of \mathcal{A}* .

5 Miscellaneous stupid notes

5.1 Observations

Let $S \in \mathbf{Set}$. We define the constant simplicial set

$$\overline{S} : \Delta^{\text{op}} \longrightarrow \mathbf{Set}$$

$$\begin{array}{ccc} [n] & \xrightarrow{\quad} & S \\ f \downarrow & & \uparrow \text{id} \\ [m] & \xrightarrow{\quad} & S \end{array}$$

It's a Kan complex. Why? Well, when you consider S as a discrete category, and take the nerve of it, you get \overline{S} . You can then either just see that it's a Kan complex (fill the horns with identities) or use the fact that the nerve of a groupoid is a Kan complex. It's surely in [8] or [9] somewhere.

Here are the original definitions of the Grothendieck equivalence (before I dualised them, just in case I did it wrong).

Let $F : \mathcal{C}^{\text{op}} \rightarrow \mathbf{Cat}$ be a pseudofunctor. Define the category X as follows: the objects of X are pairs (c, x) , with $c \in \mathcal{C}$, $x \in F(c)$. A map $(c, x) \rightarrow (d, y)$ is a pair (f, u) , where $f : c \rightarrow d$ is a morphism in \mathcal{C} , and $u : x \rightarrow (Ff)(y)$ is a morphism in $F(c)$. For an object $(c, x) \in X$, the identity morphism is given by

$$(\text{id}_c, F_x^0 : x \rightarrow F(\text{id}_c)(x)),$$

where F^0 is the natural isomorphism $\text{id}_{F(c)} \cong F(\text{id}_c)$. Further, given two maps $(f, u) : (c, x) \rightarrow (d, y)$ and $(g, v) : (d, y) \rightarrow (e, z)$, their composition $(g, v) \circ (f, u)$ is given by $g \circ f$, together with the map

$$x \xrightarrow{u} (Ff)(y) \xrightarrow{(Ff)(v)} (Ff \circ Fg)(z) \xrightarrow{(F_{f,g}^2)_z} (F(g \circ f))(z),$$

where $F_{f,g}^2$ is the natural isomorphism $Ff \circ Fg \cong F(g \circ f)$ (recall that the domain of F is \mathcal{C}^{op} , while f, g are morphisms in \mathcal{C}). One can show that the forgetful functor $X \rightarrow \mathcal{C}$ is a fibration over \mathcal{C} .

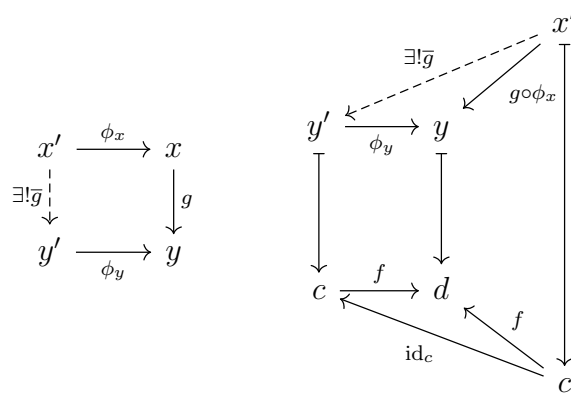
Now, let $p : X \rightarrow \mathcal{C}$ be a fibration over \mathcal{C} . Define a pseudofunctor

$$\begin{aligned} \mathcal{C}^{\text{op}} &\rightarrow \mathbf{Cat} \\ c &\mapsto p^{-1}\{c\}. \end{aligned}$$

For a map $f : c \rightarrow d$ in \mathcal{C} , we define a functor

$$f^* : p^{-1}\{d\} \rightarrow p^{-1}\{c\},$$

which sends $x \in p^{-1}\{d\}$ to the source x' of a cartesian edge $\phi_x : x' \rightarrow x$ lying over f . Now, let $f^*(y) = y'$, with $\phi_y : y' \rightarrow y$, and let $g : x \rightarrow y$ be a morphism in $p^{-1}\{d\}$ (that is, a morphism in X lying over id_c). Then, since $\phi_y : y' \rightarrow y$ is a cartesian edge, there is a unique lift $\bar{g} : x' \rightarrow y'$ of id_c making the square on the left commute, as shown in the diagram on the right.



We thus define $f^*g = \bar{g}$ in $p^{-1}\{c\}$.

It can be shown that these data assemble into a pseudofunctor $\mathcal{C} \rightarrow \mathbf{Cat}$.

Here is the dualisation of the forward direction, which I think was correct, but actually not what I wanted:

Let $F : \mathcal{C} \rightarrow \mathbf{Cat}$ be a pseudofunctor. Define the category X as follows: the objects of X are pairs (c, x) , with $c \in \mathcal{C}$, $x \in F(c)$. A map $(c, x) \rightarrow (d, y)$ is a pair (f, u) , where $f : c \rightarrow d$ is

a morphism in \mathcal{C} , and $u : y \rightarrow (Ff)(x)$ is a morphism in $F(d)$. For an object $(c, x) \in X$, the identity morphism is given by

$$(\text{id}_c, F_x^0 : x \rightarrow F(\text{id}_c)(x)),$$

where F^0 is the natural isomorphism $\text{id}_{F(c)} \cong F(\text{id}_c)$. Further, given two maps $(f, u) : (c, x) \rightarrow (d, y)$ and $(g, v) : (d, y) \rightarrow (e, z)$, their composition $(g, v) \circ (f, u)$ is given by $g \circ f$, together with the map

$$z \xrightarrow{v} (Fg)(y) \xrightarrow{(Fg)(u)} (Fg \circ Ff)(x) \xrightarrow{(F_{g,f}^2)_x} (F(g \circ f))(x),$$

where $F_{g,f}^2$ is the natural isomorphism $Fg \circ Ff \cong F(g \circ f)$. One can show that the forgetful functor $X \rightarrow \mathcal{C}$ is an opfibration over \mathcal{C} .

5.2 Questions

Questions:

- ...what *is* \mathbf{Grpd}_∞ ?

5.3 Equivalent definitions

\mathbf{Grpd}_∞

An algebraic category

An equivalence of ∞ -categories

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