

| Concept                        | 1-Categorical construction  | $\infty$ -Categorical construction  | Intuition   |
|--------------------------------|---|---|---|
| $F$ -Cartesian edge            | <p><math>F : X \rightarrow S</math> a functor, <math>f : x \rightarrow y</math> a morphism in <math>X</math> is <math>F</math>-cartesian if the induced map</p> $X_{/f} \rightarrow X_{/y} \times_{S_{/F(y)}} S_{/F(f)}$ <p>is an isomorphism of categories. ([nLa25a], Prop 2.4)</p> | <p><math>F : X \rightarrow S</math> an inner fibration of simplicial sets, <math>f : x \rightarrow y</math> an edge in <math>X</math> is <math>F</math>-cartesian if the induced map</p> $X_{/f} \rightarrow X_{/y} \times_{S_{/F(y)}} S_{/F(f)}$ <p>is a trivial Kan fibration. ([Lur09], Def 2.4.1.1)</p> | <p>In the model structure on <b>sSet</b>, the fibrations are Kan fibrations and the weak equivalences are weak homotopy equivalences ([Lur09], A.2.7). A trivial fibration in a model category is a map which is both a fibration and a weak equivalence, which in <b>sSet</b> is equivalent to the definition given in this table. Thus, being related by a Kan fibration is a higher categorical notion of ‘sameness’. <b>Why not a categorical equivalence?</b> [Lur09] Rem 1.2.5.5 implies this is stronger, which would match more with the fact that the 1-categorical version is defined in terms of an isomorphism (not equivalence) of categories.</p> |
| Category                       | Collection of objects $\mathcal{C}$ , set $\text{Hom}(X, Y)$ for every $X, Y \in \mathcal{C}$ , associative composition and identity morphisms  | Simplicial set $\mathcal{C} : \Delta^{\text{op}} \rightarrow \mathbf{Set}$ satisfying the inner horn filling condition. ([Lur09], Def 1.1.2.4)  | Category with objects $\mathcal{C}_0$ , morphisms $\mathcal{C}_1$ , morphisms between morphisms $\mathcal{C}_2$ , etc. Inner horn filling defines a non-unique composition.   |
| Colimit                        | A colimit for $F : J \rightarrow \mathcal{C}$ is an initial cone on $F$ .   | A colimit for $F : X \rightarrow \mathcal{C}$ ( $X$ a simplicial set, $\mathcal{C}$ an $\infty$ -category) is an initial object of $\mathcal{C}_{F/}$ . ([Lur09], Def 1.2.13.4)   | ??  |
| Essentially surjective functor | $F : \mathcal{C} \rightarrow \mathcal{D}$ is essentially surjective if for every $D \in \mathcal{D}$ , there exists some $C \in \mathcal{C}$ with $FC \cong D$ .  | $F : \mathcal{C} \rightarrow \mathcal{D}$ is essentially surjective if $\text{h}F : \text{h}\mathcal{C} \rightarrow \text{h}\mathcal{D}$ is essentially surjective. ([Lur09], Def 1.2.10.1)   | Essentially surjective up to homotopy.  |
| Faithful functor               | $F : \mathcal{C} \rightarrow \mathcal{D}$ is faithful if $\text{Hom}(X, Y) \rightarrow \text{Hom}(FX, FY)$ is injective for all $X, Y \in \mathcal{C}$ .  | $F : \mathcal{C} \rightarrow \mathcal{D}$ is faithful if $\text{h}F : \text{h}\mathcal{C} \rightarrow \text{h}\mathcal{D}$ is faithful. ([Lur09], Def 1.2.10.1)   | Faithful up to homotopy.  |
| Final object                   | Object $C \in \mathcal{C}$ such that for any other object $C' \in \mathcal{C}$ , there exists a unique morphism $C' \rightarrow C$ .  | Object $C \in \mathcal{C}$ such that $C$ is final in $\text{h}\mathcal{C}$ , regarded as an enriched category over $\mathcal{H}$ . ([Lur09], Def 1.2.12.1)  | Object $C \in \mathcal{C}$ such that for any other object $C' \in \mathcal{C}$ , there exists a unique (up to homotopy) morphism $C' \rightarrow C$ .   |
| Full functor                   | $F : \mathcal{C} \rightarrow \mathcal{D}$ is full if $\text{Hom}(X, Y) \rightarrow \text{Hom}(FX, FY)$ is surjective for all $X, Y \in \mathcal{C}$ .   | $F : \mathcal{C} \rightarrow \mathcal{D}$ is full if $\text{h}F : \text{h}\mathcal{C} \rightarrow \text{h}\mathcal{D}$ is full. ([Lur09], Def 1.2.10.1)   | Full up to homotopy.  |
| Functor                        | Functor.  | Natural transformation of simplicial sets. ([Lur09], 1.2.7)   | -   |
| Groupoid                       | Category whose morphisms are all invertible.  | Kan complex.  | <p>Not only can you find (non-unique) ‘composites’, but you can also fill in diagrams like</p>  |
| Initial object                 | Object $C \in \mathcal{C}$ such that for any other object $C' \in \mathcal{C}$ , there exists a unique morphism $C \rightarrow C'$ .  | Object $C \in \mathcal{C}$ such that $C$ is initial in $\text{h}\mathcal{C}$ , regarded as an enriched category over $\mathcal{H}$ . ([Lur09], Def 1.2.12.1)  | Object $C \in \mathcal{C}$ such that for any other object $C' \in \mathcal{C}$ , there exists a unique (up to homotopy) morphism $C \rightarrow C'$ .   |

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| Join   | $\mathcal{C} \star \mathcal{D}$ has objects $\text{ob } \mathcal{C} \sqcup \text{ob } \mathcal{D}$ , and $\text{Hom}_{\mathcal{C} \star \mathcal{D}}(X, Y)$ is given by: $\begin{cases} \text{Hom}_{\mathcal{C}}(X, Y) & X, Y \in \mathcal{C}, \\ \text{Hom}_{\mathcal{D}}(X, Y) & X, Y \in \mathcal{D}, \\ \emptyset & X \in \mathcal{D}, Y \in \mathcal{C}, \\ * & X \in \mathcal{C}, Y \in \mathcal{D}. \end{cases}$ ([Lur09], 1.2.8)   | $\mathcal{C} \star \mathcal{D}$ has $n$ -simplices $(\mathcal{C} \star \mathcal{D}) = \mathcal{C}_n \cup \mathcal{D}_n \cup \bigcup_{i+j=n-1} \mathcal{C}_i \times \mathcal{D}_j$ . The $i$ th boundary map $d_i : (\mathcal{C} \star \mathcal{D})_n \rightarrow (\mathcal{C} \star \mathcal{D})_{n-1}$ is defined on $\mathcal{C}_n$ and $\mathcal{D}_n$ using the $i$ th boundary map on $\mathcal{C}$ and $\mathcal{D}$ . Given $\sigma \in S_j, \tau \in T_k$ , $d_i(\sigma, \tau)$ is given by $\begin{cases} (d_i\sigma, \tau) & i \leq j, j \neq 0, \\ (\sigma, d_{i-j-1}\tau) & i > j, k \neq 0. \end{cases}$ If $j = 0$ , then $d_0(\sigma, \tau) = \tau$ , and if $k = 0$ , then $d_n(\sigma, \tau) = \sigma$ . ([Lur09], Def 1.2.8.1 / [nLa25b]) | Objects are in both cases disjoint unions of objects from the two categories being joined. Morphisms are also exactly the same in both cases (you get all the morphisms from $\mathcal{C}$ and $\mathcal{D}$ , plus a morphism from $c \rightarrow d$ for every pair $(c, d) \in \mathcal{C}_0 \times \mathcal{D}_0$ ). Whenever you have an $n$ -simplex in $\mathcal{C}$ and an $m$ -simplex in $\mathcal{D}$ , you get an $(m+n+1)$ -simplex in $\mathcal{C} \star \mathcal{D}$ , so in particular $\Delta^n \star \Delta^m \cong \Delta^{m+n+1}$ . |
| Left cone  | $\mathcal{C}^{\triangleleft} := [0] \star \mathcal{C}$ .   | $\mathcal{C}^{\triangleleft} := \Delta^0 \star \mathcal{C}$ . ([Lur09], Not 1.2.8.4)  | ??   |
| Left Kan extension (along the inclusion of a full subcategory) | Given a commutative diagram $\begin{array}{ccc} \mathcal{C}^0 & \xrightarrow{F_0} & \mathcal{D} \\ \downarrow \iota & \nearrow F & \\ \mathcal{C} & & \end{array}$ , $F$ is a left Kan extension of $F_0$ along $\iota$ if there is a natural transformation $\eta : F_0 \rightarrow F\iota$ such that for any other pair $(G : \mathcal{C} \rightarrow \mathcal{D}, \gamma : F_0 \rightarrow G\iota)$ , there exists a unique natural transformation $\alpha : F \rightarrow G$ such that $\gamma = (\alpha * \iota) \circ \eta$ . ([Rie16], Def 6.1.1) | Given a commutative diagram $\begin{array}{ccc} \mathcal{C}^0 & \xrightarrow{F_0} & \mathcal{D} \\ \downarrow \iota & \nearrow F & \\ \mathcal{C} & & \end{array}$ , $F$ is a left Kan extension of $F_0$ along $\iota$ if for all $C \in \mathcal{C}$ , the induced diagram $\begin{array}{ccc} \mathcal{C}_{/C}^0 & \xrightarrow{F_C} & \mathcal{D} \\ \downarrow & \nearrow & \\ (\mathcal{C}_{/C}^0)^{\triangleright} & & \end{array}$ exhibits $FC$ as a colimit of $F_C$ . ([Lur09], Def 4.3.2.2)   | ??   |
| Limit  | A limit for $F : J \rightarrow \mathcal{C}$ is a terminal cone on $F$ .  | A limit for $F : X \rightarrow \mathcal{C}$ ( $X$ a simplicial set, $\mathcal{C}$ an $\infty$ -category) is a final object of $\mathcal{C}_{/F}$ . ([Lur09], Def 1.2.13.4)  | ??   |
| Opposite category  | $\mathcal{C}^{\text{op}}$ has the same objects as $\mathcal{C}$ , and $\text{Hom}_{\mathcal{C}^{\text{op}}}(X, Y) = \text{Hom}_{\mathcal{C}}(Y, X)$ .  | $\mathcal{C}_n^{\text{op}} = \mathcal{C}([n]^{\text{op}})$ , where $\{0 < 1 < \dots < n\}^{\text{op}} = \{0 > 1 > \dots > n\}$ . ([Lur09], 1.2.1)   | A map $x \rightarrow y$ is an edge $\Delta^1 \rightarrow \mathcal{C}$ where $0 \mapsto x$ and $1 \mapsto y$ . In $\mathcal{C}^{\text{op}}$ 0 and 1 swap roles, so we instead get a map $y \rightarrow x$ .   |
| Overcategory   | For $C \in \mathcal{C}$ , the category $\mathcal{C}_{/C}$ satisfies the following universal property: for any category $\mathcal{D}$ , there is a bijection $\text{Hom}(\mathcal{D}, \mathcal{C}_{/C}) \simeq \text{Hom}_{\mathcal{C}}(\mathcal{D} \star [0], \mathcal{C}),$ where the subscript on the right indicates that we consider only those functors $\mathcal{D} \star [0] \rightarrow \mathcal{C}$ whose restriction to $[0]$ coincides with $C$ . ([Lur09], 1.2.9)  | For $f : S \rightarrow \mathcal{C}$ , $S$ a simplicial set and $\mathcal{C}$ an $\infty$ -category, the $\infty$ -category $\mathcal{C}_{/f}$ satisfies the following universal property: for any simplicial set $X$ , there is a bijection $\text{Hom}(X, \mathcal{C}_{/f}) \simeq \text{Hom}_f(X \star S, \mathcal{C}),$ where the subscript on the right indicates that we consider only those functors $X \star S \rightarrow \mathcal{C}$ whose restriction to $S$ coincides with $f$ . ([Lur09], Prop 1.2.9.2)  | ??   |
| Right cone   | $\mathcal{C}^{\triangleright} := \mathcal{C} \star [0]$ .  | $\mathcal{C}^{\triangleright} := \mathcal{C} \star \Delta^0$ . ([Lur09], Not 1.2.8.4)   | ??   |

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| Subcategory   | Subcategory $\mathcal{C}' \subseteq \mathcal{C}$ .   | Subsimplicial set $\mathcal{C}' \subseteq \mathcal{C}$ arising as<br>$\begin{array}{ccc} \mathcal{C}' & \longrightarrow & \mathcal{C} \\ \downarrow & \lrcorner & \downarrow \\ N(\mathbf{h}\mathcal{C})' & \longrightarrow & N(\mathbf{h}\mathcal{C}) \end{array}$ a pullback<br>where $(\mathbf{h}\mathcal{C})' \subseteq \mathbf{h}\mathcal{C}$ is a subcategory.<br>([Lur09], 1.2.11)   | ?? |
| Undercategory | For $C \in \mathcal{C}$ , the category $\mathcal{C}_{C/}$ satisfies the following universal property: for any category $\mathcal{D}$ , there is a bijection<br>$\mathrm{Hom}(\mathcal{D}, \mathcal{C}_{C/}) \simeq \mathrm{Hom}_{\mathcal{C}}([0] \star \mathcal{D}, \mathcal{C})$ ,<br>where the subscript on the right indicates that we consider only those functors $[0] \star \mathcal{D} \rightarrow \mathcal{C}$ whose restriction to $[0]$ consists with $C$ .<br>([Lur09], 1.2.9) | For $f : S \rightarrow \mathcal{C}$ , $S$ a simplicial set and $\mathcal{C}$ an $\infty$ -category, the $\infty$ -category $\mathcal{C}_{f/}$ satisfies the following universal property: for any simplicial set $X$ , there is a bijection<br>$\mathrm{Hom}(X, \mathcal{C}_{f/}) \simeq \mathrm{Hom}_f(S \star X, \mathcal{C})$ ,<br>where the subscript on the right indicates that we consider only those functors $S \star X \rightarrow \mathcal{C}$ whose restriction to $S$ consists with $f$ .<br>([Lur09], Prop 1.2.9.2) | ?? |

| Equivalences                |   |   |
|-----------------------------|---|---|
| Name                        | Between   | Definition  |
| Strong equivalence          | Topological categories $\mathcal{C}, \mathcal{D}$ | $\mathcal{C} \rightarrow \mathcal{D}$ is an equivalence in the sense of enriched category theory. ([Lur09], Def 1.1.3.1)                                      |
| (Weak) equivalence          | Topological categories $\mathcal{C}, \mathcal{D}$ | The induced functor $\mathbf{h}\mathcal{C} \rightarrow \mathbf{h}\mathcal{D}$ is an equivalence of $\mathcal{H}$ -enriched categories. ([Lur09], Def 1.1.3.6) |
| Categorical equivalence     | Simplicial sets $X, S$                            | The induced functor $\mathbf{h}X \rightarrow \mathbf{h}S$ is an equivalence of $\mathcal{H}$ -enriched categories. ([Lur09], Def 1.1.5.14)                    |
| Weak (homotopy) equivalence | Simplicial sets $X, S$                            | The induced map $ X  \rightarrow  S $ is a weak homotopy equivalence of topological spaces. ([Lur09], 1.1.4)  |
| Equivalence                 | Simplicial categories $\mathcal{C}, \mathcal{D}$  | The induced functor $\mathbf{h}\mathcal{C} \rightarrow \mathbf{h}\mathcal{D}$ is an equivalence of $\mathcal{H}$ -enriched categories. ([Lur09], Def 1.1.4.4) |

| Fibrations and anodyne morphisms |  |  |
|----------------------------------|--|--|
| Name                             | Describes                                    | Definition   |
| Acyclic Kan fibration            | $f : X \rightarrow S$ map of simplicial sets | see: trivial Kan fibration. ([nLa23])  |
| Anodyne                          | $f : X \rightarrow S$ map of simplicial sets | For every solid arrow diagram as below, with $p : Y \rightarrow T$ a Kan fibration,<br>$\begin{array}{ccc} X & \longrightarrow & Y \\ f \downarrow & \nearrow & \downarrow p \\ S & \longrightarrow & T \end{array}$ there exists a dotted lift. ([Lur09], Ex 2.0.0.1)           |
| Cartesian fibration              | $f : X \rightarrow S$ map of simplicial sets | $F$ is an inner fibration such that for every edge $g : x \rightarrow y$ of $S$ and every vertex $\tilde{y}$ of $X$ with $f(\tilde{y}) = y$ , there exists an $f$ -cartesian edge $\tilde{g} : \tilde{x} \rightarrow \tilde{y}$ with $f(\tilde{g}) = g$ . ([Lur09], Def 2.4.2.1) |

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| Cofibration     | $f : X \rightarrow S$ map of simplicial sets                          | $f$ is a monomorphism. ([Lur09], A.2.7)  |
| Inner anodyne   | $f : X \rightarrow S$ map of simplicial sets                          | <p>For every solid arrow diagram as below, with <math>p : Y \rightarrow T</math> an inner fibration,</p> $\begin{array}{ccc} X & \longrightarrow & Y \\ f \downarrow & \nearrow & \downarrow p \\ S & \longrightarrow & T \end{array}$ <p>there exists a dotted lift. ([Lur09], Def 2.0.0.3)</p>                       |
| Inner Fibration | $f : X \rightarrow S$ map of simplicial sets                          | <p>For every solid arrow diagram as below, with <math>0 &lt; i &lt; n</math>,</p> $\begin{array}{ccc} \Lambda_i^n & \longrightarrow & X \\ \downarrow & \nearrow & \downarrow f \\ \Delta^n & \longrightarrow & S \end{array}$ <p>there exists a dotted lift.</p>  |
| Isofibration    | $F : \mathcal{C} \rightarrow \mathcal{D}$ map of $\infty$ -categories | $F$ is an inner fibration such that for all $C \in \mathcal{C}$ and every isomorphism $u : D \rightarrow FC$ in $\mathcal{D}$ (i.e. $[u]$ is an isomorphism in $\mathbf{h}\mathcal{D}$ ) there exists an isomorphism $\bar{u} : \bar{D} \rightarrow C$ in $\mathcal{C}$ such that $F(\bar{u}) = u$ . [Lur25, Def 01EN] |
| (Kan) fibration | $f : X \rightarrow S$ map of simplicial sets                          | <p>For every solid arrow diagram as below, with <math>0 \leq i \leq n</math>,</p> $\begin{array}{ccc} \Lambda_i^n & \longrightarrow & X \\ \downarrow & \nearrow & \downarrow f \\ \Delta^n & \longrightarrow & S \end{array}$ <p>there exists a dotted lift. ([Lur09], A.2.7)</p>                                     |
| Left anodyne    | $f : X \rightarrow S$ map of simplicial sets                          | <p>For every solid arrow diagram as below, with <math>p : Y \rightarrow T</math> a left fibration,</p> $\begin{array}{ccc} X & \longrightarrow & Y \\ f \downarrow & \nearrow & \downarrow p \\ S & \longrightarrow & T \end{array}$ <p>there exists a dotted lift. ([Lur09], Def 2.0.0.3)</p>                         |
| Left fibration  | $f : X \rightarrow S$ map of simplicial sets                          | <p>For every solid arrow diagram as below, with <math>0 \leq i &lt; n</math>,</p> $\begin{array}{ccc} \Lambda_i^n & \longrightarrow & X \\ \downarrow & \nearrow & \downarrow f \\ \Delta^n & \longrightarrow & S \end{array}$ <p>there exists a dotted lift. ([Lur09], Def 2.0.0.3)</p>                               |
| Right anodyne   | $f : X \rightarrow S$ map of simplicial sets                          | <p>For every solid arrow diagram as below, with <math>p : Y \rightarrow T</math> a right fibration,</p> $\begin{array}{ccc} X & \longrightarrow & Y \\ f \downarrow & \nearrow & \downarrow p \\ S & \longrightarrow & T \end{array}$ <p>there exists a dotted lift. ([Lur09], Def 2.0.0.3)</p>                        |

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| Right fibration         | $f : X \rightarrow S$ map of simplicial sets    | <p>For every solid arrow diagram as below, with <math>0 &lt; i \leq n</math>,</p> $\begin{array}{ccc} \Lambda_i^n & \longrightarrow & X \\ \downarrow & \nearrow & \downarrow f \\ \Delta^n & \longrightarrow & S \end{array}$ <p>there exists a dotted lift. ([Lur09], Def 2.0.0.3)</p> |
| Serre fibration         | $f : Y \rightarrow Z$ map of topological spaces | <p>For every solid arrow diagram as below,</p> $\begin{array}{ccc} \{0\} \times  \Delta^n  & \longrightarrow & Y \\ \downarrow & \nearrow & \downarrow f \\ [0, 1] \times  \Delta^n  & \longrightarrow & Z \end{array}$ <p>there exists a dotted lift. [Lur25, Def 021R]</p>             |
| Trivial (Kan) fibration | $f : X \rightarrow S$ map of simplicial sets    | <p>For every solid arrow diagram as below,</p> $\begin{array}{ccc} \partial\Delta^n & \longrightarrow & X \\ \downarrow & \nearrow & \downarrow f \\ \Delta^n & \longrightarrow & S \end{array}$ <p>there exists a dotted lift. ([Lur25, Def 006W]/[Lur09], Def 2.0.0.2)</p>             |

| Nerves            |                                    |  |
|-------------------|------------------------------------|--|
| Name              | Domain object                      | Definition   |
| Nerve             | Category $\mathcal{C}$             | $(N\mathcal{C})_n = \{n\text{-composable strings of morphisms in } \mathcal{C}\}$ .  |
| Simplicial nerve  | Simplicial category $\mathcal{C}$  | $(N\mathcal{C})_n = \text{Hom}_{\mathbf{Cat}_\Delta}(\mathfrak{C}[\Delta^n], \mathcal{C})$ , where $\mathfrak{C}[\Delta^n]$ is the category whose objects are the same as $[n]$ , and $\text{Hom}_{\mathfrak{C}[\Delta^n]}(i, j) = \emptyset$ for $i < j$ and $N(P_{ij})$ for $i \geq j$ (where $P_{ij} = \{I \subseteq [n] : (i, j \in I) \wedge (\forall k \in I, i \leq k \leq j)\}$ ). |
| Topological nerve | Topological category $\mathcal{C}$ | The simplicial nerve of $\text{Sing } \mathcal{C}$ .   |

| Homotopy categories                |  |
|------------------------------------|--|
| Domain object                      | Definition   |
| $\infty$ -Category $\mathcal{C}$   | The objects of $\text{h}\mathcal{C}$ are the vertices of $\mathcal{C}$ , and $\text{Hom}_{\text{h}\mathcal{C}}(X, Y)$ is the set of homotopy classes of edges $X \rightarrow Y$ in $\mathcal{C}$ . ([Lur09], Prop 1.2.3.9) |
| Simplicial category $\mathcal{C}$  | $\text{h} \mathcal{C} $ . ([Lur09], 1.1.4)   |
| Topological category $\mathcal{C}$ | $\text{h}\mathcal{C}$ has the same objects as $\mathcal{C}$ , and $\text{Hom}_{\text{h}\mathcal{C}}(X, Y) = [\text{Hom}_{\mathcal{C}}(X, Y)]$ . ([Lur09], 1.1.3)   |

## References

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