∞ -Categorical construction

Intuition

1-Categorical construction

Concept

Def 1.2.10.1)

¹Lurie introduces the term κ -continuous for such functors, but in ordinary category theory this generally means a functor which preserves κ -small limits; a functor preserving κ -filtered colimits is called κ -finitary. I have thus steered clear of this term.

Faithful functor	$F: \mathcal{C} \to \mathcal{D}$ is faithful if $\operatorname{Hom}(X,Y) \to \operatorname{Hom}(FX,FY)$ is injective for all $X,Y \in \mathcal{C}$.	$F: \mathcal{C} \to \mathcal{D}$ is faithful if $hF: h\mathcal{C} \to h\mathcal{D}$ is faithful. ([Lur09], Def 1.2.10.1)	Faithful up to homotopy.
κ -Filtered category	For a regular cardinal κ , \mathcal{C} is κ -filtered if, for every κ -small category J and every functor F : $J \to \mathcal{C}$, there exists a cocone on F .	For a regular cardinal κ , \mathcal{C} is κ -filtered if, for every κ -small simplicial set X and every map $f: X \to \mathcal{C}$, there exists a map $\overline{f}: K^{\triangleright} \to \mathcal{C}$ extending f . ([Lur09], Def 5.3.1.7)	A cocone on F is a collection of compatible maps $(\lambda_j : F(j) \to C)$. Define $\overline{F} : J \star [0] \to C$ to be F on J , send the cone point to C , and send the unique morphisms $*_j$ from $j \in J$ to the cone point to the λ_j . Conversely, if you have some \overline{F} extending F , define $\lambda_j := F(*_j)$.
Final object	Object $C \in \mathcal{C}$ such that for any other object $C' \in \mathcal{C}$, there exists a unique morphism $C' \to C$.	Object $C \in \mathcal{C}$ such that C is final in $h\mathcal{C}$, regarded as an enriched category over \mathcal{H} . ([Lur09], Def 1.2.12.1)	Object $C \in \mathcal{C}$ such that for any other object $C' \in \mathcal{C}$, there exists a unique (up to homotopy) morphism $C' \to C$.
Full functor	$F: \mathcal{C} \to \mathcal{D}$ is full if $\operatorname{Hom}(X,Y) \to \operatorname{Hom}(FX,FY)$ is surjective for all $X,Y \in \mathcal{C}$.	$F: \mathcal{C} \to \mathcal{D}$ is full if $hF: h\mathcal{C} \to h\mathcal{D}$ is full. ([Lur09], Def 1.2.10.1)	Full up to homotopy.
Functor	Functor.	Natural transformation of simplicial sets. ([Lur09], 1.2.7)	-
Groupoid	Category whose morphisms are all invertible.	Kan complex.	Not only can you find (non-unique) 'composites', $C \xrightarrow{\mathrm{id}} C$ but you can also fill in diagrams like f D $C \xrightarrow{\mathrm{id}} D$ $C \xrightarrow{f} C$
Initial object	Object $C \in \mathcal{C}$ such that for any other object $C' \in \mathcal{C}$, there exists a unique morphism $C \to C'$.	Object $C \in \mathcal{C}$ such that C is initial in $h\mathcal{C}$, regarded as an enriched category over \mathcal{H} . ([Lur09], Def 1.2.12.1)	Object $C \in \mathcal{C}$ such that for any other object $C' \in \mathcal{C}$, there exists a unique (up to homotopy) morphism $C \to C'$.
Join	$ \begin{array}{c cccc} \mathcal{C} & \star & \mathcal{D} & \text{has} & \text{objects} & \text{ob} \mathcal{C} & \sqcup & \text{ob} \mathcal{D}, \\ \text{and} & \operatorname{Hom}_{\mathcal{C}\star\mathcal{D}}(X,Y) & \text{is} & \text{given} & \text{by:} \\ & \operatorname{Hom}_{\mathcal{C}}(X,Y) & X,Y \in \mathcal{C}, \\ \operatorname{Hom}_{\mathcal{D}}(X,Y) & X,Y \in \mathcal{D}, \\ \emptyset & X \in \mathcal{D},Y \in \mathcal{C}, \\ * & X \in \mathcal{C},Y \in \mathcal{D}. \\ \end{array} $	$\begin{array}{c} \mathcal{C}\star\mathcal{D} \text{ has } n\text{-simplicies } (\mathcal{C}\star\mathcal{D}) = \mathcal{C}_n\cup\mathcal{D}_n\cup\\ \bigcup_{i+j=n-1}\mathcal{C}_i\times\mathcal{D}_j. \text{ The } i\text{th boundary map } d_i:\\ (\mathcal{C}\star\mathcal{D})_n\to (\mathcal{C}\star\mathcal{D})_{n-1} \text{ is defined on } \mathcal{C}_n \text{ and }\\ \mathcal{D}_n \text{ using the } i\text{th boundary map on } \mathcal{C} \text{ and } \mathcal{D}.\\ \text{Given } \sigma\in S_j, \tau\in T_k, \ d_i(\sigma,\tau) \text{ is given by} \\ \begin{cases} (d_i\sigma,\tau) & i\leq j,\ j\neq 0,\\ (\sigma,d_{i-j-1}\tau) & i>j,\ k\neq 0. \end{cases} \\ \text{If } j=0, \text{ then } d_0(\sigma,\tau)=\tau, \text{ and if } k=0, \text{ then }\\ d_n(\sigma,\tau)=\sigma. \ ([\text{Lur09}], \text{ Def } 1.2.8.1\ / \ [\text{nLa25c}]) \end{cases}$	Objects are in both cases disjoint unions of objects from the two categories being joined. Morphisms are also exactly the same in both cases (you get all the morphisms from \mathcal{C} and \mathcal{D} , plus a morphism from $c \to d$ for every pair $(c,d) \in \mathcal{C}_0 \times \mathcal{D}_0$). Whenever you have an n -simplex in \mathcal{C} and an m -simplex in \mathcal{D} , you get an $(m+n+1)$ -simplex in $\mathcal{C} \star \mathcal{D}$, so in particular $\Delta^n \star \Delta^m \cong \Delta^{m+n+1}$.
Left cone	$\mathcal{C}^{\triangleleft} := [0] \star \mathcal{C}.$	$\mathcal{C}^{\triangleleft} := \Delta^{0} \star \mathcal{C}. \text{ ([Lur09], Not 1.2.8.4)}$	\mathcal{C} with extra vertex (cone point) added, as well as a map from that cone point to every other vertex in \mathcal{C} (plus obligatory degenerate simplicies).

Left dualisable object	Object $C \in \mathcal{C}$ such that there exists some $C^* \in \mathcal{C}$ and maps $e: C \otimes C^* \to \mathbb{1}$, $c: \mathbb{1} \to C^* \otimes C$ such that the composites $C \to C \otimes C^* \otimes C \to C$ and $C^* \to C^* \otimes C \otimes C^* \to C^*$ are equal to the identity. ²	Object $C \in \mathcal{C}$ such that there exists some $C^* \in \mathcal{C}$ and maps $e: C \otimes C^* \to \mathbb{1}$, $c: \mathbb{1} \to C^* \otimes C$ such that the composites $C \to C \otimes C^* \otimes C \to C$ and $C^* \to C^* \otimes C \otimes C^* \to C^*$ are homotopic to the identity.	C has a left dual up to homotopy.
Left Kan extension (along the inclusion of a full subcategory)	Given a commutative diagram $C^0 \xrightarrow{F_0} \mathcal{D}$, C F is a left Kan extension of F_0 along ι if there is a natural transformation $\eta: F_0 \to F\iota$ such that for any other pair $(G: \mathcal{C} \to \mathcal{D}, \gamma: F_0 \to G\iota)$, there exists a unique natural transformation $\alpha: F \to G$ such that $\gamma = (\alpha * \iota) \circ \eta$. ([Rie16], Def 6.1.1)	Given a commutative diagram $C^0 \xrightarrow{F_0} \mathcal{D}$, C F is a left Kan extension of F_0 along ι if for all $C \in \mathcal{C}$, the induced diagram $C^0_{/C} \xrightarrow{F_C} \mathcal{D}$ exhibits FC as a colimit of $(C^0_{/C})^{\triangleright}$	In the 1-categorical case, the colimits of $C_{/C}^0 \rightarrow C^0 \xrightarrow{F_0} \mathcal{D}$ for each $C \in \mathcal{C}$ (if they all exist) define the left Kan extension of F_0 along ι ([Rie16], Thm 6.2.1) ³ . This is the case if and only if F is a pointwise Kan extension ([Rie16], Thm 6.3.7), so really the higher categorical version generalises pointwise left Kan extensions (along the inclusion of a full subcategory).
Limit	A limit for $F: J \to \mathcal{C}$ is a terminal cone on F .	F_C . ([Lur09], Def 4.3.2.2) A limit for $F: X \to \mathcal{C}$ (X a simplicial set, \mathcal{C} an ∞ -category) is a final object of $\mathcal{C}_{/F}$. ([Lur09], Def 1.2.13.4)	The obvious extension of the definition of the over- category $\mathcal{C}_{/C}$ for $C: \{*\} \to \mathcal{C}$ to $\mathcal{C}_{/F}$ for an arbitrary functor $F: J \to \mathcal{C}$ ends up being exactly $\mathbf{Cone}(F)$.
Locally small category	For every $X, Y \in \mathcal{C}$, $\text{Hom}(X, Y)$ is a set.	For every $X, Y \in \mathcal{C}$, the space $\operatorname{Hom}(X, Y)$ is essentially small. ([Lur09], Prop 5.4.1.7)	-

²The opposite convention (i.e. that this defines a *right* dualisable object) is taken in [Eti+15], and by many other authors. I'm attempting to follow Lurie's conventions as far as possible. ³I think Lurie is saying exactly the same thing in the ∞-categorical case, just in a slightly confusing way.

Monoidal category	\parallel Opfibration of categories $p:\mathcal{C}^{\otimes} \to \Delta^{\mathrm{op}}$ such
1.101101441 0400801,	that for each n > 0 the associated fund
	$\operatorname{tors} \mathcal{C}_{\mathbb{R}_{+}}^{\mathbb{R}_{+}} \to \mathcal{C}_{\mathbb{R}_{+}+1}^{\mathbb{R}_{+}}$ determine an equivalence
	that, for each $n \geq 0$, the associated functions $\mathcal{C}_{[n]}^{\otimes} \to \mathcal{C}_{\{i,i+1\}}^{\otimes}$ determine an equivalence of categories $\mathcal{C}_{[n]}^{\otimes} \to \mathcal{C}_{\{0,1\}}^{\otimes} \times \times \mathcal{C}_{\{n-1,n\}}^{\otimes} \simeq (\mathcal{C}_{[1]}^{\otimes})^n$.
	$\left\ (\mathcal{C}_{[1]}^{\otimes})^n . \right\ $

Cocartesian fibration of simplicial sets $p: \mathcal{C}^{\otimes} \to N(\Delta)^{\mathrm{op}}$ such that, for each $n \geq 0$, the associated functors $\mathcal{C}^{\otimes}_{[n]} \to \mathcal{C}^{\otimes}_{\{i,i+1\}}$ determine an equivalence⁴ of ∞ -categories

$$\mathcal{C}_{[n]}^{\otimes} \to \mathcal{C}_{\{0,1\}}^{\otimes} \times \dots \times \mathcal{C}_{\{n-1,n\}}^{\otimes} \simeq (\mathcal{C}_{[1]}^{\otimes})^n$$
([Lur07], Def 1.1.2)⁵

In the 1-categorical case, you recover the original category by setting $\mathcal{C} := \mathcal{C}_{[1]}^{\otimes}$. The unit is $\mathcal{C}_{[0]}^{\otimes}$, the tensor product \otimes is induced by the outer inclusion $\{0 < 2\} = [1] \subseteq [2]$, the unitors and associators come from the commutative diagrams

 $\downarrow \qquad \qquad \downarrow \qquad \qquad \downarrow \qquad \qquad \downarrow \qquad \qquad \qquad \{0,2,3\} \longrightarrow \{0,1,2,3\}$

in Δ . Conversely, given a monoidal category \mathcal{C} , define \mathcal{C}^{\otimes} to have objects finite sequences $[C_1,...,C_n]$ of objects of \mathcal{C} , and a morphism $[C_1,...,C_n] \to [C'_1,...,C'_m]$ to be a map $[m] \to [n]$ and a collection of morphisms $C_{f(i-1)+1} \otimes \cdots \otimes C_{f(i)} \to C'_i$ for $1 \leq i \leq m$. Then the forgetful functor to Δ^{op} is the required cocartesian fibration. ([Lur07], p5-6)

⁴Weak or categorical?

⁵I consider this a temporary definition, because I can't yet reconcile it with [Lur17], Def 4.1.1.10.

⁶You "tensor along the gap", if that makes any sense.

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	• For every finite set I , every I -indexed collection of objects $\{X_i\}_{i\in I}$ of \mathcal{O} , and every $Y\in\mathcal{O}$,	categories which satisfies the following conditions:	
	a set $\operatorname{Hom}(\{X_i\}_{i\in I}, Y)$.	• For every inert morphism $f: \langle m \rangle \to \langle n \rangle$ in	
	• For every map of finite sets $I \to J$ having fibres $\{I_j\}_{j\in J}$, every finite collection of	$N(\mathbf{Fin}_*)$ and every object $C \in \mathcal{O}_{\langle m \rangle}^{\otimes}$, there exists a p -cocartesian morphism $\overline{f}: C \to C'$	
	objects $\{X_i\}_{i\in I}$, every finite collection of ob-	$\int \operatorname{in} \mathcal{O}^{\otimes} \operatorname{lifting} f.$	
	jects $\{Y_j\}_{j\in J}$, and every object $Z\in\mathcal{O}$, a composition map $\prod_j \operatorname{Hom}(\{X_i\}_{i\in I_j}, Y_j)$ ×	• Let $C \in \mathcal{O}_{\langle m \rangle}^{\otimes}$ and $C' \in \mathcal{O}_{\langle n \rangle}^{\otimes}$ be objects, let	
	$\ \operatorname{Hom}(\{Y_j\}_j, Z) \to \operatorname{Hom}(\{X_i\}_i, Z), \text{ which is as-} \ $	$f: \langle m \rangle \to \langle n \rangle$ be a morphism in \mathbf{Fin}_* , and let $\mathrm{Hom}_{\mathcal{O}\otimes}^f(C,C')$ be the union of those con-	
	sociative. • Units $id_X \in Hom(\{X\}, X)$. ([Lur17], Def	nected components of $\operatorname{Hom}_{\mathcal{O}^{\otimes}}(C,C')$ which	
	2.1.1.1)	lie over $f \in \operatorname{Hom}_{\mathbf{Fin}_*}(\langle m \rangle, \langle n \rangle)$. Choose p -cocartesian morphisms $C' \to C'_i$ lying over the	
		inert morphisms $\rho^i : \langle n \rangle \to \langle 1 \rangle$ for $1 \le i \le n$. Then the induced map	
		$\operatorname{Hom}_{\mathcal{O}^{\otimes}}^{f}(C,C') \to \prod_{1 \le i \le n} \operatorname{Hom}_{\mathcal{O}^{\otimes}}^{\rho^{i}f}(C,C'_{i})$	
		is a homotopy equivalence.	
		• For every finite collection of objects $C_1,, C_n \in \mathcal{O}_{\langle 1 \rangle}^{\otimes}$, there exists an object $C \in$	
		$\mathcal{O}_{\langle n \rangle}^{\otimes}$ and a collection of <i>p</i> -cocartesian mor-	
(C 1 1) 1	[t 1]	phisms $C \to C_i$ covering $\rho^i : \langle n \rangle \to \langle 1 \rangle$.	[, 1]
(Coloured) operad map	[todo]	[todo]	[todo]
Opposite category	C^{op} has the same objects as C , and	$C_n^{\text{op}} = \mathcal{C}([n]^{\text{op}}), \text{ where } \{0 < 1 < \dots < n\}^{\text{op}} = \{0 > 1 > \dots > n\}. ([Lur09], 1.2.1)$	A map $x \to y$ is an edge $\Delta^1 \to \mathcal{C}$ where $0 \mapsto x$ and
	$\operatorname{Hom}_{\mathcal{C}^{\operatorname{op}}}(X,Y) = \operatorname{Hom}_{\mathcal{C}}(Y,X).$		$1 \mapsto y$. In \mathcal{C}^{op} 0 and 1 swap roles, so we instead get a map $y \to x$.
Overcategory	For $C \in \mathcal{C}$, the category $\mathcal{C}_{/C}$ satisfies the following universal property: for any category \mathcal{D} ,	For $f: S \to \mathcal{C}$, S a simplicial set and \mathcal{C} an ∞ -category, the ∞ -category $\mathcal{C}_{/f}$ satisfies the	If $S = \Delta^0$, writing $C \in \mathcal{C}$ for the object picked out by f , we have $(\mathcal{C}_{/C})_n = \operatorname{Hom}_C(\Delta^n \star \Delta^0, \mathcal{C}) \cong$
	there is a bijection	following universal property: for any simpli-	$ \operatorname{Hom}_C(\Delta^{n+1},\mathcal{C}) $ (where the subscript indicates that
	$\operatorname{Hom}(\mathcal{D}, \mathcal{C}_{/C}) \simeq \operatorname{Hom}_{C}(\mathcal{D} \star [0], \mathcal{C}),$	cial set X , there is a bijection	we only consider morphisms sending the $(n + 1)$ st vertex to C). In other words, the objects are maps
	,	$\operatorname{Hom}(X, \mathcal{C}_{/f}) \simeq \operatorname{Hom}_f(X \star S, \mathcal{C}),$	to C , the morphisms are commuting triangles over
	where the subscript on the right indicates that we consider only those functors $\mathcal{D} \star [0] \rightarrow$	where the subscript on the right indicates that	C, and so on; these are exactly the objects and morphisms in the 1-categorical case.
	\mathcal{C} whose restriction to [0] consides with C . ([Lur09], 1.2.9)	we consider only those functors $X \star S \to \mathcal{C}$ whose restriction to S consides with f . Ex-	1
	([Euros], 1.2.5)	plicitly,	
		$(\mathcal{C}_{/f})_n := \operatorname{Hom}_f(\Delta^n \star S, \mathcal{C}).$	
		([Lur09], Prop 1.2.9.2)	
Presentable category	[todo]	[todo]	[todo]
Presheaf	[todo]	[todo]	[todo]

Representable func-	[todo]	[todo]	[todo]
tor			
Right cone	$\mathcal{C}^{\triangleright} := \mathcal{C} \star [0].$	$\mathcal{C}^{\triangleright} := \mathcal{C} \star \Delta^{0}$. ([Lur09], Not 1.2.8.4)	\mathcal{C} with extra vertex (cone point) added, as well as a
			map from every other vertex in \mathcal{C} to that cone point
			(plus obligatory degenerate simplicies).
Right dualisable ob-	Object $C \in \mathcal{C}$ such that there exists some ${}^*C \in$	Object $C \in \mathcal{C}$ such that there exists some ${}^*C \in$	C has a right dual up to homotopy.
ject	\mathcal{C} and maps $e: {}^*C \otimes C \to \mathbb{1}, c: \mathbb{1} \to C \otimes {}^*C$	\mathcal{C} and maps $e: {}^*C \otimes C \to \mathbb{1}, c: \mathbb{1} \to C \otimes {}^*C$	
	such that the composites $C \to C \otimes^* C \otimes C \to C$	such that the composites $C \to C \otimes^* C \otimes C \to C$	
	and $^*C \to ^*C \otimes C \otimes ^*C \to ^*C$ are equal to the	and $^*C \to ^*C \otimes C \otimes ^*C \to ^*C$ are homotopic	
	identity.	to the identity.	
Small category	\mathcal{C} has a set's worth of objects, and between	\mathcal{C} has a set's worth of nondegenerate simpli-	_
Sman category	any two objects, a set's worth of morphisms.	cies. ⁷ [Lur25, Section 03PP]	_
Subcategory	Subcategory $C' \subseteq C$.	Subsimplicial set $C' \subseteq C$ arising as a pullback	Expected definition: Subsimplicial set $\mathcal{C}' \subseteq \mathcal{C}$ satisfy-
Subcategory	Subcategory $C \subseteq C$.		ing inner horn filling. These are actually equivalent:
		$C' \longrightarrow C$	
		where $(h\mathcal{C})' \subseteq h\mathcal{C}$ is a	suppose we have a such a subsimplicial set. Then
		· · · · · · · · · · · · · · · · · · ·	$\begin{array}{cccccccccccccccccccccccccccccccccccc$
		$N(\mathrm{h}\mathcal{C})' \longrightarrow N(\mathrm{h}\mathcal{C})$	is a pullback. Conversely,
		subcategory. ([Lur09], $1.2.11$)	· •
			` '
			left. Then the diagram
			$\Delta^n \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \$
			$C' \longrightarrow C$
			\ <u>'</u> \ \ \
			7 +
			$N(hC)' \longrightarrow N(hC)$
			shows that C' also satisfies inner horn filling, where
			the outer two dotted maps come from horn filling for
			\mathcal{C} and $N(h\mathcal{C})'$, and the inner dotted map comes from
			the fact that the square is a pullback.
		subcategory. ([Lur09], 1.2.11)	$N(h\mathcal{C}') \longrightarrow N(h\mathcal{C})$ suppose we're given a pullback as in the boleft. Then the diagram $ \Delta^n \longleftrightarrow \Lambda^n_i \\ \mathcal{C}' \longrightarrow \mathcal{C} \\ N(h\mathcal{C})' \longrightarrow N(h\mathcal{C}) $ shows that \mathcal{C}' also satisfies inner horn filling the outer two dotted maps come from horn fi \mathcal{C} and $N(h\mathcal{C})'$, and the inner dotted map com

⁷In other words, \mathcal{C} is a simplicial set. So, by our definition, "∞-category" means the same as "small ∞-category"

Symmetric monoidal category	Opfibration of categories $p: \mathcal{C}^{\otimes} \to \mathbf{Fin}_*$ such that, for each $n \geq 0$, the associated functors $\mathcal{C}^{\otimes}_{\langle n \rangle} \to \mathcal{C}^{\otimes}_{\{*,i\}}$ determine an equivalence of categories $\mathcal{C}^{\otimes}_{\langle n \rangle} \to \mathcal{C}^{\otimes}_{\{*,1\}} \times \times \mathcal{C}^{\otimes}_{\{*,n\}} \simeq (\mathcal{C}^{\otimes}_{\langle 1 \rangle})^n$.	Cocartesian fibration of simplicial sets $p: \mathcal{C}^{\otimes} \to N(\mathbf{Fin}_*)$ such that, for each $n \geq 0$, the associated functors $\mathcal{C}^{\otimes}_{\langle 1 \rangle} \to \mathcal{C}^{\otimes}_{\{*,i\}}$ determine an equivalence ⁸ of ∞ -categories $\mathcal{C}^{\otimes}_{\langle n \rangle} \to \mathcal{C}^{\otimes}_{\{*,1\}} \times \ldots \times \mathcal{C}^{\otimes}_{\{*,n\}} \simeq (\mathcal{C}^{\otimes}_{\langle 1 \rangle})^n$	In the 1-categorical case, you recover the original category by setting $\mathcal{C} := \mathcal{C}_{\langle 1 \rangle}^{\otimes}$. The unit is $\mathcal{C}_{\langle 0 \rangle}^{\otimes}$, the tensor product \otimes is induced by $\langle 2 \rangle \to \langle 1 \rangle$, the symmetric isomorphism comes from the swapping map $\langle 2 \rangle \to \langle 2 \rangle$, the unitors and associators come from similar commutative diagrams to those in the intu-
		([Lur17], Def 2.0.0.7)	ition for a monoidal ∞ -category. Conversely, given a symmetric monoidal category \mathcal{C} , define \mathcal{C}^{\otimes} to have objects finite sequences $[C_1,,C_n]$ of objects of \mathcal{C} , and a morphism $[C_1,,C_n] \to [C'_1,,C'_m]$ to be a map $\langle n \rangle \to \langle m \rangle$ and a collection of morphisms $\bigotimes_{i \in \alpha^{-1}\{j\}} C_i \to C'_j$ for $1 \leq j \leq m$. Then the forgetful functor to \mathbf{Fin}_* is the required cocartesian fibration. ([Lur17], p165-168)
Symmetric monoidal functor	[todo]	Let $p: \mathcal{C}^{\otimes} \to N(\mathbf{Fin}_*), q: \mathcal{D} \to N(\mathbf{Fin}_*)$ be symmetric monoidal ∞ -categories. Then an ∞ -operad map $f \in \mathbf{Alg}_{\mathcal{C}}(\mathcal{D})$ is a symmetric monoidal functor if it carries p -cocartesian morphisms to q -cocartesian morphisms. ([Lur17], Def 2.1.3.7)	[todo]
Topos	[todo]	[todo]	[todo]
Undercategory	For $C \in \mathcal{C}$, the category $\mathcal{C}_{C/}$ satisfies the following universal property: for any category \mathcal{D} , there is a bijection $\operatorname{Hom}(\mathcal{D},\mathcal{C}_{C/}) \simeq \operatorname{Hom}_{C}([0] \star \mathcal{D},\mathcal{C}),$ where the subscript on the right indicates that we consider only those functors $[0] \star \mathcal{D} \to \mathcal{C}$ whose restriction to $[0]$ consides with C . ([Lur09], 1.2.9)	For $f: S \to \mathcal{C}$, S a simplicial set and \mathcal{C} an ∞ -category, the ∞ -category $\mathcal{C}_{f/}$ satisfies the following universal property: for any simplicial set X , there is a bijection $\operatorname{Hom}(X,\mathcal{C}_{f/}) \simeq \operatorname{Hom}_f(S \star X,\mathcal{C}),$ where the subscript on the right indicates that we consider only those functors $S \star X \to \mathcal{C}$ whose restriction to S consides with f . Explicitly,	If $S = \Delta^0$, writing $C \in \mathcal{C}$ for the object picked out by f , we have $(\mathcal{C}_{C/})_n = \operatorname{Hom}_C(\Delta^0 \star \Delta^n, \mathcal{C}) \cong \operatorname{Hom}_C(\Delta^{n+1}, \mathcal{C})$ (where the subscript indicates that we only consider morphisms sending the 0th vertex to C). In other words, the objects are maps from C , the morphisms are commuting triangles under C , and so on; these are exactly the objects and morphisms in the 1-categorical case.
		$(\mathcal{C}_{f/})_n := \operatorname{Hom}_f(S \star \Delta^n, \mathcal{C}).$	
		([Lur09], Prop 1.2.9.2)	

	Equivalences			
Name	Between	Definition		
Strong equivalence	Topological categories \mathcal{C}, \mathcal{D}	$\mathcal{C} \to \mathcal{D}$ is an equivalnce in the sense of enriched		
		category theory. ([Lur09], Def 1.1.3.1)		
(Weak) equivalence	Topological categories \mathcal{C}, \mathcal{D}	The induced functor $h\mathcal{C} \to h\mathcal{D}$ is an equiva-		
		lence of \mathcal{H} -enriched categories. ([Lur09], Def		
		1.1.3.6)		
Categorical equivalence	Simplicial sets X, S	The induced functor $hX \to hS$ is an equiva-		
		lence of \mathcal{H} -enriched categories. ([Lur09], Def		
		1.1.5.14)		
Weak (homotopy) equivalence	Simplicial sets X, S	The induced map $ X \rightarrow S $ is a weak		
		homotopy equivalence of topological spaces.		
		([Lur09], 1.1.4)		
Equivalence	Simplicial categories \mathcal{C}, \mathcal{D}	The induced functor $h\mathcal{C} \to h\mathcal{D}$ is an equiva-		
		lence of \mathcal{H} -enriched categories. ([Lur09], Def		
		1.1.4.4)		

Fibrations and anodyne morphisms			
Name	Describes	Definition	
Acyclic Kan fibration	$f: X \to S$ map of simplicial sets	see: trivial Kan fibration. ([nLa23])	
Anodyne	$f: X \to S$ map of simplicial sets	For every solid arrow diagram as below, with	
		$p: Y \to T$ a Kan fibration,	
		$X \longrightarrow Y$	
		$ \begin{array}{ccc} f \downarrow & \downarrow p \\ S & \longrightarrow T \end{array} $	
		$S \xrightarrow{\sim} T$	
		1 1 11 11 (T 00) F 2001)	
Cartesian fibration	$f: X \to S$ map of simplicial sets	there exists a dotted lift. ([Lur09], Ex 2.0.0.1) f is an inner fibration such that for every edge	
Cartesian infration	$J: X \to S$ map of simplicial sets	$g: x \to y$ of S and every vertex \tilde{y} of X with	
		$f(\tilde{y}) = y$, there exists an f-cartesian edge \tilde{g} :	
		$\tilde{x} \to \tilde{y}$ with $f(\tilde{g}) = g$. ([Lur09], Def 2.4.2.1)	
Categorical fibration	$f: X \to S$ map of simplicial sets	For every solid arrow diagram as below, with	
		$p: Y \to T$ both a cofibration and a categorical	
		equivalence,	
		$V \longrightarrow V$	
		$Y \longrightarrow X$ $\downarrow f$ $T \longrightarrow S$	
		$T \longrightarrow S$	
		there exists a dotted lift. ([Lur09], p90)	
Cocartesian fibration	$f: X \to S$ map of simplicial sets	f is an inner fibration such that for every edge	
		$g: x \to y$ of S and every vertex \tilde{x} of X with	
		$f(\tilde{x}) = x$, there exists an f-cocartesian edge	
Cofibration	$f: X \to S$ map of simplicial sets	$\tilde{g}: \tilde{x} \to \tilde{y} \text{ with } f(\tilde{g}) = g. \text{ ([Lur09], Def 2.4.2.1)}$ $f \text{ is a monomorphism. ([Lur09], A.2.7)}$	
Inner anodyne	$f: X \to S$ map of simplicial sets	For every solid arrow diagram as below, with	
inner anoughe	y 111 / S map of simplicial sees	$p: Y \to T$ an inner fibration,	
		,	
		$X \longrightarrow Y$	
		$f \downarrow p$	
		$S \xrightarrow{\checkmark} T$	
		there exists a dotted lift. ([Lur09], Def 2.0.0.3)	

$ \begin{array}{c ccccccccccccccccccccccccccccccccccc$	Inner fibration	$f: X \to S$ map of simplicial sets	For every solid arrow diagram as below, with $0 < i < n$,
$F: \mathcal{C} \to \mathcal{D} \text{ map of } \infty\text{-categories} \qquad F \text{ is an inner fibration such that for all } \mathcal{C} \in \mathcal{C} \text{ and every isomorphism } u: \mathcal{D} \to \mathcal{F}\mathcal{C} \text{ in } \mathcal{D} \cap \mathcal{C} \cap \cap C$			
$F: \mathcal{C} \to \mathcal{D} \text{ map of } \infty\text{-categories} \qquad F \text{ is an inner fibration such that for all } \mathcal{C} \in \mathcal{C} \text{ and every isomorphism } u: \mathcal{D} \to \mathcal{F}\mathcal{C} \text{ in } \mathcal{D} \cap \mathcal{C} \cap \cap C$			there exists a dotted lift
	Isofibration	$F:\mathcal{C}\to\mathcal{D}$ map of ∞ -categories	F is an inner fibration such that for all $C \in \mathcal{C}$ and every isomorphism $u:D \to FC$ in \mathcal{D} (i.e. $[u]$ is an isomorphism in $h\mathcal{D}$) there exists an isomorphism $\overline{u}:\overline{D}\to C$ in \mathcal{C} such that
Left anodyne $f: X \to S \text{ map of simplicial sets} \\ f: X \to S map o$	(Kan) fibration	$f: X \to S$ map of simplicial sets	For every solid arrow diagram as below, with $0 \le i \le n$,
Left anodyne $f: X \to S \text{ map of simplicial sets} \qquad \text{For every solid arrow diagram as below, with } \\ p: Y \to T \text{ a left fibration}, \\ X \longrightarrow Y \\ \downarrow \downarrow \\ S \longrightarrow T \\ \text{there exists a dotted lift. ([Lur09], Def 2.0.0.3)} \\ \text{For every solid arrow diagram as below, with } \\ 0 \le i < n, \\ A_i^n \longrightarrow X \\ \downarrow A_n^n \longrightarrow S \\ \text{there exists a dotted lift. ([Lur09], Def 2.0.0.3)} \\ \text{For every solid arrow diagram as below, with } \\ p: Y \to T \text{ a right fibration}, \\ X \longrightarrow Y \\ \downarrow \downarrow I \\ S \longrightarrow T \\ \text{there exists a dotted lift. ([Lur09], Def 2.0.0.3)} \\ \text{For every solid arrow diagram as below, with } \\ p: Y \to T \text{ a right fibration}, \\ X \longrightarrow Y \\ \downarrow I \\ S \longrightarrow T \\ \text{there exists a dotted lift. ([Lur09], Def 2.0.0.3)} \\ \text{Right fibration} \\ \text{For every solid arrow diagram as below, with } \\ 0 < i \le n, \\ A_i^n \longrightarrow X \\ \downarrow I \\ A_i^n \longrightarrow X \\ \downarrow I \\ A_i^n \longrightarrow S \\ Anomaly of the properties of the p$			$ \begin{array}{ccc} \Lambda_i^n & \longrightarrow X \\ \downarrow & & \downarrow f \\ \Delta^n & \longrightarrow S \end{array} $
Left anodyne $f: X \to S \text{ map of simplicial sets} \qquad \text{For every solid arrow diagram as below, with } \\ p: Y \to T \text{ a left fibration}, \\ X \longrightarrow Y \\ \downarrow \downarrow \\ S \longrightarrow T \\ \text{there exists a dotted lift. ([Lur09], Def 2.0.0.3)} \\ \text{For every solid arrow diagram as below, with } \\ 0 \le i < n, \\ A_i^n \longrightarrow X \\ \downarrow A_n^n \longrightarrow S \\ \text{there exists a dotted lift. ([Lur09], Def 2.0.0.3)} \\ \text{For every solid arrow diagram as below, with } \\ p: Y \to T \text{ a right fibration}, \\ X \longrightarrow Y \\ \downarrow \downarrow I \\ S \longrightarrow T \\ \text{there exists a dotted lift. ([Lur09], Def 2.0.0.3)} \\ \text{For every solid arrow diagram as below, with } \\ p: Y \to T \text{ a right fibration}, \\ X \longrightarrow Y \\ \downarrow I \\ S \longrightarrow T \\ \text{there exists a dotted lift. ([Lur09], Def 2.0.0.3)} \\ \text{Right fibration} \\ \text{For every solid arrow diagram as below, with } \\ 0 < i \le n, \\ A_i^n \longrightarrow X \\ \downarrow I \\ A_i^n \longrightarrow X \\ \downarrow I \\ A_i^n \longrightarrow S \\ Anomaly of the properties of the p$			there exists a dotted lift. ([Lur09], A.2.7)
Left fibration $f: X \to S \text{ map of simplicial sets} \qquad \text{For every solid arrow diagram as below, with } 0 \leq i < n, \\ $	Left anodyne	$f: X \to S$ map of simplicial sets	For every solid arrow diagram as below, with $p: Y \to T$ a left fibration,
Left fibration $f: X \to S \text{ map of simplicial sets} \qquad \text{For every solid arrow diagram as below, with } 0 \le i < n, \\ & & & & & & \\ & & & & & \\ & & & & & $			$X \longrightarrow Y$ $f \downarrow \qquad \qquad \downarrow p$ $S \longrightarrow T$
Left fibration $f: X \to S \text{ map of simplicial sets} \qquad \text{For every solid arrow diagram as below, with } 0 \le i < n, \\ & & & & & & \\ & & & & & \\ & & & & & $			there exists a dotted lift. ([Lur09], Def 2.0.0.3)
$\begin{array}{cccccccccccccccccccccccccccccccccccc$	Left fibration	$f: X \to S$ map of simplicial sets	$0 \le i < n,$
Right anodyne			$\begin{array}{ccc} & & \downarrow^f \\ \Delta^n & \longrightarrow & S \end{array}$
$p:Y\to T \text{ a right fibration},$ $X \longrightarrow Y \\ f \downarrow \qquad \qquad \downarrow p$ $S \longrightarrow T$ there exists a dotted lift. ([Lur09], Def 2.0.0.3) $f:X\to S \text{ map of simplicial sets}$ For every solid arrow diagram as below, with $0< i \leq n,$ $\Lambda^n_i \longrightarrow X \\ \downarrow f \\ \Delta^n \longrightarrow S$			
$f \mapsto f \mapsto f$ there exists a dotted lift. ([Lur09], Def 2.0.0.3) $f: X \to S \text{ map of simplicial sets} \qquad For every solid arrow diagram as below, with } 0 < i \le n,$ $A_i^n \longrightarrow X \\ \downarrow f \\ \Delta^n \longrightarrow S$	Right anodyne	$f: X \to S$ map of simplicial sets	, ,
Right fibration $f: X \to S \text{ map of simplicial sets} \qquad \text{For every solid arrow diagram as below, with} \\ 0 < i \le n, \\ & & & & & \\ & & & & \\ & & & & \\ & & & & \\ & & & & \\ & & & & \\ & & & & \\ & & & & \\ & & & & \\ & & & & \\ &$			$ \begin{array}{ccc} X & \longrightarrow Y \\ \downarrow f & & \downarrow p \\ S & \longrightarrow T \end{array} $
Right fibration $f: X \to S \text{ map of simplicial sets} \qquad \text{For every solid arrow diagram as below, with} \\ 0 < i \le n, \\ & & & & & \\ & & & & \\ & & & & \\ & & & & \\ & & & & \\ & & & & \\ & & & & \\ & & & & \\ & & & & \\ & & & & \\ &$			there exists a dotted lift. ([Lur09], Def 2.0.0.3)
	Right fibration	$f: X \to S$ map of simplicial sets	For every solid arrow diagram as below, with $0 < i \le n$,
there exists a dotted lift. ([Lur09], Def 2.0.0.3)			$ \begin{array}{ccc} \Lambda_i^n & \longrightarrow X \\ \downarrow f \\ \Delta^n & \longrightarrow S \end{array} $
			there exists a dotted lift. ([Lur09], Def 2.0.0.3)

Serre fibration	$f: Y \to Z$ map of topological	For every solid arrow diagram as below,
	spaces	
		$\{0\} \times \Delta^n \longrightarrow Y$
		$\{0\} \times \Delta^n Y$ $\downarrow \qquad \qquad \downarrow f$
		$[0,1] \times \Delta^n \longrightarrow Z$
		there exists a dotted lift. [Lur25, Def 021R]
Trivial (Kan) fibration	$f: X \to S$ map of simplicial sets	For every solid arrow diagram as below,
		0 A n W
		$\partial \Delta^n \longrightarrow X$
		$\downarrow \qquad \downarrow f$
		$\Delta^n S$
		there exists a dotted lift. ([Lur25, Def
		006W]/[Lur09], Def 2.0.0.2)

Nerves		
Name	Domain object	Definition
Nerve	Category \mathcal{C}	$(NC)_n = \{n\text{-composable strings of morphisms in } C\}.$ ([Lur09], p9)
Simplicial nerve	Simplicial category $\mathcal C$	$(NC)_n = \operatorname{Hom}_{\mathbf{Cat}_{\Delta}}(\mathfrak{C}[\Delta^n], \mathcal{C}), \text{ where } \mathfrak{C}[\Delta^n] \text{ is }$ the category whose objects are the same as $[n]$, and $\operatorname{Hom}_{\mathfrak{C}[\Delta^n]}(i,j) = \emptyset$ for $i < j$ and $N(P_{ij})$ for $i \geq j$ (where $P_{ij} = \{I \subseteq [n] : (i,j \in I) \land (\forall k \in I, i \leq k \leq j)\}$). ([Lur09], Def 1.1.5.5)
Topological nerve	Topological category \mathcal{C}	The simplicial nerve of Sing \mathcal{C} . ([Lur09], Def 1.1.5.5)

Homotopy categories		
Domain object	Definition	
∞ -Category \mathcal{C}	The objects of hC are the vertices of C , and	
	$\operatorname{Hom}_{\mathrm{h}\mathcal{C}}(X,Y)$ is the set of homotopy classes of edges	
	$X \to Y \text{ in } \mathcal{C}. \ ([\text{Lur09}], \text{Prop } 1.2.3.9)$	
Simplicial category \mathcal{C}	h C . ([Lur09], 1.1.4)	
Topological category \mathcal{C}	$h\mathcal{C}$ has the same objects as \mathcal{C} , and $\operatorname{Hom}_{h\mathcal{C}}(X,Y) = 0$	
	$[Hom_{\mathcal{C}}(X,Y)].$ ([Lur09], 1.1.3)	

Objects		
Name	Definition	
$\mathbf{Alg}_{\mathcal{C}}(\mathcal{D})$ $(p : \mathcal{C}^{\otimes} \rightarrow \mathcal{O}^{\otimes}, q : \mathcal{C}^{\otimes})$	[todo]	
$\mathcal{D}^{\otimes} \to \mathcal{O}^{\otimes} \infty$ -operads)		
Assoc (the associative operad)	The coloured operad with a single object \mathfrak{a} , and for every finite set I , $\operatorname{Hom}(\{\mathfrak{a}\}_i,\mathfrak{a})$ is the set of linear orderings on I . Given a map of finite sets $\alpha:I\to J$ together with operations $\phi_j\in \operatorname{Hom}(\{\mathfrak{a}\}_{\alpha(i)=j},\mathfrak{a})$ and $\psi\in \operatorname{Hom}(\{\mathfrak{a}_j,\mathfrak{a}\})$, we identify each ϕ_j with a linear ordering \leq_j on the set $\alpha^{-1}\{j\}$ and ψ with a linear ordering \leq' on the set J . The composition of ψ with $\{\phi_j\}$ corresponds to the linear ordering \leq on the set I which is defined by: $i\leq i'$ if either $\alpha(i)<_j\alpha(i')$ or $\alpha(i)=j=\alpha(i')$ and $i\leq_j i'$. ([Lur17], Def 4.1.1.1)	

Assoc $^{\otimes}$ (the associative ∞ -	$N(\mathbf{Assoc}^{\otimes})$. ([Lur17], Def 4.1.1.3)
operad)	
\mathbf{Assoc}^{\otimes}	The category whose objects are the objects of \mathbf{Fin}_* , and a mor-
	phism $m \to n$ is given by a map $\alpha : \langle m \rangle \to \langle n \rangle$ in \mathbf{Fin}_* together
	with a collection of linear orderings \leq_j on $\alpha^{-1}\{j\}$, for $1 \leq j \leq n$.
	Composition of morphisms is determined by the composition laws
	on \mathbf{Fin}_* and on \mathbf{Assoc} . [Lur17], Def 4.1.1.3
\mathbf{Fin}_*	The category whose objects are the sets $\langle n \rangle = \{*, 1, 2,, n\}$, and a
	morphism $\langle m \rangle \to \langle n \rangle$ is a map $\alpha : \langle m \rangle \to \langle n \rangle$ such that $\alpha(*) = *$.
Kan	The full subcategory of sSet spanned by the collection of small
	Kan complexes. ([Lur09], Def 1.2.16.1)
KAN	The category of all Kan complexes. ([Lur09], Rem 5.1.6.1)
\mathcal{S} (the ∞ -category of spaces)	The simplicial 9 nerve $N(\mathbf{Kan})$. ([Lur09], Def 1.2.16.1)
$\widehat{\mathcal{S}}$	The simplicial nerve $N(KAN)$. ([Lur09], Rem 5.1.6.1)

 $^{{}^9\}mathbf{sSet}$ is a simplicial category, with $\mathrm{Hom}(X,S)_n = \mathrm{Hom}_{\mathbf{sSet}}(\Delta^n \times X,S)$. The subcategory **Kan** inherits this structure.

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