

# Symmetric Monoidal $\infty$ -Categories

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## 1 Introduction

- Note: we will call  $(\infty, 1)$ -categories  $\infty$ -categories.
- We start with an example of a pseudofunctor to motivate Grothendieck opfibrations. We explain the relationship between pseudofunctors into **Cat** and Grothendieck opfibrations and how to pass between the two.
- We define a symmetric monoidal category  $(\mathcal{C}, \otimes)$  in the usual way, then note that it can be written as a pseudofunctor to **Cat**, and thus as a Grothendieck opfibration. We construct  $\mathcal{C}^{\otimes}$  using the process outlined earlier, and compare it to Lurie's construction.
- We define symmetric monoidal functors in the usual way, and then construct the correct definitions in terms of morphisms of opfibrations.
- We introduce inner fibrations and prove that they are stable under pullbacks and that the fibres are  $\infty$ -categories. We introduce (co)cartesian fibrations and prove that the nerve of a functor is a (co)cartesian fibration if and only if the original functor was an (op)fibration.
- We finally define symmetric monoidal  $\infty$ -categories, and functors between them. We give the trivial examples: the nerve of a symmetric monoidal category, and the symmetric monoidal (co)cartesian structure on an  $\infty$ -category with finite products.
- We give an interesting example: the derived category.
- References: [1], [2], [3], [4], [5], [6], [7], [8], [9], [10]. Referencing Kerodon test: [9, Example 01UB].

## 2 Symmetric monoidal 1-categories

### 2.1 The Grothendieck construction

(Something about motivation and the pain of higher coherences. Below we give an example to complement [3].)

Let  $\mathcal{C}$  be a category with pullbacks. Recall that for a map  $f : C \rightarrow D$  in  $\mathcal{C}$ , we may define a pullback functor

$$\begin{aligned} f^* : \mathcal{C}_{/D} &\rightarrow \mathcal{C}_{/C}, \\ (h : X \rightarrow D) &\mapsto (f^*h : P \rightarrow C), \end{aligned}$$

where we have formed a pullback

$$\begin{array}{ccc} P & \xrightarrow{h^*f} & X \\ f^*h \downarrow \lrcorner & & \downarrow h \\ C & \xrightarrow{f} & D \end{array}$$

in  $\mathcal{C}$ . For any map

$$\begin{array}{ccc} X & \xrightarrow{\phi} & X' \\ & \searrow h & \swarrow h' \\ & D & \end{array}$$

from  $h$  to  $h'$  in  $\mathcal{C}_{/D}$ , we define  $f^*\phi$  to be the unique map making the diagram below commute.

$$\begin{array}{ccccc} X & \xrightarrow{\phi} & X' & & \\ & \searrow h & \swarrow h' & & \\ & D & & & \\ \uparrow h^*f & & \uparrow h'^*f & & \\ P & \xrightarrow{f^*\phi} & P' & & \\ & \searrow f^*h & \swarrow f^*h' & & \\ & C & & & \end{array}$$

Now, we may wish to define a functor

$$\begin{aligned} F : \mathcal{C}^{\text{op}} &\rightarrow \mathbf{Cat}, \\ C &\mapsto \mathcal{C}_{/C}, \end{aligned}$$

which sends a map  $f : C \rightarrow D$  in  $\mathcal{C}$  to a pullback functor  $f^* : \mathcal{C}_{/D} \rightarrow \mathcal{C}_{/C}$ . However a problem arises when we check that  $F$  respects composition: suppose  $f : C \rightarrow D$ ,  $g : D \rightarrow E$  are maps in  $\mathcal{C}$ . Then

$$F(g \circ f)(h : X \rightarrow E) = (g \circ f)^*h : P \rightarrow C,$$

corresponding to the pullback

$$\begin{array}{ccc} P & \xrightarrow{h^*(g \circ f)} & X \\ (g \circ f)^*h \downarrow \lrcorner & & \downarrow h \\ C & \xrightarrow{g \circ f} & E \end{array}$$

in  $\mathcal{C}$ . On the other hand,

$$(F(g) \circ F(f))(h : X \rightarrow E) = f^*(g^*h) : P'' \rightarrow C,$$

which corresponds to the diagram below.

$$\begin{array}{ccccc} P'' & \xrightarrow{(g^*h)^*f} & P' & \xrightarrow{h^*g} & X \\ f^*(g^*h) \downarrow \lrcorner & & g^*h \downarrow \lrcorner & & \downarrow h \\ C & \xrightarrow{f} & D & \xrightarrow{g} & E \end{array}$$

The outer square is indeed a pullback square, since the inner two squares are, so we have a unique isomorphism  $P \cong P''$ . However, we do not in general have equality. This is because pullbacks are only unique up to unique isomorphism, and in defining a pullback functor we made arbitrary (and not necessarily compatible) choices of  $P, P'$  and  $P''$ . Thus, we have not defined a functor  $\mathcal{C}^{\text{op}} \rightarrow \mathbf{Cat}$ , rather, we have defined what is known as a *pseudofunctor*; that is, a weak functor between 2-categories.

The above example is one way in which pseudofunctors into  $\mathbf{Cat}$  naturally arise; another common example is the pseudofunctor

$$\begin{aligned} \mathbf{CRing} &\rightarrow \mathbf{Cat} \\ R &\mapsto R\text{-}\mathbf{Mod}, \end{aligned}$$

which sends a ring homomorphism  $\phi : R \rightarrow S$  to the functor  $- \otimes_R S : R\text{-}\mathbf{Mod} \rightarrow S\text{-}\mathbf{Mod}$  (extension of scalars). However, to give the data of a pseudofunctor  $F : \mathcal{C} \rightarrow \mathbf{Cat}$ , we must specify not only the functions  $\mathrm{ob}(\mathcal{C}) \rightarrow \mathrm{ob}(\mathbf{Cat})$  and  $\mathrm{Hom}_{\mathcal{C}}(X, Y) \rightarrow \mathrm{Hom}_{\mathbf{Cat}}(F(X), F(Y))$  for each  $X, Y \in \mathcal{C}$ , but also natural isomorphisms

$$F(\mathrm{id}_X) \cong \mathrm{id}_{F(X)}, \quad F(g \circ f) \cong F(g) \circ F(f).$$

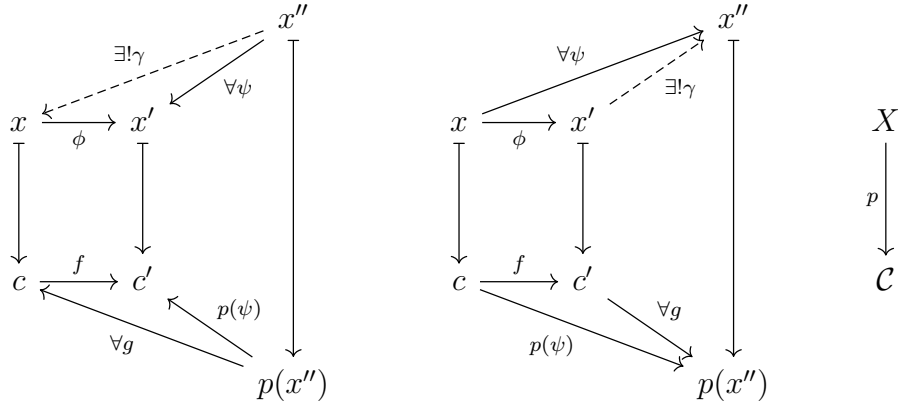
(This problem only becomes worse as we consider functors between higher categories... Let's use Grothendieck opfibrations instead.)

**DEFINITION 2.1.1.** Let  $p : X \rightarrow \mathcal{C}$  be a functor, let  $f : c \rightarrow c'$  be a morphism in  $\mathcal{C}$ , and let  $\phi : x \rightarrow x'$  be a morphism in  $X$  lying over  $f$ .

We say that  $\phi$  is *p-cartesian* if for any other morphism  $\psi : x'' \rightarrow x'$  in  $X$ , and for any morphism  $g : p(x'') \rightarrow c$  in  $\mathcal{C}$  satisfying  $f \circ g = p(\psi)$ , there exists a unique morphism  $\gamma : x'' \rightarrow x$  such that  $p(\gamma) = g$  and  $\psi = \gamma \circ \phi$ .

Dually,  $\phi$  is *p-cocartesian* if for any other morphism  $\psi : x \rightarrow x''$  in  $X$ , and for any morphism  $g : c' \rightarrow p(x'')$  in  $\mathcal{C}$  satisfying  $g \circ f = p(\psi)$ , there exists a unique morphism  $\gamma : x' \rightarrow x''$  such that  $p(\gamma) = g$  and  $\psi = \gamma \circ \phi$ .

The left diagram below corresponds to a *p-cartesian* morphism, and the right diagram corresponds to a *p-cocartesian* morphism.



**DEFINITION 2.1.2.** Let  $p : X \rightarrow \mathcal{C}$  be a functor. Then  $p$  is a *Grothendieck fibration* if for any morphism of  $\mathcal{C}$  and any lift of its target, there is a *p-cartesian* morphism with that target lying over it. Dually,  $p$  is a *Grothendieck opfibration* if for any morphism of  $\mathcal{C}$  and any lift of its source, there is a *p-cocartesian* morphism with that source lying over it.

We will usually refer to Grothendieck (op)fibrations as just (op)fibrations for brevity. Note that in [3] and [8] these are referred to as *(co)cartesian fibrations*; we reserve this term for the  $\infty$ -category analogue.

**REMARK 2.1.3.** A functor  $p : X \rightarrow \mathcal{C}$  is an opfibration if and only if  $p^{\mathrm{op}} : X^{\mathrm{op}} \rightarrow \mathcal{C}^{\mathrm{op}}$  is a fibration. We give the definition of an opfibration explicitly, since we will be working with these more often than fibrations.

(Slightly confusing thing: Theorem 8.3.1 of [2] says fibrations (not opfibrations) into  $\mathcal{C}$  are “the same” as pseudofunctors  $\mathcal{C} \rightarrow \mathbf{Cat}$ . nLab says the same thing but for pseudofunctors  $\mathcal{C}^{\text{op}} \rightarrow \mathbf{Cat}$ . Definition 2.2.7 of [6] gives the definition of an opfibration  $p : E \rightarrow B$  as a fibration  $E^{\text{op}} \rightarrow B^{\text{op}}$ , which I believe; they also say in Theorem 2.2.3 that pseudofunctors into  $\mathcal{C}$  are the same as pseudofunctors  $\mathcal{C}^{\text{op}} \rightarrow \mathbf{Cat}$ , which means that opfibrations into  $\mathcal{C}$  (which are fibrations into  $\mathcal{C}^{\text{op}}$ ) are pseudofunctors  $\mathcal{C} \rightarrow \mathbf{Cat}$ . Um?)

([2] states the theorem I need nicely, but doesn’t give a proof or even a reference for the non-discrete case. Another source is [10], which seems to give a proof (though I haven’t checked it).)

THEOREM 2.1.4 ([10], Thm 2.4). There is an equivalence of 2-categories

$$\mathbf{Psd}[\mathcal{C}^{\text{op}}, \mathbf{Cat}] \simeq \mathbf{Fib}(\mathcal{C}),$$

where  $\mathbf{Psd}[\mathcal{C}^{\text{op}}, \mathbf{Cat}]$  is the 2-category of pseudofunctors  $\mathcal{C}^{\text{op}} \rightarrow \mathbf{Cat}$ , and  $\mathbf{Fib}(\mathcal{C}) \hookrightarrow \mathbf{Cat}_{/\mathcal{C}}$  is the 2-category of fibrations into  $\mathcal{C}$ .

REMARK 2.1.5. Combining Remark 2.1.3 with Theorem 2.1.4 gives us an equivalence of 2-categories

$$\mathbf{Psd}[\mathcal{C}, \mathbf{Cat}] \simeq \mathbf{opFib}(\mathcal{C}).$$

We will not prove the theorems above (see [10] for more details) but we will describe how to pass between the two 2-categories. (Actually maybe chapter 10 of [5] is a better reference, because the functions it talks about actually compose.)

Let  $F : \mathcal{C} \rightarrow \mathbf{Cat}$  be a pseudofunctor. Define the category  $X$  as follows: the objects of  $X$  are pairs  $(c, x)$ , with  $c \in \mathcal{C}$ ,  $x \in F(c)$ . A map  $(c, x) \rightarrow (d, y)$  is a pair  $(f, u)$ , where  $f : c \rightarrow d$  is a morphism in  $\mathcal{C}$ , and  $u : y \rightarrow (Ff)(x)$  is a morphism in  $F(d)$ . For an object  $(c, x) \in X$ , the identity morphism is given by

$$(\text{id}_c, F_x^0 : x \rightarrow F(\text{id}_c)(x)),$$

where  $F^0$  is the natural isomorphism  $\text{id}_{F(c)} \cong F(\text{id}_c)$ . Further, given two maps  $(f, u) : (c, x) \rightarrow (d, y)$  and  $(g, v) : (d, y) \rightarrow (e, z)$ , their composition  $(g, v) \circ (f, u)$  is given by  $g \circ f$ , together with the map

$$z \xrightarrow{v} (Fg)(y) \xrightarrow{(Fg)(u)} (Fg \circ Ff)(x) \xrightarrow{(F_{g,f}^2)_x} (F(g \circ f))(x),$$

where  $F_{g,f}^2$  is the natural isomorphism  $Fg \circ Ff \cong F(g \circ f)$ . One can show that the forgetful functor  $X \rightarrow \mathcal{C}$  is an opfibration over  $\mathcal{C}$ .

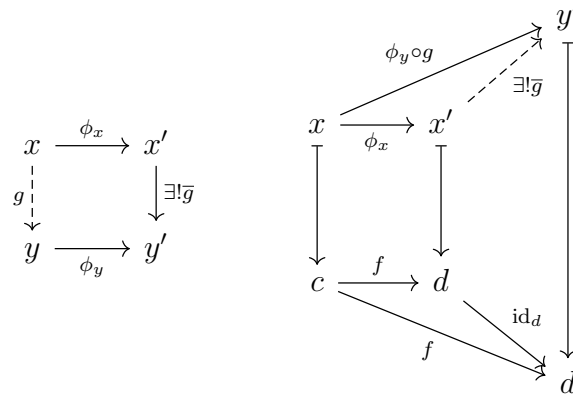
Now, let  $p : X \rightarrow \mathcal{C}$  be an opfibration over  $\mathcal{C}$ . Define a pseudofunctor

$$\begin{aligned} \mathcal{C} &\rightarrow \mathbf{Cat} \\ c &\mapsto p^{-1}\{c\}. \end{aligned}$$

For a map  $f : c \rightarrow d$  in  $\mathcal{C}$ , we define a functor

$$f_* : p^{-1}\{c\} \rightarrow p^{-1}\{d\},$$

which sends  $x \in p^{-1}\{c\}$  to the target  $x'$  of a cocartesian edge  $\phi_x : x \rightarrow x'$  lying over  $f$ . Now, let  $f^*(y) = y'$ , with  $\phi_y : y \rightarrow y'$ , and let  $g : x \rightarrow y$  be a morphism in  $p^{-1}\{c\}$  (that is, a morphism in  $X$  lying over  $\text{id}_c$ ). Then, since  $\phi_y : y \rightarrow y'$  is a cocartesian edge, there is a unique lift  $\bar{g} : x' \rightarrow y'$  of  $\text{id}_d$  making the square on the left commute, as shown in the diagram on the right.



We thus define  $f_*g = \bar{g}$  in  $p^{-1}\{d\}$ .

It can be shown that these data assemble into a pseudofunctor  $\mathcal{C} \rightarrow \mathbf{Cat}$ .

## 2.2 Symmetric monoidal categories and functors

DEFINITION 2.2.1. A *symmetric monoidal category* is a category  $\mathcal{C}$  equipped with a bifunctor

$$\otimes : \mathcal{C} \times \mathcal{C} \rightarrow \mathcal{C},$$

an object  $\mathbf{1} \in \mathcal{C}$ , and natural isomorphisms

$$\alpha : \otimes \circ (\otimes \times \text{id}) \xrightarrow{\sim} \otimes \circ (\text{id} \times \otimes), \quad l : \otimes \circ (\mathbf{1} \times \text{id}) \xrightarrow{\sim} \text{id}, \quad r : \otimes \circ (\text{id} \times \mathbf{1}) \xrightarrow{\sim} \text{id},$$

$$\tau : \otimes \xrightarrow{\sim} \sigma \circ \otimes,$$

where  $\sigma : \mathcal{C} \times \mathcal{C}$  swaps the order of the factors. These isomorphisms are subject to the condition that  $\tau^2 = \text{id}$ , and that the diagrams below commute.

$$\begin{array}{ccc}
& ((U \otimes V) \otimes W) \otimes X & \\
\alpha_{U \otimes V, W, X} \swarrow & & \searrow \alpha_{U, V, W \otimes X} \\
(U \otimes V) \otimes (W \otimes X) & & (U \otimes (V \otimes W)) \otimes X \\
\alpha_{U, V, W \otimes X} \downarrow & & \downarrow \alpha_{U, V \otimes W, X} \\
U \otimes (V \otimes (W \otimes X)) & \xleftarrow{\text{id} \otimes \alpha_{V, W, X}} & U \otimes ((V \otimes W) \otimes X)
\end{array}$$
  

$$\begin{array}{ccc}
(V \otimes \mathbf{1}) \otimes W & \xrightarrow{\alpha_{V, \mathbf{1}, W}} & V \otimes (\mathbf{1} \otimes W) \\
r_V \otimes \text{id}_W \searrow & & \swarrow \text{id}_V \otimes l_W \\
& V \otimes W &
\end{array}
\quad
\begin{array}{ccc}
V \otimes \mathbf{1} & \xrightarrow{\tau_{V, \mathbf{1}}} & \mathbf{1} \otimes V \\
r_V \searrow & & \swarrow l_V \\
& V &
\end{array}$$

$$\begin{array}{ccc}
& (U \otimes V) \otimes W & \\
\alpha_{U,V,W} \swarrow & & \searrow \tau_{U,V} \otimes \text{id}_W \\
U \otimes (V \otimes W) & & (V \otimes U) \otimes W \\
\tau_{U,V} \otimes \text{id}_W \downarrow & & \downarrow \alpha_{V,U,W} \\
(V \otimes W) \otimes U & & V \otimes (U \otimes W) \\
\alpha_{V,W,U} \searrow & & \swarrow \text{id}_V \otimes \tau_{U,W} \\
& V \otimes (W \otimes U) &
\end{array}$$

Now that we know how to move between pseudofunctors to **Cat** and opfibrations, let's write the data of a symmetric monoidal category as a pseudofunctor to **Cat**.

Let  $(\mathcal{C}, \otimes)$  be a symmetric monoidal category. Define a pseudofunctor

$$\begin{aligned}
F : \text{Fin}_* &\rightarrow \mathbf{Cat} \\
\langle n \rangle &\mapsto \mathcal{C}^n
\end{aligned}$$

Let  $f : \langle n \rangle \rightarrow \langle m \rangle$  be a morphism in  $\text{Fin}_*$ . This induces a morphism

$$f^* : (C_1, \dots, C_n) \mapsto (C'_1, \dots, C'_m),$$

where

$$C'_i = \bigotimes_{j \in f^{-1}\{i\}} C_j.$$

- The above implies there's some category  $\mathcal{D}$  such that opfibrations  $\mathcal{D} \rightarrow \text{Fin}_*$  are the same as symmetric monoidal categories. Let's see what  $\mathcal{D}$  is.
- Possibly mention swapping out **Fin**<sub>\*</sub> for  $\Delta^{\text{op}}$  gives a monoidal category rather than a symmetric monoidal category. (How do we get a braided monoidal category? Apparently there is no base 1-category we can look at opfibrations into, because the correct formulation is with  $E_2$ , which has higher homotopy groups on the mapping spaces.)
- Correspondence of symmetric monoidal functors with morphisms of opfibrations.

The construction below is in [7].

Let  $(\mathcal{C}, \otimes)$  be a symmetric monoidal category. We define a new category  $\mathcal{C}^{\otimes}$ , whose objects are finite (possibly empty) sequences of objects of  $\mathcal{C}$ , denoted by  $[C_1, \dots, C_n]$ . A morphism

$$[C_1, \dots, C_n] \rightarrow [C'_1, \dots, C'_m]$$

consists of a subset  $S \subseteq \{1, \dots, n\}$ , a map of finite sets  $\alpha : S \rightarrow \{1, \dots, m\}$ , and a collection of morphisms  $\{f_j : \bigotimes_{i \in \alpha^{-1}\{j\}} C_i \rightarrow C'_j\}_{1 \leq j \leq m}$  in  $\mathcal{C}$ .

For two morphisms  $f : [C_1, \dots, C_n] \rightarrow [C'_1, \dots, C'_m]$  and  $g : [C'_1, \dots, C'_m] \rightarrow [C''_1, \dots, C''_l]$ , determining two subsets  $S \subseteq \{1, \dots, n\}$  and  $T \subseteq \{1, \dots, m\}$  and maps  $\alpha : S \rightarrow \{1, \dots, m\}$ ,  $\beta : T \rightarrow \{1, \dots, l\}$ , the composition  $g \circ f$  is given by the subset  $U = \alpha^{-1}T \subseteq \{1, \dots, n\}$ , the map  $\beta \circ \alpha : U \rightarrow \{1, \dots, l\}$  and the maps

$$\left\{ \bigotimes_{i \in (\beta \circ \alpha)^{-1}\{k\}} C_i \cong \bigotimes_{j \in \beta^{-1}\{k\}} \bigotimes_{i \in \alpha^{-1}\{j\}} C_i \rightarrow \bigotimes_{j \in \beta^{-1}\{k\}} C'_j \rightarrow C''_k \right\}_{1 \leq k \leq l}.$$

For example, let

$$f : [C_1, C_2, C_3, C_4] \rightarrow [C'_1, C'_2, C'_3]$$

be a morphism in  $\mathcal{C}^\otimes$  consisting of the subset  $\{1, 2, 3\} \subseteq \{1, 2, 3, 4\}$ , the map

$$\begin{aligned} \alpha : \{1, 2, 3\} &\rightarrow \{1, 2, 3\}, \\ 1 &\mapsto 1, \\ 2 &\mapsto 2, \\ 3 &\mapsto 3, \end{aligned}$$

and morphisms

$$f_1 : C_1 \rightarrow C'_1, \quad f_2 : C_2 \otimes C_3 \rightarrow C'_2, \quad f_3 : \mathbf{1} \rightarrow C'_3,$$

and let

$$g : [C'_1, C'_2, C'_3] \rightarrow [C''_1, C''_2, C''_3]$$

be a morphism in  $\mathcal{C}^\otimes$  consisting of the subset  $\{1, 2, 3\} \subseteq \{1, 2, 3\}$ , the map

$$\begin{aligned} \beta : \{1, 2, 3\} &\rightarrow \{1, 2, 3\}, \\ 1, 2, 3 &\mapsto 3, \end{aligned}$$

and morphisms

$$g_1 : \mathbf{1} \rightarrow C''_1, \quad g_2 : \mathbf{1} \rightarrow C''_2, \quad g_3 : C'_1 \otimes C'_2 \otimes C'_3 \rightarrow C''_3.$$

Then the composition  $g \circ f$  consists of the subset  $\alpha^{-1}\{1, 2, 3\} = \{1, 2, 3\} \subseteq \{1, 2, 3, 4\}$ , the map

$$\begin{aligned} \beta \circ \alpha : \{1, 2, 3\} &\rightarrow \{1, 2, 3\}, \\ 1, 2, 3 &\mapsto 3, \end{aligned}$$

and the morphisms

$$(g \circ f)_1 = g_1, \quad (g \circ f)_2 = g_2, \quad (g \circ f)_3 = g_3 \circ (f_1 \otimes f_2 \otimes f_3).$$

(really?)

(some intuition on this, tensor along the fibres, etc)

Claim: the forgetful functor

$$\begin{aligned} p : \mathcal{C}^\otimes &\rightarrow \mathbf{Fin}_*, \\ [C_1, \dots, C_n] &\mapsto \langle n \rangle_* \end{aligned}$$

is an opfibration. (It almost tautologically is).

PROPOSITION 2.2.2 ([4], Prop 4.26). If  $(\mathcal{C}, \otimes)$  is a symmetric monoidal category, then the forgetful functor  $p : \mathcal{C}^\otimes \rightarrow \mathbf{Fin}_*$  given above is an opfibration. Moreover,  $p$  satisfies the *Segal condition*; that is, the Segal maps

$$(\rho_!^1, \dots, \rho_!^n) : \mathcal{C}_{\langle n \rangle}^\otimes \rightarrow \mathcal{C}^{\times n}$$

are equivalences. Conversely any Grothendieck opfibration  $p : \mathcal{C} \rightarrow \mathbf{Fin}_*$  satisfying the Segal condition gives rise to a symmetric monoidal structure on  $\mathcal{C}_{\langle 1 \rangle}$ .

We've said that symmetric monoidal categories are special opfibrations to  $\mathbf{Fin}_*$ . Now, what are symmetric monoidal functors?

The definition below is basically definition 3.3 of [3].

DEFINITION 2.2.3. Let  $p : X \rightarrow \mathcal{C}$  and  $q : Y \rightarrow \mathcal{C}$  be two Grothendieck (op)fibrations. A functor  $F : X \rightarrow Y$  is a *morphism of (op)fibrations* from  $p$  to  $q$  if the diagram below commutes,

$$\begin{array}{ccc} X & \xrightarrow{F} & Y \\ & \searrow p & \swarrow q \\ & \mathcal{C} & \end{array}$$

and  $F$  sends  $p$ -(co)cartesian morphisms to  $q$ -(co)cartesian morphisms.

### 3 Generalisation to $\infty$ -categories

- A functor  $p : D \rightarrow C$  between ordinary categories is a Grothendieck opfibration if and only if the induced functor  $N(p) : N(D) \rightarrow N(C)$  on nerves is a cocartesian fibration
- Nerve of a symmetric monoidal category
- If an  $\infty$ -category has finite (co)products, there is a (co)cartesian monoidal structure on  $\mathcal{C}$ . And we would have hoped so, because it's definitely true for 1-categories!
- Might be cool to try to look at  $E_k$  algebras, to resolve the earlier mystery of how to write braided monoidal categories.

We first need an  $\infty$ -categorical analogue of Grothendieck opfibrations. We start by requiring that our functor is what's known as an *inner fibration*; there is no 1-categorical analogue of this, since all functors between 1-categories are automatically inner fibrations under the nerve functor (see [Example 3.0.2](#)). Think of it as a ‘minimum niceness condition’ – we want the fibres to be  $\infty$ -categories in much the same way as we want the fibres of ordinary functors to be categories themselves.

DEFINITION 3.0.1 ([1], Def 2.1). A functor  $p : X \rightarrow Y$  between simplicial sets is an *inner fibration* if for all  $n \geq 2$ , all  $0 < k < n$ , and any solid arrow commutative square

$$\begin{array}{ccc} \Lambda_k^n & \longrightarrow & X \\ \downarrow & \nearrow & \downarrow p \\ \Delta^n & \longrightarrow & Y \end{array}$$

there exists a dotted lift.

EXAMPLE 3.0.2. Let  $\mathcal{C}, \mathcal{D}$  be categories, and  $p : \mathcal{C} \rightarrow \mathcal{D}$  be a functor between them. Then  $N(p) : N\mathcal{C} \rightarrow N\mathcal{D}$  is an inner fibration.

The following proposition is stated without proof in Section 2.3 of [8].

PROPOSITION 3.0.3. Let  $p : X \rightarrow Y$  be an inner fibration, and suppose that the diagram below is a pullback square in **sSet**.

$$\begin{array}{ccc} X' & \xrightarrow{f} & X \\ p' \downarrow \lrcorner & & \downarrow p \\ Y' & \xrightarrow{g} & Y \end{array}$$

Then  $p'$  is also an inner fibration.



PROOF. Consider the (commutative) solid arrow diagram below.

$$\begin{array}{ccccc}
\Lambda_k^n & \xrightarrow{\lambda} & X' & \xrightarrow{f} & X \\
\downarrow \iota & & \downarrow p' & \dashrightarrow & \downarrow p \\
\Delta^n & \xrightarrow{\delta} & Y' & \xrightarrow{g} & Y
\end{array}$$

Since  $p$  is a fibration, there exists a dotted lift  $\phi$  of  $g\delta$ ; that is,  $p\phi = g\delta$  and  $\phi\iota = f\lambda$ . Further, since the right square is a pullback diagram, there exists a unique map  $\phi' : \Delta^n \rightarrow X'$  making the diagram below commute.

$$\begin{array}{ccccc}
\Delta^n & & & & \\
\downarrow \delta & \searrow \phi' & & \searrow \phi & \\
& X' & \xrightarrow{f} & X & \\
& \downarrow p' & \dashrightarrow & \downarrow p & \\
& Y' & \xrightarrow{g} & Y &
\end{array}$$

It remains to show that the triangle below commutes.

$$\begin{array}{ccc}
\Lambda_k^n & \xrightarrow{\lambda} & X' \\
\downarrow \iota & \searrow \phi' & \\
\Delta^n & &
\end{array}$$

Again, using the universal property of pullbacks, we see that there exist unique dotted maps such that the diagrams below commute.

$$\begin{array}{ccc}
\Lambda_k^n & \xrightarrow{f\lambda} & X \\
\downarrow \delta\iota & \searrow \phi' & \downarrow p \\
& X' & \xrightarrow{f} \\
& \downarrow p' & \dashrightarrow \\
& Y' & \xrightarrow{g} Y
\end{array}
\qquad
\begin{array}{ccc}
\Lambda_k^n & \xrightarrow{f\phi'\iota} & X \\
\downarrow \delta\iota & \searrow \phi' & \downarrow p \\
& X' & \xrightarrow{f} \\
& \downarrow p' & \dashrightarrow \\
& Y' & \xrightarrow{g} Y
\end{array}$$

The maps  $\lambda$  and  $\phi'\iota$  make the left and right diagrams commute respectively. Further, we note that  $f\phi' = \phi$  (by the second diagram) and  $\phi\iota = f\lambda$  (since  $p$  is an inner fibration), so  $f\phi'\iota = f\lambda$ . Therefore, the above two diagrams are identical. Thus, by the uniqueness property of pullbacks,  $\lambda = \phi'\iota$ .  $\square$

(Stupid note to self, very obvious but I forget it every now and again):

- If  $X : \Delta^{\text{op}} \rightarrow \mathbf{Set}$  is a simplicial set, and  $\Delta^0 : \Delta^{\text{op}} \rightarrow \mathbf{Set} := \text{Hom}(-, [0])$ , then a map  $F : X \rightarrow \Delta^0$  is a natural transformation  $(F_n : X_n \rightarrow *)_{n \in \mathbb{N}_0}$ . That is, such a natural transformation is a family of maps down to a point. In other words, there's only really one natural transformation, so we really \*can\* view  $\Delta^0$  as a point.
- If  $Y$  is a simplicial set, and  $y \in Y_0$  is a vertex of  $Y$ , we can view  $\{y\}$  as a copy of  $\Delta^0$ . Why is this? We can view  $\{y\}$  as the constant simplicial set, sending everything to  $y$ . Then a natural isomorphism  $\Delta^0 \cong \{y\}$  is a collection of isomorphisms  $(* \rightarrow *)$ , of which there is exactly one. Why is it natural? Well, there's only one map from a one-point set to another one-point set, so the square always commutes.

EXAMPLE 3.0.4 ([1], Ex 2.2). Let  $p : X \rightarrow \Delta^0$  be the canonical map, and suppose we have the diagram below, such that the outer square commutes.

$$\begin{array}{ccc} \Lambda_k^n & \longrightarrow & X \\ \downarrow & \nearrow & \downarrow p \\ \Delta^n & \longrightarrow & \Delta^0 \end{array}$$

The lower triangle commutes automatically, so the statement that  $p$  is an inner fibration is equivalent to the statement that for all  $n \geq 2$ , all  $0 < k < n$ , and any map  $\Lambda_k^n \rightarrow X$ , there exists a dotted lift.

$$\begin{array}{ccc} \Lambda_k^n & \longrightarrow & X \\ \downarrow & \nearrow & \\ \Delta^n & & \end{array}$$

That is,  $X$  is an  $\infty$ -category.

Now, combining the above argument with Proposition 3.0.3, we see that for any inner fibration  $p : X \rightarrow Y$ , each fibre  $X \times_Y \{y\}$  is an  $\infty$ -category.

DEFINITION 3.0.5 ([1], Def 3.1). Let  $p : X \rightarrow Y$  be an inner fibration. An edge  $f : \Delta^1 \rightarrow X$  of  $X$  is *p-cocartesian* if for all  $n \geq 2$ , any extension

$$\begin{array}{ccc} \Delta^{\{0,1\}} & \xrightarrow{f} & X \\ \downarrow & \nearrow F & \\ \Lambda_0^n & & \end{array}$$

and any solid arrow commutative diagram

$$\begin{array}{ccc} \Lambda_0^n & \xrightarrow{F} & X \\ \downarrow & \nearrow & \downarrow p \\ \Delta^n & \longrightarrow & Y \end{array}$$

a dotted lift exists.

DEFINITION 3.0.6. Let  $p : X \rightarrow Y$  be an inner fibration. Then  $p$  is a cocartesian fibration if for any edge  $\phi : y \rightarrow y'$  in  $Y_1$ , and for every  $x \in X_0$  lying over  $y$ , there exists a  $p$ -cocartesian edge  $f : x \rightarrow x'$  of  $X$  lying over  $\phi$ .

The following proposition tells us that the above definition is a reasonable generalisation of Definition 2.1.2. It is also stated without proof in [8], which did not do wonders for my ego.

PROPOSITION 3.0.7 ([8], Rmk 2.4.2.2). Let  $\mathcal{C}, \mathcal{D}$  be categories, and let  $p : \mathcal{C} \rightarrow \mathcal{D}$  be a functor between them. Then  $p$  is a Grothendieck opfibration if and only if the induced map  $N(p) : N\mathcal{C} \rightarrow N\mathcal{D}$  is a cocartesian fibration of simplicial sets.

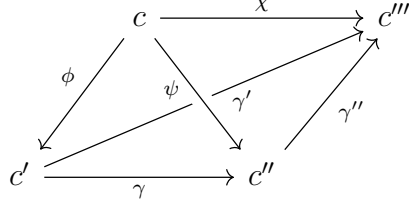
PROOF. Let  $f : d \rightarrow d'$  be a morphism of  $\mathcal{D}$ , and let  $c$  lie over  $d$ .

Suppose  $p$  is a Grothendieck opfibration, let  $F : \Lambda_0^n \rightarrow N\mathcal{C}$  be an extension of  $f$ , and let

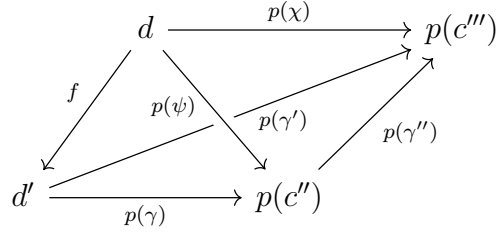
$$\begin{array}{ccc} \Lambda_0^n & \xrightarrow{F} & N\mathcal{C} \\ \downarrow & & \downarrow N(p) \\ \Delta^n & \longrightarrow & N\mathcal{D} \end{array}$$

be a commutative diagram. If  $n = 2$ , it follows immediately from the fact that  $p$  is an opfibration that a dotted lift exists. Further, if  $n > 3$ , there is nothing to check, since an  $n$ -simplex in a category commutes if and only if all of its triangles commute, which is guaranteed for any extension  $F : \Lambda_0^n \rightarrow N\mathcal{C}$ . We thus prove the proposition for  $n = 3$ .

Suppose we have an extension  $F : \Lambda_0^3 \rightarrow N\mathcal{C}$  of  $f$ ; that is, a tetrahedron



such that all faces containing the vertex  $c$  commute. Let

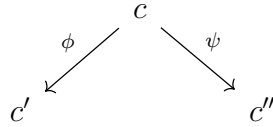


be a commutative tetrahedron in  $\mathcal{D}$ . We claim that the tetrahedron in  $\mathcal{C}$  commutes. First, note that  $\gamma'' \circ \gamma$  is a lift of  $p(\gamma')$ , since  $p(\gamma') = p(\gamma'') \circ p(\gamma) = p(\gamma'' \circ \gamma)$ . Further,

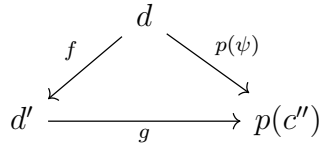
$$\begin{aligned} (\gamma \circ \gamma'') \circ \phi &= \gamma'' \circ \psi \\ &= \chi. \end{aligned}$$

Thus, by the uniqueness in the universal property of  $\phi$ , we have that  $\gamma' = \gamma'' \circ \gamma$ , as required.

Now, suppose  $N(p)$  is a cocartesian fibration. Then there exists a lift  $\phi : c \rightarrow c'$  of  $f$ , and, in particular, for any diagram

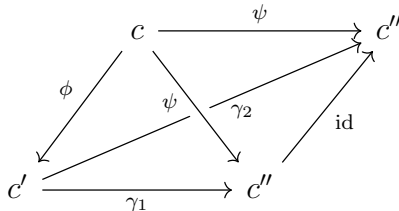


in  $\mathcal{C}$ , and any commutative diagram



in  $\mathcal{D}$ , there exists a map  $\gamma : c' \rightarrow c''$  such that  $\gamma$  lies over  $g$  and  $\gamma \circ \phi = \psi$ . It remains to show that  $\gamma$  is unique.

Suppose that there were two maps  $\gamma_1, \gamma_2 : c' \rightarrow c''$  lying over  $g$  and satisfying  $\gamma_1 \circ \phi = \gamma_2 \circ \phi = \psi$ . Then we would have a tetrahedron



where all faces containing the vertex  $c$  commute. The image of this tetrahedron under  $p$  commutes in  $\mathcal{D}$ , so the original tetrahedron must commute in  $\mathcal{C}$ ; that is,  $\gamma_1 = \gamma_2$ .  $\square$

DEFINITION 3.0.8 ([8], Def 2.0.0.7). A *symmetric monoidal  $\infty$ -category* is a cocartesian fibration of simplicial sets  $p : X^\otimes \rightarrow N(\mathbf{Fin}_*)$  such that for each  $n \geq 0$ , the maps

$$\{\rho^i : \langle n \rangle \rightarrow \langle 1 \rangle\}_{1 \leq i \leq n}$$

induce functors  $\rho^i : X_{\langle n \rangle}^\otimes \rightarrow X_{\langle 1 \rangle}^\otimes$  which determine an equivalence  $X_{\langle n \rangle}^\otimes \simeq \left(X_{\langle 1 \rangle}^\otimes\right)^n$ .

EXAMPLE 3.0.9. Let  $(\mathcal{C}, \otimes)$  be a symmetric monoidal category. Then  $p : N(\mathcal{C}^\otimes) \rightarrow N(\mathbf{Fin}_*)$  is a symmetric monoidal  $\infty$ -category.

## 4 A nontrivial example

Throughout this section,  $\mathcal{A}$  is an abelian category, and  $\mathcal{A}_{\text{proj}}$  is the full subcategory of  $\mathcal{A}$  spanned by the projective objects.

DEFINITION 4.0.1 ([7], Def 1.2.3.1). A *chain complex* with values in  $\mathcal{A}$  is a composable sequence of morphisms

$$\cdots \rightarrow A_2 \xrightarrow{d(2)} A_1 \xrightarrow{d(1)} A_0 \xrightarrow{d(0)} A_{-1} \rightarrow \cdots$$

in  $\mathcal{A}$  such that  $d(n-1) \circ d(n) = 0$  for all  $n \in \mathbb{Z}$ . The collection of chain complexes with values in  $\mathcal{A}$  is an additive category,  $\text{Ch}(\mathcal{A})$ .

DEFINITION 4.0.2 ([7], Not 1.3.2.6).  $\text{Ch}^-(\mathcal{A})$  is the full subcategory of  $\text{Ch}(\mathcal{A})$  spanned by those chain complexes  $M_*$  such that  $M_n \simeq 0$  for  $n \ll 0$ .

DEFINITION 4.0.3 ([7], Def 1.3.2.7). Suppose  $\mathcal{A}$  has enough projective objects. We let  $\mathcal{D}^-(\mathcal{A})$  denote the  $\infty$ -category  $N_{\text{dg}}(\text{Ch}^-(\mathcal{A}_{\text{proj}}))$ . We refer to  $\mathcal{D}^-(\mathcal{A})$  as the *derived  $\infty$ -category of  $\mathcal{A}$* .

## 5 Miscellaneous stupid notes

### 5.1 Observations

Let  $S \in \mathbf{Set}$ . We define the constant simplicial set

$$\overline{S} : \Delta^{\text{op}} \longrightarrow \mathbf{Set}$$

$$\begin{array}{ccc} [n] & \xrightarrow{\quad} & S \\ f \downarrow & & \uparrow \text{id} \\ [m] & \xrightarrow{\quad} & S \end{array}$$

It's a Kan complex. Why? Well, when you consider  $S$  as a discrete category, and take the nerve of it, you get  $\overline{S}$ . You can then either just see that it's a Kan complex (fill the horns with identities) or use the fact that the nerve of a groupoid is a Kan complex. It's surely in [8] or [9] somewhere.

Here are the original definitions of the Grothendieck equivalence (before I dualised them, just in case I did it wrong).

Let  $F : \mathcal{C}^{\text{op}} \rightarrow \mathbf{Cat}$  be a pseudofunctor. Define the category  $X$  as follows: the objects of  $X$  are pairs  $(c, x)$ , with  $c \in \mathcal{C}$ ,  $x \in F(c)$ . A map  $(c, x) \rightarrow (d, y)$  is a pair  $(f, u)$ , where  $f : c \rightarrow d$  is a morphism in  $\mathcal{C}$ , and  $u : x \rightarrow (Ff)(y)$  is a morphism in  $F(c)$ . For an object  $(c, x) \in X$ , the identity morphism is given by

$$(\text{id}_c, F_x^0 : x \rightarrow F(\text{id}_c)(x)),$$

where  $F^0$  is the natural isomorphism  $\text{id}_{F(c)} \cong F(\text{id}_c)$ . Further, given two maps  $(f, u) : (c, x) \rightarrow (d, y)$  and  $(g, v) : (d, y) \rightarrow (e, z)$ , their composition  $(g, v) \circ (f, u)$  is given by  $g \circ f$ , together with the map

$$x \xrightarrow{u} (Ff)(y) \xrightarrow{(Ff)(v)} (Ff \circ Fg)(z) \xrightarrow{(F_{f,g}^2)_z} (F(g \circ f))(z),$$

where  $F_{f,g}^2$  is the natural isomorphism  $Ff \circ Fg \cong F(g \circ f)$  (recall that the domain of  $F$  is  $\mathcal{C}^{\text{op}}$ , while  $f, g$  are morphisms in  $\mathcal{C}$ ). One can show that the forgetful functor  $X \rightarrow \mathcal{C}$  is a fibration over  $\mathcal{C}$ .

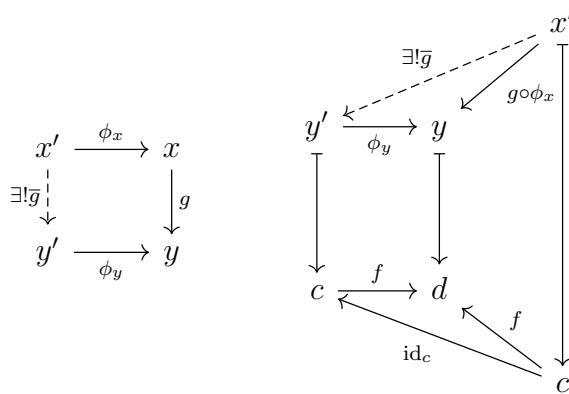
Now, let  $p : X \rightarrow \mathcal{C}$  be a fibration over  $\mathcal{C}$ . Define a pseudofunctor

$$\begin{aligned} \mathcal{C}^{\text{op}} &\rightarrow \mathbf{Cat} \\ c &\mapsto p^{-1}\{c\}. \end{aligned}$$

For a map  $f : c \rightarrow d$  in  $\mathcal{C}$ , we define a functor

$$f^* : p^{-1}\{d\} \rightarrow p^{-1}\{c\},$$

which sends  $x \in p^{-1}\{d\}$  to the source  $x'$  of a cartesian edge  $\phi_x : x' \rightarrow x$  lying over  $f$ . Now, let  $f^*(y) = y'$ , with  $\phi_y : y' \rightarrow y$ , and let  $g : x \rightarrow y$  be a morphism in  $p^{-1}\{d\}$  (that is, a morphism in  $X$  lying over  $\text{id}_c$ ). Then, since  $\phi_y : y' \rightarrow y$  is a cartesian edge, there is a unique lift  $\bar{g} : x' \rightarrow y'$  of  $\text{id}_c$  making the square on the left commute, as shown in the diagram on the right.



We thus define  $f^*g = \bar{g}$  in  $p^{-1}\{c\}$ .

It can be shown that these data assemble into a pseudofunctor  $\mathcal{C} \rightarrow \mathbf{Cat}$ .

## 5.2 Questions

Questions:

- ...what  $\ast$  is  $\mathbf{Grpd}_\infty$ ?

## 5.3 Equivalent definitions

$\mathbf{Grpd}_\infty$

An algebraic category

An equivalence of  $\infty$ -categories

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