

Symmetric Monoidal ∞ -Categories

- Note: we will call $(\infty, 1)$ -categories ∞ -categories.
- References: [1], [2], [3], [4], [5], [6], [7], [8], [9], [10]. Referencing Kerodon test: [9, Example 01UB].

1 Pseudofunctors, opfibrations, and symmetric monoidal categories

1.1 Pseudofunctors to **Cat** are Grothendieck opfibrations

- Grothendieck op-fibrations correspond to pseudofunctors to **Cat**.

The below gives an example to complement [3].

Example of a pseudofunctor to **Cat**: Let \mathcal{C} be a category with pullbacks. Recall that for a map $f : C \rightarrow D$ in \mathcal{C} , we define a pullback functor

$$\begin{aligned} f^* : \mathcal{C}_{/D} &\rightarrow \mathcal{C}_{/C}, \\ (h : X \rightarrow D) &\mapsto (f^*h : P \rightarrow C), \end{aligned}$$

where we have formed a pullback

$$\begin{array}{ccc} P & \xrightarrow{h^*f} & X \\ f^*h \downarrow & & \downarrow h \\ C & \xrightarrow{f} & D \end{array}$$

in \mathcal{C} . For any map

$$\begin{array}{ccc} X & \xrightarrow{\phi} & X' \\ & \searrow h & \swarrow h' \\ & D & \end{array}$$

from h to h' in $\mathcal{C}_{/D}$, we define $f^*\phi$ to be the unique map making the diagram below commute.

$$\begin{array}{ccccc} X & \xrightarrow{\phi} & & X' & \\ & \searrow h & & \swarrow h' & \\ & D & & & \\ & \uparrow f & & & \\ P & \xrightarrow{f^*\phi} & P' & & \\ & \searrow f^*h & & \swarrow f^*h' & \\ & C & & & \end{array}$$

Now, we may wish to define a functor

$$\begin{aligned} F : \mathcal{C}^{\text{op}} &\rightarrow \mathbf{Cat}, \\ C &\mapsto \mathcal{C}_{/C}, \end{aligned}$$

which sends a map $f : C \rightarrow D$ in \mathcal{C} to a pullback functor $f^* : \mathcal{C}_{/D} \rightarrow \mathcal{C}_{/C}$. However a problem arises when we check that F respects composition: suppose $f : C \rightarrow D$, $g : D \rightarrow E$ are maps in \mathcal{C} . Then

$$F(g \circ f)(h : X \rightarrow E) = (g \circ f)^*h : P \rightarrow C,$$

corresponding to the pullback

$$\begin{array}{ccc} P & \xrightarrow{h^*(g \circ f)} & X \\ (g \circ f)^*h \downarrow & & \downarrow h \\ C & \xrightarrow{g \circ f} & E \end{array}$$

in \mathcal{C} . On the other hand,

$$(F(g) \circ F(f))(h : X \rightarrow E) = f^*(g^*h) : P'' \rightarrow C,$$

which corresponds to the diagram below.

$$\begin{array}{ccccc} P'' & \xrightarrow{(g^*h)^*f} & P' & \xrightarrow{h^*g} & X \\ f^*(g^*h) \downarrow & & \downarrow g^*h & & \downarrow h \\ C & \xrightarrow{f} & D & \xrightarrow{g} & E \end{array}$$

The outer square is indeed a pullback square, since the inner two squares are, so we have a unique isomorphism $P \cong P''$. However, we do not in general have equality. This is because pullbacks are only unique up to unique isomorphism, and in defining a pullback functor we made arbitrary (and not necessarily compatible) choices of P, P' and P'' . Thus, we have not defined a functor $\mathcal{C}^{\text{op}} \rightarrow \mathbf{Cat}$, rather, we have defined what is known as a *pseudofunctor*; that is, a weak functor between 2-categories.

The above example is one way in which pseudofunctors into \mathbf{Cat} naturally arise; another common example is the pseudofunctor

$$\begin{aligned} \mathbf{CRing} &\rightarrow \mathbf{Cat} \\ R &\mapsto R\text{-}\mathbf{Mod}, \end{aligned}$$

which sends a ring homomorphism $\phi : R \rightarrow S$ to the functor $- \otimes_R S : R\text{-}\mathbf{Mod} \rightarrow S\text{-}\mathbf{Mod}$ (extension of scalars). However, to give the data of a pseudofunctor $F : \mathcal{C} \rightarrow \mathbf{Cat}$, we must specify not only the functions $\text{ob}(\mathcal{C}) \rightarrow \text{ob}(\mathbf{Cat})$ and $\text{Hom}_{\mathcal{C}}(X, Y) \rightarrow \text{Hom}_{\mathbf{Cat}}(F(X), F(Y))$ for each $X, Y \in \mathcal{C}$, but also natural isomorphisms

$$F(\text{id}_X) \cong \text{id}_{F(X)}, \quad F(g \circ f) \cong F(g) \circ F(f).$$

This problem only becomes worse as we consider functors between higher categories...

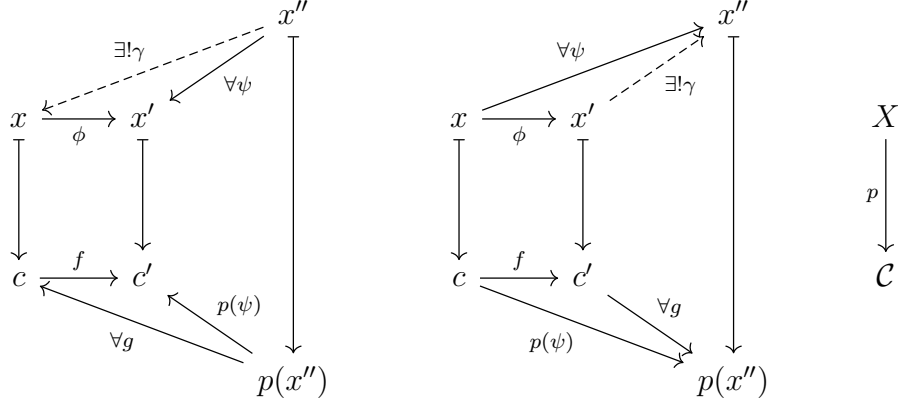
Let's use Grothendieck opfibrations instead.

DEFINITION 1.1.1. Let $p : X \rightarrow \mathcal{C}$ be a functor, let $f : c \rightarrow c'$ be a morphism in \mathcal{C} , and let $\phi : x \rightarrow x'$ be a morphism in X lying over f .

ϕ is *p-cartesian* if for any other morphism $\psi : x'' \rightarrow x'$ in X , and for any morphism $g : p(x'') \rightarrow c$ in \mathcal{C} satisfying $f \circ g = p(\psi)$, there exists a unique morphism $\gamma : x'' \rightarrow x$ such that $p(\gamma) = g$ and $\psi = \gamma \circ \phi$.

Dually, ϕ is *p-cocartesian* if for any other morphism $\psi : x \rightarrow x''$ in X , and for any morphism $g : c' \rightarrow p(x'')$ in \mathcal{C} satisfying $g \circ f = p(\psi)$, there exists a unique morphism $\gamma : x' \rightarrow x''$ such that $p(\gamma) = g$ and $\psi = \gamma \circ \phi$.

The left diagram below corresponds to a *p-cartesian* morphism, and the right diagram corresponds to *p-cocartesian* morphism.



DEFINITION 1.1.2. Let $p : X \rightarrow \mathcal{C}$ be a functor. Then p is a *Grothendieck fibration* if for any morphism of \mathcal{C} and any lift of its target, there is a *p-cartesian* morphism with that target lying over it. Dually, p is a *Grothendieck opfibration* if for any morphism of \mathcal{C} and any lift of its source, there is a *p-cocartesian* morphism with that source lying over it.

We will usually refer to Grothendieck (op)fibrations as just (op)fibrations for brevity. Note that in [3] and [8] these are referred to as *(co)cartesian fibrations*; we reserve this term for the ∞ -category analogue.

REMARK 1.1.3. A functor $p : X \rightarrow \mathcal{C}$ is an opfibration if and only if $p^{\text{op}} : X^{\text{op}} \rightarrow \mathcal{C}^{\text{op}}$ is a fibration. We give the definition of an opfibration explicitly, since we will be working with these more often than fibrations.

Slightly confusing thing: Theorem 8.3.1 of [2] says fibrations (not opfibrations) into \mathcal{C} are “the same” as pseudofunctors $\mathcal{C} \rightarrow \mathbf{Cat}$. nLab says the same thing but for pseudofunctors $\mathcal{C}^{\text{op}} \rightarrow \mathbf{Cat}$. Definition 2.2.7 of [6] gives the definition of an opfibration $p : E \rightarrow B$ as a fibration $E^{\text{op}} \rightarrow B^{\text{op}}$, which I believe; they also say in Theorem 2.2.3 that pseudofunctors into \mathcal{C} are the same as pseudofunctors $\mathcal{C}^{\text{op}} \rightarrow \mathbf{Cat}$, which means that opfibrations into \mathcal{C} (which are fibrations into \mathcal{C}^{op}) are pseudofunctors $\mathcal{C} \rightarrow \mathbf{Cat}$. Um?

[2] states the theorem I need nicely, but doesn’t give a proof or even a reference for the non-discrete case. Another source is [10], which seems to give a proof (though I haven’t checked it).

THEOREM 1.1.4 ([10], Thm 2.4). There is an equivalence of 2-categories

$$\mathbf{Psd}[\mathcal{C}^{\text{op}}, \mathbf{Cat}] \simeq \mathbf{Fib}(\mathcal{C}),$$

where $\mathbf{Psd}[\mathcal{C}^{\text{op}}, \mathbf{Cat}]$ is the 2-category of pseudofunctors $\mathcal{C}^{\text{op}} \rightarrow \mathbf{Cat}$, and $\mathbf{Fib}(\mathcal{C}) \hookrightarrow \mathbf{Cat}_{/\mathcal{C}}$ is the 2-category of fibrations into \mathcal{C} .

REMARK 1.1.5. Combining [Remark 1.1.3](#) with [Theorem 1.1.4](#) gives us an equivalence of 2-categories

$$\mathbf{Psd}[\mathcal{C}, \mathbf{Cat}] \simeq \mathbf{opFib}(\mathcal{C}).$$

We will not prove the theorems above (see [\[10\]](#) for more details) but we will describe how to pass between the two 2-categories. Actually maybe chapter 10 of [\[5\]](#) is a better reference, because the functions it talks about actually compose.

Let $F : \mathcal{C}^{\text{op}} \rightarrow \mathbf{Cat}$ be a pseudofunctor. Define the category X as follows: the objects of X are pairs (c, x) , with $c \in \mathcal{C}$, $x \in F(c)$. A map $(c, x) \rightarrow (d, y)$ is a pair (f, u) , where $f : c \rightarrow d$ is a morphism in \mathcal{C} , and $u : x \rightarrow (Ff)(y)$ is a morphism in $F(c)$. For an object $(c, x) \in X$, the identity morphism is given by

$$(\text{id}_c, F_x^0 : x \rightarrow F(\text{id}_c)(x)),$$

where F^0 is the natural isomorphism $\text{id}_{F(c)} \cong F(\text{id}_c)$. Further, given two maps $(f, u) : (c, x) \rightarrow (d, y)$ and $(g, v) : (d, y) \rightarrow (e, z)$, their composition $(g, v) \circ (f, u)$ is given by

$$x \xrightarrow{u} (Ff)(y) \xrightarrow{(Ff)(v)} (Ff \circ Fg)(z) \xrightarrow{(F_{f,g}^2)_z} (F(g \circ f))(z),$$

where $F_{f,g}^2$ is the natural isomorphism $Ff \circ Fg \cong F(g \circ f)$ (recall that the domain of F is \mathcal{C}^{op} , while f, g are morphisms in \mathcal{C}). One can show that the forgetful functor $X \rightarrow \mathcal{C}$ is a fibration over \mathcal{C} .

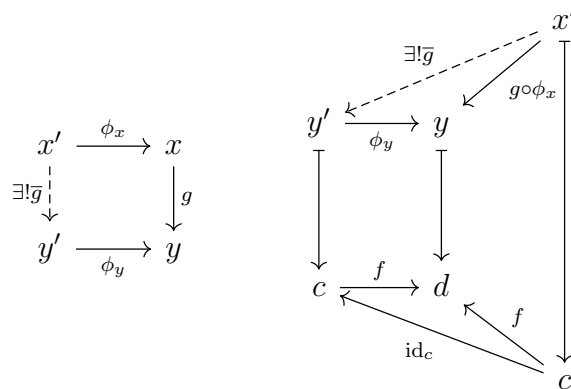
Now, let $p : X \rightarrow \mathcal{C}$ be a fibration over \mathcal{C} . Define a pseudofunctor

$$\begin{aligned} \mathcal{C}^{\text{op}} &\rightarrow \mathbf{Cat} \\ c &\mapsto p^{-1}\{c\}. \end{aligned}$$

For a map $f : c \rightarrow d$ in \mathcal{C} , we define a functor

$$f^* : p^{-1}\{d\} \rightarrow p^{-1}\{c\},$$

which sends $x \in p^{-1}\{d\}$ to the source x' of a cartesian edge $\phi_x : x' \rightarrow x$ lying over f . Now, let $f^*(y) = y'$, with $\phi_y : y' \rightarrow y$, and let $g : x \rightarrow y$ be a morphism in $p^{-1}\{d\}$ (that is, a morphism in X lying over id_c). Then, since $\phi_y : y' \rightarrow y$ is a cartesian edge, there is a unique lift $\bar{g} : x' \rightarrow y'$ of id_c making the square on the left commute, as shown in the diagram on the right.



We thus define $f^*g = \bar{g}$ in $p^{-1}\{c\}$.

It can be shown that these data assemble into a pseudofunctor $\mathcal{C}^{\text{op}} \rightarrow \mathbf{Cat}$.

I should dualise the above to make it not ridiculous, but I'm already so many circles into op hell now, and it's also past 1am.

1.2 Symmetric monoidal categories are special pseudofunctors to \mathbf{Cat}

DEFINITION 1.2.1. A *symmetric monoidal category* is a category \mathcal{C} equipped with a bifunctor

$$\otimes : \mathcal{C} \times \mathcal{C} \rightarrow \mathcal{C},$$

an object $\mathbf{1} \in \mathcal{C}$, and natural isomorphisms

$$\alpha : \otimes \circ (\otimes \times \text{id}) \xrightarrow{\sim} \otimes \circ (\text{id} \times \otimes), \quad l : \otimes \circ (\mathbf{1} \times \text{id}) \xrightarrow{\sim} \text{id}, \quad r : \otimes \circ (\text{id} \times \mathbf{1}) \xrightarrow{\sim} \text{id},$$

$$\tau : \otimes \xrightarrow{\sim} \sigma \circ \otimes,$$

where $\sigma : \mathcal{C} \times \mathcal{C}$ swaps the order of the factors. These isomorphisms are subject to the condition that $\tau^2 = \text{id}$, and that the diagrams below commute.

$$\begin{array}{ccc} & ((U \otimes V) \otimes W) \otimes X & \\ \alpha_{U \otimes V, W, X} \swarrow & & \searrow \alpha_{U, V, W \otimes X} \\ (U \otimes V) \otimes (W \otimes X) & & (U \otimes (V \otimes W)) \otimes X \\ \alpha_{U, V, W \otimes X} \downarrow & & \downarrow \alpha_{U, V \otimes W, X} \\ U \otimes (V \otimes (W \otimes X)) & \xleftarrow{\text{id} \otimes \alpha_{V, W, X}} & U \otimes ((V \otimes W) \otimes X) \end{array}$$

$$\begin{array}{ccccc} (V \otimes \mathbf{1}) \otimes W & \xrightarrow{\alpha_{V, \mathbf{1}, W}} & V \otimes (\mathbf{1} \otimes W) & & V \otimes \mathbf{1} \xrightarrow{\tau_{V, \mathbf{1}}} \mathbf{1} \otimes V \\ & \searrow r_V \otimes \text{id}_W & \swarrow \text{id}_V \otimes l_W & & \searrow r_V & \swarrow l_V \\ & & V \otimes W & & & V \end{array}$$

$$\begin{array}{ccc} & (U \otimes V) \otimes W & \\ \alpha_{U, V, W} \swarrow & & \searrow \tau_{U, V} \otimes \text{id}_W \\ U \otimes (V \otimes W) & & (V \otimes U) \otimes W \\ \tau_{U, V \otimes W} \downarrow & & \downarrow \alpha_{V, U, W} \\ (V \otimes W) \otimes U & & V \otimes (U \otimes W) \\ \alpha_{V, W, U} \searrow & & \swarrow \text{id}_V \otimes \tau_{U, W} \\ & V \otimes (W \otimes U) & \end{array}$$

Now that we know how to move between pseudofunctors to \mathbf{Cat} and opfibrations, let's write the data of a symmetric monoidal category as a pseudofunctor to \mathbf{Cat} .

Let (\mathcal{C}, \otimes) be a symmetric monoidal category. Define a pseudofunctor

$$F : \text{Fin}_* \rightarrow \mathbf{Cat}$$

$$\langle n \rangle \mapsto \mathcal{C}^n$$

Let $f : \langle n \rangle \rightarrow \langle m \rangle$ be a morphism in Fin_* . This induces a morphism

$$f^* : (C_1, \dots, C_n) \mapsto (C'_1, \dots, C'_m),$$

where

$$C'_i = \bigotimes_{j \in f^{-1}\{i\}} C_j.$$

1.3 ...which are special opfibrations

- The above implies there's some category \mathcal{D} such that opfibrations $\mathcal{D} \rightarrow \mathbf{Fin}_*$ are the same as symmetric monoidal categories. Let's see what \mathcal{D} is.
- (Usual) definition of symmetric monoidal category and translation into the language of op-fibrations into \mathbf{Fin}_* , running example of \mathbf{Vect}_k with \otimes or \times .
- Possibly mention swapping out \mathbf{Fin}_* for Δ^{op} gives a monoidal category rather than a symmetric monoidal category. (How do we get a braided monoidal category? Apparently there is no base 1-category we can look at opfibrations into, because the correct formulation is with E_2 , which has higher homotopy groups on the mapping spaces.)
- Correspondence of symmetric monoidal functors with morphisms of opfibrations.

The construction below is in [7].

Let (\mathcal{C}, \otimes) be a symmetric monoidal category. We define a new category \mathcal{C}^{\otimes} , whose objects are finite (possibly empty) sequences of objects of \mathcal{C} , denoted by $[C_1, \dots, C_n]$. A morphism

$$[C_1, \dots, C_n] \rightarrow [C'_1, \dots, C'_m]$$

consists of a subset $S \subseteq \{1, \dots, n\}$, a map of finite sets $\alpha : S \rightarrow \{1, \dots, m\}$, and a collection of morphisms $\{f_j : \bigotimes_{i \in \alpha^{-1}\{j\}} C_i \rightarrow C'_j\}_{1 \leq j \leq m}$ in \mathcal{C} .

For two morphisms $f : [C_1, \dots, C_n] \rightarrow [C'_1, \dots, C'_m]$ and $g : [C'_1, \dots, C'_m] \rightarrow [C''_1, \dots, C''_l]$, determining two subsets $S \subseteq \{1, \dots, n\}$ and $T \subseteq \{1, \dots, m\}$ and maps $\alpha : S \rightarrow \{1, \dots, m\}$, $\beta : T \rightarrow \{1, \dots, l\}$, the composition $g \circ f$ is given by the subset $U = \alpha^{-1}T \subseteq \{1, \dots, n\}$, the map $\beta \circ \alpha : U \rightarrow \{1, \dots, l\}$ and the maps

$$\left\{ \bigotimes_{i \in (\beta \circ \alpha)^{-1}\{k\}} C_i \cong \bigotimes_{j \in \beta^{-1}\{k\}} \bigotimes_{i \in \alpha^{-1}\{j\}} C_i \rightarrow \bigotimes_{j \in \beta^{-1}\{k\}} C'_j \rightarrow C''_k \right\}_{1 \leq k \leq l}.$$

For example, let

$$f : [C_1, C_2, C_3, C_4] \rightarrow [C'_1, C'_2, C'_3]$$

be a morphism in \mathcal{C}^{\otimes} consisting of the subset $\{1, 2, 3\} \subseteq \{1, 2, 3, 4\}$, the map

$$\begin{aligned} \alpha : \{1, 2, 3\} &\rightarrow \{1, 2, 3\}, \\ 1 &\mapsto 1, \\ 2 &\mapsto 2, \\ 3 &\mapsto 3, \end{aligned}$$

and morphisms

$$f_1 : C_1 \rightarrow C'_1, \quad f_2 : C_2 \otimes C_3 \rightarrow C'_2, \quad f_3 : \mathbf{1} \rightarrow C'_3,$$

and let

$$g : [C'_1, C'_2, C'_3] \rightarrow [C''_1, C''_2, C''_3]$$

be a morphism in \mathcal{C}^{\otimes} consisting of the subset $\{1, 2, 3\} \subseteq \{1, 2, 3\}$, the map

$$\begin{aligned} \beta : \{1, 2, 3\} &\rightarrow \{1, 2, 3\}, \\ 1, 2, 3 &\mapsto 3, \end{aligned}$$

and morphisms

$$g_1 : \mathbf{1} \rightarrow C''_1, \quad g_2 : \mathbf{1} \rightarrow C''_2, \quad g_3 : C'_1 \otimes C'_2 \otimes C'_3 \rightarrow C''_3.$$

Then the composition $g \circ f$ consists of the subset $\alpha^{-1}\{1, 2, 3\} = \{1, 2, 3\} \subseteq \{1, 2, 3, 4\}$, the map

$$\begin{aligned}\beta \circ \alpha : \{1, 2, 3\} &\rightarrow \{1, 2, 3\}, \\ 1, 2, 3 &\mapsto 3,\end{aligned}$$

and the morphisms

$$(g \circ f)_1 = g_1, \quad (g \circ f)_2 = g_2, \quad (g \circ f)_3 = g_3 \circ (f_1 \otimes f_2 \otimes f_3).$$

(really?)

(some intuition on this, tensor along the fibres, etc)

Claim: the forgetful functor

$$\begin{aligned}p : \mathcal{C}^{\otimes} &\rightarrow \mathbf{Fin}_*, \\ [C_1, \dots, C_n] &\mapsto \langle n \rangle_*\end{aligned}$$

is an opfibration. (It almost tautologically is).

PROPOSITION 1.3.1 ([4], Prop 4.26). If (\mathcal{C}, \otimes) is a symmetric monoidal category, then the forgetful functor $p : \mathcal{C}^{\otimes} \rightarrow \mathbf{Fin}_*$ given above is an opfibration. Moreover, p satisfies the *Segal condition*; that is, the Segal maps

$$(\rho_!^1, \dots, \rho_!^n) : \mathcal{C}_{\langle n \rangle}^{\otimes} \rightarrow \mathcal{C}^{\times n}$$

are equivalences. Conversely any Grothendieck opfibration $p : \mathcal{C} \rightarrow \mathbf{Fin}_*$ satisfying the Segal condition gives rise to a symmetric monoidal structure on $\mathcal{C}_{\langle 1 \rangle}$.

We've said that symmetric monoidal categories are special opfibrations to \mathbf{Fin}_* . Now, what are symmetric monoidal functors?

The definition below is basically definition 3.3 of [3].

DEFINITION 1.3.2. Let $p : X \rightarrow \mathcal{C}$ and $q : Y \rightarrow \mathcal{C}$ be two Grothendieck (op)fibrations. A functor $F : X \rightarrow Y$ is a *morphism of (op)fibrations* from p to q if the diagram below commutes,

$$\begin{array}{ccc} X & \xrightarrow{F} & Y \\ & \searrow p & \swarrow q \\ & \mathcal{C} & \end{array}$$

and F sends p -(co)cartesian morphisms to q -(co)cartesian morphisms.

2 Generalisation to ∞ -categories

- Translation of the above into ∞ -categorical language.
- A functor $p : D \rightarrow C$ between ordinary categories is a Grothendieck opfibration if and only if the induced functor $N(p) : N(D) \rightarrow N(C)$ on nerves is a cocartesian fibration – I *think* I have finally managed to prove this!
- Some examples (nerve of an ordinary symmetric monoidal category, currently trying to find more examples – many people talk about **Sp**, but it seems like I'd need a lot of background to understand this).

- If an ∞ -category has finite (co)products, there is a (co)cartesian monoidal structure on \mathcal{C} . And we would have hoped so, because it's definitely true for 1-categories!
- Algebra objects in monoidal (∞ -)categories
- Possibly generalisation to ∞ -operads, depending on how much the above comes to or if I find anything fun to do with symmetric monoidal ∞ -categories.
- Might be cool to try to look at E_k algebras, to resolve the earlier mystery of how to write braided monoidal categories.

We first need an ∞ -categorical analogue of Grothendieck opfibrations. We start by requiring that our functor is what's known as an *inner fibration*; there is no 1-categorical analogue of this, since all functors between 1-categories are automatically inner fibrations under the nerve functor (see [Example 2.0.2](#)). Think of it as a ‘minimum niceness condition’ – we want the fibres to be ∞ -categories in much the same way as we want the fibres of ordinary functors to be categories themselves.

DEFINITION 2.0.1 ([1], Def 2.1). A functor $p : X \rightarrow Y$ between simplicial sets is an *inner fibration* if for all $n \geq 2$, all $0 < k < n$, and any solid arrow commutative square

$$\begin{array}{ccc} \Lambda_k^n & \longrightarrow & X \\ \downarrow & \nearrow & \downarrow p \\ \Delta^n & \longrightarrow & Y \end{array}$$

there exists a dotted lift.

EXAMPLE 2.0.2. Let \mathcal{C}, \mathcal{D} be categories, and $p : \mathcal{C} \rightarrow \mathcal{D}$ be a functor between them. Then $N(p) : N\mathcal{C} \rightarrow N\mathcal{D}$ is an inner fibration.

The following proposition is stated without proof in Section 2.3 of [8].

PROPOSITION 2.0.3. Let $p : X \rightarrow Y$ be an inner fibration, and suppose that the diagram below is a pullback square in **sSet**.

$$\begin{array}{ccc} X' & \xrightarrow{f} & X \\ p' \downarrow & \lrcorner & \downarrow p \\ Y' & \xrightarrow{g} & Y \end{array}$$

Then p' is also an inner fibration.

PROOF. Consider the (commutative) solid arrow diagram below.

$$\begin{array}{ccccc} \Lambda_k^n & \xrightarrow{\lambda} & X' & \xrightarrow{f} & X \\ \downarrow \iota & & \downarrow p' & \nearrow \phi & \downarrow p \\ \Delta^n & \xrightarrow{\delta} & Y' & \xrightarrow{g} & Y \end{array}$$

Since p is a fibration, there exists a dotted lift ϕ of $g\delta$; that is, $p\phi = g\delta$ and $\phi\iota = f\lambda$. Further, since the right square is a pullback diagram, there exists a unique map $\phi' : \Delta^n \rightarrow X'$ making

the diagram below commute.

$$\begin{array}{ccccc}
 \Delta^n & & \xrightarrow{\phi} & & X \\
 & \searrow \phi' & & \searrow f & \\
 & & X' & \xrightarrow{\quad} & X \\
 & \searrow \delta & \downarrow p' & \lrcorner & \downarrow p \\
 & & Y' & \xrightarrow{\quad g \quad} & Y
 \end{array}$$

It remains to show that the triangle below commutes.

$$\begin{array}{ccc}
 \Lambda_k^n & \xrightarrow{\lambda} & X' \\
 \downarrow \iota & \searrow \phi' & \\
 \Delta^n & &
 \end{array}$$

Again, using the universal property of pullbacks, we see that there exist unique dotted maps such that the diagrams below commute.

$$\begin{array}{ccc}
 \Lambda_k^n & \xrightarrow{f\lambda} & X \\
 \searrow \delta\iota & & \downarrow p \\
 X' & \xrightarrow{\quad f \quad} & X \\
 \downarrow p' & \lrcorner & \\
 Y' & \xrightarrow{\quad g \quad} & Y
 \end{array}
 \qquad
 \begin{array}{ccc}
 \Lambda_k^n & \xrightarrow{f\phi'\iota} & X \\
 \searrow \delta\iota & & \downarrow p \\
 X' & \xrightarrow{\quad f \quad} & X \\
 \downarrow p' & \lrcorner & \\
 Y' & \xrightarrow{\quad g \quad} & Y
 \end{array}$$

The maps λ and $\phi'\iota$ make the left and right diagrams commute respectively. Further, we note that $f\phi' = \phi$ (by the second diagram) and $\phi\iota = f\lambda$ (since p is an inner fibration), so $f\phi'\iota = f\lambda$. Therefore, the above two diagrams are identical. Thus, by the uniqueness property of pullbacks, $\lambda = \phi'\iota$. \square

(Stupid note to self, very obvious but I forget it every now and again):

- If $X : \Delta^{\text{op}} \rightarrow \mathbf{Set}$ is a simplicial set, and $\Delta^0 : \Delta^{\text{op}} \rightarrow \mathbf{Set} := \text{Hom}(-, [0])$, then a map $F : X \rightarrow \Delta^0$ is a natural transformation $(F_n : X_n \rightarrow *)_{n \in \mathbb{N}_0}$. That is, such a natural transformation is a family of maps down to a point. In other words, there's only really one natural transformation, so we really *can* view Δ^0 as a point.
- If Y is a simplicial set, and $y \in Y_0$ is a vertex of Y , we can view $\{y\}$ as a copy of Δ^0 . Why is this? We can view $\{y\}$ as the constant simplicial set, sending everything to y . Then a natural isomorphism $\Delta^0 \cong \{y\}$ is a collection of isomorphisms $(* \rightarrow *)$, of which there is exactly one. Why is it natural? Well, there's only one map from a one-point set to another one-point set, so the square always commutes.

EXAMPLE 2.0.4 ([1], Ex 2.2). Let $p : X \rightarrow \Delta^0$ be the canonical map, and suppose we have the diagram below, such that the outer square commutes.

$$\begin{array}{ccc}
 \Lambda_k^n & \longrightarrow & X \\
 \downarrow & \nearrow & \downarrow p \\
 \Delta^n & \longrightarrow & \Delta^0
 \end{array}$$

The lower triangle commutes automatically, so the statement that p is an inner fibration is equivalent to the statement that for all $n \geq 2$, all $0 < k < n$, and any map $\Lambda_k^n \rightarrow X$, there

exists a dotted lift.

$$\begin{array}{ccc} \Lambda_k^n & \longrightarrow & X \\ \downarrow & \nearrow \text{dotted} & \\ \Delta^n & & \end{array}$$

That is, X is an ∞ -category.

Now, combining the above argument with [Proposition 2.0.3](#), we see that for any inner fibration $p : X \rightarrow Y$, each fibre $X \times_Y \{y\}$ is an ∞ -category.

DEFINITION 2.0.5 ([1], Def 3.1). Let $p : X \rightarrow Y$ be an inner fibration. An edge $f : \Delta^1 \rightarrow X$ of X is *p-cocartesian* if for all $n \geq 2$, any extension

$$\begin{array}{ccc} \Delta^{\{0,1\}} & \xrightarrow{f} & X \\ \downarrow & \nearrow F & \\ \Lambda_0^n & & \end{array}$$

and any solid arrow commutative diagram

$$\begin{array}{ccc} \Lambda_0^n & \xrightarrow{F} & X \\ \downarrow & \nearrow \text{dotted} & \downarrow p \\ \Delta^n & \longrightarrow & Y \end{array}$$

a dotted lift exists.

DEFINITION 2.0.6. Let $p : X \rightarrow Y$ be an inner fibration. Then p is a cocartesian fibration if for any edge $\phi : y \rightarrow y'$ in Y_1 , and for every $x \in X_0$ lying over y , there exists a p -cocartesian edge $f : x \rightarrow x'$ of X lying over ϕ .

The following proposition tells us that the above definition is a reasonable generalisation of [Definition 1.1.2](#). It is also stated without proof in [8], which did not do wonders for my ego.

PROPOSITION 2.0.7 ([8], Rmk 2.4.2.2). Let \mathcal{C}, \mathcal{D} be categories, and let $p : \mathcal{C} \rightarrow \mathcal{D}$ be a functor between them. Then p is a Grothendieck opfibration if and only if the induced map $N(p) : N\mathcal{C} \rightarrow N\mathcal{D}$ is a cocartesian fibration of simplicial sets.

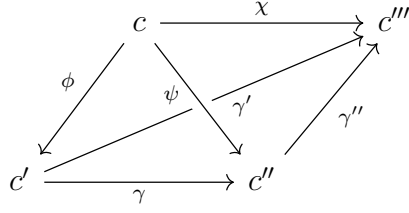
PROOF. Let $f : d \rightarrow d'$ be a morphism of \mathcal{D} , and let c lie over d .

Suppose p is a Grothendieck opfibration, let $F : \Lambda_0^n \rightarrow N\mathcal{C}$ be an extension of f , and let

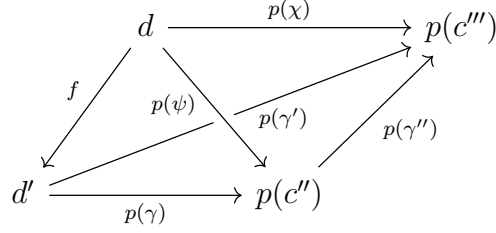
$$\begin{array}{ccc} \Lambda_0^n & \xrightarrow{F} & N\mathcal{C} \\ \downarrow & & \downarrow N(p) \\ \Delta^n & \longrightarrow & N\mathcal{D} \end{array}$$

be a commutative diagram. If $n = 2$, it follows immediately from the fact that p is an opfibration that a dotted lift exists. Further, if $n > 3$, there is nothing to check, since an n -simplex in a category commutes if and only if all of its triangles commute, which is guaranteed for any extension $F : \Lambda_0^n \rightarrow N\mathcal{C}$. We thus prove the proposition for $n = 3$.

Suppose we have an extension $F : \Lambda_0^3 \rightarrow N\mathcal{C}$ of f ; that is, a tetrahedron



such that all faces containing the vertex c commute. Let

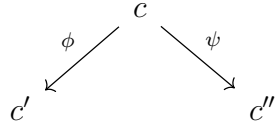


be a commutative tetrahedron in \mathcal{D} . We claim that the tetrahedron in \mathcal{C} commutes. First, note that $\gamma'' \circ \gamma$ is a lift of $p(\gamma')$, since $p(\gamma') = p(\gamma'') \circ p(\gamma) = p(\gamma'' \circ \gamma)$. Further,

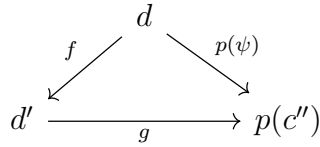
$$\begin{aligned} (\gamma \circ \gamma'') \circ \phi &= \gamma'' \circ \psi \\ &= \chi. \end{aligned}$$

Thus, by the uniqueness in the universal property of ϕ , we have that $\gamma' = \gamma'' \circ \gamma$, as required.

Now, suppose $N(p)$ is a cocartesian fibration. Then there exists a lift $\phi : c \rightarrow c'$ of f , and, in particular, for any diagram

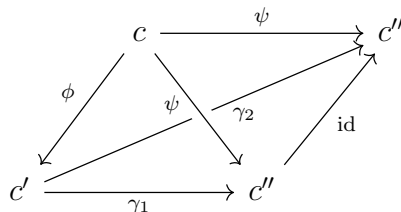


in \mathcal{C} , and any commutative diagram



in \mathcal{D} , there exists a map $\gamma : c' \rightarrow c''$ such that γ lies over g and $\gamma \circ \phi = \psi$. It remains to show that γ is unique.

Suppose that there were two maps $\gamma_1, \gamma_2 : c' \rightarrow c''$ lying over g and satisfying $\gamma_1 \circ \phi = \gamma_2 \circ \phi = \psi$. Then we would have a tetrahedron



where all faces containing the vertex c commute. The image of this tetrahedron under p commutes in \mathcal{D} , so the original tetrahedron must commute in \mathcal{C} ; that is, $\gamma_1 = \gamma_2$. \square

DEFINITION 2.0.8 ([8], Def 2.0.0.7). A *symmetric monoidal ∞ -category* is a cocartesian fibration of simplicial sets $p : X^\otimes \rightarrow N(\mathbf{Fin}_*)$ such that for each $n \geq 0$, the maps

$$\{\rho^i : \langle n \rangle \rightarrow \langle 1 \rangle\}_{1 \leq i \leq n}$$

induce functors $\rho_i^! : X_{\langle n \rangle}^\otimes \rightarrow X_{\langle 1 \rangle}^\otimes$ which determine an equivalence $X_{\langle n \rangle}^\otimes \simeq \left(X_{\langle 1 \rangle}^\otimes\right)^n$.

EXAMPLE 2.0.9. Let (\mathcal{C}, \otimes) be a symmetric monoidal category. Then $p : N(\mathcal{C}^\otimes) \rightarrow N(\mathbf{Fin}_*)$ is a symmetric monoidal ∞ -category.

3 A nontrivial example

Throughout this section, \mathcal{A} is an abelian category, and $\mathcal{A}_{\text{proj}}$ is the full subcategory of \mathcal{A} spanned by the projective objects.

DEFINITION 3.0.1 ([7], Def 1.2.3.1). A *chain complex* with values in \mathcal{A} is a composable sequence of morphisms

$$\cdots \rightarrow A_2 \xrightarrow{d(2)} A_1 \xrightarrow{d(1)} A_0 \xrightarrow{d(0)} A_{-1} \rightarrow \cdots$$

in \mathcal{A} such that $d(n-1) \circ d(n) = 0$ for all $n \in \mathbb{Z}$. The collection of chain complexes with values in \mathcal{A} is an additive category, $\text{Ch}(\mathcal{A})$.

DEFINITION 3.0.2 ([7], Not 1.3.2.6). $\text{Ch}^-(\mathcal{A})$ is the full subcategory of $\text{Ch}(\mathcal{A})$ spanned by those chain complexes M_* such that $M_n \simeq 0$ for $n < 0$.

DEFINITION 3.0.3 ([7], Def 1.3.2.7). Suppose \mathcal{A} has enough projective objects. We let $\mathcal{D}^-(\mathcal{A})$ denote the ∞ -category $N_{\text{dg}}(\text{Ch}^-(\mathcal{A}_{\text{proj}}))$. We refer to $\mathcal{D}^-(\mathcal{A})$ as the *derived ∞ -category of \mathcal{A}* .

4 Miscellaneous stupid notes

4.1 Observations

Let $S \in \mathbf{Set}$. We define the constant simplicial set

$$\overline{S} : \Delta^{\text{op}} \longrightarrow \mathbf{Set}$$

$$\begin{array}{ccc} [n] & \longrightarrow & S \\ f \downarrow & & \uparrow \text{id} \\ [m] & \longrightarrow & S \end{array}$$

It's a Kan complex. Why? Well, when you consider S as a discrete category, and take the nerve of it, you get \overline{S} . You can then either just see that it's a Kan complex (fill the horns with identities) or use the fact that the nerve of a groupoid is a Kan complex. It's surely in [8] or [9] somewhere.

4.2 Questions

Questions:

- ...what $\ast\text{is}\ast \text{ Grpd}_\infty$?

4.3 Equivalent definitions

\mathbf{Grpd}_∞

An algebraic category

An equivalence of ∞ -categories

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