Stable Homotopy Groups of Spheres [DRAFT]

Contents

1 Introduction

- Define homotopy groups
- The Eilenberg-MacLane space is $K(G,n)$, and it has the property that

$$\pi_i(K(G,n)) = \begin{cases} \mathbb{Z} & i = n, \\ 0 & i \neq n. \end{cases}$$

  They're unique up to weak homotopy equivalence (i.e. if you have another one X, there's a map between them which descends to an isomorphism on homotopy groups)
- Define suspension of a topological space
- Freudenthal's suspension theorem: if $\pi_i(X) = 0$ for $i \leq k$ (i.e. X is k-connected) then the map

$$\begin{aligned} \pi_n(X) &\to \pi_{n+1}(\Sigma X) \\ [\gamma : S^n \to X] &\mapsto [\Sigma\gamma : \Sigma S^n = S^{n+1} \to \Sigma X] \end{aligned}$$

  is an isomorphism for $n \leq 2k$ and surjective for $n = 2k+1$
- This implies $\pi_{n+k}(S^n)$ depends only on k for $n \geq k+2$
- (Obviously be careful with basepoints above)
- Suppose X is k-connected. Then, for $k \geq 0$, $0 = \pi_k(X) \cong \pi_{k+1}(\Sigma X)$, so whenever a space is k-connected its suspension is $k+1$-connected.
- As you take suspensions, then, your successive bounds are $n \leq 2k$, $n+1 \leq 2k+2 \implies n \leq 2k+1$, $n \leq 2k+2$, etc ... so the sequence $\pi_n(X) \to \pi_{n+1}(\Sigma X) \to \cdots$ will eventually stabilise.
- Thus, if you take the colimit of that direct system, it'll just equal the stable value, with the higher legs just being the inverse isomorphisms.
- [1], Cor 1.9 [not 100% convinced of how this follows, but believing it for now]: if X is a CW complex of dimension d and Y a $(k-1)$-connected space, then the suspension homomorphism $[X,Y] \to [\Sigma X, \Sigma Y]$ is bijective if $d < 2k-1$ and surjective if $d = 2k-1$.

Miscellaneous facts I might need later:

- Cohomology [possibly only of pointed CW complexes] is representable, and its representing object is the Eilenberg-MacLane space. i.e. $H^n(-;G) \cong \mathrm{Hom}(-, K(G,n))$.
- There is an adjunction $\Sigma \dashv \Omega$, where Ω is the loop functor.
- $\mathscr{A}_2$ is generated as an algebra by elements Sq^{2^k} ([2], Prop 4L.8).
- The map $\mathscr{A}_2 \to \tilde{H}^*(K(\mathbb{Z}/2\mathbb{Z},n);\mathbb{Z}/2\mathbb{Z})$, $Sq^I \mapsto Sq^I(\iota_n)$ is an isomorphism from the degree d part of $\mathscr{A}_2$ onto $H^{n+d}(K(\mathbb{Z}/2\mathbb{Z},n);\mathbb{Z}/2\mathbb{Z})$ for $d \geq n$. In particular, the admissible monomials Sq^I form an additive basis for $\mathscr{A}_2$. Thus, $\mathscr{A}_2$ is exactly the algebra of all $\mathbb{Z}/2\mathbb{Z}$ cohomology operations that are stable, commuting with suspension ([3], Cor 5.38).

Algebraic background:

Definition 1.0.1. Let M, N be modules over a ring R. A *free resolution* F of M is an exact sequence

$$\cdots \to F_2 \to F_1 \to F_0 \to M \to 0,$$

with each F_i a free R-module.

Applying $\mathrm{Hom}_R(-, N)$ gives us a chain complex

$$\cdots \leftarrow \mathrm{Hom}_R(F_2, N) \leftarrow \mathrm{Hom}_R(F_1, N) \leftarrow \mathrm{Hom}_R(F_0, N) \leftarrow \mathrm{Hom}_R(M, N) \leftarrow 0.$$

Dropping the term $\mathrm{Hom}_R(M, N)$ [why?] we get the sequence

$$\cdots \leftarrow \mathrm{Hom}_R(F_2, N) \leftarrow \mathrm{Hom}_R(F_1, N) \leftarrow \mathrm{Hom}_R(F_0, N) \leftarrow 0,$$

and we define $\mathrm{Ext}^n_R(M, N)$ to be the nth homology group of this chain complex.

[1], [4], [2]

2 The Steenrod algebra

The following is from [2] 4L.

- There are maps $Sq^i : H^n(-; \mathbb{Z}/2\mathbb{Z}) \to H^{n+i}(-; \mathbb{Z}/2\mathbb{Z})$ for each i, and they satisfy the following properties:
  1. $Sq^i_X(f^*(\alpha)) = f^*(Sq^i_Y(\alpha))$ for $f : X \to Y$ (i.e. Sq^i is a natural transformation).
  2. $Sq^i_X(\alpha + \beta) = Sq^i_X(\alpha) + Sq^i_X(\beta)$ (i.e. Sq^i_X respects the group operation for all X).
  3. $Sq^i(\alpha \smile \beta) = \sum_{0 \leq j \leq i} (Sq^j(\alpha) \smile Sq^{i-j}(\beta))$ (the Cartan formula)
  4. $Sq^i(\sigma(\alpha)) = \sigma(Sq^i(\alpha))$ where $\sigma : H^n(X; \mathbb{Z}/2\mathbb{Z}) \to H^{n+1}(\Sigma X; \mathbb{Z}/2\mathbb{Z})$ is the "suspension isomorphism given by reduced cross product with a generator of $H^1(S^1; \mathbb{Z}/2\mathbb{Z})$"
  5. $Sq^i(\alpha) = \alpha^2$ if $i = |\alpha|$ and $Sq^i(\alpha) = 0$ if $i > |\alpha|$.
  6. $Sq^0 = \mathrm{id}$.
  7. Sq^1 is the "$\mathbb{Z}/2\mathbb{Z}$ Bockstein homomorphism β associated with the coefficient sequence $0 \to \mathbb{Z}/2\mathbb{Z} \to \mathbb{Z}/4\mathbb{Z} \to \mathbb{Z}/2\mathbb{Z} \to 0$".
- Define $Sq := Sq^0 + Sq^1 + \cdots$. Then $Sq(\alpha \smile \beta) = Sq(\alpha) \smile Sq(\beta)$ (since $(Sq(\alpha \smile \beta))_n = \sum_i Sq^i(\alpha) \smile Sq^{n-i}(\beta) = (Sq(\alpha) \smile Sq(\beta))_n$). Thus, Sq is a ring homomorphism.
- Adem relations:

  $$Sq^a Sq^b = \sum_j \binom{b-j-1}{a-2j} Sq^{a+b-j} Sq^j \quad \text{if } a < 2b,$$

  where $\binom{m}{n}$ is zero if m or n is negative, or $m < n$, and $\binom{m}{0} = 1$ for $m \geq 0$.
- The Steenrod algebra $\mathscr{A}_2$ is the algebra over $\mathbb{Z}/2\mathbb{Z}$ that is the quotient of the algebra of polynomials in the noncommuting variables $Sq^1, Sq^2, \ldots$ by the two-sided ideal generated by the Adem relations. Thus, for every space X, $H^*(X; \mathbb{Z}/2\mathbb{Z})$ is a module over $\mathscr{A}_2$, via $\alpha \cdot f = f(\alpha)$.
- $\mathscr{A}_2$ is graded, and its elements of degree k are those that map $H^n(X; \mathbb{Z}/2\mathbb{Z})$ to $H^{n_k}(X, \mathbb{Z}/2\mathbb{Z})$ for all n. [Presumably you've fixed a space X while you're doing all this?]

[5], [6], [1], [7], [2]

3 Spectra may not be your friends, but I can introduce you

- [1]: There is a category $\mathcal{H}$ of finite [because the corollary wanted f.d. CW complexes] based CW complexes, with $\mathrm{Hom}(X, Y) =: [X, Y]$ the set of homotopy classes of base-point preserving maps $X \to Y$.
- There is a category $\mathbf{St}(\mathcal{H})$ of finite[?] based CW complexes, with $\mathrm{Hom}(X, Y) =: \{X, Y\}$ the set $\mathrm{colim}_i[\Sigma^i X, \Sigma^i Y]$ [it's just a colimit of sets, and **Set** is cocomplete, so we should be fine. [1] says it's a group?]
- There is a functor $\mathcal{H} \to \mathbf{St}(\mathcal{H})$. [1] doesn't say what this is but it's presumably the one that is the identity on objects and sends $[f : X \to Y] \in [\Sigma^0 X, \Sigma^0 Y]$ to whatever it gets sent to in $\{X, Y\}$ using the universal property of the colimit. Uniqueness makes it functorial, etc.
- We have a fully faithful functor $\mathbf{St}(\mathcal{H}) \to \mathbf{St}(\mathcal{H})$ given by the suspension on objects, and the unique isomorphism $\{X, Y\} \to \{\Sigma X, \Sigma Y\}$ on maps (such an isomorphism exists, since both of those things are colimits for $[\Sigma^i X, \Sigma^i Y]$ - one of the sequences is cut off at the beginning, but it doesn't matter because both reach the stable value (see above discussion and [1] 1.9), aka the colimit).
- It's not an equivalence, because not every object is isomorphic to a suspension (e.g. anything not connected, since suspensions always connected [?])
- We can formally adjoin desuspensions $\Sigma^{-n} X$ for all n [does this mean just putting the objects there and defining $\mathrm{Hom}(Y, \Sigma^{-n} X) := \mathrm{Hom}(\Sigma^n Y, X)$ and $\mathrm{Hom}(\Sigma^{-n} X, Y) := \mathrm{Hom}(X, \Sigma^n Y)$?], but this category does not have weak colimits (i.e. colimits w/o uniqueness property). [why does it not, and why do we even want that?]
- We instead consider formal sequences of desuspensions $X_0 \to \Sigma^{-1} X_1 \to \cdots$, or sequences (X_n) and maps $\Sigma X_n \to X_{n+1}$, i.e. spectra. [and this fixes the problem?]

[5], [6], [1], [7]

4 The Adams spectral sequence

[8], [5], [6]

5 $\mathrm{Ext}^s_A(\mathbb{F}_2, \mathbb{F}_2)_t$

[5], [6], [1]

6 Methods of resolving ambiguities

[5], [6]

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