

Symmetric Monoidal ∞ -Categories

- Note: we will call $(\infty, 1)$ -categories ∞ -categories.
- Main references: [4], [5], [2], [1], [3], [6]. How does [6, Example 01UB] work?

1 Pseudofunctors, opfibrations, and symmetric monoidal categories

1.1 Pseudofunctors to **Cat** are Grothendieck opfibrations

- Grothendieck op-fibrations correspond to pseudofunctors to **Cat**.

The below gives an example to complement [2].

Example of a pseudofunctor to **Cat**: Let \mathcal{C} be a category with pullbacks. Recall that for a map $f : C \rightarrow D$ in \mathcal{C} , we define a pullback functor

$$\begin{aligned} f^* : \mathcal{C}_{/D} &\rightarrow \mathcal{C}_{/C}, \\ (h : X \rightarrow D) &\mapsto (f^*h : P \rightarrow C), \end{aligned}$$

where we have formed a pullback

$$\begin{array}{ccc} P & \xrightarrow{h^*f} & X \\ f^*h \downarrow & & \downarrow h \\ C & \xrightarrow{f} & D \end{array}$$

in \mathcal{C} . For any map

$$\begin{array}{ccc} X & \xrightarrow{\phi} & X' \\ & \searrow h & \swarrow h' \\ & D & \end{array}$$

from h to h' in $\mathcal{C}_{/D}$, we define $f^*\phi$ to be the unique map making the diagram below commute.

$$\begin{array}{ccccc} X & \xrightarrow{\phi} & X' & & \\ & \searrow h & \swarrow h' & & \\ & D & & & \\ & \uparrow f & & & \\ P & \xrightarrow{f^*\phi} & P' & & \\ & \searrow f^*h & \swarrow f^*h' & & \\ & C & & & \end{array}$$

Now, we may wish to define a functor

$$\begin{aligned} F : \mathcal{C}^{\text{op}} &\rightarrow \mathbf{Cat}, \\ C &\mapsto \mathcal{C}_{/C}, \end{aligned}$$

which sends a map $f : C \rightarrow D$ in \mathcal{C} to a pullback functor $f^* : \mathcal{C}_{/D} \rightarrow \mathcal{C}_{/C}$. However a problem arises when we check that F respects composition: suppose $f : C \rightarrow D$, $g : D \rightarrow E$ are maps in \mathcal{C} . Then

$$F(g \circ f)(h : X \rightarrow E) = (g \circ f)^* h : P \rightarrow C,$$

corresponding to the pullback

$$\begin{array}{ccc} P & \xrightarrow{h^*(g \circ f)} & X \\ (g \circ f)^* h \downarrow & & \downarrow h \\ C & \xrightarrow{g \circ f} & E \end{array}$$

in \mathcal{C} . On the other hand,

$$(F(g) \circ F(f))(h : X \rightarrow E) = f^*(g^* h) : P'' \rightarrow C,$$

which corresponds to the diagram below.

$$\begin{array}{ccccc} P'' & \xrightarrow{(g^* h)^* f} & P' & \xrightarrow{h^* g} & X \\ f^*(g^* h) \downarrow & & \downarrow g^* h & & \downarrow h \\ C & \xrightarrow{f} & D & \xrightarrow{g} & E \end{array}$$

The outer square is indeed a pullback square, since the inner two squares are, so we have a unique isomorphism $P \cong P''$. However, we do not in general have equality. This is because pullbacks are only unique up to unique isomorphism, and in defining a pullback functor we made arbitrary (and not necessarily compatible) choices of P, P' and P'' . Thus, we have not defined a functor $\mathcal{C}^{\text{op}} \rightarrow \mathbf{Cat}$, rather, we have defined what is known as a *pseudofunctor*.

(definition of a pseudofunctor)

The above example is one way in which pseudofunctors into \mathbf{Cat} naturally arise; another common example is the pseudofunctor

$$\begin{aligned} \mathbf{CRing} &\rightarrow \mathbf{Cat} \\ R &\mapsto R\text{-}\mathbf{Mod}, \end{aligned}$$

which sends a ring homomorphism $\phi : R \rightarrow S$ to the functor $- \otimes_R S : R\text{-}\mathbf{Mod} \rightarrow S\text{-}\mathbf{Mod}$ (extension of scalars). However, to give the data of a pseudofunctor $F : \mathcal{C} \rightarrow \mathbf{Cat}$, we must specify not only the functions $\text{ob}(\mathcal{C}) \rightarrow \text{ob}(\mathbf{Cat})$ and $\text{Hom}_{\mathcal{C}}(X, Y) \rightarrow \text{Hom}_{\mathbf{Cat}}(F(X), F(Y))$ for each $X, Y \in \mathcal{C}$, but also natural isomorphisms

$$F(\text{id}_X) \cong \text{id}_{F(X)}, \quad F(g \circ f) \cong F(g) \circ F(f).$$

That's a pain, let's use Grothendieck opfibrations instead.

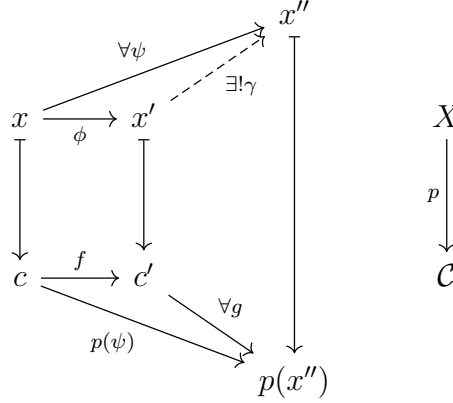
The definitions below are from [1].

DEFINITION 1.1.1. Let $p : X \rightarrow \mathcal{C}$ be a functor, and let $f : c \rightarrow d$ be a morphism in \mathcal{C} . A morphism Φ in X lying over f is *p-cocartesian* if for any other morphism Ψ of X lying over f , there exists a morphism I of X such that $p(I) = \text{id}_d$ and $\Psi = I \circ \Phi$.

DEFINITION 1.1.2. Let $p : X \rightarrow \mathcal{C}$ be a functor. Then p is a *Grothendieck opfibration* if for any morphism of \mathcal{C} and any lift of its source, there is a p -cocartesian morphism with that source lying over it.

And the reconstructed definitions that actually (hopefully) work are below:

DEFINITION 1.1.3. Let $p : X \rightarrow \mathcal{C}$ be a functor, and let $f : c \rightarrow c'$ be a morphism in \mathcal{C} . A morphism $\phi : x \rightarrow x'$ in X lying over f is *p-cocartesian* if for any other morphism $\psi : x \rightarrow x''$ in X , and for any morphism $g : c' \rightarrow p(x'')$ in \mathcal{C} satisfying $g \circ f = p(\psi)$, there exists a unique morphism $\gamma : x' \rightarrow x''$ such that $p(\gamma) = g$ and $\psi = \gamma \circ \phi$.



DEFINITION 1.1.4. Let $p : X \rightarrow \mathcal{C}$ be a functor. Then p is a *Grothendieck opfibration* if for any morphism of \mathcal{C} and any lift of its source, there is a p -cocartesian morphism with that source lying over it.

1.2 Symmetric monoidal categories are special pseudofunctors to **Cat**

- Now that we know how to move between pseudofunctors to **Cat** and opfibrations, let's write the data of a symmetric monoidal category as a pseudofunctor to **Cat**.

1.3 ...which are special opfibrations

- The above implies there's some category \mathcal{D} such that opfibrations $\mathcal{D} \rightarrow \mathbf{Fin}_*$ are the same as symmetric monoidal categories. Let's see what \mathcal{D} is.
- (Usual) definition of symmetric monoidal category and translation into the language of op-fibrations into \mathbf{Fin}_* , running example of \mathbf{Vect}_k with \otimes or \times .
- Possibly mention swapping out \mathbf{Fin}_* for Δ^{op} gives a monoidal category rather than a symmetric monoidal category (how do we get a braided monoidal category?)
- Correspondence of symmetric monoidal functors with morphisms of opfibrations.

The construction below is in [4].

Let (\mathcal{C}, \otimes) be a symmetric monoidal category. We define a new category \mathcal{C}^{\otimes} , whose objects are finite (possibly empty) sequences of objects of \mathcal{C} , denoted by $[C_1, \dots, C_n]$. A morphism

$$[C_1, \dots, C_n] \rightarrow [C'_1, \dots, C'_m]$$

consists of a subset $S \subseteq \{1, \dots, n\}$, a map of finite sets $\alpha : S \rightarrow \{1, \dots, m\}$, and a collection of morphisms $\{f_j : \bigotimes_{i \in \alpha^{-1}\{j\}} C_i \rightarrow C'_j\}_{1 \leq j \leq m}$ in \mathcal{C} .

For two morphisms $f : [C_1, \dots, C_n] \rightarrow [C'_1, \dots, C'_m]$ and $g : [C'_1, \dots, C'_m] \rightarrow [C''_1, \dots, C''_l]$, determining two subsets $S \subseteq \{1, \dots, n\}$ and $T \subseteq \{1, \dots, m\}$ and maps $\alpha : S \rightarrow \{1, \dots, m\}$, $\beta : T \rightarrow \{1, \dots, l\}$, the composition $g \circ f$ is given by the subset $U = \alpha^{-1}T \subseteq \{1, \dots, n\}$, the map $\beta \circ \alpha : U \rightarrow \{1, \dots, l\}$ and the maps

$$\left\{ \bigotimes_{i \in (\beta \circ \alpha)^{-1}\{k\}} C_i \cong \bigotimes_{j \in \beta^{-1}\{k\}} \bigotimes_{i \in \alpha^{-1}\{j\}} C_i \rightarrow \bigotimes_{j \in \beta^{-1}\{k\}} C'_j \rightarrow C''_k \right\}_{1 \leq k \leq l}.$$

For example, let

$$f : [C_1, C_2, C_3, C_4] \rightarrow [C'_1, C'_2, C'_3]$$

be a morphism in \mathcal{C}^\otimes consisting of the subset $\{1, 2, 3\} \subseteq \{1, 2, 3, 4\}$, the map

$$\begin{aligned} \alpha : \{1, 2, 3\} &\rightarrow \{1, 2, 3\}, \\ 1 &\mapsto 1, \\ 2 &\mapsto 2, \\ 3 &\mapsto 3, \end{aligned}$$

and morphisms

$$f_1 : C_1 \rightarrow C'_1, \quad f_2 : C_2 \otimes C_3 \rightarrow C'_2, \quad f_3 : \mathbf{1} \rightarrow C'_3,$$

and let

$$g : [C'_1, C'_2, C'_3] \rightarrow [C''_1, C''_2, C''_3]$$

be a morphism in \mathcal{C}^\otimes consisting of the subset $\{1, 2, 3\} \subseteq \{1, 2, 3\}$, the map

$$\begin{aligned} \beta : \{1, 2, 3\} &\rightarrow \{1, 2, 3\}, \\ 1, 2, 3 &\mapsto 3, \end{aligned}$$

and morphisms

$$g_1 : \mathbf{1} \rightarrow C''_1, \quad g_2 : \mathbf{1} \rightarrow C''_2, \quad g_3 : C'_1 \otimes C'_2 \otimes C'_3 \rightarrow C''_3.$$

Then the composition $g \circ f$ consists of the subset $\alpha^{-1}\{1, 2, 3\} = \{1, 2, 3\} \subseteq \{1, 2, 3, 4\}$, the map

$$\begin{aligned} \beta \circ \alpha : \{1, 2, 3\} &\rightarrow \{1, 2, 3\}, \\ 1, 2, 3 &\mapsto 3, \end{aligned}$$

and the morphisms

$$(g \circ f)_1 = g_1, \quad (g \circ f)_2 = g_2, \quad (g \circ f)_3 = g_3 \circ (f_1 \otimes f_2 \otimes f_3).$$

(really?)

(some intuition on this, tensor along the fibres, etc)

Claim: the forgetful functor

$$\begin{aligned} p : \mathcal{C}^\otimes &\rightarrow \mathbf{Fin}_*, \\ [C_1, \dots, C_n] &\mapsto \langle n \rangle_* \end{aligned}$$

is an opfibration. (It almost tautologically is).

1.4 Generalisation to ∞ -categories

- Translation of the above into ∞ -categorical language.
- A functor $p : D \rightarrow C$ between ordinary categories is a Grothendieck opfibration if and only if the induced functor $N(p) : N(D) \rightarrow N(C)$ on nerves is a cocartesian fibration – I *think* I have finally managed to prove this!
- Some examples (nerve of an ordinary symmetric monoidal category, currently trying to find more examples – many people talk about **Sp**, but it seems like I'd need a lot of background to understand this).
- If an ∞ -category has finite (co)products, there is a (co)cartesian monoidal structure on \mathcal{C} . And we would have hoped so, because it's definitely true for 1-categories!
- Algebra objects in monoidal (∞ -)categories
- Possibly generalisation to ∞ -operads, depending on how much the above comes to or if I find anything fun to do with symmetric monoidal ∞ -categories.

DEFINITION 1.4.1 ([1], Def 2.1). A functor $p : X \rightarrow Y$ between simplicial sets is an *inner fibration* if for all $n \geq 2$, all $0 < k < n$, and any solid arrow commutative square

$$\begin{array}{ccc} \Lambda_k^n & \longrightarrow & X \\ \downarrow & \nearrow & \downarrow p \\ \Delta^n & \longrightarrow & Y \end{array}$$

there exists a dotted lift.

The following proposition is stated without proof in Section 2.3 of [5].

PROPOSITION 1.4.2. Let $p : X \rightarrow Y$ be an inner fibration, and suppose that the diagram below is a pullback square in **sSet**.

$$\begin{array}{ccc} X' & \xrightarrow{f} & X \\ p' \downarrow \lrcorner & & \downarrow p \\ Y' & \xrightarrow{g} & Y \end{array}$$

Then p' is also an inner fibration.

PROOF. Consider the (commutative) solid arrow diagram below.

$$\begin{array}{ccccc} \Lambda_k^n & \xrightarrow{\lambda} & X' & \xrightarrow{f} & X \\ \downarrow \iota & & \downarrow \lrcorner & \nearrow & \downarrow p \\ \Delta^n & \xrightarrow{\delta} & Y' & \xrightarrow{g} & Y \end{array}$$

Since p is a fibration, there exists a dotted lift ϕ of $g\delta$; that is, $p\phi = g\delta$ and $\phi\iota = f\lambda$. Further, since the right square is a pullback diagram, there exists a unique map $\phi' : \Delta^n \rightarrow X'$ making the diagram below commute.

$$\begin{array}{ccccc} \Delta^n & & & & \\ & \searrow \phi' & & \nearrow \phi & \\ & X' & \xrightarrow{f} & X & \\ & p' \downarrow \lrcorner & & \downarrow p & \\ & Y' & \xrightarrow{g} & Y & \end{array}$$

It remains to show that the triangle below commutes.

$$\begin{array}{ccc} \Lambda_k^n & \xrightarrow{\lambda} & X' \\ \downarrow \iota & \nearrow \phi' & \\ \Delta^n & & \end{array}$$

Again, using the universal property of pullbacks, we see that there exist unique dotted maps such that the diagrams below commute.

$$\begin{array}{ccc} \Lambda_k^n & \xrightarrow{f\lambda} & X \\ \downarrow \delta\iota & \nearrow \text{dotted} & \downarrow p \\ X' & \xrightarrow{f} & X \\ \downarrow p' & \lrcorner & \downarrow p \\ Y' & \xrightarrow{g} & Y \end{array} \quad \begin{array}{ccc} \Lambda_k^n & \xrightarrow{f\phi'\iota} & X \\ \downarrow \delta\iota & \nearrow \text{dotted} & \downarrow p \\ X' & \xrightarrow{f} & X \\ \downarrow p' & \lrcorner & \downarrow p \\ Y' & \xrightarrow{g} & Y \end{array}$$

The maps λ and $\phi'\iota$ make the left and right diagrams commute respectively. Further, we note that $f\phi' = \phi$ (by the second diagram) and $\phi\iota = f\lambda$ (since p is an inner fibration), so $f\phi'\iota = f\lambda$. Therefore, the above two diagrams are identical. Thus, by the uniqueness property of pullbacks, $\lambda = \phi'\iota$. \square

(Stupid note to self, very obvious but I forget it every now and again):

- If $X : \Delta^{\text{op}} \rightarrow \mathbf{Set}$ is a simplicial set, and $\Delta^0 : \Delta^{\text{op}} \rightarrow \mathbf{Set} := \text{Hom}(-, [0])$, then a map $F : X \rightarrow \Delta^0$ is a natural transformation $(F_n : X_n \rightarrow *)_{n \in \mathbb{N}_0}$. That is, such a natural transformation is a family of maps down to a point. In other words, there's only really one natural transformation, so we really *can* view Δ^0 as a point.
- If Y is a simplicial set, and $y \in Y_0$ is a vertex of Y , we can view $\{y\}$ as a copy of Δ^0 . Why is this? We can view $\{y\}$ as the constant simplicial set, sending everything to y . Then a natural isomorphism $\Delta^0 \cong \{y\}$ is a collection of isomorphisms $(* \rightarrow *)$, of which there is exactly one. Why is it natural? Well, there's only one map from a one-point set to another one-point set, so the square always commutes.

EXAMPLE 1.4.3 ([1], Ex 2.2). Let $p : X \rightarrow \Delta^0$ be the canonical map, and suppose we have the diagram below, such that the outer square commutes.

$$\begin{array}{ccc} \Lambda_k^n & \longrightarrow & X \\ \downarrow & \nearrow & \downarrow p \\ \Delta^n & \longrightarrow & \Delta^0 \end{array}$$

The lower triangle commutes automatically, so the statement that p is an inner fibration is equivalent to the statement that for all $n \geq 2$, all $0 < k < n$, and any map $\Lambda_k^n \rightarrow X$, there exists a dotted lift.

$$\begin{array}{ccc} \Lambda_k^n & \longrightarrow & X \\ \downarrow & \nearrow & \\ \Delta^n & & \end{array}$$

That is, X is an ∞ -category.

Now, combining the above argument with [Proposition 1.4.2](#), we see that for any inner fibration $p : X \rightarrow Y$, each fibre $X \times_Y \{y\}$ is an ∞ -category.

DEFINITION 1.4.4 ([1], Def 3.1). Let $p : X \rightarrow Y$ be an inner fibration. An edge $f : \Delta^1 \rightarrow X$ of X is *p-cocartesian* if for all $n \geq 2$, any extension

$$\begin{array}{ccc} \Delta^{\{0,1\}} & \xrightarrow{f} & X \\ \downarrow & \nearrow F & \\ \Lambda_0^n & & \end{array}$$

and any solid arrow commutative diagram

$$\begin{array}{ccc} \Lambda_0^n & \xrightarrow{F} & X \\ \downarrow & \nearrow \text{dotted} & \downarrow p \\ \Delta^n & \longrightarrow & Y \end{array}$$

a dotted lift exists.

DEFINITION 1.4.5. Let $p : X \rightarrow Y$ be an inner fibration. Then p is a cocartesian fibration if for any edge $\phi : y \rightarrow y'$ in Y_1 , and for every $x \in X_0$ lying over y , there exists a p -cocartesian edge $f : x \rightarrow x'$ of X lying over ϕ .

The following proposition tells us that the above definition is a reasonable generalisation of Definition 1.1.4. It is also stated without proof in [5], which did not do wonders for my ego.

PROPOSITION 1.4.6 ([5], Rmk 2.4.2.2). Let \mathcal{C}, \mathcal{D} be categories, and let $p : \mathcal{C} \rightarrow \mathcal{D}$ be a functor between them. Then p is a Grothendieck opfibration if and only if the induced map $N(p) : N\mathcal{C} \rightarrow N\mathcal{D}$ is a cocartesian fibration of simplicial sets.

PROOF. Let $f : d \rightarrow d'$ be a morphism of \mathcal{D} , and let c lie over d .

Suppose p is a Grothendieck opfibration, let $F : \Lambda_0^n \rightarrow N\mathcal{C}$ be an extension of f , and let

$$\begin{array}{ccc} \Lambda_0^n & \xrightarrow{F} & N\mathcal{C} \\ \downarrow & & \downarrow N(p) \\ \Delta^n & \longrightarrow & N\mathcal{D} \end{array}$$

be a commutative diagram. If $n = 2$, it follows immediately from the fact that p is an opfibration that a dotted lift exists. Further, if $n > 3$, there is nothing to check, since an n -simplex in a category commutes if and only if all of its triangles commute, which is guaranteed for any extension $F : \Lambda_0^n \rightarrow N\mathcal{C}$. We thus prove the proposition for $n = 3$.

Suppose we have an extension $F : \Lambda_0^3 \rightarrow N\mathcal{C}$ of f ; that is, a tetrahedron

$$\begin{array}{ccccc} & c & \xrightarrow{\chi} & c''' & \\ & \phi \searrow & & \nearrow \gamma' & \\ c' & & \psi \searrow & & \\ & \gamma \searrow & & \nearrow \gamma'' & \\ & c'' & & & \end{array}$$

such that all faces containing the vertex c commute. Let

$$\begin{array}{ccccc} & d & \xrightarrow{p(\chi)} & p(c''') & \\ & f \searrow & & \nearrow p(\gamma') & \\ d' & & p(\psi) \searrow & & \\ & p(\gamma) \searrow & & \nearrow p(\gamma'') & \\ & p(c'') & & & \end{array}$$

be a commutative tetrahedron in \mathcal{D} . We claim that the tetrahedron in \mathcal{C} commutes. First, note that $\gamma'' \circ \gamma$ is a lift of $p(\gamma')$, since $p(\gamma') = p(\gamma'') \circ p(\gamma) = p(\gamma'' \circ \gamma)$. Further,

$$\begin{aligned} (\gamma \circ \gamma'') \circ \phi &= \gamma'' \circ \psi \\ &= \chi. \end{aligned}$$

Thus, by the uniqueness in the universal property of ϕ , we have that $\gamma' = \gamma'' \circ \gamma$, as required.

Now, suppose $N(p)$ is a cocartesian fibration. Then there exists a lift $\phi : c \rightarrow c'$ of f , and, in particular, for any diagram

$$\begin{array}{ccc} & c & \\ \phi \swarrow & & \searrow \psi \\ c' & & c'' \end{array}$$

in \mathcal{C} , and any commutative diagram

$$\begin{array}{ccc} & d & \\ f \swarrow & & \searrow p(\psi) \\ d' & \xrightarrow{g} & p(c'') \end{array}$$

in \mathcal{D} , there exists a map $\gamma : c' \rightarrow c''$ such that γ lies over g and $\gamma \circ \phi = \psi$. It remains to show that γ is unique.

Suppose that there were two maps $\gamma_1, \gamma_2 : c' \rightarrow c''$ lying over g and satisfying $\gamma_1 \circ \phi = \gamma_2 \circ \phi = \psi$. Then we would have a tetrahedron

$$\begin{array}{ccc} & c & \xrightarrow{\psi} c'' \\ \phi \swarrow & & \searrow \psi \\ c' & \xrightarrow{\gamma_1} c'' & \end{array} \quad \begin{array}{ccc} & & \nearrow \gamma_2 \\ & & \searrow \text{id} \end{array}$$

where all faces containing the vertex c commute. The image of this tetrahedron under p commutes in \mathcal{D} , so the original tetrahedron must commute in \mathcal{C} ; that is, $\gamma_1 = \gamma_2$. \square

References

- [1] Clark Barwick and Jay Shah. ‘Fibrations in ∞ -Category Theory’. In: *2016 MATRIX Annals*. Springer Cham, 2018, pp. 17–42.
- [2] Julius Frank. *Symmetric monoidal ∞ -categories*. 2021. URL: <https://homepages.abdn.ac.uk/markus.upmeier/pages/JuliusMay13.pdf>.
- [3] Moritz Groth. *A Short Course on ∞ -Categories*. 2015. arXiv: [1007.2925](https://arxiv.org/abs/1007.2925) [math.AT].
- [4] Jacob Lurie. *Higher Algebra*. 2017. URL: <https://www.math.ias.edu/~lurie/papers/HA.pdf>.
- [5] Jacob Lurie. *Higher Topos Theory*. 2008. arXiv: [math/0608040](https://arxiv.org/abs/math/0608040) [math.CT].
- [6] Jacob Lurie. *Kerodon*. <https://kerodon.net>. 2024.