

# Symmetric Monoidal $\infty$ -Categories

- Note: we will call  $(\infty, 1)$ -categories  $\infty$ -categories.
- Main references: [6], [7], [3], [1], [4], [8]. Referencing Kerodon test: [8, Example 01UB].

## 1 Pseudofunctors, opfibrations, and symmetric monoidal categories

### 1.1 Pseudofunctors to **Cat** are Grothendieck opfibrations

- Grothendieck op-fibrations correspond to pseudofunctors to **Cat**.

The below gives an example to complement [3].

Example of a pseudofunctor to **Cat**: Let  $\mathcal{C}$  be a category with pullbacks. Recall that for a map  $f : C \rightarrow D$  in  $\mathcal{C}$ , we define a pullback functor

$$\begin{aligned} f^* : \mathcal{C}_{/D} &\rightarrow \mathcal{C}_{/C}, \\ (h : X \rightarrow D) &\mapsto (f^*h : P \rightarrow C), \end{aligned}$$

where we have formed a pullback

$$\begin{array}{ccc} P & \xrightarrow{h^*f} & X \\ f^*h \downarrow & & \downarrow h \\ C & \xrightarrow{f} & D \end{array}$$

in  $\mathcal{C}$ . For any map

$$\begin{array}{ccc} X & \xrightarrow{\phi} & X' \\ & \searrow h & \swarrow h' \\ & D & \end{array}$$

from  $h$  to  $h'$  in  $\mathcal{C}_{/D}$ , we define  $f^*\phi$  to be the unique map making the diagram below commute.

$$\begin{array}{ccccc} X & \xrightarrow{\phi} & X' & & \\ & \searrow h & \swarrow h' & & \\ & D & & & \\ & \uparrow f & & & \\ P & \xrightarrow{f^*\phi} & P' & & \\ & \searrow f^*h & \swarrow f^*h' & & \\ & C & & & \end{array}$$

Now, we may wish to define a functor

$$\begin{aligned} F : \mathcal{C}^{\text{op}} &\rightarrow \mathbf{Cat}, \\ C &\mapsto \mathcal{C}_{/C}, \end{aligned}$$

which sends a map  $f : C \rightarrow D$  in  $\mathcal{C}$  to a pullback functor  $f^* : \mathcal{C}_{/D} \rightarrow \mathcal{C}_{/C}$ . However a problem arises when we check that  $F$  respects composition: suppose  $f : C \rightarrow D$ ,  $g : D \rightarrow E$  are maps in  $\mathcal{C}$ . Then

$$F(g \circ f)(h : X \rightarrow E) = (g \circ f)^*h : P \rightarrow C,$$

corresponding to the pullback

$$\begin{array}{ccc} P & \xrightarrow{h^*(g \circ f)} & X \\ (g \circ f)^*h \downarrow & & \downarrow h \\ C & \xrightarrow{g \circ f} & E \end{array}$$

in  $\mathcal{C}$ . On the other hand,

$$(F(g) \circ F(f))(h : X \rightarrow E) = f^*(g^*h) : P'' \rightarrow C,$$

which corresponds to the diagram below.

$$\begin{array}{ccccc} P'' & \xrightarrow{(g^*h)^*f} & P' & \xrightarrow{h^*g} & X \\ f^*(g^*h) \downarrow & & \downarrow g^*h & & \downarrow h \\ C & \xrightarrow{f} & D & \xrightarrow{g} & E \end{array}$$

The outer square is indeed a pullback square, since the inner two squares are, so we have a unique isomorphism  $P \cong P''$ . However, we do not in general have equality. This is because pullbacks are only unique up to unique isomorphism, and in defining a pullback functor we made arbitrary (and not necessarily compatible) choices of  $P, P'$  and  $P''$ . Thus, we have not defined a functor  $\mathcal{C}^{\text{op}} \rightarrow \mathbf{Cat}$ , rather, we have defined what is known as a *pseudofunctor*; that is, a weak functor between 2-categories.

The above example is one way in which pseudofunctors into  $\mathbf{Cat}$  naturally arise; another common example is the pseudofunctor

$$\begin{aligned} \mathbf{CRing} &\rightarrow \mathbf{Cat} \\ R &\mapsto R\text{-}\mathbf{Mod}, \end{aligned}$$

which sends a ring homomorphism  $\phi : R \rightarrow S$  to the functor  $- \otimes_R S : R\text{-}\mathbf{Mod} \rightarrow S\text{-}\mathbf{Mod}$  (extension of scalars). However, to give the data of a pseudofunctor  $F : \mathcal{C} \rightarrow \mathbf{Cat}$ , we must specify not only the functions  $\text{ob}(\mathcal{C}) \rightarrow \text{ob}(\mathbf{Cat})$  and  $\text{Hom}_{\mathcal{C}}(X, Y) \rightarrow \text{Hom}_{\mathbf{Cat}}(F(X), F(Y))$  for each  $X, Y \in \mathcal{C}$ , but also natural isomorphisms

$$F(\text{id}_X) \cong \text{id}_{F(X)}, \quad F(g \circ f) \cong F(g) \circ F(f).$$

This problem only becomes worse as we consider functors between higher categories...

Let's use Grothendieck opfibrations instead.

**DEFINITION 1.1.1.** Let  $p : X \rightarrow \mathcal{C}$  be a functor, and let  $f : c \rightarrow c'$  be a morphism in  $\mathcal{C}$ . A morphism  $\phi : x \rightarrow x'$  in  $X$  lying over  $f$  is *p-cocartesian* if for any other morphism  $\psi : x \rightarrow x''$



- Possibly mention swapping out  $\mathbf{Fin}_*$  for  $\Delta^{\text{op}}$  gives a monoidal category rather than a symmetric monoidal category. (How do we get a braided monoidal category? Apparently there is no base 1-category we can look at opfibrations into, because the correct formulation is with  $E_2$ , which has higher homotopy groups on the mapping spaces.)
- Correspondence of symmetric monoidal functors with morphisms of opfibrations.

The construction below is in [6].

Let  $(\mathcal{C}, \otimes)$  be a symmetric monoidal category. We define a new category  $\mathcal{C}^{\otimes}$ , whose objects are finite (possibly empty) sequences of objects of  $\mathcal{C}$ , denoted by  $[C_1, \dots, C_n]$ . A morphism

$$[C_1, \dots, C_n] \rightarrow [C'_1, \dots, C'_m]$$

consists of a subset  $S \subseteq \{1, \dots, n\}$ , a map of finite sets  $\alpha : S \rightarrow \{1, \dots, m\}$ , and a collection of morphisms  $\{f_j : \bigotimes_{i \in \alpha^{-1}\{j\}} C_i \rightarrow C'_j\}_{1 \leq j \leq m}$  in  $\mathcal{C}$ .

For two morphisms  $f : [C_1, \dots, C_n] \rightarrow [C'_1, \dots, C'_m]$  and  $g : [C'_1, \dots, C'_m] \rightarrow [C''_1, \dots, C''_l]$ , determining two subsets  $S \subseteq \{1, \dots, n\}$  and  $T \subseteq \{1, \dots, m\}$  and maps  $\alpha : S \rightarrow \{1, \dots, m\}$ ,  $\beta : T \rightarrow \{1, \dots, l\}$ , the composition  $g \circ f$  is given by the subset  $U = \alpha^{-1}T \subseteq \{1, \dots, n\}$ , the map  $\beta \circ \alpha : U \rightarrow \{1, \dots, l\}$  and the maps

$$\left\{ \bigotimes_{i \in (\beta \circ \alpha)^{-1}\{k\}} C_i \cong \bigotimes_{j \in \beta^{-1}\{k\}} \bigotimes_{i \in \alpha^{-1}\{j\}} C_i \rightarrow \bigotimes_{j \in \beta^{-1}\{k\}} C'_j \rightarrow C''_k \right\}_{1 \leq k \leq l}.$$

For example, let

$$f : [C_1, C_2, C_3, C_4] \rightarrow [C'_1, C'_2, C'_3]$$

be a morphism in  $\mathcal{C}^{\otimes}$  consisting of the subset  $\{1, 2, 3\} \subseteq \{1, 2, 3, 4\}$ , the map

$$\begin{aligned} \alpha : \{1, 2, 3\} &\rightarrow \{1, 2, 3\}, \\ 1 &\mapsto 1, \\ 2 &\mapsto 2, \\ 3 &\mapsto 3, \end{aligned}$$

and morphisms

$$f_1 : C_1 \rightarrow C'_1, \quad f_2 : C_2 \otimes C_3 \rightarrow C'_2, \quad f_3 : \mathbf{1} \rightarrow C'_3,$$

and let

$$g : [C'_1, C'_2, C'_3] \rightarrow [C''_1, C''_2, C''_3]$$

be a morphism in  $\mathcal{C}^{\otimes}$  consisting of the subset  $\{1, 2, 3\} \subseteq \{1, 2, 3\}$ , the map

$$\begin{aligned} \beta : \{1, 2, 3\} &\rightarrow \{1, 2, 3\}, \\ 1, 2, 3 &\mapsto 3, \end{aligned}$$

and morphisms

$$g_1 : \mathbf{1} \rightarrow C''_1, \quad g_2 : \mathbf{1} \rightarrow C''_2, \quad g_3 : C'_1 \otimes C'_2 \otimes C'_3 \rightarrow C''_3.$$

Then the composition  $g \circ f$  consists of the subset  $\alpha^{-1}\{1, 2, 3\} = \{1, 2, 3\} \subseteq \{1, 2, 3, 4\}$ , the map

$$\begin{aligned} \beta \circ \alpha : \{1, 2, 3\} &\rightarrow \{1, 2, 3\}, \\ 1, 2, 3 &\mapsto 3, \end{aligned}$$

and the morphisms

$$(g \circ f)_1 = g_1, \quad (g \circ f)_2 = g_2, \quad (g \circ f)_3 = g_3 \circ (f_1 \otimes f_2 \otimes f_3).$$

(really?)

(some intuition on this, tensor along the fibres, etc)

Claim: the forgetful functor

$$\begin{aligned} p : \mathcal{C}^{\otimes} &\rightarrow \mathbf{Fin}_*, \\ [C_1, \dots, C_n] &\mapsto \langle n \rangle_* \end{aligned}$$

is an opfibration. (It almost tautologically is).

## 1.4 Generalisation to $\infty$ -categories

- Translation of the above into  $\infty$ -categorical language.
- A functor  $p : D \rightarrow C$  between ordinary categories is a Grothendieck opfibration if and only if the induced functor  $N(p) : N(D) \rightarrow N(C)$  on nerves is a cocartesian fibration – I \*think\* I have finally managed to prove this!
- Some examples (nerve of an ordinary symmetric monoidal category, currently trying to find more examples – many people talk about **Sp**, but it seems like I’d need a lot of background to understand this).
- If an  $\infty$ -category has finite (co)products, there is a (co)cartesian monoidal structure on  $\mathcal{C}$ . And we would have hoped so, because it’s definitely true for 1-categories!
- Algebra objects in monoidal ( $\infty$ -)categories
- Possibly generalisation to  $\infty$ -operads, depending on how much the above comes to or if I find anything fun to do with symmetric monoidal  $\infty$ -categories.
- Might be cool to try to look at  $E_k$  algebras, to resolve the earlier mystery of how to write braided monoidal categories.

DEFINITION 1.4.1 ([1], Def 2.1). A functor  $p : X \rightarrow Y$  between simplicial sets is an *inner fibration* if for all  $n \geq 2$ , all  $0 < k < n$ , and any solid arrow commutative square

$$\begin{array}{ccc} \Lambda_k^n & \longrightarrow & X \\ \downarrow & \nearrow & \downarrow p \\ \Delta^n & \longrightarrow & Y \end{array}$$

there exists a dotted lift.

EXAMPLE 1.4.2. Let  $\mathcal{C}, \mathcal{D}$  be categories, and  $p : \mathcal{C} \rightarrow \mathcal{D}$  be a functor between them. Then  $N(p) : N\mathcal{C} \rightarrow N\mathcal{D}$  is an inner fibration.

The following proposition is stated without proof in Section 2.3 of [7].

PROPOSITION 1.4.3. Let  $p : X \rightarrow Y$  be an inner fibration, and suppose that the diagram below is a pullback square in **sSet**.

$$\begin{array}{ccc} X' & \xrightarrow{f} & X \\ p' \downarrow \lrcorner & & \downarrow p \\ Y' & \xrightarrow{g} & Y \end{array}$$

Then  $p'$  is also an inner fibration.

PROOF. Consider the (commutative) solid arrow diagram below.

$$\begin{array}{ccccc}
 \Lambda_k^n & \xrightarrow{\lambda} & X' & \xrightarrow{f} & X \\
 \downarrow \iota & & \downarrow p' & \dashrightarrow & \downarrow p \\
 \Delta^n & \xrightarrow{\delta} & Y' & \xrightarrow{g} & Y
 \end{array}$$

Since  $p$  is a fibration, there exists a dotted lift  $\phi$  of  $g\delta$ ; that is,  $p\phi = g\delta$  and  $\phi\iota = f\lambda$ . Further, since the right square is a pullback diagram, there exists a unique map  $\phi' : \Delta^n \rightarrow X'$  making the diagram below commute.

$$\begin{array}{ccccc}
 \Delta^n & & & & \\
 \delta \searrow & \phi' \dashrightarrow & X' & \xrightarrow{f} & X \\
 & & \downarrow p' & \dashrightarrow & \downarrow p \\
 & & Y' & \xrightarrow{g} & Y
 \end{array}$$

It remains to show that the triangle below commutes.

$$\begin{array}{ccc}
 \Lambda_k^n & \xrightarrow{\lambda} & X' \\
 \downarrow \iota & \dashrightarrow & \downarrow \\
 \Delta^n & & 
 \end{array}$$

Again, using the universal property of pullbacks, we see that there exist unique dotted maps such that the diagrams below commute.

$$\begin{array}{ccc}
 \Lambda_k^n & \xrightarrow{f\lambda} & X \\
 \delta \searrow & \dashrightarrow & \downarrow p \\
 & & Y' \xrightarrow{g} Y
 \end{array}
 \quad
 \begin{array}{ccc}
 \Lambda_k^n & \xrightarrow{f\phi'\iota} & X \\
 \delta \searrow & \dashrightarrow & \downarrow p \\
 & & Y' \xrightarrow{g} Y
 \end{array}$$

The maps  $\lambda$  and  $\phi'\iota$  make the left and right diagrams commute respectively. Further, we note that  $f\phi' = \phi$  (by the second diagram) and  $\phi\iota = f\lambda$  (since  $p$  is an inner fibration), so  $f\phi'\iota = f\lambda$ . Therefore, the above two diagrams are identical. Thus, by the uniqueness property of pullbacks,  $\lambda = \phi'\iota$ .  $\square$

(Stupid note to self, very obvious but I forget it every now and again):

- If  $X : \Delta^{\text{op}} \rightarrow \mathbf{Set}$  is a simplicial set, and  $\Delta^0 : \Delta^{\text{op}} \rightarrow \mathbf{Set} := \text{Hom}(-, [0])$ , then a map  $F : X \rightarrow \Delta^0$  is a natural transformation  $(F_n : X_n \rightarrow *)_{n \in \mathbb{N}_0}$ . That is, such a natural transformation is a family of maps down to a point. In other words, there's only really one natural transformation, so we really \*can\* view  $\Delta^0$  as a point.
- If  $Y$  is a simplicial set, and  $y \in Y_0$  is a vertex of  $Y$ , we can view  $\{y\}$  as a copy of  $\Delta^0$ . Why is this? We can view  $\{y\}$  as the constant simplicial set, sending everything to  $y$ . Then a natural isomorphism  $\Delta^0 \cong \{y\}$  is a collection of isomorphisms  $(* \rightarrow *)$ , of which there is exactly one. Why is it natural? Well, there's only one map from a one-point set to another one-point set, so the square always commutes.

EXAMPLE 1.4.4 ([1], Ex 2.2). Let  $p : X \rightarrow \Delta^0$  be the canonical map, and suppose we have the diagram below, such that the outer square commutes.

$$\begin{array}{ccc} \Lambda_k^n & \longrightarrow & X \\ \downarrow & \nearrow & \downarrow p \\ \Delta^n & \longrightarrow & \Delta^0 \end{array}$$

The lower triangle commutes automatically, so the statement that  $p$  is an inner fibration is equivalent to the statement that for all  $n \geq 2$ , all  $0 < k < n$ , and any map  $\Lambda_k^n \rightarrow X$ , there exists a dotted lift.

$$\begin{array}{ccc} \Lambda_k^n & \longrightarrow & X \\ \downarrow & \nearrow & \\ \Delta^n & & \end{array}$$

That is,  $X$  is an  $\infty$ -category.

Now, combining the above argument with Proposition 1.4.3, we see that for any inner fibration  $p : X \rightarrow Y$ , each fibre  $X \times_Y \{y\}$  is an  $\infty$ -category.

DEFINITION 1.4.5 ([1], Def 3.1). Let  $p : X \rightarrow Y$  be an inner fibration. An edge  $f : \Delta^1 \rightarrow X$  of  $X$  is *p-cocartesian* if for all  $n \geq 2$ , any extension

$$\begin{array}{ccc} \Delta^{\{0,1\}} & \xrightarrow{f} & X \\ \downarrow & \nearrow F & \\ \Lambda_0^n & & \end{array}$$

and any solid arrow commutative diagram

$$\begin{array}{ccc} \Lambda_0^n & \xrightarrow{F} & X \\ \downarrow & \nearrow & \downarrow p \\ \Delta^n & \longrightarrow & Y \end{array}$$

a dotted lift exists.

DEFINITION 1.4.6. Let  $p : X \rightarrow Y$  be an inner fibration. Then  $p$  is a cocartesian fibration if for any edge  $\phi : y \rightarrow y'$  in  $Y_1$ , and for every  $x \in X_0$  lying over  $y$ , there exists a  $p$ -cocartesian edge  $f : x \rightarrow x'$  of  $X$  lying over  $\phi$ .

The following proposition tells us that the above definition is a reasonable generalisation of Definition 1.1.2. It is also stated without proof in [7], which did not do wonders for my ego.

PROPOSITION 1.4.7 ([7], Rmk 2.4.2.2). Let  $\mathcal{C}, \mathcal{D}$  be categories, and let  $p : \mathcal{C} \rightarrow \mathcal{D}$  be a functor between them. Then  $p$  is a Grothendieck opfibration if and only if the induced map  $N(p) : N\mathcal{C} \rightarrow N\mathcal{D}$  is a cocartesian fibration of simplicial sets.

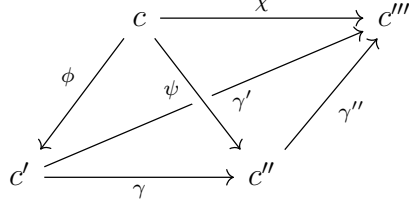
PROOF. Let  $f : d \rightarrow d'$  be a morphism of  $\mathcal{D}$ , and let  $c$  lie over  $d$ .

Suppose  $p$  is a Grothendieck opfibration, let  $F : \Lambda_0^n \rightarrow N\mathcal{C}$  be an extension of  $f$ , and let

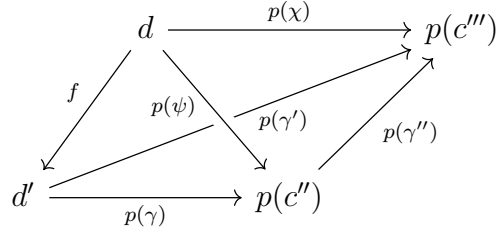
$$\begin{array}{ccc} \Lambda_0^n & \xrightarrow{F} & N\mathcal{C} \\ \downarrow & & \downarrow N(p) \\ \Delta^n & \longrightarrow & N\mathcal{D} \end{array}$$

be a commutative diagram. If  $n = 2$ , it follows immediately from the fact that  $p$  is an opfibration that a dotted lift exists. Further, if  $n > 3$ , there is nothing to check, since an  $n$ -simplex in a category commutes if and only if all of its triangles commute, which is guaranteed for any extension  $F : \Lambda_0^n \rightarrow N\mathcal{C}$ . We thus prove the proposition for  $n = 3$ .

Suppose we have an extension  $F : \Lambda_0^3 \rightarrow N\mathcal{C}$  of  $f$ ; that is, a tetrahedron



such that all faces containing the vertex  $c$  commute. Let

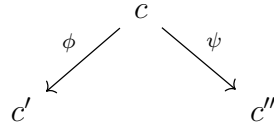


be a commutative tetrahedron in  $\mathcal{D}$ . We claim that the tetrahedron in  $\mathcal{C}$  commutes. First, note that  $\gamma'' \circ \gamma$  is a lift of  $p(\gamma')$ , since  $p(\gamma') = p(\gamma'') \circ p(\gamma) = p(\gamma'' \circ \gamma)$ . Further,

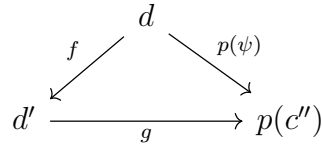
$$\begin{aligned} (\gamma \circ \gamma'') \circ \phi &= \gamma'' \circ \psi \\ &= \chi. \end{aligned}$$

Thus, by the uniqueness in the universal property of  $\phi$ , we have that  $\gamma' = \gamma'' \circ \gamma$ , as required.

Now, suppose  $N(p)$  is a cocartesian fibration. Then there exists a lift  $\phi : c \rightarrow c'$  of  $f$ , and, in particular, for any diagram

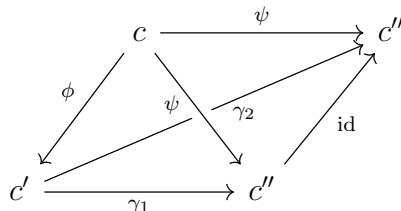


in  $\mathcal{C}$ , and any commutative diagram



in  $\mathcal{D}$ , there exists a map  $\gamma : c' \rightarrow c''$  such that  $\gamma$  lies over  $g$  and  $\gamma \circ \phi = \psi$ . It remains to show that  $\gamma$  is unique.

Suppose that there were two maps  $\gamma_1, \gamma_2 : c' \rightarrow c''$  lying over  $g$  and satisfying  $\gamma_1 \circ \phi = \gamma_2 \circ \phi = \psi$ . Then we would have a tetrahedron





where all faces containing the vertex  $c$  commute. The image of this tetrahedron under  $p$  commutes in  $\mathcal{D}$ , so the original tetrahedron must commute in  $\mathcal{C}$ ; that is,  $\gamma_1 = \gamma_2$ .  $\square$

## 1.5 A nontrivial example

Throughout this section,  $\mathcal{A}$  is an abelian category, and  $\mathcal{A}_{\text{proj}}$  is the full subcategory of  $\mathcal{A}$  spanned by the projective objects.

DEFINITION 1.5.1 ([6], Def 1.2.3.1). A *chain complex* with values in  $\mathcal{A}$  is a composable sequence of morphisms

$$\cdots \rightarrow A_2 \xrightarrow{d(2)} A_1 \xrightarrow{d(1)} A_0 \xrightarrow{d(0)} A_{-1} \rightarrow \cdots$$

in  $\mathcal{A}$  such that  $d(n-1) \circ d(n) = 0$  for all  $n \in \mathbb{Z}$ . The collection of chain complexes with values in  $\mathcal{A}$  is an additive category,  $\text{Ch}(\mathcal{A})$ .

DEFINITION 1.5.2 ([6], Not 1.3.2.6).  $\text{Ch}^-(\mathcal{A})$  is the full subcategory of  $\text{Ch}(\mathcal{A})$  spanned by those chain complexes  $M_*$  such that  $M_n \simeq 0$  for  $n < 0$ .

DEFINITION 1.5.3 ([6], Def 1.3.2.7). Suppose  $\mathcal{A}$  has enough projective objects. We let  $\mathcal{D}^-(\mathcal{A})$  denote the  $\infty$ -category  $N_{\text{dg}}(\text{Ch}^-(\mathcal{A}_{\text{proj}}))$ . We refer to  $\mathcal{D}^-(\mathcal{A})$  as the *derived  $\infty$ -category of  $\mathcal{A}$* .

## References

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