Symmetric Monoidal ∞-Categories

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1 Introduction

In the first half of this report, we focus on ordinary 1-categories and 2-categories; we first consider some natural examples of pseudofunctors to **Cat**, to motivate the definitions of Grothendieck fibrations and opfibrations. We then outline in detail the relationship between pseudofunctors into **Cat** and Grothendieck opfibrations, using the (covariant) Grothendieck construction.

Next, we consider symmetric monoidal 1-categories, giving first the traditional definition, and then showing that it can be written as a pseudofunctor to **Cat** (and thus a Grothendieck opfibration). We construct the category $\mathcal{C}^{\otimes}$ using the process outlined earlier, and recover Lurie's construction in Chapter 2 of [5]. We then define symmetric monoidal functors in the usual way, and construct the definitions in terms of opfibrations, which will be more useful when working with ∞-categories. We mostly follow the approaches taken by [2] and [3].

In section 3, we move to higher category theory. We first introduce inner fibrations, proving that they are stable under pullbacks and that their fibres are ∞-categories. Then, we introduce cartesian and cocartesian fibrations, and show that the nerve of a functor is a (co)cartesian fibration of ∞-categories if and only if the original functor was an (op)fibration of ordinary categories.

Finally, we define symmetric monoidal ∞-categories and symmetric monoidal functors between them. We give two examples that justify the definition as an appropriate generalisation of ordinary symmetric monoidal categories: the nerve of a symmetric monoidal category, and any ∞-category with finite (co)products.

2 Symmetric monoidal 1-categories

2.1 The Grothendieck construction

Let $\mathcal{C}$ be a category with pullbacks. Recall that for a map $f : C \to D$ in $\mathcal{C}$, we may define a pullback functor

$$\begin{aligned} f^* : \quad \mathcal{C}_{/D} &\to \mathcal{C}_{/C}, \\ (h : X \to D) &\mapsto (f^*h : P \to C), \end{aligned}$$

where we have formed a pullback

$$\begin{tikzcd} P \arrow[r, "h^*f"] \arrow[d, "f^*h"'] \arrow[dr, phantom, "\lrcorner", very near start] & X \arrow[d, "h"] \\ C \arrow[r, "f"'] & D \end{tikzcd}$$

in $\mathcal{C}$. For any map

$$\begin{tikzcd}
X \arrow[rr, "\phi"] \arrow[dr, "h"'] & & X' \arrow[dl, "h'"] \\
& D &
\end{tikzcd}$$

from h to h' in $\mathcal{C}_{/D}$, we define $f^*\phi$ to be the unique map making the diagram below commute.

$$\begin{tikzcd}
X \arrow[rr, "\phi"] \arrow[dr, "h"'] & & X' \arrow[dl, "h'"] \\
& D & \\
P \arrow[uu, "h^*f"] \arrow[rr, dashed, "f^*\phi"] \arrow[dr, "f^*h"'] & & P' \arrow[uu, "h'^*f"'] \arrow[dl, "f^*h'"] \\
& C \arrow[uu, "f"'] &
\end{tikzcd}$$

Now, we may wish to define a functor

$$\begin{aligned} F : \mathcal{C}^{\mathrm{op}} &\to \mathbf{Cat}, \\ C &\mapsto \mathcal{C}_{/C}, \end{aligned}$$

which sends a map $f : C \to D$ in $\mathcal{C}$ to a pullback functor $f^* : \mathcal{C}_{/D} \to \mathcal{C}_{/C}$. However a problem arises when we check that F respects composition: suppose $f : C \to D$, $g : D \to E$ are maps in $\mathcal{C}$. Then

$$F(g \circ f)(h : X \to E) = (g \circ f)^*h : P \to C,$$

corresponding to the pullback

$$\begin{tikzcd}
P \arrow[r, "h^*(g\circ f)"] \arrow[d, "(g\circ f)^*h"'] \arrow[dr, phantom, "\lrcorner", very near start] & X \arrow[d, "h"] \\
C \arrow[r, "g\circ f"'] & E
\end{tikzcd}$$

in $\mathcal{C}$. On the other hand,

$$(F(g) \circ F(f))(h : X \to E) = f^*(g^*h) : P'' \to C,$$

which corresponds to the diagram below.

$$\begin{tikzcd}
P'' \arrow[r, "(g^*h)^*f"] \arrow[d, "f^*(g^*h)"'] \arrow[dr, phantom, "\lrcorner", very near start] & P' \arrow[r, "h^*g"] \arrow[d, "g^*h"'] \arrow[dr, phantom, "\lrcorner", very near start] & X \arrow[d, "h"] \\
C \arrow[r, "f"'] & D \arrow[r, "g"'] & E
\end{tikzcd}$$

The outer square is indeed a pullback square, since the inner two squares are, so we have a unique isomorphism $P \cong P''$. However, we do not in general have equality. This is because pullbacks are only unique up to unique isomorphism, and in defining a pullback functor we made arbitrary (and not necessarily compatible) choices of P, P' and P''. Thus, we have not defined a functor $\mathcal{C}^{\mathrm{op}} \to \mathbf{Cat}$, rather, we have defined what is known as a *pseudofunctor*; that is, a weak functor between 2-categories.

The above example is just one way in which pseudofunctors into **Cat** naturally arise; another common example is the pseudofunctor

$$\begin{aligned} \mathbf{CRing} &\to \mathbf{Cat} \\ R &\mapsto R\text{-}\mathbf{Mod}, \end{aligned}$$

which sends a ring homomorphism $\phi : R \to S$ to the functor $- \otimes_R S : R\text{-}\mathbf{Mod} \to S\text{-}\mathbf{Mod}$; this is known as *extension of scalars*. However, to give the data of a pseudofunctor $F : \mathcal{C} \to \mathbf{Cat}$, we must specify not only the functions $\mathrm{ob}(\mathcal{C}) \to \mathrm{ob}(\mathbf{Cat})$ and $\mathrm{Hom}_{\mathcal{C}}(X, Y) \to \mathrm{Hom}_{\mathbf{Cat}}(F(X), F(Y))$ for each $X, Y \in \mathcal{C}$, but also natural isomorphisms

$$F(\mathrm{id}_X) \cong \mathrm{id}_{F(X)}, \quad F(g \circ f) \cong F(g) \circ F(f).$$

Thus, as we move into the realm of higher categories, the amount of data needed to define the appropriate notion of a functor grows rapidly. Since we intend to generalise these concepts to ∞-categories, we will turn to a different method of encoding the same information - Grothendieck fibrations and opfibrations.

Definition 2.1.1. Let $p : X \to \mathcal{C}$ be a functor, let $f : c \to c'$ be a morphism in $\mathcal{C}$, and let $\phi : x \to x'$ be a morphism in X lying over f.

We say that ϕ is *p-cartesian* if for any other morphism $\psi : x'' \to x'$ in X, and for any morphism $g : p(x'') \to c$ in $\mathcal{C}$ satisfying $f \circ g = p(\psi)$, there exists a unique morphism $\gamma : x'' \to x$ such that $p(\gamma) = g$ and $\psi = \gamma \circ \phi$.

Dually, ϕ is *p-cocartesian* if for any other morphism $\psi : x \to x''$ in X, and for any morphism $g : c' \to p(x'')$ in $\mathcal{C}$ satisfying $g \circ f = p(\psi)$, there exists a unique morphism $\gamma : x' \to x''$ such that $p(\gamma) = g$ and $\psi = \gamma \circ \phi$.

The left diagram below corresponds to a p-cartesian morphism, and the right diagram corresponds to a p-cocartesian morphism.

$$\begin{tikzcd}
 & & x'' \arrow[dll, dashed, "\exists!\gamma"'] \arrow[dl, "\forall\psi"] \arrow[ddd] \\
x \arrow[r, "\phi"'] \arrow[d] & x' \arrow[d] & \\
c \arrow[r, "f"] & c' & \\
 & & p(x'') \arrow[ull, "\forall g"] \arrow[ul, "p(\psi)"']
\end{tikzcd}
\qquad
\begin{tikzcd}
 & & x'' \arrow[ddd] \\
x \arrow[urr, "\forall\psi"] \arrow[r, "\phi"'] \arrow[d] & x' \arrow[ur, dashed, "\exists!\gamma"'] \arrow[d] & \\
c \arrow[r, "f"] \arrow[drr, "p(\psi)"'] & c' \arrow[dr, "\forall g"] & \\
 & & p(x'')
\end{tikzcd}
\qquad
\begin{tikzcd}
X \arrow[d, "p"'] \\
\mathcal{C}
\end{tikzcd}$$

Definition 2.1.2. Let $p : X \to \mathcal{C}$ be a functor. Then p is a *Grothendieck fibration* if for any morphism of $\mathcal{C}$ and any lift of its target, there is a p-cartesian morphism with that target lying over it. Dually, p is a *Grothendieck opfibration* if for any morphism of $\mathcal{C}$ and any lift of its source, there is a p-cocartesian morphism with that source lying over it.

We will usually refer to Grothendieck (op)fibrations as just (op)fibrations for brevity. Note that in [2] and [7] these are referred to as *(co)cartesian fibrations*; we reserve this term for the ∞-category analogue. We will also often write (co)cartesian rather than p-(co)cartesian when it is clear what the functor p is.

REMARK 2.1.3. A functor $p : X \to \mathcal{C}$ is an opfibration if and only if $p^{\mathrm{op}} : X^{\mathrm{op}} \to \mathcal{C}^{\mathrm{op}}$ is a fibration. We give the definition of an opfibration explicitly above, since we will be working with these more often than fibrations.

THEOREM 2.1.4 ([8], Thm 2.4). There is an equivalence of 2-categories

$$\mathbf{Psd}(\mathcal{C}^{\mathrm{op}}, \mathbf{Cat}) \simeq \mathbf{Fib}(\mathcal{C}),$$

where $\mathbf{Psd}(\mathcal{C}^{\mathrm{op}}, \mathbf{Cat})$ is the 2-category of pseudofunctors $\mathcal{C}^{\mathrm{op}} \to \mathbf{Cat}$, and $\mathbf{Fib}(\mathcal{C}) \hookrightarrow \mathbf{Cat}_{/\mathcal{C}}$ is the 2-category of fibrations into $\mathcal{C}$.

REMARK 2.1.5. Combining Remark 2.1.3 with Theorem 2.1.4 gives us a chain of equivalences

$$\mathbf{Psd}(\mathcal{C}, \mathbf{Cat}) \xrightarrow{\mathrm{op}\,\circ\, -} \mathbf{Psd}(\mathcal{C}, \mathbf{Cat}) \simeq \mathbf{Psd}((\mathcal{C}^{\mathrm{op}})^{\mathrm{op}}, \mathbf{Cat}) \simeq \mathbf{Fib}(\mathcal{C}^{\mathrm{op}}) \simeq \mathbf{opFib}(\mathcal{C}),$$

where the first functor $\mathbf{Psd}(\mathcal{C}, \mathbf{Cat}) \to \mathbf{Psd}(\mathcal{C}, \mathbf{Cat})$ is given by postcomposition with $\mathrm{op} : \mathbf{Cat} \to \mathbf{Cat}$.

We will not prove Theorem 2.1.4 (see [8] or [4] for more details) but we will describe how to pass between $\mathbf{Psd}[\mathcal{C}, \mathbf{Cat}]$ and $\mathbf{opFib}(\mathcal{C})$ by dualising[1] the constructions in these sources.

Let $F : \mathcal{C} \to \mathbf{Cat}$ be a pseudofunctor. Define the category X as follows: the objects of X are pairs (c, x), with $c \in \mathcal{C}$, $x \in F(c)$. A map $(c, x) \to (d, y)$ is a pair (f, u), where $f : c \to d$ is a morphism in $\mathcal{C}$, and $u : (Ff)(x) \to y$ is a morphism in $F(d)$. For an object $(c, x) \in X$, the identity morphism is given by

$$(\mathrm{id}_c, F^0_x : F(\mathrm{id}_c)(x) \to x),$$

where F^0 is the natural isomorphism $F(\mathrm{id}_c) \cong \mathrm{id}_{F(c)}$. Further, given two maps $(f, u) : (c, x) \to (d, y)$ and $(g, v) : (d, y) \to (e, z)$, their composition $(g, v) \circ (f, u)$ is given by $g \circ f$, together with the map

$$(F(g \circ f))(x) \xrightarrow{(F^2_{g,f})_x} (Fg \circ Ff)(x) \xrightarrow{(Fg)(u)} (Fg)(y) \xrightarrow{v} z,$$

where $F^2_{g,f}$ is the natural isomorphism $F(g \circ f) \cong Fg \circ Ff$.

LEMMA 2.1.6. The forgetful functor $p : X \to \mathcal{C}$ is an opfibration over $\mathcal{C}$.

PROOF. Let $f : c \to d$ be a morphism in $\mathcal{C}$, and let (c, x) be a lift of its source. Consider the map $(f, \mathrm{id}_{(Ff)(x)}) : (c, x) \to (d, (Ff)(x))$. Let $(g, \psi) : (c, x) \to (e, z)$ be a map in X, and let $h : d \to e$ be a map in $\mathcal{C}$ such that $g = h \circ f$. We wish to find some map $\gamma : (Fh \circ Ff)(x) \to z$ such that the composition

$$F(h \circ f)(x) \xrightarrow{F^2_{h,f}} (Fh \circ Ff)(x) \xrightarrow{(Fh)(\mathrm{id}_{(Ff)(x)})} (Fh \circ Ff)(x) \xrightarrow{\gamma} z$$

is equal to the map

$$(Fg)(x) \xrightarrow{\psi} z.$$

Since $h \circ f = g$, and $(Fh)(\mathrm{id}_{(Ff)(x)}) = \mathrm{id}_{(Fh \circ Ff)(x)}$, we see that $\psi = \gamma \circ F^2_{h,f}$. Since F^2 is a natural isomorphism, we have $\gamma = \psi \circ (F^2_{h,f})^{-1}$, so γ exists and is unique. Therefore, for every morphism in $\mathcal{C}$ and every lift of its source, there is a cocartesian morphism lying over it; that is, $p : X \to \mathcal{C}$ is a an opfibration. □

[1] We use the equivalences in Remark 2.1.5; in particular we postcompose with the equivalence $\mathrm{op} : \mathbf{Cat} \to \mathbf{Cat}$. This is to allow us to define the pseudofunctor $F : \mathcal{C} \to \mathbf{Cat}$ to send $c \in \mathcal{C}$ to $p^{-1}\{c\}$ rather than $(p^{\mathrm{op}})^{-1}\{c\} \simeq (p^{-1}\{c\})^{\mathrm{op}}$.

Now, let $p : X \to \mathcal{C}$ be an opfibration over $\mathcal{C}$. Define a pseudofunctor

$$\begin{aligned} \mathcal{C} &\to \mathbf{Cat} \\ c &\mapsto p^{-1}\{c\}. \end{aligned}$$

For a map $f : c \to d$ in $\mathcal{C}$, we define a functor

$$f_* : p^{-1}\{c\} \to p^{-1}\{d\},$$

which sends $x \in p^{-1}\{c\}$ to the target x' of a cocartesian edge $\phi_x : x \to x'$ lying over f. Now, let $f^*(y) = y'$, with $\phi_y : y \to y'$, and let $g : x \to y$ be a morphism in $p^{-1}\{c\}$ (that is, a morphism in X lying over id_c). Then, since $\phi_y : y \to y'$ is a cocartesian edge, there is a unique lift $\overline{g} : x' \to y'$ of id_d making the square on the left commute, as shown in the diagram on the right.

$$\begin{tikzcd}
x \arrow[r, "\phi_x"] \arrow[d, dashed, "g"'] & x' \arrow[d, "\exists!\overline{g}"] \\
y \arrow[r, "\phi_y"'] & y'
\end{tikzcd}
\qquad
\begin{tikzcd}
 & & y' \arrow[ddd, mapsto] \\
x \arrow[r, "\phi_x"'] \arrow[urr, "\phi_y \circ g"] \arrow[d, mapsto] & x' \arrow[ur, dashed, "\exists!\overline{g}"'] \arrow[d, mapsto] & \\
c \arrow[r, "f"] \arrow[drr, "f"'] & d \arrow[dr, "\mathrm{id}_d"] & \\
 & & d
\end{tikzcd}$$

We thus define $f_* g = \overline{g}$ in $p^{-1}\{d\}$. It can be shown that these data assemble into a pseudofunctor $\mathcal{C} \to \mathbf{Cat}$.

2.2 Symmetric monoidal categories and functors

We now move onto the main object of our consideration: symmetric monoidal categories.

Definition 2.2.1. A *symmetric monoidal category* is a category $\mathcal{C}$ equipped with a bifunctor

$$\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C},$$

an object $\mathbf{1} \in \mathcal{C}$ (the unit), and natural isomorphisms

$$\alpha : \otimes \circ (\otimes \times \mathrm{id}) \xrightarrow{\sim} \otimes \circ (\mathrm{id} \times \otimes), \quad \tau : \otimes \xrightarrow{\sim} \sigma \circ \otimes$$

$$l : \otimes \circ (\mathbf{1} \times \mathrm{id}) \xrightarrow{\sim} \mathrm{id}, \quad r : \otimes \circ (\mathrm{id} \times \mathbf{1}) \xrightarrow{\sim} \mathrm{id},$$

where $\sigma : \mathcal{C} \times \mathcal{C}$ swaps the order of the factors. These isomorphisms (encoding associativity, symmetry, and left and right unitality respectively) are subject to the condition that $\tau^2 = \mathrm{id}$, and that the diagrams below commute.

$$\begin{tikzcd}
 & ((U \otimes V) \otimes W) \otimes X \arrow[dl, "\alpha_{U \otimes V, W, X}"'] \arrow[dr, "\alpha_{U,V,W} \otimes \mathrm{id}_X"] & \\
(U \otimes V) \otimes (W \otimes X) \arrow[d, "\alpha_{U,V,W \otimes X}"'] & & (U \otimes (V \otimes W)) \otimes X \arrow[d, "\alpha_{U, V \otimes W, X}"] \\
U \otimes (V \otimes (W \otimes X)) & & U \otimes ((V \otimes W) \otimes X) \arrow[ll, "\mathrm{id} \otimes \alpha_{V,W,X}"]
\end{tikzcd}$$

$$\begin{tikzcd}
(V \otimes \mathbf{1}) \otimes W \arrow[rr, "\alpha_{V,\mathbf{1},W}"] \arrow[dr, "r_V \otimes \mathrm{id}_W"'] & & V \otimes (\mathbf{1} \otimes W) \arrow[dl, "\mathrm{id}_V \otimes l_W"] \\
& V \otimes W &
\end{tikzcd}
\qquad
\begin{tikzcd}
V \otimes \mathbf{1} \arrow[rr, "\tau_{V,\mathbf{1}}"] \arrow[dr, "r_V"'] & & \mathbf{1} \otimes V \arrow[dl, "l_V"] \\
& V &
\end{tikzcd}$$

$$\begin{tikzcd}
& (U \otimes V) \otimes W \arrow[dl, "\alpha_{U,V,W}"'] \arrow[dr, "\tau_{U,V} \otimes \mathrm{id}_W"] & \\
U \otimes (V \otimes W) \arrow[d, "\tau_{U,V \otimes W}"'] & & (V \otimes U) \otimes W \arrow[d, "\alpha_{V,U,W}"] \\
(V \otimes W) \otimes U \arrow[dr, "\alpha_{V,W,U}"'] & & V \otimes (U \otimes W) \arrow[dl, "\mathrm{id}_V \otimes \tau_{U,W}"] \\
& V \otimes (W \otimes U) &
\end{tikzcd}$$

EXAMPLE 2.2.2. Any category $\mathcal{C}$ with finite products has a symmetric monoidal structure (the *cartesian monoidal structure*) given by the categorical product, with unit the terminal object of $\mathcal{C}$. Dually, any category with finite coproducts has a symmetric monoidal structure (the *cocartesian monoidal structure*) given by the categorical coproduct, with unit the initial object.

EXAMPLE 2.2.3. The category $\mathbf{Vect}_k$ has an additional symmetric monoidal structure given by the tensor product of vector spaces, and unit object k.

Note the increase in complexity between the definition of a commutative monoid and that of a symmetric monoidal category; in the former case, we require that the product be strictly associative, unital, and commutative, while relaxing these conditions for the latter forced us to specify extra data (the natural isomorphisms α, l, r, τ) which then had to satisfy higher coherences. Much like specifying the data for a pseudofunctor, this quickly becomes overwhelming when considering higher categories. We thus use the tools of the previous section, writing the data of a symmetric monoidal category as a pseudofunctor to **Cat**, which we can transform into an opfibration using the Grothendieck construction.

Let $(\mathcal{C}, \otimes)$ be a symmetric monoidal category, and let $\mathbf{Fin}_*$ be the category of finite pointed sets[2], with objects denoted by $\langle n \rangle := \{1, 2, ..., n, *\}$. Define a pseudofunctor

$$\begin{aligned} F : \mathbf{Fin}_* &\to \mathbf{Cat} \\ \langle n \rangle &\mapsto \mathcal{C}^{\times n}. \end{aligned}$$

Let $f : \langle n \rangle \to \langle m \rangle$ be a morphism in $\mathbf{Fin}_*$. This induces a morphism

$$f^* : (C_1, ..., C_n) \mapsto (C'_1, ..., C'_m),$$

where

$$C'_i = \bigotimes_{j \in f^{-1}\{i\}} C_j.$$

[2]Replacing $\mathbf{Fin}_*$ by Δ^{op} encodes the data of an ordinary monoidal category, since the total ordering on the finite sets removes the permutations that give rise to the natural isomorphism τ of a symmetric monoidal category. See [3] for this formulation.

Using the pseudofunctor above, along with the Grothendieck construction, we can obtain a category $\mathcal{C}^{\otimes}$ and an opfibration $p : \mathcal{C}^{\otimes} \to \mathbf{Fin}_*$ corresponding to the pseudofunctor encoding $(\mathcal{C}, \otimes)$. Unravelling the definitions gives exactly the construction below, presented in Chapter 2 of [5].

Let $(\mathcal{C}, \otimes)$ be a symmetric monoidal category. We define a new category $\mathcal{C}^{\otimes}$, whose objects are finite (possibly empty) sequences of objects of $\mathcal{C}$, denoted by $[C_1, ..., C_n]$. A morphism

$$[C_1, ..., C_n] \to [C'_1, ..., C'_m]$$

consists of a subset $S \subseteq \{1, ..., n\}$ and a map of finite sets[3] $\alpha : S \to \{1, ..., m\}$, along with a collection of morphisms $\{f_j : \bigotimes_{i \in \alpha^{-1}\{j\}} C_i \to C'_j\}_{1 \leq j \leq m}$ in $\mathcal{C}$.

For two morphisms $f : [C_1, ..., C_n] \to [C'_1, ..., C'_m]$ and $g : [C'_1, ..., C'_m] \to [C''_1, ..., C''_l]$, determining two subsets $S \subseteq \{1, ..., n\}$ and $T \subseteq \{1, ..., m\}$ and maps $\alpha : S \to \{1, ..., m\}$, $\beta : T \to \{1, ..., l\}$, the composition $g \circ f$ is given by the subset $U = \alpha^{-1}T \subseteq \{1, ..., n\}$, the map $\beta \circ \alpha : U \to \{1, ..., l\}$ and the maps

$$\left\{ \bigotimes_{i \in (\beta \circ \alpha)^{-1}\{k\}} C_i \cong \bigotimes_{j \in \beta^{-1}\{k\}} \bigotimes_{i \in \alpha^{-1}\{j\}} C_i \to \bigotimes_{j \in \beta^{-1}\{k\}} C'_j \to C''_k \right\}_{1 \leq k \leq l}.$$

For example, let

$$f : [C_1, C_2, C_3, C_4] \to [C'_1, C'_2, C'_3]$$

be a morphism in $\mathcal{C}^{\otimes}$ consisting of the subset $\{1, 2, 3\} \subseteq \{1, 2, 3, 4\}$, the map

$$\begin{aligned} \alpha : \{1, 2, 3\} &\to \{1, 2, 3\}, \\ 1 &\mapsto 1, \\ 2 &\mapsto 2, \\ 3 &\mapsto 2, \end{aligned}$$

and morphisms

$$f_1 : C_1 \to C'_1, \quad f_2 : C_2 \otimes C_3 \to C'_2, \quad f_3 : \mathbf{1} \to C'_3,$$

and let

$$g : [C'_1, C'_2, C'_3] \to [C''_1, C''_2, C''_3]$$

be a morphism in $\mathcal{C}^{\otimes}$ consisting of the subset $\{1, 2, 3\} \subseteq \{1, 2, 3\}$, the map

$$\begin{aligned} \beta : \{1, 2, 3\} &\to \{1, 2, 3\}, \\ 1, 2, 3 &\mapsto 3, \end{aligned}$$

and morphisms

$$g_1 : \mathbf{1} \to C''_1, \quad g_2 : \mathbf{1} \to C''_2, \quad g_3 : C'_1 \otimes C'_2 \otimes C'_3 \to C''_3.$$

Then the composition $g \circ f$ consists of the subset $\alpha^{-1}\{1, 2, 3\} = \{1, 2, 3\} \subseteq \{1, 2, 3, 4\}$, the map

$$\begin{aligned} \beta \circ \alpha : \{1, 2, 3\} &\to \{1, 2, 3\}, \\ 1, 2, 3 &\mapsto 3, \end{aligned}$$

[3]Note that this is exactly the same data as a map of pointed finite sets, since the subset S determines the elements of $\langle n \rangle$ which are not mapped to the point $* \in \langle m \rangle$. Also, note that the objects of $\mathcal{C}^{\otimes}$ would technically be pairs $(\langle n \rangle, [C_1, ..., C_n])$, but the first factor can clearly be recovered from the second, so we omit it for simplicity.

and the morphisms

$$(g \circ f)_1 = g_1, \quad (g \circ f)_2 = g_2, \quad (g \circ f)_3 = g_3 \circ (f_1 \otimes f_2 \otimes f_3).$$

For each i, j satisfying $1 \leq i \leq n$, define $\rho^i \langle n \rangle \to \langle 1 \rangle$ by

$$\rho^i(j) = \begin{cases} 1 & \text{if } i = j \\ * & \text{otherwise.} \end{cases}$$

REMARK 2.2.4. Let $\mathcal{C}^{\otimes}_{\langle n \rangle}$ denote the fibre $p^{-1}\{\langle n \rangle\}$. Then every morphism $\langle n \rangle \to \langle m \rangle$ in $\mathbf{Fin}_*$ induces a functor between the fibres $\mathcal{C}^{\otimes}_{\langle n \rangle} \to \mathcal{C}^{\otimes}_{\langle m \rangle}$. In particular, the maps ρ^i defined above induce projections, the *Segal maps*, $\rho^i_! : \mathcal{C}^{\otimes}_{\langle n \rangle} \to \mathcal{C}^{\otimes}_{\langle 1 \rangle} \simeq \mathcal{C}$.

PROPOSITION 2.2.5 ([3], Prop 4.26). If $(\mathcal{C}, \otimes)$ is a symmetric monoidal category, then the forgetful functor

$$\begin{array}{rccc} p: & \mathcal{C}^{\otimes} & \to & \mathbf{Fin}_* \\ & [C_1, ..., C_n] & \mapsto & \langle n \rangle_* \end{array}$$

is a Grothendieck opfibration. Moreover, p satisfies the *Segal condition*; that is, the Segal maps

$$(\rho^1_!, ..., \rho^n_!) : \mathcal{C}^{\otimes}_{\langle n \rangle} \to \mathcal{C}^{\times n}$$

are equivalences. Conversely any Grothendieck opfibration $p : \mathcal{D} \to \mathbf{Fin}_*$ satisfying the Segal condition gives rise to a symmetric monoidal structure on $\mathcal{D}_{\langle 1 \rangle}$.

We do not prove the proposition above, but sketch how to recover a symmetric monoidal category from an opfibration satisfying the Segal condition.

Let $p : \mathcal{D} \to \mathbf{Fin}_*$ be an opfibration which satisfies the Segal condition. Then $\mathcal{D}_{\langle 0 \rangle} \simeq \mathcal{D}^{\times 0}_{\langle 1 \rangle} \simeq \{*\}$, so $\mathcal{D}_{\langle 0 \rangle}$ has a single object up to equivalence. The unique map $\langle 0 \rangle \to \langle 1 \rangle$ in $\mathbf{Fin}_*$ determines a map $\{*\} \to \mathcal{D}$, which picks out an object $\mathbf{1} \in \mathcal{D}$; this is the unit of $\mathcal{D}$. Now, let $a : \langle 2 \rangle \to \langle 1 \rangle$ be the map in $\mathbf{Fin}_*$ which sends $1, 2 \in \langle 2 \rangle$ to $1 \in \langle 1 \rangle$. Then there are maps

$$\mathcal{D}_{\langle 1 \rangle} \times \mathcal{D}_{\langle 1 \rangle} \simeq \mathcal{D}_{\langle 2 \rangle} \to \mathcal{D}_{\langle 1 \rangle}, \tag{1}$$

where the first equivalence is induced by ρ^1 and ρ^2, and the second map is induced by a. This gives us, up to canonical isomorphism, a map $\otimes : \mathcal{D}_{\langle 1 \rangle} \times \mathcal{D}_{\langle 1 \rangle} \to \mathcal{D}_{\langle 1 \rangle}$; the product of the monoidal category.

The natural isomorphisms are obtained as follows: let $s : \langle 2 \rangle \to \langle 2 \rangle$ be the map in $\mathbf{Fin}_*$ which permutes 1 and 2. Then $a \circ s = a$, and $(\rho^1 \times \rho^2) \circ s = \rho^2 \circ \rho^1$. Using the maps in (1), we obtain a canonical isomorphism $\otimes \xrightarrow{\sim} \sigma \circ \otimes$. Now, consider the commutative diagrams

$$\begin{tikzcd} \langle 1 \rangle \arrow[r, "\mathrm{id}"] \arrow[d, "\mathrm{id}"'] & \langle 1 \rangle \arrow[d, "f"] \\ \langle 1 \rangle & \langle 2 \rangle \arrow[l, "a"] \end{tikzcd} \qquad \begin{tikzcd} \langle 1 \rangle \arrow[r, "\mathrm{id}"] \arrow[d, "\mathrm{id}"'] & \langle 1 \rangle \arrow[d, "g"] \\ \langle 1 \rangle & \langle 2 \rangle \arrow[l, "a"] \end{tikzcd}$$

where $f : \langle 1 \rangle \to \langle 2 \rangle$ sends $1 \mapsto 1$, and $g : \langle 1 \rangle \to \langle 2 \rangle$ sends $1 \mapsto 2$. These determine diagrams

$$\begin{tikzcd} \mathcal{D}_{\langle 1 \rangle} \arrow[r, "\mathrm{id}_!"] \arrow[d, "\mathrm{id}_!"'] & \mathcal{D}_{\langle 1 \rangle} \arrow[d, "f_!"] \\ \mathcal{D}_{\langle 1 \rangle} & \mathcal{D}_{\langle 2 \rangle} \arrow[l, "a_!"] \end{tikzcd} \qquad \begin{tikzcd} \mathcal{D}_{\langle 1 \rangle} \arrow[r, "\mathrm{id}_!"] \arrow[d, "\mathrm{id}_!"'] & \mathcal{D}_{\langle 1 \rangle} \arrow[d, "g_!"] \\ \mathcal{D}_{\langle 1 \rangle} & \mathcal{D}_{\langle 2 \rangle} \arrow[l, "a_!"] \end{tikzcd}$$

which, combined with the equivalences $\mathcal{D}_{\langle n\rangle} \simeq \mathcal{D}_{\langle 1\rangle}^{\times n}$, give rise to the natural isomorphisms r and l respectively. In a similar way, the commutative diagram

$$\begin{tikzcd} \langle 3\rangle \arrow[r, "\tau_1^3"] \arrow[d, "\tau_2^3"'] & \langle 2\rangle \arrow[d, "\tau_1^2"] \\ \langle 2\rangle \arrow[r, "\tau_1^2"] & \langle 1\rangle \end{tikzcd}$$

where

$$\tau_i^n(j) = \begin{cases} j & \text{if } 1 \leq j \leq i \\ j-1 & \text{if } i < j \leq n \\ * & \text{if } j = * \end{cases}$$

determines the natural isomorphism $\alpha : \otimes \circ (\otimes \times \mathrm{id}) \xrightarrow{\sim} \otimes \circ (\mathrm{id} \times \otimes)$ (see Chapter 2 of [5] for details).

Proposition 2.2.5 allows us to make the following alternative definition of a symmetric monoidal category.

Definition 2.2.6. A *symmetric monoidal category* is a Grothendieck opfibration $p : \mathcal{D} \to \mathbf{Fin}_*$ such that the Segal maps

$$(\rho_!^1, ..., \rho_!^n) : \mathcal{D}_{\langle n\rangle} \to \mathcal{D}_{\langle 1\rangle}^{\times n},$$

are equivalences.

It is worth pausing to summarise what we have achieved in the last two sections. First, we noticed that pseudofunctors $\mathcal{C} \to \mathbf{Cat}$ encode the same information as opfibrations into $\mathcal{C}$. We also noticed that symmetric monoidal categories could be written as special pseudofunctors $\mathbf{Fin}_* \to \mathbf{Cat}$, which means they are special opfibrations into $\mathbf{Fin}_*$. We looked at the corresponding construction of $\mathcal{C}^{\otimes}$ (via the Grothendieck construction). Thus, symmetric monoidal categories, certain pseudofunctors $\mathbf{Fin}_* \to \mathbf{Cat}$, and opfibrations into $\mathbf{Fin}_*$ satisfying the Segal condition all encode the same data, but the latter package allows us to 'hide' the unwieldy higher coherences in the opfibration itself, which will be incredibly useful when we begin to consider symmetric monoidal ∞-categories.

We conclude by similarly rewriting the definition of a symmetric monoidal functor.

Definition 2.2.7. Let $(\mathcal{C}, \otimes)$, $(\mathcal{D}, \otimes')$ be symmetric monoidal categories (in the sense of Definition 2.2.1), with units $\mathbf{1}$ and $\mathbf{1}'$ respectively. A *symmetric monoidal functor* $F : \mathcal{C} \to \mathcal{D}$ is a functor $\mathcal{C} \to \mathcal{D}$ equipped with natural isomorphisms

$$\phi : \mathbf{1}' \xrightarrow{\sim} F(\mathbf{1}), \quad J_{X,Y} : F(X) \otimes' F(Y) \to F(X \otimes Y),$$

such that the diagrams below commute.

$$\begin{tikzcd} (F(X) \otimes' F(Y)) \otimes' F(Z) \arrow[rr, "\alpha'_{F(X),F(Y),F(Z)}"] \arrow[d, "J_{X,Y} \otimes \mathrm{id}_{F(Z)}"'] & & F(X) \otimes' (F(Y) \otimes' F(Z)) \arrow[d, "\mathrm{id}_{F(X)} \otimes J_{Y,Z}"] \\ F(X \otimes Y) \otimes' F(Z) \arrow[d, "J_{X \otimes Y, Z}"'] & & F(X) \otimes' F(Y \otimes Z) \arrow[d, "J_{X, Y \otimes Z}"] \\ F((X \otimes Y) \otimes Z) \arrow[rr, "F(\alpha_{X,Y,Z})"] & & F(X \otimes (Y \otimes Z)) \end{tikzcd}$$

$$\begin{tikzcd}
\mathbf{1}' \otimes' F(X) \arrow[r, "\phi\otimes\mathrm{id}_{F(X)}"] \arrow[d, "l'_{F(X)}"'] & F(\mathbf{1}) \otimes' F(X) \arrow[d, "J_{\mathbf{1},X}"] \\
F(X) \arrow[r, "F(l_X)^{-1}"'] & F(\mathbf{1} \otimes X)
\end{tikzcd}
\qquad
\begin{tikzcd}
F(X) \otimes \mathbf{1}' \arrow[r, "\mathrm{id}_{F(X)}\otimes\phi"] \arrow[d, "r'_{F(X)}"'] & F(X) \otimes' F(\mathbf{1}) \arrow[d, "J_{X,\mathbf{1}}"] \\
F(X) \arrow[r, "F(r_X)^{-1}"'] & F(X \otimes \mathbf{1})
\end{tikzcd}$$

$$\begin{tikzcd}
F(X) \otimes' F(Y) \arrow[r, "J_{X,Y}"] \arrow[d, "\tau'_{X,Y}"'] & F(X \otimes Y) \arrow[d, "F(\tau_{X,Y})"] \\
F(Y) \otimes F(X) \arrow[r, "J_{Y,X}"'] & F(Y \otimes X)
\end{tikzcd}$$

We end this section by briefly defining a notion of maps between opfibrations, and use this to find the appropriate definition of a symmetric monoidal functor in a form compatible with Definition 2.2.6. Further detail may be found in [2].

Definition 2.2.8 ([2], Def 3.3). Let $p : X \to \mathcal{C}$ and $q : Y \to \mathcal{C}$ be two Grothendieck (op)fibrations. We say that a functor $F : X \to Y$ is a *morphism of (op)fibrations* from p to q if the diagram below commutes,

$$\begin{tikzcd}
X \arrow[rr, "F"] \arrow[dr, "p"'] & & Y \arrow[dl, "q"] \\
& \mathcal{C} &
\end{tikzcd}$$

and F sends p-(co)cartesian morphisms to q-(co)cartesian morphisms.

Proposition 2.2.9 ([2], Prop 3.4). Under the identification of symmetric monoidal categories and opfibrations over $\mathbf{Fin}_*$ given above, symmetric monoidal functors correspond to morphisms of opfibrations.

3 Generalisation to ∞-categories

3.1 Cocartesian fibrations

We first need an ∞-categorical analogue of Grothendieck opfibrations. We begin by requiring that our map between ∞-categories (or, more generally, simplicial sets) is what is known as an *inner fibration*; there is no 1–categorical analogue of this, since all functors between 1-categories are automatically inner fibrations under the nerve functor (see Example 3.1.2). This condition may be thought of as a 'minimum niceness condition' – we want the fibres to be ∞-categories in much the same way as we want the fibres of ordinary functors to be categories themselves.

Definition 3.1.1 ([1], Def 2.1). A functor $p : X \to Y$ between simplicial sets is an *inner fibration* if for all $n \geq 2$, all $0 < k < n$, and any solid arrow commutative square

$$\begin{tikzcd}
\Lambda^n_k \arrow[r] \arrow[d, hook] & X \arrow[d, "p"] \\
\Delta^n \arrow[r] \arrow[ur, dashed] & Y
\end{tikzcd}$$

there exists a dotted lift.

Example 3.1.2. Let $\mathcal{C}, \mathcal{D}$ be categories, and $p : \mathcal{C} \to \mathcal{D}$ be a functor between them. Then $N(p) : N\mathcal{C} \to N\mathcal{D}$ is an inner fibration.

The following proposition tells us that inner fibrations are stable under pullbacks; it is stated without proof in Section 2.3 of [6].

Proposition 3.1.3. Let $p : X \to Y$ be an inner fibration, and suppose that the diagram below is a pullback square in **sSet**.

$$\begin{tikzcd} X' \arrow[r, "f"] \arrow[d, "p'"'] \arrow[dr, phantom, "\lrcorner", very near start] & X \arrow[d, "p"] \\ Y' \arrow[r, "g"'] & Y \end{tikzcd}$$

Then p' is also an inner fibration.

Proof. Consider the (commutative) solid arrow diagram below.

$$\begin{tikzcd} \Lambda^n_k \arrow[r, "\lambda"] \arrow[d, hook, "\iota"'] & X' \arrow[r, "f"] \arrow[d, "p'"] \arrow[dr, phantom, "\lrcorner", very near start] & X \arrow[d, "p"] \\ \Delta^n \arrow[r, "\delta"'] \arrow[urr, dashed, "\phi"] & Y' \arrow[r, "g"'] & Y \end{tikzcd}$$

Since p is a fibration, there exists a dotted lift ϕ of $g\delta$; that is, $p\phi = g\delta$ and $\phi\iota = f\lambda$. Further, since the right square is a pullback diagram, there exists a unique map $\phi' : \Delta^n \to X'$ making the diagram below commute.

$$\begin{tikzcd} \Delta^n \arrow[dr, dashed, "\phi'"] \arrow[drr, bend left, "\phi"] \arrow[ddr, bend right, "\delta"'] & & \\ & X' \arrow[r, "f"] \arrow[d, "p'"'] \arrow[dr, phantom, "\lrcorner", very near start] & X \arrow[d, "p"] \\ & Y' \arrow[r, "g"'] & Y \end{tikzcd}$$

It remains to show that the triangle below commutes.

$$\begin{tikzcd} \Lambda^n_k \arrow[r, "\lambda"] \arrow[d, hook, "\iota"'] & X' \\ \Delta^n \arrow[ur, dashed, "\phi'"'] & \end{tikzcd}$$

Again, using the universal property of pullbacks, we see that there exist unique dotted maps such that the diagrams below commute.

$$\begin{tikzcd} \Lambda^n_k \arrow[dr, dashed] \arrow[drr, bend left, "f\lambda"] \arrow[ddr, bend right, "\delta\iota"'] & & \\ & X' \arrow[r, "f"] \arrow[d, "p'"'] \arrow[dr, phantom, "\lrcorner", very near start] & X \arrow[d, "p"] \\ & Y' \arrow[r, "g"'] & Y \end{tikzcd} \qquad \begin{tikzcd} \Lambda^n_k \arrow[dr, dashed] \arrow[drr, bend left, "f\phi'\iota"] \arrow[ddr, bend right, "\delta\iota"'] & & \\ & X' \arrow[r, "f"] \arrow[d, "p'"'] \arrow[dr, phantom, "\lrcorner", very near start] & X \arrow[d, "p"] \\ & Y' \arrow[r, "g"'] & Y \end{tikzcd}$$

The maps λ and $\phi'\iota$ make the left and right diagrams commute respectively. Further, we note that $f\phi' = \phi$ (by the second diagram) and $\phi\iota = f\lambda$ (since p is an inner fibration), so $f\phi'\iota = f\lambda$. Therefore, the above two diagrams are identical. Thus, by the uniqueness property of pullbacks, $\lambda = \phi'\iota$. $\square$

Example 3.1.4 ([1], Ex 2.2). Let $p : X \to \Delta^0$ be the canonical map, and suppose we have the diagram below, such that the outer square commutes.

$$\begin{tikzcd} \Lambda^n_k \arrow[r] \arrow[d, hook] & X \arrow[d, "p"] \\ \Delta^n \arrow[r] \arrow[ur, dashed] & \Delta^0 \end{tikzcd}$$

The lower triangle commutes automatically, so the statement that p is an inner fibration is equivalent to the statement that for all $n \geq 2$, all $0 < k < n$, and any map $\Lambda^n_k \to X$, there exists a dotted lift.

$$\begin{tikzcd} \Lambda^n_k \arrow[r] \arrow[d, hook] & X \\ \Delta^n \arrow[ur, dashed] & \end{tikzcd}$$

That is, X is an ∞-category.

Now, combining the above argument with Proposition 3.1.3, we see that for any inner fibration $p : X \to Y$, each fibre $X \times_Y \{y\}$ is an ∞-category.

Definition 3.1.5 ([1], Def 3.1). Let $p : X \to Y$ be an inner fibration. An edge $f : \Delta^1 \to X$ of X is *p-cocartesian* if for all $n \geq 2$, any extension

$$\begin{tikzcd} \Delta^{\{0,1\}} \arrow[r, "f"] \arrow[d, hook] & X \\ \Lambda^n_0 \arrow[ur, "F"'] & \end{tikzcd}$$

and any solid arrow commutative diagram

$$\begin{tikzcd} \Lambda^n_0 \arrow[r, "F"] \arrow[d, hook] & X \arrow[d, "p"] \\ \Delta^n \arrow[r] \arrow[ur, dashed] & Y \end{tikzcd}$$

a dotted lift exists.

Definition 3.1.6. Let $p : X \to Y$ be an inner fibration. Then p is a cocartesian fibration if for any edge $\phi : y \to y'$ in Y_1, and for every $x \in X_0$ lying over y, there exists a p-cocartesian edge $f : x \to x'$ of X lying over ϕ.

The following proposition (the dual of which is stated as Remark 2.4.2.2 in [6]) shows us that the above definition is a reasonable generalisation of Definition 2.1.2.

Proposition 3.1.7. Let $\mathcal{C}$, $\mathcal{D}$ be categories, and let $p : \mathcal{C} \to \mathcal{D}$ be a functor between them. Then p is a Grothendieck opfibration if and only if the induced map $N(p) : N\mathcal{C} \to N\mathcal{D}$ is a cocartesian fibration of simplicial sets.

Proof. Let $f : d \to d'$ be a morphism of $\mathcal{D}$, and let c lie over d.

Suppose p is a Grothendieck opfibration, let $F : \Lambda^n_0 \to N\mathcal{C}$ be an extension of f, and let

$$\begin{tikzcd} \Lambda^n_0 \arrow[r, "F"] \arrow[d, hook] & N\mathcal{C} \arrow[d, "N(p)"] \\ \Delta^n \arrow[r] & N\mathcal{D} \end{tikzcd}$$

be a commutative diagram. If $n = 2$, it follows immediately from the fact that p is an opfibration that a dotted lift exists. Further, if $n > 3$, there is nothing to check, since an n-simplex in a category commutes if and only if all of its triangles commute, which is guaranteed for any extension $F : \Lambda^n_0 \to N\mathcal{C}$. We thus prove the proposition for $n = 3$.

Suppose we have an extension $F : \Lambda^3_0 \to N\mathcal{C}$ of f; that is, a tetrahedron

$$\begin{tikzcd}
 & c \arrow[rr, "\chi"] \arrow[dl, "\phi"'] \arrow[dr, "\psi"] & & c''' \\
c' \arrow[rr, "\gamma"'] \arrow[urrr, "\gamma'"] & & c'' \arrow[ur, "\gamma''"'] &
\end{tikzcd}$$

such that all faces containing the vertex c commute. Let

$$\begin{tikzcd}
 & d \arrow[rr, "p(\chi)"] \arrow[dl, "f"'] \arrow[dr, "p(\psi)"] & & p(c''') \\
d' \arrow[rr, "p(\gamma)"'] \arrow[urrr, "p(\gamma')"] & & p(c'') \arrow[ur, "p(\gamma'')"'] &
\end{tikzcd}$$

be a commutative tetrahedron in $\mathcal{D}$. We claim that the tetrahedron in $\mathcal{C}$ commutes. First, note that $\gamma'' \circ \gamma$ is a lift of $p(\gamma')$, since $p(\gamma') = p(\gamma'') \circ p(\gamma) = p(\gamma'' \circ \gamma)$. Further,

$$\begin{aligned}
(\gamma \circ \gamma'') \circ \phi &= \gamma'' \circ \psi \\
&= \chi.
\end{aligned}$$

Thus, by the uniqueness in the universal property of ϕ, we have that $\gamma' = \gamma'' \circ \gamma$, as required.

Now, suppose $N(p)$ is a cocartesian fibration. Then there exists a lift $\phi : c \to c'$ of f, and, in particular, for any diagram

$$\begin{tikzcd}
 & c \arrow[dl, "\phi"'] \arrow[dr, "\psi"] & \\
c' & & c''
\end{tikzcd}$$

in $\mathcal{C}$, and any commutative diagram

$$\begin{tikzcd}
 & d \arrow[dl, "f"'] \arrow[dr, "p(\psi)"] & \\
d' \arrow[rr, "g"'] & & p(c'')
\end{tikzcd}$$

in $\mathcal{D}$, there exists a map $\gamma : c' \to c''$ such that γ lies over g and $\gamma \circ \phi = \psi$. It remains to show that γ is unique.

Suppose that there were two maps $\gamma_1, \gamma_2 : c' \to c''$ lying over g and satisfying $\gamma_1 \circ \phi = \gamma_2 \circ \phi = \psi$. Then we would have a tetrahedron

$$\begin{tikzcd}
 & c \arrow[rr, "\psi"] \arrow[dl, "\phi"'] \arrow[dr, "\psi"] & & c'' \\
c' \arrow[rr, "\gamma_1"'] \arrow[urrr, "\gamma_2"] & & c'' \arrow[ur, "\mathrm{id}"'] &
\end{tikzcd}$$

where all faces containing the vertex c commute. The image of this tetrahedron under p commutes in $\mathcal{D}$, so the original tetrahedron must commute in $\mathcal{C}$; that is, $\gamma_1 = \gamma_2$. $\square$

3.2 Symmetric monoidal ∞-categories

We are finally ready to define symmetric monoidal ∞-categories; the following definitions are now almost identical to those for symmetric monoidal 1-categories, and thus this section will be surprisingly brief.

Definition 3.2.1 ([5], Def 2.0.0.7). A *symmetric monoidal ∞-category* is a cocartesian fibration of simplicial sets $p : X^{\otimes} \to N(\mathbf{Fin}_*)$ such that for each $n \geq 0$, the maps

$$\{\rho^i : \langle n \rangle \to \langle 1 \rangle\}_{1 \leq i \leq n}$$

induce functors $\rho^i_! : X^{\otimes}_{\langle n \rangle} \to X^{\otimes}_{\langle 1 \rangle}$ which determine an equivalence $X^{\otimes}_{\langle n \rangle} \simeq \left(X^{\otimes}_{\langle 1 \rangle}\right)^n$.

The following examples are evidence that this definition is a reasonable generalisation of ordinary symmetric monoidal categories.

Example 3.2.2. Let $(\mathcal{C}, \otimes)$ be a symmetric monoidal category. Then $p : N(\mathcal{C}^{\otimes}) \to N(\mathbf{Fin}_*)$ is a symmetric monoidal ∞-category. This follows from Proposition 3.1.7, and the fact that the nerve of an equivalence is an equivalence of ∞-categories (see [7, Example 01E1]).

Example 3.2.3. If an ∞-category X has finite (co)products, there is a (co)cartesian monoidal structure on X (see [5], sections 2.4.2 and 2.4.3).

Definition 3.2.4. Let $X \xrightarrow{p} \mathbf{Fin}_*$, $Y \xrightarrow{q} \mathbf{Fin}_*$ be symmetric monoidal ∞-categories. A symmetric monoidal functor is a commutative (up to homotopy) diagram

$$\begin{tikzcd} X \arrow[rr, "F"] \arrow[dr, "p"'] & & Y \arrow[dl, "q"] \\ & N(\mathbf{Fin}_*) & \end{tikzcd}$$

such that F preserves cocartesian edges.

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