Something True and Beautiful [DRAFT]

Contents

Key:

To do (likely straightforward)

To do (likely difficult)

Problems

1 Introduction

- Define homotopy groups

THEOREM 1.0.1 ([11], Thm 1.1.4, Freudenthal suspension theorem). If $\pi_i(X) = 0$ for $i \leq k$ (i.e. X is k-connected) then the map

$$\begin{aligned} \pi_n(X) &\to \pi_{n+1}(\Sigma X) \\ [\gamma : S^n \to X] &\mapsto [\Sigma\gamma : \Sigma S^n = S^{n+1} \to \Sigma X] \end{aligned}$$

is an isomorphism for $n \leq 2k$ and surjective for $n = 2k+1$

- This implies $\pi_{n+k}(S^n)$ depends only on k for $n \geq k+2$
- (Obviously be careful with basepoints above)
- Suppose X is k-connected. Then, for $k \geq 0$, $0 = \pi_k(X) \cong \pi_{k+1}(\Sigma X)$, so whenever a space is k-connected its suspension is $k+1$-connected.
- As you take suspensions, then, your successive bounds are $n \leq 2k$, $n+1 \leq 2k+2 \implies n \leq 2k+1$, $n \leq 2k+2$, etc ... so the sequence $\pi_n(X) \to \pi_{n+1}(\Sigma X) \to \cdots$ will eventually stabilise.
- Thus, if you take the colimit of that direct system, it'll just equal the stable value, with the higher legs just being the inverse isomorphisms.

THEOREM 1.0.2 ([11], Thm 1.1.8). $\pi_{n+k}(S^n)$ is finite for $k > 0$ except when $n = 2m$, $k = 2m-1$.

COROLLARY 1.0.3. π_i^S is finite for $i > 0$.

[13], [5]

2 The Steenrod algebra

[Intro. Sources: [5]]

PROPOSITION 2.0.1 ([5], p489). There are maps $Sq^i : H^n(-;\mathbb{Z}/2\mathbb{Z}) \to H^{n+i}(-;\mathbb{Z}/2\mathbb{Z})$ for each i, and they satisfy the following properties:

1. $Sq^i(f^*(\alpha)) = f^*(Sq^i(\alpha))$ for $f : X \to Y$ (i.e. Sq^i is a natural transformation).
2. $Sq^i(\alpha + \beta) = Sq^i(\alpha) + Sq^i_X(\beta)$ (i.e. Sq^i respects the group operation for all X).
3. $Sq^i(\alpha \smile \beta) = \sum_{0 \leq j \leq i} (Sq^j(\alpha) \smile Sq^{i-j}(\beta))$ (the Cartan formula)
4. $Sq^i(\sigma(\alpha)) = \sigma(Sq^i(\alpha))$ where $\sigma : H^n(X;\mathbb{Z}/2\mathbb{Z}) \to H^{n+1}(\Sigma X;\mathbb{Z}/2\mathbb{Z})$ is the suspension isomorphism given by reduced cross product with a generator of $H^1(S^1;\mathbb{Z}/2\mathbb{Z})$.

5. $Sq^i(\alpha) = \alpha^2$ if $i = \deg(\alpha)$ and $Sq^i(\alpha) = 0$ if $i > \deg(\alpha)$.
6. $Sq^0 = \mathrm{id}$.

Define $Sq := Sq^0 + Sq^1 + \cdots$. Then $Sq(\alpha \smile \beta) = Sq(\alpha) \smile Sq(\beta)$ (since $(Sq(\alpha \smile \beta))_n = \sum_i Sq^i(\alpha) \smile Sq^{n-i}(\beta) = (Sq(\alpha) \smile Sq(\beta))_n$). Thus, Sq is a ring homomorphism.

PROPOSITION 2.0.2 ([5], p496). The Steenrod squares satisfy the following relations, known as the *Adem relations*:

$$Sq^a Sq^b = \sum_j \binom{b-j-1}{a-2j} Sq^{a+b-j} Sq^j \quad \text{if } a < 2b,$$

where $\binom{m}{n}$ is zero if m or n is negative, or $m < n$, and $\binom{m}{0} = 1$ for $m \geq 0$.

DEFINITION 2.0.3. The *Steenrod algebra* $\mathscr{A}_2$ is the algebra over $\mathbb{Z}/2\mathbb{Z}$ that is the quotient of the algebra of polynomials in the noncommuting variables $Sq^1, Sq^2, \ldots$ by the two-sided ideal generated by the Adem relations. Thus, for every space X, $H^*(X; \mathbb{Z}/2\mathbb{Z})$ is a module over $\mathscr{A}_2$.

Note that $\mathscr{A}_2$ is graded, with elements of degree k those that map $H^n(X; \mathbb{Z}/2\mathbb{Z})$ to $H^{n+k}(X, \mathbb{Z}/2\mathbb{Z})$ for all n.

DEFINITION 2.0.4. Write Sq^I for the monomial $Sq^{i_1} Sq^{i_2} \cdots Sq^{i_n}$. Then Sq^I is *admissible* if $i_j \geq 2i_{j+1}$ for all $0 \leq j < n$.

The admissible monomials are exactly those to which no Adem relations can be applied. Thus, $\mathscr{A}_2$ is generated as an $\mathbb{F}_2$ module by admissible monomials.

3 Spectra may not be your friends, but I can introduce you

[Intro. Sources: [13], [6], [8], [7]]

3.1 Definitions and examples

[Some intro, below follows [6], Section 5.2. Some intuition on spectra and why we're doing this.]

DEFINITION 3.1.1. A *spectrum* is a collection of pointed topological spaces $\{X_n\}_{n \in \mathbb{N}}$, together with basepoint-preserving maps $\sigma_n : \Sigma X_n \to X_{n+1}$.

EXAMPLE 3.1.2. Let X be a topological space. The *suspension spectrum* of X, denoted by $\Sigma^\infty X$, has $X_n = \Sigma^n X$ and $\sigma_n = \mathrm{id} : \Sigma X_n \to X_{n+1}$.

We write $\mathbb{S}$ for the suspension spectrum $\Sigma^\infty S^0$, and call $\mathbb{S}$ the *sphere spectrum*. For $i > 0$, we write $\mathbb{S}^i$ for $\Sigma^\infty S^i$.

EXAMPLE 3.1.3. An *Eilenberg-MacLane spectrum* $\mathbb{K}(G, m)$ has $(\mathbb{K}(G, m))_n$ a CW complex $K(G, m+n)$, and can be constructed inductively by attaching cells to $\Sigma K(G, m+n))$ to kill $\pi_i(\Sigma K(G, m+n))$ for $i > m+n+1$. By Theorem 1.0.1, $\pi_i(K(G, m+n)) \cong \pi_{i+1}(\Sigma K(G, m+n))$ for $i \leq 2m+2n-2$, so the cells attached can be taken to have dimension $\geq 2m+2n-1$. The maps σ_n are inclusions of subcomplexes.

Definition 3.1.4. Let $X = \{X_n\}$ be a spectrum. We define $\pi_i(X) = \operatorname{colim}_n \pi_{i+n}(X_n)$, where the map $\pi_{i+n}(X_n) \to \pi_{i+n+1}(X_{n+1})$ is given by the composition

$$\pi_{i+n}(X_n) \xrightarrow{\Sigma} \pi_{i+n+1}(\Sigma X_n) \xrightarrow{(\sigma_n)_*} \pi_{i+n+1}(X_{n+1}).$$

Example 3.1.5. If X is a topological space, then $\pi_i(\Sigma^\infty X) = \pi_i^S(X)$, the ith stable homotopy group of X.

Definition 3.1.6. A CW spectrum is a spectrum X consisting of CW complexes X_n with the maps $\Sigma X_n \hookrightarrow X_{n+1}$ inclusions of subcomplexes.

Definition 3.1.7. Let X be a CW spectrum. Then the k-*cells* of X are the equivalence classes of non-basepoint $(k+n)$-cells in X_n, where two cells are equivalent if one is an m-fold suspension of the other, for some $m > 0$.

Definition 3.1.8. A CW spectrum X is *connective* if it has no cells below a given dimension, *finite* if it has only finitely many cells, and *of finite type* if it has only finitely many cells in each dimension.

Example 3.1.9. If X is a finite (resp. finite type) CW complex, then Σ^∞ is a finite (resp. finite type) CW spectrum. In particular, $\mathbb{S}$ is a finite CW spectrum with a unique cell in dimension 1.

Example 3.1.10. For each m, the Eilenberg-MacLane spectrum $\mathbb{K}(G,m)$ constructed in Example 3.1.3 has finite type. This follows from the fact that the dimension of the cells added to $\Sigma K(G, n+m)$ is eventually larger than $n+i$ for any i, so $\mathbb{K}(G,m)$ only has finitely many i-cells.

Lemma 3.1.11. Let X be a connective spectrum of finite type. Then the groups $\pi_{i+n}(X_n)$ eventually stabilise; i.e. the maps $\pi_{i+n}(X_n) \xrightarrow{(\sigma_n)_* \circ \Sigma} \pi_{i+n+1}(X_{n+1})$ are isomorphisms for large enough n.

Proof. First, note that that maps $\pi_{i+n}(X_n) \xrightarrow{\Sigma} \pi_{i+n+1}(\Sigma X_n)$ are eventually isomorphisms by Theorem 1.0.1.

Recall that whenever $(X_{n+1}, \Sigma X_n)$ are such that $X_{n+1} \setminus X_n$ has no cells in dimension $\leq k$, the map $\pi_i(\Sigma X_n) \to \pi_i(X_{n+1})$ induced by the inclusion is an isomorphism ([5], Cor 4.12 and a long exact sequence). Thus, if $(\sigma_n)_* : \pi_{i+n+1}(\Sigma X_n) \to \pi_{i+n+1}$ never stabilises, there must be infinitely many natural numbers N_j such that $(X_{N_j+1}, \Sigma X_{N_j})$ is not $(i + N_j + 1)$-connected, and thus that $X_{N_j+1} \setminus \Sigma X_{N_j}$ has cells of dimension $\leq i + N_j + 2$. By connectivity, there is some fixed l such that these cells are of dimension $N_j + k + 1$ for $-l \leq k \leq i+1$. Thus, there must be some k such that infinitely many of the X_{N_j+1} have a $(k + N_j + 1)$-cell not included in ΣX_{N_j}. This then contradicts the assumption that X is of finite type, since it has infinitely many k-cells.

Thus, the maps $(\sigma_n)_* : \pi_{i+n+1}(\Sigma X_n) \to \pi_{i+n+1}$ are also eventually isomorphisms, so the groups $\pi_{i+n}(X_n)$ do stabilise. $\square$

3.2 Homology and cohomology

Recall that $C_i^{\mathrm{cell}}(X_n; G)$ is the free abelian group on the i-cells of X_n. We have an injection

$$\begin{aligned} C_i^{\mathrm{cell}}(X_n; G) &\to C_{i+1}^{\mathrm{cell}}(\Sigma X_n; G) \\ e_\alpha^i &\mapsto \Sigma e_\alpha^i, \end{aligned}$$

and an injection $C_{i+1}^{\mathrm{cell}}(\Sigma X_n; G) \to C_{i+1}^{\mathrm{cell}}(X_{n+1}; G)$ induced by the structure map σ_n, so we get an injection $C_i^{\mathrm{cell}}(X_n; G) \hookrightarrow C_{i+1}^{\mathrm{cell}}(X_{n+1}; G)$.

We define

$$C_n(X; G) := \bigcup_{i \in \mathbb{Z}} C_{i+n}^{\mathrm{cell}}(X_i; G).$$

Note that there is a G summand for every $i + n$ cell of X_i up to treating suspensions of cells as equivalent to the cells themselves, i.e. a G summand for every n-cell of X. We define $H^*(X; G)$ and $H_*(X; G)$ to be the cohomology and homology of this chain complex, respectively.

Lemma 3.2.1. Let X be a connective CW spectrum of finite type. Then $H_i(X; G)$, $H^i(X; G)$, and $\pi_i(X)$ are finitely generated for all i.

Proof. First, note that $H_i(X; G) = H_{i+n}(X_n; G)$ for sufficiently large n, since for large enough n, X_n contains all the cells of dimension $\leq i$. Similarly, $H^i = H^{i+n}(X_n; G)$ for sufficiently large n. Each $H_{i+n}(X_n; G)$ is finitely generated, since X_n has only finitely many cells in each dimension, and thus each $H^{i+n}(X_n; G)$ is also finitely generated ([5] Cor 3.3). Thus, $H_i(X; G), H^i(X; G)$ are finitely generated.

Now, $\pi_i(X) = \operatorname{colim}_n \pi_{i+n}(X_n)$, and the groups $\pi_{i+n}(X_n)$ stabilise by Lemma 3.1.11. The X_n must eventually be simply-connected, since X is connective. A simply-connected space has finitely generated homotopy groups if and only if it has finitely generated homology groups (see e.g. [5], Thm 5.7), and we have just seen that the $H_{i+n}(X_n; G)$ are finitely generated, so $\pi_i(X) = \pi_{i+n}(X_n)$ is finitely generated. □

Example 3.2.2. Recall that $\mathbb{S}$ is a finite spectrum. We thus have

$$\begin{aligned} H^i(\mathbb{S}; \mathbb{F}_2) &= \lim_{\leftarrow n} H^{i+n}(S^n; \mathbb{F}_2) \\ &= \begin{cases} \mathbb{F}_2 & i = 0, \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

Definition 3.2.3. Let $X = \{X_n\}$ be a CW spectrum. A *subspectrum* X' of X is a sequence of subcomplexes $\{X_n' \subseteq X_n\}$ satisfying $\Sigma X_n' \subseteq X_{n+1}'$. The subspectrum X' is *cofinal* if, for each n and each cell e_α^i of X_n, the cell $\Sigma^k e_\alpha^i$ belongs to X_{n+k}' for all sufficiently large k.

Note that if $\Sigma^k e_\alpha^i$ belongs to X_{n+k}' then $\Sigma^{k+1} e_\alpha^i$ belongs to $\Sigma X_{n+k}' \subseteq X_{n+k+1}' \subseteq X_{k+k+2}' \subseteq \cdots$. Thus, if X', X'' are cofinal spectra of X with $\Sigma^k e_\alpha^i$ a cell of X_{n+k}' and $\Sigma^l e_\alpha^i$ a cell of X_{n+l}'' (with $l \geq k$) then $\Sigma^l e_\alpha^i$ is a cell of X_{n+l}' and therefore of $X_{n+l}' \cap X_{n+l}''$. In other words, the intersection of two cofinal spectra is a cofinal spectrum.

Definition 3.2.4. Let X, Y be CW spectra. A *strict map* $f : X \to Y$ is a sequence of cellular maps $f_n : X_n \to Y_n$ such that the diagram below commutes.

$$\begin{tikzcd} \Sigma X_n \arrow[r, "\sigma_n"] \arrow[d, "\Sigma f_n"'] & X_{n+1} \arrow[d, "f_{n+1}"] \\ \Sigma Y_n \arrow[r, "\sigma_n"'] & Y_{n+1} \end{tikzcd}$$

Taking strict maps to be our notion of maps between spectra, however, turns out to be too strong. For instance, a strict map $\mathbb{S}^i \to \Sigma^\infty X$ would be given simply by a map $S^i \to X$, whereas if we want to know about the stable homotopy groups of X, we should also consider maps $S^{i+n} \to \Sigma^n X$ which cannot necessarily by desuspended. We will therefore relax the definition of maps between spectra to include maps that are 'defined eventually', in the following sense.

Definition 3.2.5. A *map of CW spectra* $f : X \to Y$ is an equivalence class of strict maps $f' : X' \to Y$ with X' a cofinal subspectrum of X, where two strict maps $f' : X' \to Y$ and $f'' : X'' \to Y$ are equivalent if they agree on some common cofinal subspectrum.

Given two maps $f : X \to Y$, $g : Y \to Z$ represented by $f' : X' \to Y$, $g' : Y' \to Z$ respectively, we compose as follows: let X'' be the subspectrum of X', where the cells of X''_n consist of the cells of X'_n mapped to Y'_n under f'_n. Then, for any cell e^i_α of X'_n, $f_n(e^i_\alpha)$ is contained in a finite union of cells of Y_n (since the image of a compact set is compact), whose k-fold suspension lies in Y'_{n+k} for large enough k. Since f' is a strict map, $\Sigma^k f'_n(e^i_\alpha) = f'_{n+k}\Sigma^k e^i_\alpha$, so $\Sigma^k e^i_\alpha$ is a cell of X'_{n+k}. Thus, X'' is cofinal in X' and hence in X. We define $gf := [X'' \xrightarrow{f'|_{X''}} Y' \xrightarrow{g'} Z]$, which is well-defined since the intersection of cofinal subspectra is again a cofinal subspectrum.

Since any strict map $f' : X' \to Y$ can be taken to be cellular, a map $f : X \to Y$ induces a well-defined map $C_*(X) \to C_*(Y)$ (by cofinality), and thus maps on homology and cohomology.

Further, any map $\Sigma^\infty S^i \to X$ can be represented by a map $S^{i+n} \to X_n$, which has compact image and thus by Proposition A.3.7 is contained in a finite subcomplex of $\overline{X}_n$. Given any map $f : X \to Y$ represented by a strict map $f' : X' \to Y$, the kth suspension of the cells of $\overline{X}_n$ lie in X'_{n+k}, and thus f induces a map $\pi_*(X) \to \pi_*(Y)$.

Definition 3.2.6. Two spectra X, Y are *equivalent* if there are maps $f : X \to Y$ and $g : Y \to X$ such that $fg = \mathrm{id}_Y$ and $gf = \mathrm{id}_X$.

Note that a spectrum is equivalent to any of its cofinal subspectra. In particular, if $X = \{X_n\}$ is a spectrum, then $X' = \{\Sigma X_{n-1}\}$ is a cofinal subspectrum of X (where we take X_{-1} to be the basepoint of X_0). We define $\Sigma^{-1}X := \{X_{n-1}\}$, noting that $\Sigma\Sigma^{-1}X = \Sigma^{-1}\Sigma = X' \simeq X$. Thus, a spectrum is always equivalent to the suspension of some other spectrum.

Definition 3.2.7. A *homotopy* of maps between spectra is a map $X \times I \to Y$, where $X \times I$ is the spectrum with $(X \times I)_n = X_n \times_{\mathrm{red}} I$.

Note that $\Sigma(X_n \times_{\mathrm{red}} I) = \Sigma X_n \times_{\mathrm{red}} I$. The set of homotopy classes of maps $X \to Y$ is denoted by $[X, Y]$.

Remark 3.2.8. For any CW spectra Z, $[\Sigma^\infty S^t, Z] = \pi_t(Z)$.

For any CW spectra X, Y, the set $[X, Y]$ can the structure of an abelian group, since X has be written as a double suspension $\Sigma^2 X'$, and each set $[\Sigma^2 X'_n, Y_n]$ has the structure of an abelian group by Remark A.1.4.

THEOREM 3.2.9. The suspension map $[X, Y] \to [\Sigma X, \Sigma Y]$ is an isomorphism of groups.

PROOF. The suspension map is a homomorphism, since it is a homomorphism on maps between CW complexes. Thus, it suffices to show it is a bijection on maps between spectra.

Recall that $\Sigma^{-1}\Sigma X = \Sigma\Sigma^{-1}X \simeq X$. For any map $f : X \to Y$ given by strict maps $f_n : X'_n \to Y_n$, define $\Sigma^{-1}f : \Sigma^{-1}X \to \Sigma^{-1}Y$ by $\{f_{n-1} : X'_{n-1} \to Y_{n-1}\}$. Then $\Sigma\Sigma^{-1}f = \{\Sigma f_{n-1}\} = \{f_n|_{\Sigma X_{n-1}}\} = f$, and similarly $\Sigma^{-1}\Sigma f = f$. Thus, we have bijections $[X, Y] \cong [\Sigma\Sigma^{-1}X, \Sigma\Sigma^{-1}Y] \cong [\Sigma^{-1}\Sigma X, \Sigma^{-1}\Sigma Y]$, so Σ has a two-sided inverse. $\square$

3.3 Cofibration sequences

DEFINITION 3.3.1. Let $X = \{X_n\}, Y = \{Y_n\}$ be spectra. Then their *wedge sum* is $X \vee Y := \{X_n \vee Y_n\}$. Note that Remark A.2.5 gives us an inclusion $\Sigma(X_n \vee Y_n) \hookrightarrow X_{n+1} \vee Y_{n+1}$.

DEFINITION 3.3.2. Let $f : X \to Y$ be a map of CW spectra, and let $f' : X' \to Y$ be a representative for f, where $X' \subseteq X$ is cofinal. The **mapping cylinder** M_f has components $(M_f)_n = M_{f'_n}$, where $M_{f'_n}$ is the reduced mapping cylinder of f'_n, and is independent of the choice of X' up to equivalence.

REMARK 3.3.3. Given any map $f : X \to Y$ of CW spectra, we have a deformation retraction of M_f onto Y. Since we will only be interested in spectra up to homotopy equivalence, by replacing Y by M_f we may assume any map $f : X \to Y$ is an inclusion.

DEFINITION 3.3.4. Let X be a spectrum, $A \subseteq X$ a subspectrum. Then A is *closed* in X if for every cell e^n_α of X_n, if $\Sigma^k e^n_\alpha \in A_{n+k}$ then $e^n_\alpha \in A_n$.

Any subspectrum is cofinal in (and thus equivalent to) its closure. We define X/A to be the CW spectrum with $(X/A)_n = X_n/A'_n$, where $A' = \{A'_n\}$ is the closure of A. Note that a quotient of connective spectra of finite type is again a connective spectrum of finite type (since the quotient has fewer cells in each dimension than the original space).

The map $X \cup CA \to X/A$ is a homotopy equivalence of spectra, since each quotient $X_n \cup CA_n \to X_n/A_n$ is, so we have a cofibration sequence

$$A \hookrightarrow X \to X \cap CA \to \Sigma A \hookrightarrow \Sigma X \to \cdots .$$

THEOREM 3.3.5. Let X, Y be spectra, and $A \subseteq X$ a subspectrum. Then there is an exact sequence

$$[Y, A] \to [Y, X] \to [Y, X/A] \to [Y, \Sigma A] \to [Y, \Sigma X] \to \cdots .$$

PROOF. It suffices to show that

$$[Y, A] \to [Y, X] \to [Y, X/A] \to [Y, \Sigma A]$$

is exact.

We first show exactness at $[Y, X/A]$. The composition $[Y, X] \to [Y, X/A] \to [Y, \Sigma A]$ is clearly zero. Now, if $Y \xrightarrow{f} X \cup CA \to \Sigma A$ is homotopic to the constant map, then f must be homotopic to a map contained entirely in X, and thus is in the image of $[Y, X] \to [Y, X/A]$.

Now, we show exactness at $[Y, X]$. Again, the composition $[Y, A] \to [Y, X] \to [Y, X/A]$ is clearly zero. Suppose $Y \xrightarrow{f} X \to X \cup CA$ is homotopic to the constant map. Then we have a map $h : CY \to X \cup CA$ making the solid diagram below commute.

$$\begin{tikzcd}
Y \arrow[r, "\mathrm{id}"] \arrow[d, dashed] & Y \arrow[r, hook] \arrow[d, "f"] & CY \arrow[r, hook] \arrow[d, "h"] & \Sigma Y \arrow[r, "\mathrm{id}"] \arrow[d, dashed] & \Sigma Y \arrow[d, dashed, "\Sigma f"] \\
A \arrow[r, hook, "i"'] & X \arrow[r, hook] & X \cup CA \arrow[r] & \Sigma A \arrow[r, hook, "\Sigma i"'] & \Sigma X
\end{tikzcd}$$

[[6] says that "we can then automatically fill in the next two vertical maps to make homotopy commutative squares". I believe that we can write down a map making the left square commute, but I have no idea why that map makes the right square commute. But if it does...]

By Theorem 3.2.9, we can take the map $\Sigma Y \to \Sigma A$ to be Σg for some $g : Y \to A$. Then $\Sigma f \simeq (\Sigma i)(\Sigma g) = \Sigma(ig)$, so $f \simeq ig$, as required. $\square$

Finally, we get the lemma below, which follows from the equivalent result for CW complexes.

LEMMA 3.3.6. Let $A \overset{f}{\hookrightarrow} X \xrightarrow{i} C_f \xrightarrow{j} \Sigma A \to \cdots$ be a cofibration, where X, A are CW spectra of finite type. Then there is a long exact sequence

$$\cdots \leftarrow H^{n-1}(\Sigma A) \leftarrow H^n(X) \xleftarrow{i^*} H^n(C_f) \xleftarrow{j^*} H^n(\Sigma A) \leftarrow H^{n+1}(X) \leftarrow \cdots.$$

3.4 Eilenberg-MacLane spectra

THEOREM 3.4.1 ([6], Prop 5.45). There are natural isomorphisms $H^m(X; G) \cong [X, K(G, m)]$ for all CW spectra.

Recall that giving a map into a product is equivalent to giving a map into each of its components. We have maps $F_i : [X, \bigvee_i \mathbb{K}(G, n_i)] \to [X, \mathbb{K}(G, n_i)]$.

PROPOSITION 3.4.2 ([6], Prop 5.46). The map $F : [X, \bigvee_i \mathbb{K}(G, n_i)] \to \prod_i [X, \mathbb{K}(G, n_i)]$ is an isomorphism if X is a connective spectrum of finite type and $n_i \to \infty$ as $i \to \infty$.

3.5 p-completion of spectra

DEFINITION 3.5.1 ([8], Def 10.1.1). Let A be an abelian group. Then its *p-adic completion* is the limit

$$A^\wedge_p = \varprojlim_n (A/p^n A).$$

If $A = \mathbb{Z}$, we instead write $\mathbb{Z}_p$ for the p-adic integers. There is a natural map $A \to A^\wedge_p$, whose component at n is reduction modulo $p^n A$.

When A is finitely generated, its p-adic completion is given by the map $A \to A \otimes \mathbb{Z}_p$; $a \mapsto a \otimes 1$.

LEMMA 3.5.2. Suppose A is finite, and write $|A| = np^r$ for $p \nmid n$. Then $A \otimes \mathbb{Z}_p \cong A/T$, where $T \subseteq A$ is the subgroup generated by all torsion coprime to p.

PROOF. Define a homomorphism $A \otimes \mathbb{Z}_p \to A/T$ sending $a \otimes z \mapsto [\hat{z}a]$, where $\hat{z} \in \mathbb{Z}$ is a lift of $q(z)$, chosen such that $0 < \hat{z} \le p^r$, and q is the projection $\mathbb{Z}_p \twoheadrightarrow \mathbb{Z}/p^r\mathbb{Z}$. Suppose $a \otimes z \mapsto 0$. Then $\hat{z}a \in T$, so $k\hat{z}a = 0$ for some k coprime to p. Write $z = \hat{z} + p^r z'$ for some

$z' \in \mathbb{Z}_p$. Then $a \otimes z = a \otimes (\hat{z} + p^r z') = \hat{z}a \otimes (1 + \frac{p^r z'}{\hat{z}}) = k\hat{z} \otimes (\frac{1}{k} + \frac{p^r z'}{k\hat{z}}) = 0$, where the second equality follows from the fact that $\hat{z}$ was chosen such that the highest power of p dividing it was less than or equal to r. Thus, $a \otimes z = 0$, so the map is injective. The map is clearly also surjective, since $a \otimes 1 \mapsto [a]$, so it is an isomorphism. $\square$

REMARK 3.5.3. If A is finite with order np^r for $p \not| n$, then $|A_p^\wedge| = p^r$, by Cauchy's theorem.

DEFINITION 3.5.4 ([7], p129). Let X be a CW spectrum. Then a *p-completion* of X is a map $f : X \to X_p^\wedge$ such that for all i, $\pi_i f$ expresses $\pi_i(X_p^\wedge)$ as the p-completion of $\pi_i(X)$.

THEOREM 3.5.5 ([7], Thm 9.1.1). If X has finite type, then it has a p-completion unique up to equivalence.

THEOREM 3.5.6 ([7], Prop 9.2.22). Let X be a connective spectrum of finite type, and let Y be p-complete. Then the map $[X_p^\wedge, Y] \to [X, Y]$ is an isomorphism. That is, given any map $X \xrightarrow{f} Y$, there exists a unique (up to homotopy) map $X_p^\wedge \xrightarrow{\overline{f}} Y$ such that f factors as $X \to X_p^\wedge \xrightarrow{\overline{f}} Y$.

4 The Adams spectral sequence

[Some intro. Following [10], [6], [14], [5]. From this point on, all homology and cohomology will be taken with $\mathbb{F}_2$ coefficients, and we will thus ease notation by writing $H^*(X)$ (resp. $H_*(X)$) for $H^*(X, \mathbb{F}_2)$ (resp. $H_*(X; \mathbb{F}_2)$).]

4.1 Spectral sequences

[Some notes from [10], C2 - just here as a placeholder/reference and I'll probably completely rewrite this bit. Maybe add some notes from [13]]

DEFINITION 4.1.1. A *differential bigraded module* E over a ring R is a collection of R-modules $\{E^{p,q}\}$, $p, q \in \mathbb{Z}$, together with a map $d : E^{p,q} \to E^{p+s,q-s+1}$ for each p, q and some fixed $s \in \mathbb{Z}$, satisfying $d^2 = 0$.

We can take the homology of (E, d):

$$H^{p,q}(E^{*,*}, d) = \ker(d : E^{p,q} \to E^{p+s,q-s+1}) / \operatorname{im}(d : E^{p-s,q+s-1} \to E^{p,q}).$$

DEFINITION 4.1.2. A *spectral sequence* (of *cohomological type*[1]) is a collection of differential bigraded R-modules $\{E_r^{*,*}, d_r\}, r \in \mathbb{N}$, with the differentials d_r of bidegree $(r, 1 - r)$. These satisfy the further condition that for all p, q, r, $E_{r+1}^{p,q} \cong H^{p,q}(E_r^{*,*}, d_r)$.

We will sometimes write $d_r^{p,q}$ for the differential $d_r : E^{p,q} \to E^{p+r,q-s+1}$.

Consider the term $E_2^{*,*}$. Define

$$Z_2^{p,q} := \ker d_2^{p,q} \quad \text{and} \quad B_2^{p,q} := \operatorname{im} d_2^{p-2,q+1}.$$

The condition $d^2 = 0$ implies that $B_2^{p,q} \subseteq Z_2^{p,q} \subseteq E_2^{p,q}$, and by definition we have $E_3^{p,q} \cong Z_2^{p,q} / B_2^{p,q}$.

[1] I'll have to rewrite this section because the Adams spectral sequence is not a cohomological or a homological spectral sequence I don't think - the grading is $d_r : E^{s,t} \to E^{s+r,t+r-1}$.

Now, write

$$Z_3^{p,q} := \ker d_3^{p,q} \quad \text{and} \quad B_3^{p,q} := \operatorname{im} d_3^{p-3,q+2}.$$

Since $Z_3^{p,q} \subseteq E_3^{p,q}$, it can be written as $\overline{Z}_3^{p,q}/B_2^{p,q}$ for some $\overline{Z}_3^{p,q} \subseteq Z_2^{p,q}$. Similarly, $B_3^{p,q} \cong \overline{B}_3^{p,q}/B_2^{p,q}$ for some $\overline{B}_3^{p,q} \subseteq Z_2^{p,q}$. Thus,

$$E_4^{p,q} \cong Z_3^{p,q}/B_3^{p,q} \cong \frac{\overline{Z}_2^{p,q}/B_2^{p,q}}{\overline{B}_3^{p,q}/B_2^{p,q}} \cong \overline{Z}_3^{p,q}/\overline{B}_3^{p,q}.$$

Iterating this process, we present the spectral sequence as an infinite tower of submodules of $E_2^{p,q}$:

$$B_2^{p,q} \subseteq \overline{B}_3^{p,q} \subseteq \cdots \subseteq \overline{B}_n^{p,q} \subseteq \cdots \subseteq \overline{Z}_n^{p,q} \subseteq \cdots \subseteq \overline{Z}_3^{p,q} \subseteq Z_2^{p,q} \subseteq E_2^{p,q},$$

with the property that $E_{n+1}^{p,q} \cong \overline{Z}_n^{p,q}/\overline{B}_n^{p,q}$. The differential $d_{n+1}^{p,q}$ can be taken as a map $\overline{Z}_n^{p,q}/\overline{B}_n^{p,q} \to \overline{Z}_n^{p,q}/\overline{B}_n^{p,q}$ with kernel $\overline{Z}_{n+1}^{p,q}/\overline{B}_n^{p,q}$ and image $\overline{B}_{n+1}^{p,q}$. The short exact sequence induced by d_{n+1},

$$0 \to \overline{Z}_{n+1}^{p,q}/\overline{B}_n^{p,q} \to \overline{Z}_n^{p,q}/\overline{B}_n^{p,q} \xrightarrow{d_{n+1}^{p,q}} \overline{B}_{n+1}^{p,q}/\overline{B}_n^{p,q} \to 0,$$

gives rise to isomorphisms $\overline{Z}_n^{p,q}/\overline{Z}_{n+1}^{p,q} \cong \overline{B}_{n+1}^{p,q}/\overline{B}_n^{p,q}$ for all n. Conversely, a tower of submodules of E_2, together with a set of isomorphisms, gives rise to a spectral sequence.

Definition 4.1.3. An element of $E_2^{p,q}$ *survives to the rth stage* if lies in $\overline{Z}_r^{p,q}$, having been in the kernel of the previous $r-2$ differentials, and is *bounded by the rth stage* if it lies in $\overline{B}_r^{p,q}$. The bigraded module $E_r^{*,*}$ is called the *E_r-term* of the spectral sequence.

We define

$$Z_\infty^{p,q} := \bigcap_n \overline{Z}_n^{p,q}, \quad B_\infty^{p,q} := \bigcup_n \overline{B}_n^{p,q}.$$

From the tower of inclusions, we see that $B_\infty^{p,q} \subseteq Z_\infty^{p,q}$, so we define $E_\infty^{p,q} := Z_\infty^{p,q}/B_\infty^{p,q}$.

Definition 4.1.4. A spectral sequence *collapses at the Nth term* if the differentials $d_r^{p,q} = 0$ for $r \geq N$.

From the short exact sequence

$$0 \to \overline{Z}_r^{p,q}/\overline{B}_{r-1}^{p,q} \to \overline{Z}_{r-1}^{p,q}/\overline{B}_{r-1}^{p,q} \xrightarrow{d_r^{p,q}} \overline{B}_r^{p,q}/\overline{B}_{r-1}^{p,q} \to 0,$$

the condition $d_r^{p,q}$ forces $\overline{Z}_r^{p,q} = \overline{Z}_{r-1}^{p,q}$ and $\overline{B}_r^{p,q} = \overline{B}_{r-1}^{p,q}$. The tower of submodules becomes

$$B_2^{p,q} \subseteq \overline{B}_3^{p,q} \subseteq \cdots \subseteq \overline{B}_{N-1}^{p,q} = B_N^{p,q} = \cdots = B_\infty^{p,q} \subseteq Z_\infty^{p,q} = \cdots = \overline{Z}_N^{p,q} = \overline{Z}_{N-1}^{p,q} \subseteq \cdots \subseteq \overline{Z}_3^{p,q} \subseteq Z_2^{p,q} \subseteq E_2^{p,q}.$$

Thus, $E_\infty^{p,q} = E_N^{p,q}$.

4.2 Exact couples

[Following [10], C2]

Definition 4.2.1. Let D, E be R-modules, and let $i : D \to D$, $j : D \to E$, $k : E \to D$ be module homomorphisms. We call $\mathcal{C} = \{D, E, i, j, k\}$ an *exact couple* if the diagram below is exact.

$$\begin{tikzcd} D \arrow[rr, "i"] & & D \arrow[dl, "j"] \\ & E \arrow[ul, "k"] & \end{tikzcd}$$

Let $d := jk$, and define the following:

$$\begin{aligned}
E' &:= H(E,d) = \ker d / \operatorname{im} d \\
D' &:= i(D) = \ker j \\
i' &:= i|_{i(D)} : D' \to D' \\
j' &:= i(x) \mapsto j(x) + dE : D' \to E' \\
k' &:= (e + dE) \mapsto k(e) : E' \to D'
\end{aligned}$$

We call $\mathcal{C}' = \{D', E'.i', j', k'\}$ the *derived couple* of $\mathcal{C}$.

PROPOSITION 4.2.2 ([10], Prop 2.7). If $\mathcal{C} = \{D, E, i, j, k\}$ is an exact couple, then $\mathcal{C}'$ is also an exact couple.

THEOREM 4.2.3 ([10], Thm 2.8). Suppose $D^{*,*} = \{D^{p,q}\}$ and $E^{*,*} = \{E^{p,q}\}$ are bigraded modules equipped with homomorphisms i of bidegree $(-1,1)$[2], j of bidegree $(0,0)$, and k of bidegree $(1,0)$, such that $\{D^{*,*}, E^{*,*}, i, j, k\}$ is an exact couple. Then these data determine a spectral sequence $\{E_r, d_r\}$ for $r \in \mathbb{Z}_+$ of cohomological type, with $E_r = (E^{*,*})^{(r-1)}$, the $(r-1)$st derived module of $E^{*,*}$ and $d_r = j^{(r-1)} \circ k^{(r-1)}$.

A bigraded exact couple may be displayed in the following diagram, known as a *staircase diagram*:

$$\begin{tikzcd}
 & \vdots \arrow[d, "i"] & & \vdots \arrow[d, "i"] & \\
\cdots \arrow[r, "k"] & D^{p+2,q-1} \arrow[r, "j"] \arrow[d, "i"] & E^{p+2,q-1} \arrow[r, "k"] & D^{p+3,q-1} \arrow[r, "j"] \arrow[d, "i"] & \cdots \\
\cdots \arrow[r, "k"] & D^{p+1,q} \arrow[r, "j"] \arrow[d, "i"] & E^{p+1,q} \arrow[r, "k"] & D^{p+2,q} \arrow[r, "j"] \arrow[d, "i"] & \cdots \\
\cdots \arrow[r, "k"] & D^{p,q+1} \arrow[r, "j"] \arrow[d, "i"] & E^{p,q+1} \arrow[r, "k"] & D^{p+1,q+1} \arrow[r, "j"] \arrow[d, "i"] & \cdots \\
 & \vdots & & \vdots &
\end{tikzcd}$$

[This is the same staircase diagram that Hatcher is talking about (set $D^{*,*} = \pi_*^S X_*$ and $E^{*,*} = \pi_* K_*$) - just modulo some tweak of i's bigrading because the bigrading is a bit weird in the Adams spectral sequence.]

4.3 Ext

Before constructing the Adams spectral sequence, we briefly recall the definition of the Ext functor and some of its basic properties, which will be of importance later. We mainly follow [14].

[2] This is the wrong index. For my purposes it should be $(-1,-1)$. So, the spectral sequence will not be of cohomological type, but hopefully the proof still goes through.

Definition 4.3.1. Let M, N be modules over a ring R. A *projective resolution* P of M is an exact sequence,

$$\cdots \to P_2 \to P_1 \to P_0 \to M \to 0.$$

where each P_i is projective. If, in addition, each P_i is free, then the resolution is called *free*.

Dually, an *injective resolution* I of M is a exact sequence

$$0 \to M \to I_0 \to I_1 \to I_2 \to \cdots .$$

The following result can be obtained from [14], Lemmas 2.2.5, 2.3.6, and Exercise 2.3.5.

Lemma 4.3.2. Every R-module M has a projective resolution and an injective resolution.

Given a projective resolution as in Definition 4.3.1, applying $\mathrm{Hom}_R(-, N)$ gives us a chain complex

$$\cdots \leftarrow \mathrm{Hom}_R(P_2, N) \leftarrow \mathrm{Hom}_R(P_1, N) \leftarrow \mathrm{Hom}_R(P_0, N) \leftarrow \mathrm{Hom}_R(M, N) \leftarrow 0.$$

Dropping the term $\mathrm{Hom}_R(M, N)$, we get the chain complex

$$\cdots \leftarrow \mathrm{Hom}_R(P_2, N) \leftarrow \mathrm{Hom}_R(P_1, N) \leftarrow \mathrm{Hom}_R(P_0, N) \leftarrow 0,$$

which we denote by $\mathrm{Hom}_R(P_\bullet, N)$.

Dually, given an injective resolution as in Definition 4.3.1, we can form the chain complex

$$\cdots \leftarrow \mathrm{Hom}_R(N, I_2) \leftarrow \mathrm{Hom}_R(N, I_1) \leftarrow \mathrm{Hom}_R(N, I_0) \leftarrow 0,$$

denoted by $\mathrm{Hom}_R(N, I_\bullet)$.

The result below is a combination of [14], Lemma 2.4.1 and Theorem 2.7.6.

Proposition 4.3.3. Let M, N be R-modules. For any projective resolution P and any injective resolution I of M, $H^*(\mathrm{Hom}_R(P_\bullet, N)) = H^*(\mathrm{Hom}_R(N, I_\bullet))$.

We define $\mathrm{Ext}^n_R(M, N) := H^n(\mathrm{Hom}_R(P_\bullet, N)) = H^n(\mathrm{Hom}_R(N, I_\bullet))$.

4.4 Setting up the Adams spectral sequence

Let X be a connective CW spectrum of finite type. Then $H^*(X)$ is an $\mathscr{A}_2$-module, since $H^i(X) \cong H^{i+n}(X_n)$ for sufficiently large n, so we can define $Sq^j : H^i(X) \to H^{i+j}(X)$ by evaluating $Sq^j : H^{i+n}(X_n) \to H^{i+j+n}(X_n)$ followed by enough suspensions. Note that we could also have first suspended $H^{i+n}(X_n)$ and $H^{i+j+n}(X_n)$ until they were both stable, and then evaluated Sq^j, but that these two $\mathscr{A}_2$-actions coincide by Proposition 2.0.1 (4).

We can pick generators α_i for $H^*(X)$ as an $\mathscr{A}_2$-module such that there are at most finitely many in each $H^n(X)$ (since each $H^n(X)$ is finitely generated by Lemma 3.2.1, and such a finite generating set would certainly also generate it as an $\mathscr{A}_2$-module). Each generator $\alpha_i \in H^{n_i}(X)$ corresponds to a map $X \to \mathbb{K}(\mathbb{F}_2, n_i)$ by Theorem 3.4.1, so putting these maps together gives an element of $\pi_i[X, \mathbb{K}(\mathbb{F}_2, n_i)]$. Now, $n_i \to \infty$ as $i \to \infty$ since there are only finitely many α_i in each $H^{n_i}(X)$, so Proposition 3.4.2 implies that we get an element of $[X, \bigvee_i \mathbb{K}(\mathbb{F}_2, n_i)]$. We write $K_0 := \bigvee_i \mathbb{K}(\mathbb{F}_2, n_i)$, and replace the map $X \to K_0$ by an inclusion (see Remark 3.3.3).

Remark 4.4.1. K_0 has finite type, which can be seen as follows: first, recall from Example 3.1.10 that each spectrum $\mathbb{K}(G, n_i)$ has finite type. Now, the j-cells of $\bigvee_i \mathbb{K}(G, n_i)$ consist of the $(j+k)$-cells of $\bigvee_i K(G, n_i + k)$ for each k, up to equivalence under suspension. However, there are only finitely many n_i with $n_i \leq j$, and if $n_i > j$ the space $K(G, n_i)$ can be taken to have no cells of dimension $\leq j$. Thus, the j-cells of $\bigvee_i \mathbb{K}(G, n_i)$ are the j-cells of the finite wedge $\bigvee_{i, n_i \leq j} \mathbb{K}(G, n_i)$, of which there are only finitely many (since a finite wedge of finite-type spectra has finite type).

Now, we set $X_1 = K_0/X$, and repeat the construction to get a diagram:

$$\begin{tikzcd}
X \arrow[r] & K_0 \arrow[rr] \arrow[dr] & & K_1 \arrow[rr] \arrow[dr] & & K_2 \arrow[rr] \arrow[dr] & & \cdots \\
& & K_0/X = X_1 \arrow[ur] & & K_1/X_1 = X_2 \arrow[ur] & & K_2/X_2 = X_3 \arrow[ur] &
\end{tikzcd}$$

Taking cohomology, we get a diagram

$$\begin{tikzcd}
0 & H^*(X) \arrow[l] & H^*(K_0) \arrow[l] \arrow[dr, leftarrow] & & H^*(K_1) \arrow[ll] \arrow[dl] \arrow[dr, leftarrow] & & H^*(K_2) \arrow[ll] \arrow[dl] \arrow[dr, leftarrow] & & \cdots \arrow[ll] \arrow[dl] \\
& & & H^*(X_1) \arrow[dl] \arrow[dr, leftarrow] & & H^*(X_2) \arrow[dl] \arrow[dr, leftarrow] & & H^*(X_3) \arrow[dl] \arrow[dr, leftarrow] & \\
& & 0 & & 0 & & 0 & & 0
\end{tikzcd} \tag{1}$$

- [Fact: The induced map
$$H^*(X) \leftarrow H^*(K_0)$$
is surjective, and thus by construction each sequence
$$0 \leftarrow H^*(X_s) \leftarrow H^*(K_s)$$
is exact.

  I'm extremely confused by the supposed surjectivity - going back to the sketch of the construction for CW complexes, the corresponding statement is supposed to be that if we choose generators α_i for $H^*(X)$ as an $\mathscr{A}_2$-module giving maps $f_i : X \to K(G, \deg(\alpha_i))$, the product of these maps induces a surjection on H^*. But I can't see how it does? So, we know by [5], Thm 4.57 that there are natural bijections $T : [X, K(G, n)] \to H^n(X; G)$ with $T[f] = f^*(\beta)$ for some class $\beta \in H^n(K(G, n); G)$. So, for any given $\alpha^i \in H^n(X; G)$, it's equal to $f^*(\beta_i)$, so definitely f^* hits it. But if I have some enormous product of EM spaces, and all these maps $f_i : X \to K(G, n_i)$ come together to give a map $f : X \to \prod_i K(G, n_i)$, then Hatcher is trying to tell me I get a surjection $H^*(\prod_i K(G, n_i)) \to H^*(X)$?? I know nothing about the cohomology of an infinite product, there's no reason it would be e.g. the product of the cohomologies.]

  [Something something reduced cohomology theories for spectra and the wedge axiom?? Suppose K_0 had finite type, which I'm not convinced it does. Then for some n, $(K_0)_n = (\bigvee_i \mathbb{K}(G, n_i))_n = \bigvee_i (\mathbb{K}(G, n_i))_n = \bigvee_i K(G, n + n_i)$, and $H^i(K_0) = H^{i+n}(\bigvee_i K(G, n + n_i)) = \prod_i H^{i+n}(K(G, n + n_i))$. Does that help?]]

Now, as with CW complexes, we have a long exact sequence

$$\cdots \leftarrow H^{n+1}(X_{s+1}) \leftarrow H^n(X_s) \leftarrow H^n(K_s) \leftarrow H^n(X_{s+1}) \leftarrow H^{n-1}(X_s) \leftarrow \cdots,$$

and surjectivity of the maps $H^*(X_s) \leftarrow H^*(K_s)$ implies that the boundary maps $H^{n+1}(X_{s+1}) \leftarrow H^n(X_s)$ are all zero (writing $X_0 := X$). We thus get short exact sequences

$$0 \leftarrow H^n(X_s) \leftarrow H^n(K_s) \leftarrow H^n(X_{s+1}) \leftarrow 0,$$

giving rise to a short exact sequence

$$0 \leftarrow H^*(X_s) \leftarrow H^*(K_s) \leftarrow H^*(X_{s+1}) \leftarrow 0,$$

for each s. This then implies that the top row of (1) is exact.

Now, each $H^*(K_s)$ is a free $\mathscr{A}_2$-module, since K_s has finite type and the cohomology of a wedge of Eilenberg-MacLane spaces $K(G, n_i)$ (with $n_i \geq n$ and only finitely many n_i of each dimension) is free below dimension $2n$ (this can be shown by combining [12] Cor 7.5.6 and the wedge axiom). Thus, the top row of (1) gives a free resolution of $H^*(X)$.

By Theorem 3.3.5, we obtain a long exact sequences

$$\cdots \to [\mathbb{S}^{t+1}, X_s] \to [\mathbb{S}^{t+1}, K_s] \to [\mathbb{S}^{t+1}, X_{s+1}] \to [\mathbb{S}^{t+1}, \Sigma X_s] \to [\mathbb{S}^{t+1}, \Sigma K_s] \to \cdots.$$

Using the isomorphism $[Y, Z] \cong [\Sigma Y, \Sigma Z]$, we get long exact sequences

$$\cdots \to \pi_{t+1}X_s \to \pi_{t+1}K_s \to \pi_{t+1}X_{s+1} \to \pi_t X_s \to \pi_t K_s \to \cdots,$$

which form the staircase diagram shown below.

$$\begin{tikzcd}
 & \vdots \arrow[d, "i"] & & \vdots \arrow[d, "i"] & & \vdots \arrow[d, "i"] & \\
\cdots \arrow[r, "k"] & \pi_{t+1}X_s \arrow[r, "j"] \arrow[d, "i"] & \pi_{t+1}K_s \arrow[r, "k"] & \pi_{t+1}X_{s+1} \arrow[r, "j"] \arrow[d, "i"] & \pi_{t+1}K_{s+1} \arrow[r, "k"] & \pi_{t+1}X_{s+2} \arrow[r, "j"] \arrow[d, "i"] & \cdots \\
\cdots \arrow[r, "k"] & \pi_t X_{s-1} \arrow[r, "j"] \arrow[d, "i"] & \pi_t K_{s-1} \arrow[r, "k"] & \pi_t X_s \arrow[r, "j"] \arrow[d, "i"] & \pi_t K_s \arrow[r, "k"] & \pi_t X_{s+1} \arrow[r, "j"] \arrow[d, "i"] & \cdots \\
\cdots \arrow[r, "k"] & \pi_{t-1}X_{s-2} \arrow[r, "j"] \arrow[d, "i"] & \pi_{t-1}K_{s-2} \arrow[r, "k"] & \pi_{t-1}X_{s-1} \arrow[r, "j"] \arrow[d, "i"] & \pi_{t-1}K_{s-1} \arrow[r, "k"] & \pi_{t-1}X_s \arrow[r, "j"] \arrow[d, "i"] & \cdots \\
 & \vdots & & \vdots & & \vdots &
\end{tikzcd} \tag{2}$$

This gives rise to a spectral sequence, by [some version of] Theorem 4.2.3.

Now, since $K_s = \bigvee_i \mathbb{K}(G, n_{s_i})$, Proposition 3.4.2, tells us that $[\mathbb{S}, K_s] \cong \prod_i [\mathbb{S}, \mathbb{K}(G, n_{s_i})]$, which is naturally isomorphic to $\prod_i H^{n_{s_i}}(\mathbb{S}; G)$. Thus, elements of $[\mathbb{S}, K_s]$ are tuples of elements of $H^*(\mathbb{S})$.

We have a map

$$\begin{aligned} [\mathbb{S}, K_s] &\to \operatorname{Hom}^0_{\mathscr{A}_2}(H^*(K_s), H^*(\mathbb{S})) \\ f &\mapsto f^*, \end{aligned}$$

since f^* is an $\mathscr{A}_2$-module homomorphism by Proposition 2.0.1 (1), and the fact that $H^*(K_s)$ is free implies that it is an isomorphism.

We thus have

$$[\Sigma^t \mathbb{S}, K_s] = \mathrm{Hom}^0_{\mathscr{A}_2}(H^*(K_s), H^*(\Sigma^t \mathbb{S})) = \mathrm{Hom}^t_{\mathscr{A}_2}(H^*(K_s), H^*(\mathbb{S})),$$

where $\mathrm{Hom}^t_{\mathscr{A}_2}(H^*(K_s), H^*(\mathbb{S}))$ is the set of algebra morphisms which lower the degree by t. In the case of CW complexes, we have $H^*(\Sigma^t X) \cong H^{*-t}(X)$. Since $\mathbb{S}$ has finite type, for i large enough we have $H^n(\Sigma^t \mathbb{S}) = H^{n+i}(\Sigma^t S^i) \cong H^{n+i-t}(S^i) = H^{n-t}(\mathbb{S})$.

Now, $E_1^{s,t} = \pi_t K_s = \mathrm{Hom}^t_{\mathscr{A}_2}(H^*(K_s), H^*(\mathbb{S}))$, since the staircase diagram comes from the exact couple

$$\begin{tikzcd} \pi_* X_* \arrow[rr, "i"] & & \pi_* X_* \arrow[dl, "j"] \\ & \pi_* K_* \arrow[ul, "k"] & \end{tikzcd}$$

where $i : \pi_{t+1} X_{s+1} \to \pi_t X_s$, $j : \pi_{t+1} X_s \to \pi_{t+1} K_s$, and $k : \pi_{t+1} X_{s+1}$ are as in (2). The differential $d_1 : \pi_t(K_s) \to \pi_t K_{s+1}$ is induced by the map $K_s \to K_{s+1}$, since it is defined to be $j \circ k$.

Further, $E_2^{s,t} = H^{s,t}(E_1^{*,*}, d_1)$, so each $E^{*,t}$ is the homology of the chain complex

$$0 \to E_1^{0,t} \to E^{1,t} \to E^{2,t} \to \cdots,$$

which is by construction the chain complex below.

$$0 \to \mathrm{Hom}^t_{\mathscr{A}_2}(H^*(K_0), H^*(\mathbb{S})) \to \mathrm{Hom}^t_{\mathscr{A}_2}(H^*(K_1), H^*(\mathbb{S})) \to \cdots.$$

The homology of this is by definition $\mathrm{Ext}^{s,t}_{\mathscr{A}_2}(H^*(X), H^*(\mathbb{S}))$, so $E_2^{s,t} = \mathrm{Ext}^{s,t}_{\mathscr{A}_2}(H^*(X), H^*(\mathbb{S}))$.

Theorem 4.4.2 ([6], Thm 5.47). There is a spectral sequence $\{E_r, d_r\}$ such that $E_2^{s,t} = \mathrm{Ext}^{s,t}_{\mathscr{A}_2}(\mathbb{F}_2, \mathbb{F}_2)$ and $\{E_r, d_r\} \implies \pi^S_{t-s}$ modulo torsion of odd order. [which is probably a lie but let's worry about that later]

4.5 First computations

We will say that a free resolution

$$\cdots \xrightarrow{f_3} F_2 \xrightarrow{f_2} F_1 \xrightarrow{f_1} F_0 \xrightarrow{f_0} H^*(X)$$

is *minimal* if $\mathrm{im}\, f_i \subseteq \mathscr{A}_2^+ F_{i-1}$ for all i, where $\mathscr{A}_2^+ \subseteq \mathscr{A}_2$ is the irrelevant ideal.

Lemma 4.5.1 ([6], Lem 5.49). For a minimal free resolution

$$\cdots \xrightarrow{f_3} F_2 \xrightarrow{f_2} F_1 \xrightarrow{f_1} F_0 \xrightarrow{f_1} H^*(X) \to 0$$

of $H^*(X)$ as an $\mathscr{A}_2$-module, we have $\mathrm{Ext}^{s,t}_{\mathscr{A}_2}(H^*(X), \mathbb{F}_2) = \mathrm{Hom}^t_{\mathscr{A}_2}(F_s, \mathbb{F}_2)$.

Proof. Let $x \in F_i$. Since $f_{i-1} f_i = 0$, we have $f_i(x) \in \ker f_{i-1} = \mathrm{im}\, f_i \subseteq \mathscr{A}_2^+ F_{i-1}$. We can thus write $f_i(x) = \sum_j a_j x_{i-1,j}$ with $a_j \in \mathscr{A}_2^+$. Now, for $g \in \mathrm{Hom}_{\mathscr{A}_2}(F_{i-1}, \mathbb{F}_2)$, we have $g f_i(x) = \sum_j a_j g(x_{i-1}, j) = 0$, since a_j acts trivially on elements of $\mathbb{F}_2$.

Thus, the boundary maps in the complex

$$\cdots \xleftarrow{-\circ f_3} \mathrm{Hom}_{\mathscr{A}_2}(F_2, \mathbb{F}_2) \xleftarrow{-\circ f_2} \mathrm{Hom}_{\mathscr{A}_2}(F_1, \mathbb{F}_2) \xleftarrow{-\circ f_1} \mathrm{Hom}_{\mathscr{A}_2}(F_0, \mathbb{F}_2) \leftarrow 0$$

are all zero, so $\mathrm{Ext}^{s,t}_{\mathscr{A}_2}(H^*(X), \mathbb{F}_2) = \mathrm{Hom}^t_{\mathscr{A}_2}(F_s, \mathbb{F}_2)$. □

Now, since $\mathbb{F}_2$ is concentrated in degree 0, the only elements of F_s which can be sent to $1 \in \mathbb{F}_2$ are the elements of degree t, so for every generator of F_s in degree t, there is an $\mathbb{F}_2$ summand in $\mathrm{Hom}^t_{\mathscr{A}_2}(F_s, \mathbb{F}_2)$.

Figure 4.1 shows part of a construction of a minimal free resolution of $H^*(\mathbb{S}) = \mathbb{F}_2$, where position $(t-s, s)$ consists of degree $t-s$ elements of F_s. Instead of inducting on s and calculating column by column, we will instead induct on $t-s$, assuming the previous rows have been computed. Note that since we will add the minimum number of generators needed in each row, the addition of new generators in later rows will not affect the induction (because a new generator will not impact the kernel of any f_i).

For the first row, at position $(0,0)$, we need a generator $\iota \in F_0$ to map to $1 \in \mathbb{F}_2$, in order to make f_0 a surjection. The kernel of f_0 thus contains every multiple of ι by an element of $\mathscr{A}_2$, which by exactness should be contained in the image of f_1. Thus, we need a new generator α_1^1 at $(0,1)$ mapping to $Sq^1\iota$. The element $Sq^1\alpha_1^1 \in F_1$ is therefore in the kernel of f_1, since $Sq^1Sq^1 = 0$, so we need a generator α_2^2 at $(0,2)$ mapping to $Sq^1\alpha_1^1$. Now, it is clear that each position $(0,s)$ will require a new generator α_s^s, since each $Sq^1\alpha_{s-1}^{s-1}$ maps to $Sq^1Sq^1\alpha_{s-2}^{s-2} = 0$, so the first row is completely determined, and $\mathrm{Ext}^{s,s}_{\mathscr{A}_2}(\mathbb{F}_2, \mathbb{F}_2) = \mathbb{F}_2$.

In the second row, a generator α_2^1 is needed in position $(1,1)$ mapping to $Sq^2\iota$, since $f_1(Sq^1\alpha_1^1) = 0$ but $Sq^2\iota \in \ker f_0$. No other generators are needed, since $Sq^1\alpha_2^1$ maps to $Sq^3\iota$ and $Sq^2\alpha_s^s$ maps to $Sq^2Sq^1\alpha_{s-1}^{s-1} \neq 0$ for all $s > 1$.

In row three, generators α_4^1, α_5^2, and α_6^3 are needed to map to $Sq^4\iota$, $Sq^4\alpha_1^1 + Sq^2Sq^1\alpha_2^1 + Sq^1\alpha_4^1$, and $Sq^4\alpha_2^2 + Sq^2\alpha_4^2 + Sq^1\alpha_5^2$ respectively, since the latter elements are in the kernel of their respective f_i's. No new generators are needed after $s = 4$, since $Sq^1\alpha_6^3$ maps to $Sq^5\alpha_2^2 + Sq^3\alpha_4^2$, $Sq^4\alpha_s^s$ maps to $Sq^4Sq^1\alpha_{s-1}^{s-1}$, and although $Sq^3Sq^1\alpha_s^s$ maps to zero, it is hit by $Sq^3\alpha^{s+1,s+1}$.

Continuing in this fashion, the computations for rows 4 and 5 are shown in 4.1, though the Adem relations required to justify them are not. Note that although to compute each row, knowledge of maps involving the next two rows is required, rows 6 and 7 do not contain all the new generators needed.

From Figure 4.2, we see that $(\pi_1^S)_2^\wedge$ has order dividing 2, $(\pi_2^S)_2^\wedge = \mathbb{Z}/2\mathbb{Z}$, $(\pi_3^S)_2^\wedge$ has order 8, and $(\pi_4^S)_2^\wedge = (\pi_5^S)_2^\wedge = 0$. However, we do not currently have the tools to determine whether or not α_2^1 survives to the E_∞ page, or the isomorphism class of $(\pi_3^S)_2^\wedge$. We will therefore spend some time describing a multiplication on the Adams spectral sequence which will allow us to resolve such ambiguities.

4.6 Multiplicative structure

[Intro. Sources: [13], [12], [5]]

4.6.1 The Yoneda product

Definition 4.6.1 ([12], Def 11.8.1). For any algebra A and A-modules L, M, N, there is a product, the *Yoneda product*

$$\circ : \mathrm{Ext}^{s,t}_A(M,N) \otimes \mathrm{Ext}^{u,v}_A(L,M) \to \mathrm{Ext}^{s+u}_A(L,N),$$

defined as follows: let

$$\cdots \xrightarrow{f_3} F_2 \xrightarrow{f_2} F_1 \xrightarrow{f_1} F_0 \xrightarrow{f_0} L \to 0,$$

$t-s$ \ s	0	1	2	3	4
0	ι	α^1_1	α^2_2	α^3_3	α^4_4
1	$Sq^1\iota$	$Sq^1\alpha^1_1$ α^1_2	$Sq^1\alpha^2_2$	$Sq^1\alpha^3_3$	$Sq^1\alpha^4_4$
2	$Sq^2\iota$	$Sq^2\alpha^1_1$ $Sq^1\alpha^1_2$	$Sq^2\alpha^2_2$ α^2_4	$Sq^2\alpha^3_3$	$Sq^2\alpha^4_4$
3	$Sq^2Sq^1\iota$ $Sq^3\iota$	$Sq^2Sq^1\alpha^1_1$ $Sq^3\alpha^1_1$ $Sq^2\alpha^1_2$ α^1_4	$Sq^2Sq^1\alpha^2_2$ $Sq^3\alpha^2_2$ $Sq^1\alpha^2_4$ α^2_5	$Sq^2Sq^1\alpha^3_3$ $Sq^3\alpha^3_3$ α^3_6	$Sq^2Sq^1\alpha^4_4$ $Sq^3\alpha^4_4$
4	$Sq^3Sq^1\iota$ $Sq^4\iota$	$Sq^3Sq^1\alpha^1_1$ $Sq^4\alpha^1_1$ $Sq^2Sq^1\alpha^1_2$ $Sq^3\alpha^1_2$ $Sq^1\alpha^1_4$	$Sq^3Sq^1\alpha^2_2$ $Sq^4\alpha^2_2$ $Sq^2\alpha^2_4$ $Sq^1\alpha^2_5$	$Sq^3Sq^1\alpha^3_3$ $Sq^4\alpha^3_3$ $Sq^1\alpha^3_6$	$Sq^3Sq^1\alpha^4_4$ $Sq^4\alpha^4_4$
5	$Sq^4Sq^1\iota$ $Sq^5\iota$	$Sq^4Sq^1\alpha^1_1$ $Sq^5\alpha^1_1$ $Sq^3Sq^1\alpha^1_2$ $Sq^4\alpha^1_2$ $Sq^2\alpha^1_4$	$Sq^4Sq^1\alpha^2_2$ $Sq^5\alpha^2_2$ $Sq^2Sq^1\alpha^2_4$ $Sq^3\alpha^2_4$ $Sq^2\alpha^2_5$	$Sq^4Sq^1\alpha^3_3$ $Sq^5\alpha^3_3$ $Sq^2\alpha^3_6$	$Sq^4Sq^1\alpha^4_4$ $Sq^5\alpha^4_4$
6	$Sq^5Sq^1\iota$ $Sq^4Sq^2\iota$ $Sq^6\iota$	$Sq^5Sq^1\alpha^1_1$ $Sq^4Sq^2\alpha^1_1$ $Sq^6\alpha^1_1$ $Sq^4Sq^1\alpha^1_2$ $Sq^5\alpha^1_2$ $Sq^2Sq^1\alpha^1_4$ $Sq^3\alpha^1_4$	$Sq^5Sq^1\alpha^2_2$ $Sq^4Sq^2\alpha^2_2$ $Sq^6\alpha^2_2$ $Sq^3Sq^1\alpha^2_4$ $Sq^4\alpha^2_4$ $Sq^2Sq^1\alpha^2_5$ $Sq^3\alpha^2_5$	$Sq^5Sq^1\alpha^3_3$ $Sq^4Sq^2\alpha^3_3$ $Sq^6\alpha^3_3$ $Sq^2Sq^1\alpha^3_6$ $Sq^3\alpha^3_6$	$Sq^5Sq^1\alpha^4_4$ $Sq^4Sq^2\alpha^4_4$ $Sq^6\alpha^4_4$
7	$Sq^4Sq^2Sq^1\iota$ $Sq^6Sq^1\iota$ $Sq^5Sq^2\iota$ $Sq^7\iota$	$Sq^4Sq^2Sq^1\alpha^1_1$ $Sq^6Sq^1\alpha^1_1$ $Sq^5Sq^2\alpha^1_1$ $Sq^7\alpha^1_1$ $Sq^5Sq^1\alpha^1_2$ $Sq^4Sq^2\alpha^1_2$ $Sq^6\alpha^1_2$ $Sq^3Sq^1\alpha^1_4$ $Sq^4\alpha^1_4$	$Sq^4Sq^2Sq^1\alpha^2_2$ $Sq^6Sq^1\alpha^2_2$ $Sq^5Sq^2\alpha^2_2$ $Sq^7\alpha^2_2$ $Sq^4Sq^1\alpha^2_4$ $Sq^5\alpha^2_4$ $Sq^3Sq^1\alpha^2_5$ $Sq^4\alpha^2_5$	$Sq^4Sq^2Sq^1\alpha^3_3$ $Sq^6Sq^1\alpha^3_3$ $Sq^5Sq^2\alpha^3_3$ $Sq^7\alpha^3_3$ $Sq^3Sq^1\alpha^3_3$ $Sq^4\alpha^3_3$	$Sq^4Sq^2Sq^1\alpha^4_4$ $Sq^6Sq^1\alpha^4_4$ $Sq^5Sq^2\alpha^4_4$ $Sq^7\alpha^4_4$

Figure 4.1: A construction of a minimal free resolution of $H^*(\mathbb{S}) = \mathbb{F}_2$. Generators are highlighted in pink.

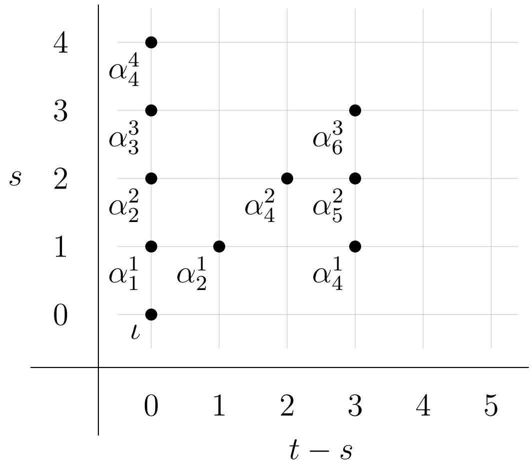


Figure 4.2: $\mathrm{Ext}_{\mathscr{A}_2}^{s,t}(\mathbb{F}_2, \mathbb{F}_2)$ for $t - s \leq 5$.

$$\cdots \xrightarrow{f_3'} F_2' \xrightarrow{f_2'} F_1' \xrightarrow{f_1'} F_0' \xrightarrow{f_0'} M \to 0$$

be free resolutions for L and M. Then, given $[g] \in \mathrm{Ext}_A^{s,t}(M, N)$, $[h] \in \mathrm{Ext}_A^{u,v}(L, M)$, we inductively construct a chain map $h_\bullet : F_{u+\bullet} \to F_\bullet'[v]$, as shown in the diagram below.

$$\begin{tikzcd}
F_{u+s} \arrow[r, "f_{u+s}"] \arrow[d, dashed, "h_s"'] & F_{u+s-1} \arrow[r, "f_{u+s-1}"] \arrow[d, dashed, "h_{s-1}"'] & \cdots \arrow[r, "f_{u+1}"] & F_u \arrow[r, "f_u"] \arrow[d, dashed, "h_0"'] \arrow[dr, "h"] & F_{u-1} \arrow[r, "f_{u-1}"] & \cdots \arrow[r, "f_1"] & F_0 \arrow[r, "f_0"] & L \\
F_s'[v] \arrow[r, "f_s'"'] \arrow[d, "g"'] & F_{s-1}'[v] \arrow[r, "f_{s-1}'"'] & \cdots \arrow[r, "f_1'"'] & F_0'[v] \arrow[r, two heads, "f_0'"'] & M[v] & & & \\
N[v+t] & & & & & & &
\end{tikzcd}$$

The map h_0 is defined as follows: let $\alpha \in F_u$ be a generator, and consider $h(\alpha) \in M[v]$. Since f_0' is surjective, there exists some $\beta \in F_0'[v]$ such that $f_0'(\beta) = h(\alpha)$. We define $h_0(\alpha) = \beta$. Now, suppose the h_i have been constructed for $i < w$, and consider the diagram below.

$$\begin{tikzcd}
F_{u+w} \arrow[r, "f_{u+w}"] \arrow[d, dashed, "h_w"'] & F_{u+w-1} \arrow[r, "f_{u+w-1}"] \arrow[d, "h_{w-1}"] & F_{u+w-2} \arrow[d, "h_{w-2}"] \\
F_w'[v] \arrow[r, "f_w'"'] & F_{w-1}'[v] \arrow[r, "f_{w-1}'"'] & F_{w-2}'[v]
\end{tikzcd}$$

Let $\alpha \in F_{u+w}$ be a generator, and consider $f_{w-1}' h_{w-1} f_{u+w}(\alpha) \in F_{w-2}'[v]$. By induction, the right square commutes, so $f_{w-1}' h_{w-1} f_{u+w}(\alpha) = h_{w-2} f_{u+w-1} f_{u+w}(\alpha) = 0$, by exactness of the top row. Thus, $h_{w-1} f_{u+w}(\alpha) \in \ker f_{w-1}' = \operatorname{im} f_w'$. Write $h_{w-1} f_{u+w}(\alpha) = f_w'(\beta)$, and define $h_w(\alpha) = \beta$.

Now, consider the diagram below.

$$\begin{tikzcd}
F_{u+s+1} \arrow[r, "f_{u+s+1}"] \arrow[d, dashed, "h_{s+1}"'] & F_{u+s} \arrow[d, "h_s"] \\
F'_{s+1}[v] \arrow[r, "f'_{s+1}"'] & F'_s[v] \arrow[d, "g"] \\
& N[v+t]
\end{tikzcd}$$

We have $gh_sf_{u+s+1} = gf'_{s+1}h_{s+1} = 0$, since $[g] \in \mathrm{Ext}_A^{s,t}(F'_s, N)$, so $[gh_s] \in \mathrm{Ext}_A^{u+s,v+t}$. We thus define $[g] \cdot [h] = [gh_s]$.

This definition is independent of the lifts chosen, which can be seen as follows. Suppose we have two chain maps $\{h_i\}, \{h'_i\}$; we will construct a chain homotopy between them. Define $k_0 : F_{u-1} \to F'_0[v]$ to be the zero map. By construction, $f'_0h_0 = f'_0h'_0 = h$, so $f'_0(h_0 - h'_0) = 0$. Thus, $\mathrm{im}(h_0 - h'_0) \subseteq \ker f'_0 = \mathrm{im}\, f'_1$, so $h_0 - h'_0 = f'_1k_1 = f'_1k_1 + k_0f_u$ for some map $k_1 : F_u \to F'_1[v]$. Now, suppose we have k_i, k_{i-1} such that $h_{i-1} - h'_{i-1} = f'_ik_i + k_{i-1}f_{u+i-1}$. Then $f'_ih_i = h_{i-1}f_{u+i}$ and $f'_ih'_i = h'_{i-1}f_{u+i}$, so $f'_i(h_i - h'_i) = (h_{i-1} - h'_{i-1})f_{u+i} = (f'_ik_i + k_{i-1}f_{u+i-1})f_{u+i} = f'_ik_if_{u+i}$, and thus we can construct k_{i+1} such that $h_i - h'_i = f'_{i+1}k_{i+1} + k_if_{u+i}$. Now, $g(h_s - h'_s) = g(f'_{s+1}k_{s+1} + k_sf_{u+s}) = gk_sf_{u+s}$, and therefore $[g(h_s - h'_s)] = [gk_sf_{u+s}] = [0]$.

Finally, if $h = lf_u$ for some $l : F_{u-1} \to M[v]$, with filling $\{l_i\}$, then $\{l_if_{u+i}\}$ is a filling for h, so $[g] \cdot [h] = [gl_sf_{u+s}] = [0]$. On the other hand, if $g = mf'_s$, then $[g] \cdot [h] = [gh_s] = [mf'_sh_s] = [mh_{s-1}f_{u+s}] = [0]$. Thus, the Yoneda product is well defined.

4.6.2 The composition product

Definition 4.6.2 ([13], p47). Let X, Y, Z be spectra. The *composition pairing* $\circ : [Y, Z]_* \otimes [X, Y]_* \to [X, Z]_*$ is defined as follows:

$$\begin{aligned} \circ :\ [Y, Z]_v \ \otimes\ [X, Y]_t \ &\to\ [X, Z]_{v+t} \\ [g : \Sigma^v Y \to Z] \otimes [f : \Sigma^t X \to Y] &\mapsto [g \circ \Sigma^v f : \Sigma^{v+t} X \to Z], \end{aligned}$$

where $[X, Y]_n = [\Sigma^n X, Y]$.

In particular, if $X = Y = Z = \mathbb{S}$, we have a product $\pi_v^S \otimes \pi_t^S \to \pi_{v+t}^S$.

Lemma 4.6.3. *Let $f, g : S^n \to S^n$ be pointed maps such that $\deg f = \deg g$. Then $f \simeq g$.*

Proof. We prove the contrapositive. Suppose $f \not\simeq g$. Then f, g represent two different elements in $\pi_n S^n \simeq \mathbb{Z}$, say $[f] = n \neq m = [g]$ for $n, m \in \mathbb{Z}$. The Hurewicz theorem then implies that for a fixed generator $u_n \in H^n(S^n)$, $f_*(u_n) \neq g_*(u_n) \in H^n(S^n)$, so $\deg f \neq \deg g$, as required. □

Lemma 4.6.4 ([5], Prop 4.56). *The composition product makes π_*^S into a graded commutative ring.*

(proof)

Lemma 4.6.5. *There is a unique ring structure on $(\pi_*^S)_2^\wedge$ which makes the completion map $c : \pi_*^S \to (\pi_*^S)_2^\wedge$ into a ring homomorphism.*

Proof. We show uniqueness first. Let $f \in (\pi_i^S)_2^\wedge$, $g \in (\pi_j^S)_2^\wedge$. If $i, j \geq 1$, then the completion map is surjective, so $f = c(\tilde{f}), g = c(\tilde{g})$ for some $\tilde{f} \in \pi_i^S, \tilde{g} \in \pi_j^S$. Then $fg = c(\tilde{f})c(\tilde{g}) = c(\tilde{f}\tilde{g})$.

If $i = 0, j \geq 1$, then let $\hat{f} \in \pi_0^S$ be a lift of $q(f) \in \pi_0^S/2^r\pi_0^S$, where 2^r is the highest power of 2 dividing the order of π_j^S. Then $f \equiv c(\hat{f}) \mod 2^r$, so $f = c(\hat{f}) + 2^r w$. We have $fg = fc(\tilde{g}) = (c(\hat{f}) + 2^r w)c(\tilde{g}) = c(\hat{f})c(\tilde{g}) + 2^r(wc(\tilde{g})) = c(\hat{f}\tilde{g}) \in (\pi_j^S)_2^\wedge$.

Finally, if $i = j = 0$, we claim that any two multiplications on $\mathbb{Z}_2$ which agree on $\mathbb{Z}$ must agree on all of $\mathbb{Z}_2$, and thus the multiplication is given by the usual product on $\mathbb{Z}_2$.

Suppose not; let $\star, \cdot$ be two products on $\mathbb{Z}_2$, agreeing on $\mathbb{Z}$, with $f \star g \neq f \cdot g$. Then there is some k such that $f \star g \not\equiv f \cdot g \mod k$. Pick integers n, m such that $n \equiv f \mod k$ and $m \equiv g \mod k$. Then, modulo k, $f \cdot g \equiv n \cdot m = n \star m \equiv f \star g$, giving a contradiction.

Now, for $i, j \geq 1$, this multiplication is well defined, since if $\tilde{f}' = \tilde{f} + t$, with $nt = 0$ for odd n, then $c(\tilde{f}'\tilde{g}) = c(\tilde{f}\tilde{g} + t\tilde{g}) = c(\tilde{f}\tilde{g})$ (since $nt\tilde{g} = (nt)\tilde{g} = 0\tilde{g} = 0$). Likewise, if $\tilde{g}' = \tilde{g} + t$, then $c(\tilde{f}\tilde{g}') = c(\tilde{f}\tilde{g})$. Note that the product for $i = 0, j \geq 1$ (and vice versa) is exactly the isomorphism in the proof of Lemma 3.5.2, and the usual product on $\mathbb{Z}_2$ is of course well-defined. Finally, associativity, distributivity, and unitality are inherited from π_*^S. □

Now, given spectra X, Y, Z, we can define a pairing $\circ : [Y, {}_2Z]_* \otimes [X, {}_2Y]_* \to [X, {}_2Z]$ as follows: let $f \in [Y, {}_2Z]_s, g \in [X, {}_2Y]_t$. By Theorem 3.5.6, there exists a unique (up to homotopy) map $\overline{f} : (\Sigma^s Y)_2^\wedge \to Z_2^\wedge$ such that f factors through $\overline{f}$. Now, note that $(\Sigma^s Y)_2^\wedge \simeq \Sigma^s Y_2^\wedge$, since $\pi_i(\Sigma^s Y) = \pi_{i-s}(Y)$. We can thus define the pairing of f and g to be $\overline{f} \circ \Sigma^s g$, as shown below.

$$\begin{tikzcd} & \Sigma^s Y \arrow[d] \arrow[dr, "f"] & \\ \Sigma^{s+t} X \arrow[r, "\Sigma^s g"'] & \Sigma^s Y_2^\wedge \arrow[r, dashed, "\overline{f}"'] & Z_2^\wedge \end{tikzcd}$$

Lemma 4.6.6. The completion map $c_* : \pi_*^S \to \pi_*(\mathbb{S}_2^\wedge)$ is a ring homomorphism. In particular, by Lemma 4.6.5, the composition product on $\pi_*(\mathbb{S}_2^\wedge)$ coincides with the product on $(\pi_*^S)_2^\wedge$ inherited from π_*^S, so the two groups are also isomorphic as rings.

Proof. Let $f : \mathbb{S}^i \to \mathbb{S}$, $g : \mathbb{S}^j \to \mathbb{S}$ be elements of π_i^S and π_j^S respectively. Then $c_*(f)c_*(g) = (cf)(cg)$ is given by factorising $cg = \overline{cg}c$ and composing to get $\overline{cg}c\Sigma^j f$. We thus have the commutative diagram below.

$$\begin{tikzcd} \mathbb{S}^{i+j} \arrow[r, "\Sigma^j f"] & \mathbb{S}^j \arrow[r, "g"] \arrow[d, "c"] & \mathbb{S} \arrow[d, "c"] \\ & \Sigma^j \mathbb{S}_2^\wedge \arrow[r, "\overline{cg}"'] & \mathbb{S}_2^\wedge \end{tikzcd}$$

The upper path is exactly $c_*(fg)$, so $c_*(f)c_*(g) = c_*(fg)$. Further, the completion map clearly preserves the identity, so it is a ring homomorphism. □

4.6.3 Multiplication on the Adams spectral sequence

Definition 4.6.7 ([13], Def 5.5). Let $\{'E_r\}, \{''E_r\}, \{E_r\}$ be three spectral sequences. A *pairing* of these spectral sequences is a sequence of homomorphisms

$$\phi_r : {}'E_r^{*,*} \otimes {}''E_r^{*,*} \to E_r^{*,*},$$

such that the Leibniz rule

$$d_r\phi_r(x\otimes y) = \phi_r(d_r(x)\otimes y) + (-1)^{\deg x}\phi_r(x\otimes d_r(y))$$

holds, and

$$\phi_{r+1}([x]\otimes[y]) = [\phi_r(x\otimes y)], \tag{3}$$

where $[x] \in {}'E^{*,*}_{r+1}$ is the homology class of a d_r-cycle $x \in {}'E^{*,*}_r$, and similarly for y and the right hand side.

A spectral sequence pairing $\{\phi_r\}$ induces a pairing

$$\phi_\infty : {}'E^{*,*}_\infty \otimes {}''E^{*,*}_\infty \to E^{*,*}_\infty.$$

THEOREM 4.6.8 ([13], Thm 5.8). Let X, Y, Z be spectra, with Y, Z connective and of finite type. There is a pairing of spectral sequences

$$E^{*,*}_r(Y,Z)\otimes E^{*,*}_r(X,Y) \to E^{*,*}_r(X,Z)$$

which agrees for $r = 2$ with the Yoneda pairing

$$\mathrm{Ext}^{*,*}_{\mathscr{A}_2}(H^*(Z),H^*(Y))\otimes \mathrm{Ext}^{*,*}_{\mathscr{A}_2}(H^*(Y),H^*(X)) \to \mathrm{Ext}^{*,*}_{\mathscr{A}_2}(H^*(Z),H^*(X))$$

and which converges to the composition pairing

$$[Y, Z^\wedge_2]_* \otimes [X, Y^\wedge_2]_* \to [X, Z^\wedge_2]_*.$$

The pairing is associative and unital.

REMARK 4.6.9. Condition (3) of Definition 4.6.7 ensures that if a product is computed on the E_2 page, and both terms survive to the E_r page for some $r > 2$, then the computation is still valid on that page.

5 Calculating stable homotopy groups

[Some intro. Sources: [13], [12]]

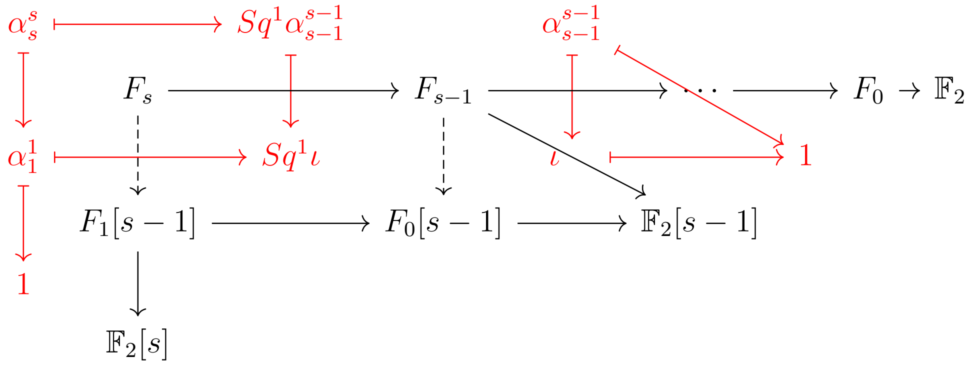
5.1 Resolving extensions

PROPOSITION 5.1.1 ([13], Cor 6.5). We have the following relations:

$$\begin{aligned}
\alpha^i_i &= (\alpha^1_1)^i\\
\alpha^2_4 &= (\alpha^1_2)^2\\
\alpha^2_5 &= \alpha^1_1\alpha^1_4\\
\alpha^3_6 &= (\alpha^1_1)^2\alpha^1_4 = (\alpha^1_2)^3.
\end{aligned}$$

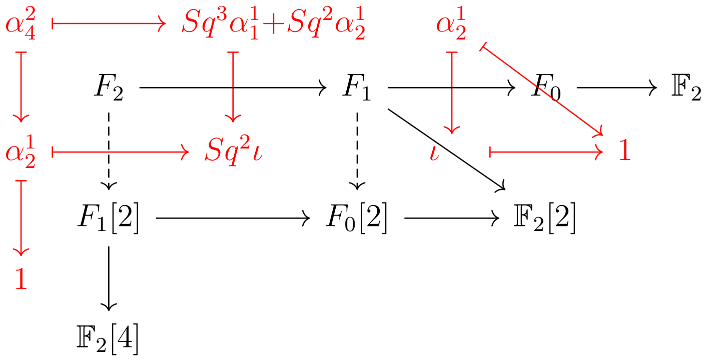
PROOF. We show the first two relations; the final two are obtained similarly.

Consider the diagram below.



Since $\alpha^{s-1}_{s-1} \in F_{s-1}$ is the only generator of degree $s-1$, to write down a lift $F_{s-1} \to F_0[s-1]$ it suffices to say where α^{s-1}_{s-1} is sent. In order for the right triangle to commute, we must send α^{s-1}_{s-1} to ι. Now, to write down a lift $F_s \to F_1[s-1]$, it again suffices to write down the image of α^s_s. In order for the left square to commute, we must send α^s_s to α^1_1. The composite map $F_s \to \mathbb{F}_2[2]$ is the unique map sending α^s_s to 1, so $\alpha^1_1 \cdot \alpha^{s-1}_{s-1} = \alpha^s_s$ for all $s > 0$. Thus, $\alpha^s_s = (\alpha^1_1)^s$.

Similarly, the calculation below shows that $\alpha^1_2 \cdot \alpha^1_2 = \alpha^2_4$.



□

From now on, we will write h_i for the generator $\alpha^1_{2^i}$.

Proposition 5.1.2. Suppose $\alpha \in (\pi^S_i)^\wedge_2$ represents $a \in E_\infty$. Then 2α represents $h_0 a$. In other words, multiplication by h_0 is induced by multiplication by 2.

Proof. Recall that $\pi^S_0 = \mathbb{Z}$, since $\pi_1 S^1 = \mathbb{Z}$ and $n = 1 \leq 2 = 2(1)$, so this lies in the stable region. Now, $E^{s,s}_r(\mathbb{S})$ converges to some filtration of $\mathbb{Z}_2$ whose quotients are all $\mathbb{Z}/2\mathbb{Z}$. The filtration must therefore be

$$\cdots \subseteq 4\mathbb{Z}_2 \subseteq 2\mathbb{Z}_2 \subseteq \mathbb{Z}_2,$$

since finite index subgroups of $\mathbb{Z}_p$ are of the form $p^k\mathbb{Z}_p$.

Thus, $\iota = [1] \in \mathbb{Z}_2/2\mathbb{Z}_2$, and by computing the Yoneda product we see that ι is a unit. We also have $h_0 = [2] \in 2\mathbb{Z}_2/4\mathbb{Z}_2$ so $h_0 = [2] = [2[1]] = [2\iota]$, and hence h_0 acts on ι by multiplication by 2. Now, for any $\kappa \in E^{s,t}_r(\mathbb{S})$, $h_0 \cdot \kappa = (\iota h_0) \cdot \kappa = 2\kappa \in E^{s+1,t+1}_r(\mathbb{S})$. □

Lemma 5.1.3. There are no nontrivial differentials for $t - s \leq 5$.

Proof. First, note that the only possible nontrivial differentials in this range are the differentials $d_r : E^{1,2}_r(\mathbb{S}) \to E^{1+r,1+r}_r(\mathbb{S})$. Now, $0 = d_r(h_0 h_1) = d_r(h_0)h_1 + h_0 d_r(h_1) = h_0 d_r(h_1)$, so $d_r(h_1) = 0$. Since $E^{1,2}_r(\mathbb{S})$ is generated by h_1, we must have $d_r = 0$. □

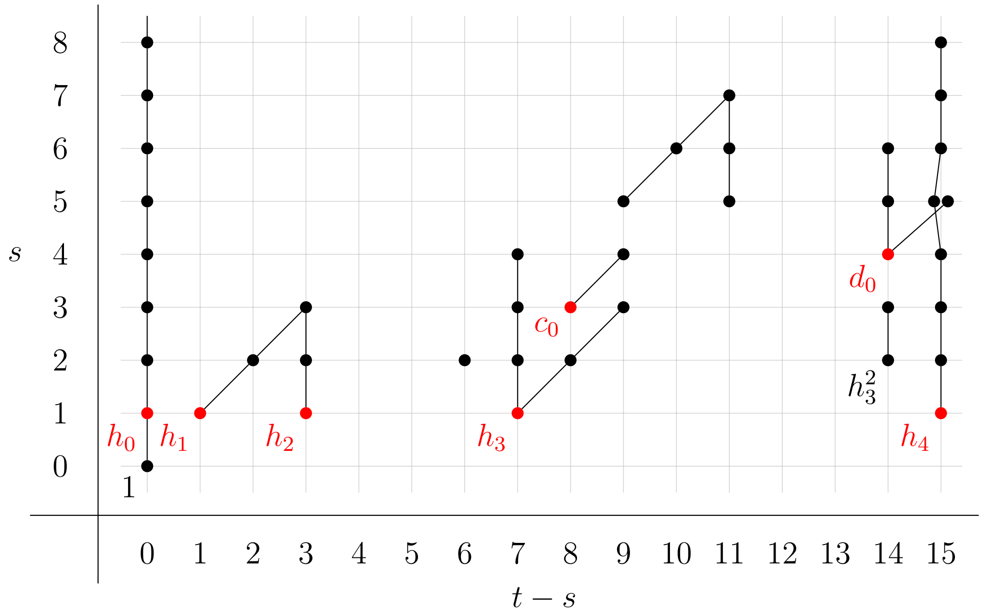


Figure 5.1: $\operatorname{Ext}_{\mathscr{A}_2}^{s,t}(\mathbb{F}_2,\mathbb{F}_2)$ for $t-s \le 15$. The vertical and diagonal lines indicate multiplication by h_0 and h_1 respectively. Some of the algebra generators are shown in red, with their standard names.

THEOREM 5.1.4.

$$(\pi_i^S)_2^\wedge = \begin{cases} \mathbb{Z}/2\mathbb{Z} & i = 1,2 \\ \mathbb{Z}/8\mathbb{Z} & i = 3 \\ 0 & i = 4,5. \end{cases}$$

5.2 The E_2 page for $t - s \le 15$

LEMMA 5.2.1. There are no nontrivial differentials for $t - s \le 13$.

PROOF. We have shown in Lemma 5.1.3 that there are no nontrivial differentials for $t - s \le 5$; the only remaining possibility is that $d_2 : E_2^{2,10}(\mathbb{S}) \to E_2^{4,11}(\mathbb{S})$ is nonzero.

From Figure 5.2, we see that $E_2^{2,10}(\mathbb{S})$ is generated by h_1h_3, and $d_2(h_1h_3) = d_2(h_1)h_3 + h_1d_2(h_3) = 0 + 0 = 0$ (the first factor is zero by Lemma 5.1.3, and the second is an element of a trivial group). □

THEOREM 5.2.2.

$$(\pi_i^S)_2^\wedge = \begin{cases} \mathbb{Z}/2\mathbb{Z} & i = 6, 10, \\ \mathbb{Z}/16\mathbb{Z} & i = 7, \\ (\mathbb{Z}/2\mathbb{Z})^2 & i = 8, \\ (\mathbb{Z}/2\mathbb{Z})^3 & i = 9, \\ \mathbb{Z}/8\mathbb{Z} & i = 11, \\ 0 & i = 12, 13. \end{cases}$$

5.3 Differentials at $14 \le t - s \le 15$

For $t - s < 14$, the computation of $(\pi_{t-s}^S)_2^\wedge$ is purely mechanical, since there are no nontrivial differentials in this range. However, the first nonzero differential will appear at $t - s = 15$,

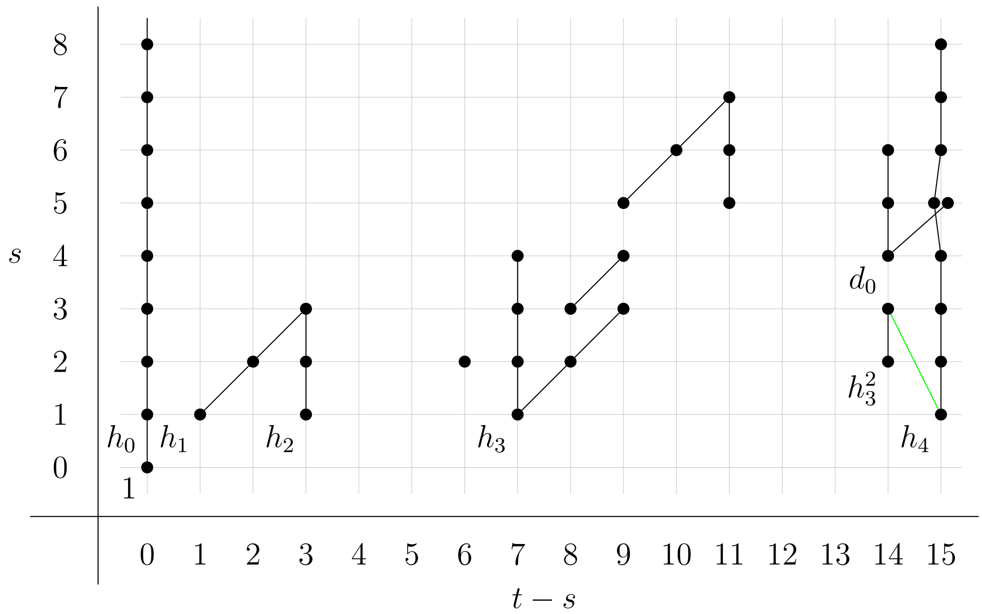


Figure 5.2: The E_2 page of the Adams spectral sequence for $\mathbb{S}$, in the range $t-s \leq 15$; the unique d_2 differential is shown in green.

and in fact there are many differentials after this point, though we will only fully compute those in the range $14 \leq t-s \leq 15$. In general, the problem of computing differentials is much harder than determining $\mathrm{Ext}^{s,t}_{\mathscr{A}_2}(\mathbb{F}_2, \mathbb{F}_2)$, and is not algorithmic.

Theorem 5.3.1 ([12], Thm 11.10.2). $d_2(h_4) = h_0 h_3^2$.

Proof. We have shown that h_0 detects $2 \in (\pi_*^S)_2^\wedge$ (i.e. 2 is a representative for h_0). Let $\sigma \in (\pi_7^S)_2^\wedge$ be a representative for h_3. Then $2\sigma^2$ is a representative for $h_0 h_3^2$. By graded commutativity of $(\pi_*^S)_2^\wedge$, $\sigma^2 = -\sigma^2$, so $2\sigma^2 = 0$, and thus $h_0 h_3^2 = 0$ in $E_\infty^{3,17}(\mathbb{S})$. Therefore, $h_0 h_3^2$ is the boundary of a differential, and the only possibility is $d_2(h_4) = h_0 h_3^2$. □

The d_2 differentials at $E_2^{2,17}(\mathbb{S}), E_2^{3,18}(\mathbb{S}), E_2^{4,19}(\mathbb{S})$ are all trivial: $d_2(h_0^n h_4) = h_0^n d_2(h_4) = h_0^{n-1}(h_0 h_3^2) = 0$.

There are two possible d_3 differentials for $t-s \leq 15$ (emanating from $E_3^{2,17}$ and $E_3^{3,18}$), and in fact it will turn out that both are nontrivial. The method of proof will be by comparison to the Adams spectral sequence of a different spectrum, so we will first state a result comparing the Adams spectral sequences of two spectra with a map between them.

Theorem 5.3.2 ([13], Cor 4.17). Let $f : Y \to Z$ be a map of connective spectra of finite type. Then there is a map

$$f_* : \{E_r(Y), d_r\}_r \to \{E_r(Z), d_r\}_r$$

of Adams spectral sequences, given at the E_2-level by the homomorphism

$$(f^*)^* : \mathrm{Ext}^{s,t}_{\mathscr{A}_2}(H^*(Y), \mathbb{F}_2) \to \mathrm{Ext}^{s,t}_{\mathscr{A}_2}(H^*(Z), \mathbb{F}_2)$$

induced by the $\mathscr{A}_2$-module homomorphism $f^* : H^*(Z) \to H^*(Y)$, converging to the homomorphism

$$f_* : \pi_*(Y) \to \pi_*(Z).$$

Remark 5.3.3. For any map $f : Y \to Z$ of connective spectra of finite type, the induced map

$$(f^*)^* : \mathrm{Ext}^{s,t}_{\mathscr{A}_2}(H^*(Y), \mathbb{F}_2) \to \mathrm{Ext}^{s,t}_{\mathscr{A}_2}(H^*(Z), \mathbb{F}_2)$$

satisfies $(f^*)^*(\alpha\beta) = (f^*)^*(\alpha)(f^*)^*(\beta)$. This follows from the definition of the Yoneda product, since f^* induces a chain homotopy between resolutions of $H^*(Z)$ and $H^*(Y)$, so both $(f^*)^*(\alpha\beta)$ and $(f^*)^*(\alpha)(f^*)^*(\beta)$ arise from chain homotopies, and thus descend to same element in $\mathrm{Ext}^{*,*}_{\mathscr{A}_2}(H^*(Z), \mathbb{F}_2)$.

Lemma 5.3.4 ([12], Table 14.1 (9)). $d_2(f_0), d_2(e_0) \neq 0$.

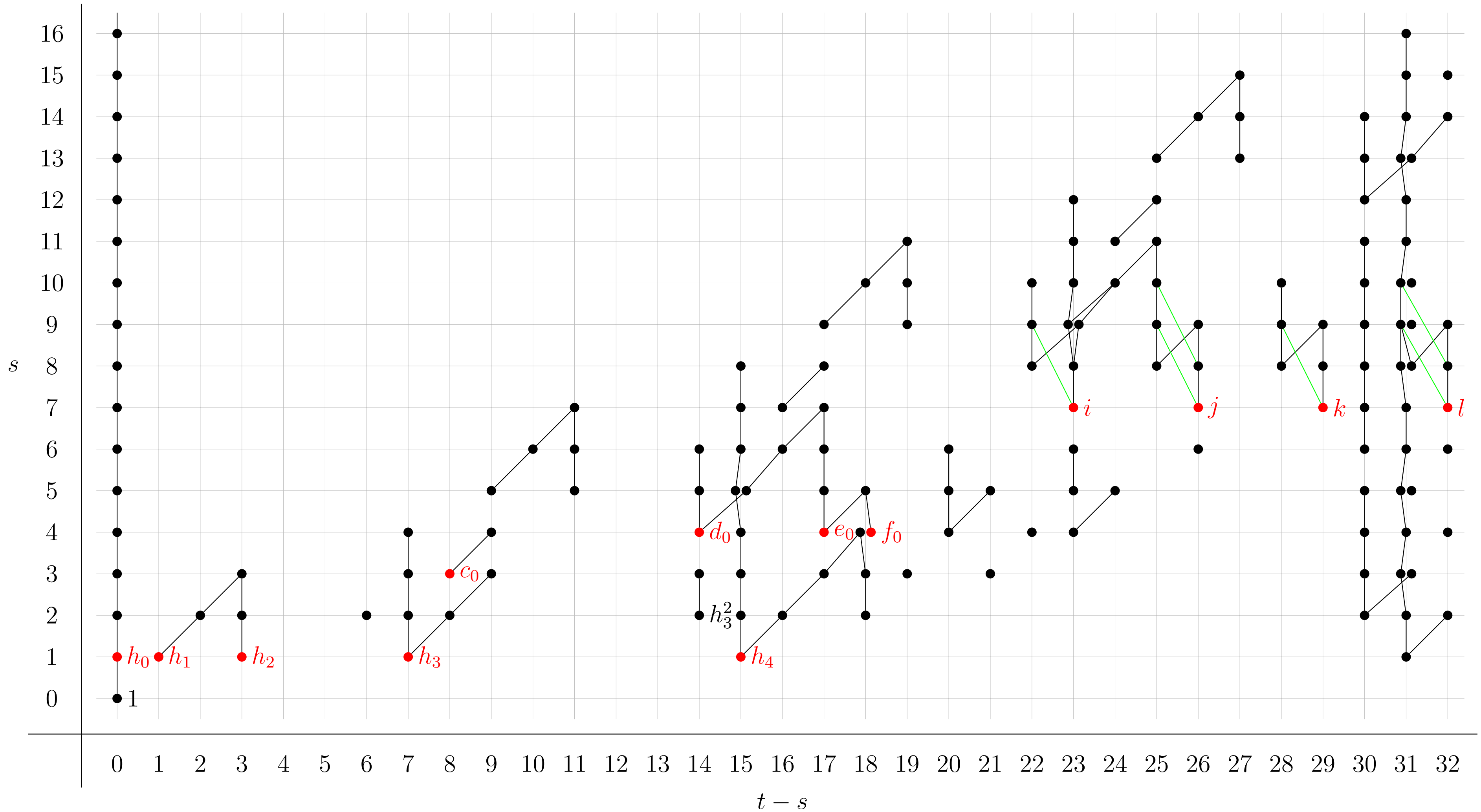


Figure 5.3: $\mathrm{Ext}^{s,t}_{\mathscr{A}_2}(\mathbb{F}_2, \mathbb{F}_2)$ for $t - s \leq 32$. The vertical and diagonal lines indicate multiplication by h_0 and h_1 respectively. Some of the algebra generators are shown in red, with naming conventions as in [12]. The d_2 differentials referenced in the proof of Lemma 5.3.4 are shown in green.

Proof. Let i, j, k, l be as shown in Figure 5.3. [Using sseq or ext, we calculate $h_4 i = 0$ and $h_0 h_3^2 i \neq 0$, using the Yoneda product]. Now, $d_2(i)$ is nontrivial, since $h_4 i = 0$ and $h_0 h_3^2 i \neq 0$, so $0 = d_2(h_4 i) = h_0 h_3^2 i + h_4 d_2(i)$ implies that $d_2(i) \neq 0$. Further, $d_2(j) \neq 0$ since $h_0 d_2(j) = d_2(h_0 j) = d_2(h_2 i) = h_2 d_2(i) \neq 0$. An almost identical argument shows that $d_2(k), d_2(l) \neq 0$, and thus $d_2(h_0 l) = h_0 d_2(l) \neq 0$. Finally, we have $h_0 l = d_0 f_0$, so $0 \neq d_2(h_0 l) = d_2(d_0 f_0) = d_0 d_2(f_0)$.

Now, $d_2(f_0) \neq 0$, so looking at Figure 5.3 we see that $0 \neq h_0 d_2(f_0) = d_2(h_0 f_0) = d_2(h_1 e_0) = h_1 d_2(e_0)$, and thus $d_2(e_0) \neq 0$. $\square$

Lemma 5.3.5 ([12], Table 14.9 (4)). Consider the cofibration

$$\mathbb{S}^7 \xrightarrow{\sigma} \mathbb{S} \xrightarrow{i} C_\sigma \xrightarrow{j} \mathbb{S}^8 \to \cdots .$$

Let $\overline{\overline{h_0^2 h_3}} \in E_2^{3,18}(C_\sigma)$ be the generator shown in Figure 5.3. Then $d_2(\overline{\overline{h_0^2 h_3}}) = \hat{\imath}(h_0 d_0)$, where $\hat{\imath} = (i^*)^* : \mathrm{Ext}_{\mathscr{A}_2}^{s,t}(\mathbb{F}_2, \mathbb{F}_2) \to \mathrm{Ext}_{\mathscr{A}_2}^{s,t}(C_\sigma, \mathbb{F}_2)$ is the map induced by $i^* : H^*(C_\sigma) \to H^*(\mathbb{S})$.

Proof. We first show that $d_2(\hat{\imath}(h_2)\overline{\overline{h_0^2 h_3}}) = d_2(\hat{\imath}(f_0))$. By Lemma 3.3.6, we have a long exact sequence

$$\cdots \leftarrow H^{n-1}(\mathbb{S}^8) \leftarrow H^n(\mathbb{S}) \xleftarrow{i^*} H^n(C_\sigma) \xleftarrow{j^*} H^n(\mathbb{S}^8) \leftarrow H^{n+1}(\mathbb{S}) \leftarrow \cdots .$$

However, any map $H^n(\mathbb{S}) \to H^{n-1}(\mathbb{S}^8)$ must be zero, so we get short exact sequences

$$0 \leftarrow H^n(\mathbb{S}) \xleftarrow{i^*} H^n(C_\sigma) \xleftarrow{j^*} H^n(\mathbb{S}^8) \leftarrow 0.$$

Taking a direct sum gives a short exact sequence

$$0 \leftarrow \mathbb{F}_2 \xleftarrow{i^*} H^*(C_\sigma) \xleftarrow{j^*} \mathbb{F}_2[8] \leftarrow 0,$$

and from this we get a short exact sequence of chain complexes

$$0 \to \mathrm{Hom}(\mathbb{F}_2, I_\bullet) \xrightarrow{i^*} \mathrm{Hom}(H^*(C_\sigma), I_\bullet) \xrightarrow{j^*} \mathrm{Hom}(\mathbb{F}_2[8], I_\bullet) \to 0,$$

for any injective resolution I, and thus the long exact sequence below.

$$\begin{array}{l} \cdots \longrightarrow \mathrm{Ext}_{\mathscr{A}_2}^{s,t}(\mathbb{F}_2, \mathbb{F}_2) \xrightarrow{i} \mathrm{Ext}_{\mathscr{A}_2}^{s,t}(H^*(C_\sigma), \mathbb{F}_2) \xrightarrow{j} \mathrm{Ext}_{\mathscr{A}_2}^{s,t-8}(\mathbb{F}_2, \mathbb{F}_2) \\ \hookrightarrow \mathrm{Ext}_{\mathscr{A}_2}^{s+1,t}(\mathbb{F}_2, \mathbb{F}_2) \xrightarrow{i} \mathrm{Ext}_{\mathscr{A}_2}^{s+1,t}(H^*(C_\sigma), \mathbb{F}_2) \xrightarrow{j} \mathrm{Ext}_{\mathscr{A}_2}^{s+1,t-8}(\mathbb{F}_2, \mathbb{F}_2) \longrightarrow \cdots \end{array}$$

Note that for $t - s < 7$, the map $\hat{\imath}$ must be injective, since $\mathrm{Ext}_{\mathscr{A}_2}^{s,t-8}(\mathbb{F}_2, \mathbb{F}_2) = 0$. In particular, $\hat{\imath}(h_0), \hat{\imath}(h_2) \neq 0$. Now, $f_0 \in \mathrm{Ext}_{\mathscr{A}_2}^{4,22}(\mathbb{F}_2, \mathbb{F}_2)$; we consider the exact sequence

$$\mathrm{Ext}_{\mathscr{A}_2}^{3,14}(\mathbb{F}_2, \mathbb{F}_2) \to \mathrm{Ext}_{\mathscr{A}_2}^{4,22}(\mathbb{F}_2, \mathbb{F}_2) \xrightarrow{i} \mathrm{Ext}_{\mathscr{A}_2}^{4,22}(H^*(C_\sigma), \mathbb{F}_2).$$

Figure 5.1 shows us that $\mathrm{Ext}_{\mathscr{A}_2}^{3,14}(\mathbb{F}_2, \mathbb{F}_2) = 0$, so $\hat{\imath}$ is injective at this point, and thus $\hat{\imath}(f_0) \neq 0$. Similarly, $\mathrm{Ext}_{\mathscr{A}_2}^{4,15}(\mathbb{F}_2, \mathbb{F}_2) = 0$ and $\mathrm{Ext}_{\mathscr{A}_2}^{0,8}(\mathbb{F}_2, \mathbb{F}_2) = 0$, so $\hat{\imath}(h_0 f_0), \hat{\imath}(h_4) \neq 0$. Since $\hat{\imath}$ respects

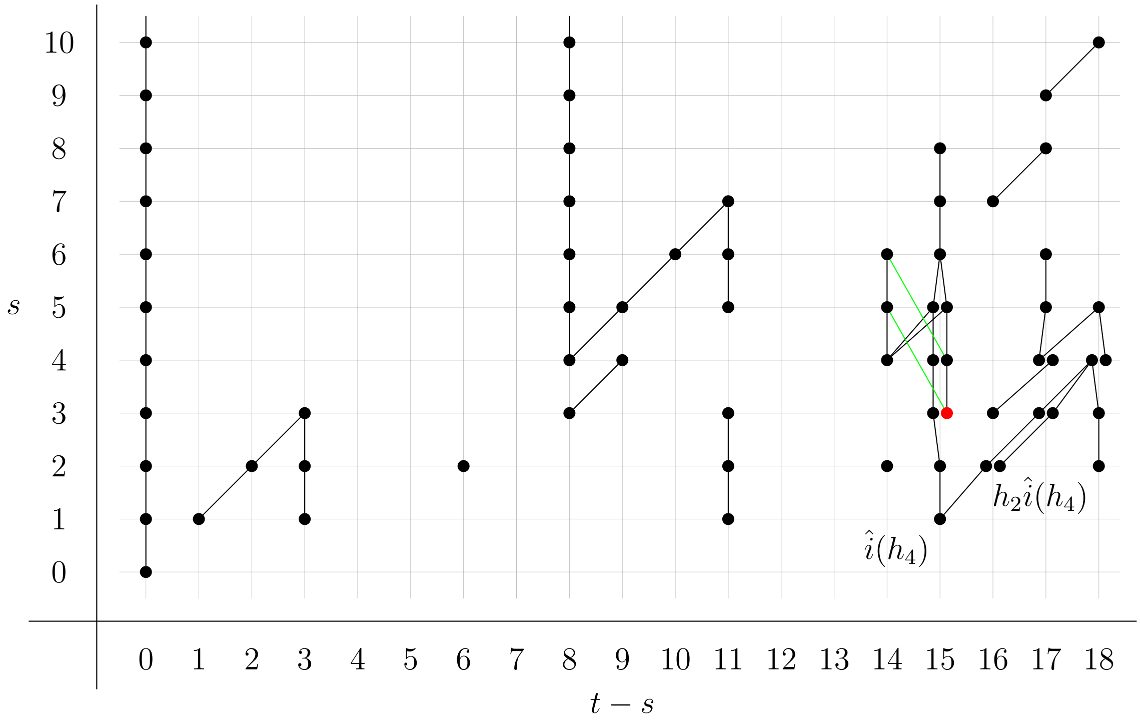


Figure 5.4: The E_2 page of the Adams spectral sequence for C_σ, in the range $t - s \leq 18$, with the generator $\overline{\overline{h_0^2 h_3}}$ shown in red, and two of the differentials shown in green.

multiplication (by Remark 5.3.3), $\hat{\imath}(h_0 f_0) = \hat{\imath}(h_0)\hat{\imath}(f_0) \neq 0$, so $\hat{\imath}(f_0)$ is equal to either $\overline{\overline{h_0^2 h_3}}$ or $\overline{\overline{h_0^2 h_3}} + \hat{\imath}(h_2)\hat{\imath}(h_4)$. Now, $d_2(h_2 h_4) = 0$, since otherwise it would be equal to e_0, and we would have $d_2^2(h_2 h_4) \neq 0$, contradicting the fact that d_2 is a differential. Thus, by linearity of d_2, we have $d_2(\hat{\imath}(f_0)) = d_2(\overline{\overline{h_0^2 h_3}})$ (since $d_2(\hat{\imath}(h_2)\hat{\imath}(h_4)) = \hat{\imath}(d_2(h_2 h_4)) = 0$).

Finally, $\hat{\imath}(h_0^2 e_0) \neq 0$, since $\mathrm{Ext}^{5,15}_{\mathscr{A}_2}(\mathbb{F}_2, \mathbb{F}_2) = 0$ (using the long exact sequence in Ext again). Thus, $\hat{\imath}(h_2) d_2(\overline{\overline{h_0^2 h_3}}) = d_2(\hat{\imath}(h_2)\overline{\overline{h_0^2 h_3}}) = d_2(\hat{\imath}(f_0)) = \hat{\imath}(d_2(f_0)) = \hat{\imath}(h_0^2 e_0) \neq 0$, as required. $\square$

THEOREM 5.3.6 ([12], Table 14.2 (10)). *$d_3(h_0 h_4) = h_0 d_0$.*

PROOF. From the cofibration

$$\mathbb{S}^7 \overset{\sigma}{\hookrightarrow} \mathbb{S} \xrightarrow{i} C_\sigma \xrightarrow{j} \mathbb{S}^8 \hookrightarrow \mathbb{S}^1 \to \cdots,$$

we get an exact sequence

$$\pi_7^S \xrightarrow{\sigma_*} \pi_{14}^S \xrightarrow{i_*} \pi_{14}(C_\sigma) \xrightarrow{j_*} \pi_6^S \to \pi_{13}^S,$$

by Theorem 3.3.5. Since these stable homotopy groups are all finite[3], this induces an exact sequence

$$(\pi_7^S)_2^\wedge \xrightarrow{\sigma_*} (\pi_{14}^S)_2^\wedge \xrightarrow{i_*} \pi_{14}(C_\sigma)_2^\wedge \xrightarrow{j_*} (\pi_6^S)_2^\wedge \xrightarrow{\sigma_*} (\pi_{13}^S)_2^\wedge = 0.$$

In $E_2(C_\sigma)$ we have $d_2(\overline{\overline{h_0^2 h_3}}) = \hat{\imath}(h_0 d_0)$ (by Lemma 5.3.5), so $\pi_{14}(C_\sigma)_2^\wedge$ has order dividing four. Let $\nu \in (\pi_3^S)_2^\wedge$ be a representative for h_2. Then $(\pi_6^S)_2^\wedge = \mathbb{Z}/2\mathbb{Z}\langle \nu^2 \rangle$, and $\nu^2\sigma = 0$. By

[3] A priori $\pi_{14}(C_\sigma)$ is only finitely generated, but from Figure 5.3 we see that its 2-completion is finite, so the group itself must be finite.

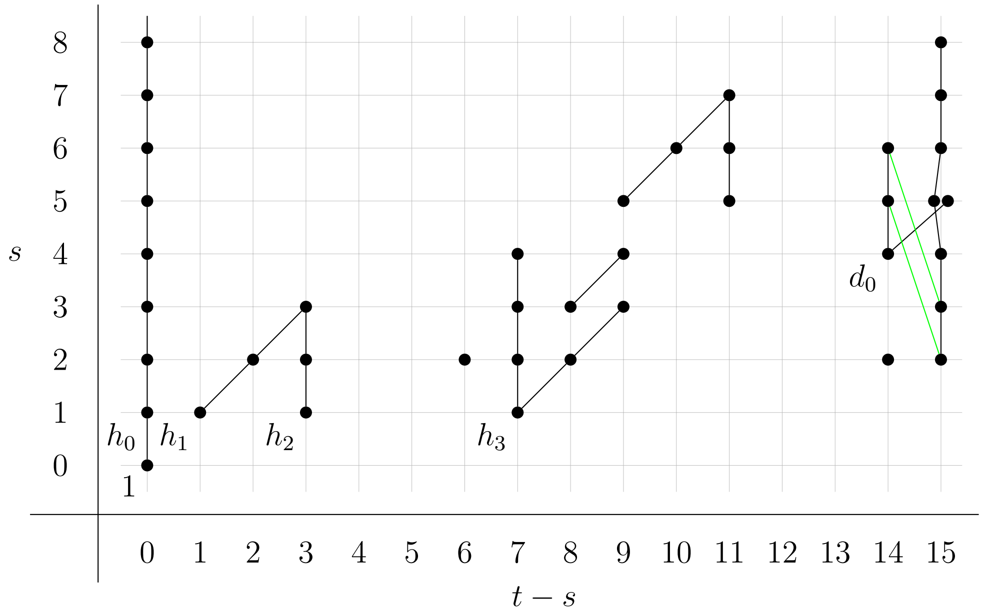


Figure 5.5: The E_3 page of the Adams spectral sequence for $\mathbb{S}$, in the range $t-s \leq 15$; the differentials are shown in green.

exactness, we see that j_* is surjective, so $(\pi_6^S)_2^\wedge \cong \pi_{14}(C_\sigma)_2^\wedge/\ker j_* = \pi_{14}(C_\sigma)_2^\wedge/\operatorname{im} i_*$. We know $\pi_{14}(C_\sigma)_2^\wedge$ has order dividing 4 and $(\pi_6^S)_2^\wedge$ has order 2, so $\operatorname{im} i_*$ has order dividing 2.

Now, $(\pi_7^S)_2^\wedge = \mathbb{Z}/16\mathbb{Z}\langle\sigma\rangle$, and $2\sigma^2 = 0$ by graded commutativity, so the first isomorphism theorem implies that $(\pi_{14}^S)_2^\wedge$ has order dividing four. Thus, h_0d_0 and $h_0^2d_0$ must be boundaries, and $d_3(h_0h_4) = h_0d_0$ is the only possibility. □

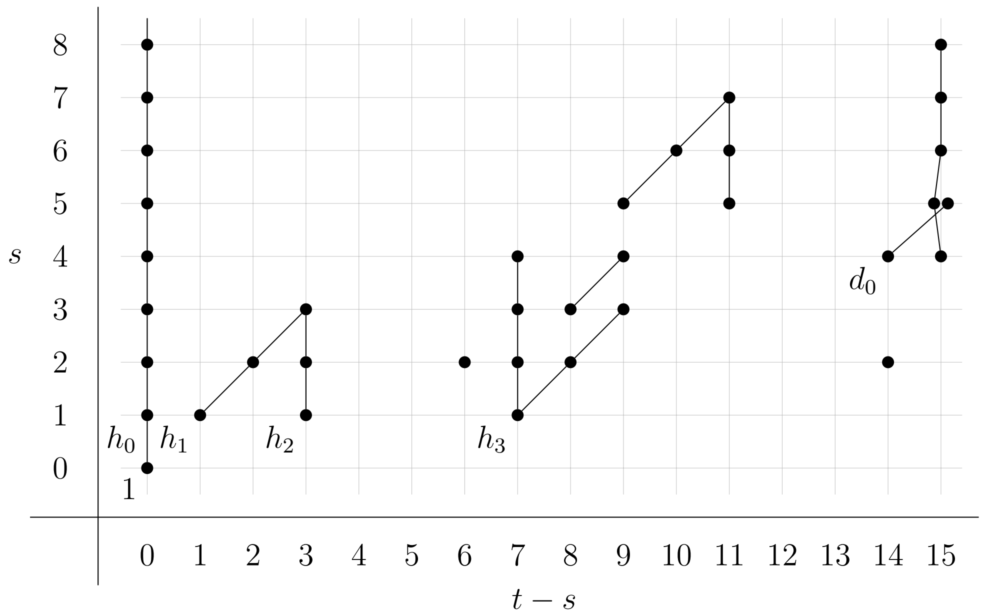


Figure 5.6: The E_4 page of the Adams spectral sequence for $\mathbb{S}$, in the range $t-s \leq 15$. There are no possible higher differentials, so this coincides with the E_∞ page for $t-s \leq 15$.

Theorem 5.3.7.

$$(\pi_i^S)_2^\wedge = \begin{cases} (\mathbb{Z}/2\mathbb{Z})^2 & i = 14, \\ \mathbb{Z}/32\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} & i = 15. \end{cases}$$

A Topology

All from [5] unless otherwise stated.

A.1 Suspension

DEFINITION 1.1.1. Let X be a topological space. The *suspension* SX is the space $(X \times I)/\sim$, where $(x,0) \sim (x',0)$ and $(x,1) \sim (x',1)$ for all $x, x' \in X$.

DEFINITION 1.1.2. Let X be a pointed topological space. The *reduced suspension* ΣX is the space $SX/\sim$, where $[x_0, t] \sim [x_0, t']$ for all $t, t' \in I$.

Given a map $f : X \to Y$, we can define $\Sigma f : \Sigma X \to \Sigma Y$ by $\Sigma f[(x,t)] = [(fx, t)]$. This makes Σ into a functor $\Sigma : \mathbf{Top} \to \mathbf{Top}$.

REMARK 1.1.3. Σ is faithful, since for any maps $f, g : X \to Y$, if $\Sigma f = \Sigma g$ then in particular $[(fx, \frac{1}{2})] = [(gx, \frac{1}{2})]$, so $fx = gx$.

[below is reconstructed from [9]]

Given pointed maps $f, g : \Sigma X \to Z$, define

$$\begin{aligned} f \star g : \Sigma X &\to Z \\ [x,t] &\mapsto \begin{cases} f[x, 2t-1] & t \geq \frac{1}{2}, \\ g[x, 2t] & t \leq \frac{1}{2}. \end{cases} \end{aligned}$$

This is well defined, since both f and g are basepoint-preserving.

REMARK 1.1.4. This defines a group structure on $[\Sigma X, Z]$, and thus $[\Sigma^i X, Z]$ is a group for all $i \geq 1$. For $i \geq 2$, these can be shown to be abelian, via the Eckmann-Hilton argument. The suspension map $[\Sigma X, Y] \to [\Sigma^2 X, \Sigma Y]$ is a homomorphism.[4]

REMARK 1.1.5. The homotopy groups $\pi_i(Z)$ are a special case of the above construction, taking $X := S^{i-1}$.

- Loops; the adjunction $\Sigma \dashv \Omega$, where Ω is the loop functor.

[5], p395:

REMARK 1.1.6. It follows that $\pi_{n+1}(X) \cong \pi_n(\Omega X)$. In particular, $\Omega K(G,n)$ is a $K(G, n-1)$.

- [5] 2.1 Ex 20 and 2.2 Ex 32: $\widetilde{H}_n(X) \cong \widetilde{H}_{n+1}(SX)$, where S is the (non-reduced) suspension. (MV?)
- Hatcher also says on p219 that $\widetilde{H}^n(X;R) \cong \widetilde{H}^{n+k}(\Sigma^k X; R)$, where Σ is reduced suspension.

A.2 Other basic constructions

DEFINITION 1.2.1. Let $(X, x_0), (Y, y_0)$ be pointed topological spaces, and consider their product $X \times Y$. The subspaces $X \times \{y_0\} \cong X$ and $\{x_0\} \times Y \cong Y$ intersect at exactly one

[4] Probably follows from the result for $\pi_*(Y)$ and induction on the cells of X, but I'll check this.

point, (x_0, y_0), and so can be identified with the wedge $X \vee Y$. We thus define the *smash product* $X \wedge Y := (X \times Y)/(X \vee Y)$, with the canonical basepoint (x_0, y_0).

EXAMPLE 1.2.2. We have $S^n \wedge S^m \cong S^{n+m}$. [is this obvious?]

REMARK 1.2.3. Note that $\Sigma X \cong X \wedge S^1$.

REMARK 1.2.4. Observe that $X \wedge (Y \wedge Z) \cong (X \wedge Y) \wedge Z$. Combining this with the remarks above, we see that $\Sigma^k X \cong X \wedge S^k$.

REMARK 1.2.5. Note that $\Sigma(X \vee Y) \cong \Sigma X \vee \Sigma Y$.

- An Eilenberg-MacLane space is $K(G, n)$, and it has the property that

$$\pi_i(K(G,n)) = \begin{cases} G & i = n, \\ 0 & i \neq n. \end{cases}$$

  They're unique up to weak homotopy equivalence (i.e. if you have another one X, there's a map between them which descends to an isomorphism on homotopy groups). They can be taken to be CW complexes.

DEFINITION 1.2.6. Let X, Y be topological spaces, where X has a basepoint x_0. Then the *reduced product* $X \times_{\text{red}} Y := (X \times Y)/(x_0 \times Y)$.

DEFINITION 1.2.7. Let $f : X \to Y$ be a map. The *mapping cylinder* M_f is defined by $((X \times I) \sqcup Y)/\sim$, where $(x, 1) \sim f(x)$ for all $x \in X$. If $(X, x_0), (Y, y_0)$ are pointed spaces, the *reduced mapping cylinder* is the quotient $M_f/\sim$, where $[x_0, t] \sim [x_0, t']$ for all $t \in I$.

REMARK 1.2.8. The mapping cylinder deformation retracts onto Y via $h : M_f \times I \to M_f$; $([x, t], s) \mapsto [x, t + s(1 - t)]$.

DEFINITION 1.2.9. Let $f : X \to Y$ be a map. The *mapping cone*[5] C_f is defined to be $Y \sqcup_f CX := (Y \sqcup CX)/(f(x) \sim [x, 1])$.

Relative Künneth Theorem:

THEOREM 1.2.10 ([5]). For CW pairs $(X, A), (Y, B)$, the cross product homomorphism $H^*(X, A; R) \otimes_R H^*(Y, B; R) \to H^*(X \times Y, A \times Y \cup X \times B; R)$ is an isomorphism of rings if $H^k(Y, B)$ is a finitely generated free R-module for each k.

In particular, for pointed spaces $(X, x_0), (Y, y_0)$, we have an isomorphism

$$\bigoplus_{i+j=n} H^i(X, x_0; R) \otimes_R H^j(Y, y_0; R) \to H^n(X \times Y, X \vee Y; R).$$

Or, in other words,

$$\bigoplus_{i+j=n} \widetilde{H}^i(X; R) \otimes_R \widetilde{H}^j(Y; R) \to \widetilde{H}^n(X \wedge Y; R).$$

Setting $Y = S^1$, we get an isomorphism

$$\widetilde{H}^{n-1}(X; R) \to \widetilde{H}^n(\Sigma X; R).$$

[5]Why does Hatcher not insist this guy is reduced, like he does with the mapping cylinders?

A.3 Cell complexes

Definition 1.3.1. Let X be a cell complex, $A \subseteq X$ a subcomplex. Then the quotient X/A has a cell complex structure, with cells the cells of $X \setminus A$ along with a basepoint (the image of A in X).

Definition 1.3.2. Let $f : X \to Y$ be a map between CW complexes. Then f is *cellular* if $f(X_{(n)}) \subseteq Y_{(n)}$ for all n, where $X_{(n)}$ is the n-skeleton of X.

Cellular approximation theorem:

Theorem 1.3.3 ([5], Thm 4.8). Let $f : X \to Y$ be a map of CW complexes. Then f is homotopic to a cellular map.

Lemma 1.3.4 ([5], Prop 0.16). Let $A \subseteq X$ be CW complexes. Then the pair (X, A) has the *homotopy extension property*; that is, for any map $f : X \to Y$ and homotopy $h : A \times I \to Y$ such that $h(a, 0) = f|_A$, there is a homotopy $\widetilde{h} : X \times I \to Y$ extending h.

- The product of cell complexes is a cell complex (maybe only if one of them is finite?)
- The smash product of (pointed?) cell complexes is a cell complex (maybe only if one is them is finite?) [[5] says "the smash product $X \wedge Y$ is a cell complex if X and Y are cell complexes with x_0 and y_0 0-cells, assuming that we give $X \times Y$ the cell-complex topology rather than the product topology in cases where these two topologies differ".]
- For a CW complex X, $SX \simeq \Sigma X$.
- The reduced suspension of a pointed cell complex (X, x_0) is another pointed cell complex ΣX with basepoint x_0 and an n-cell for each non-basepoint $n - 1$ cell e^{n-1}_α of X.

Definition 1.3.5. Let X is a topological space. A *CW approximation* to X is a CW complex Z equipped with a weak homotopy equivalence $f : Z \to X$.

Theorem 1.3.6 ([5], Prop 4.13). Every space X has a CW approximation $f : Z \to X$.

- In particular, $\Omega K(G, n)$ has a CW approximation $Z \to \Omega K(G, n)$, and since $\Omega K(G, n)$ is a $K(G, n - 1)$, so is Z.

Any finite CW complex is compact.

Proposition 1.3.7 ([5], A.1). A compact subspace of a CW complex is contained in a finite subcomplex.

B Notes to self

B.1 Vague problems and questions....

B.1.1 ...that probably don't matter

- On p588 of [6], he says "every CW spectrum is equivalent to a suspension spectrum". Does he actually mean that, or does he mean 'equivalent to the suspension of a spectrum'? The former seems way too strong, although in fairness I still don't know what an equivalence of spectra actually *is*.
- On p586 of [6], Hatcher says "If X is of finite type then for each i there is an n such that X_n contains all the i-cells of X. It follows that $H_i(X; G) = H_i(X_n; G)$ for all sufficiently

large n, and the same is true for cohomology." But from the way he set up H_* and H^* earlier, shouldn't this be $H_i(X;G) = H_{i+n}(X_n;G)$? Because $H_i(X;G) = \varinjlim H_{i+n}(X_n)$, and he talks about things stabilising in the next sentence, so shouldn't the stable point be at some H_{i+n}?

- I write $\mathscr{A}_2$ where Hatcher writes $\mathscr{A}$. We mean the same thing, right...?

B.1.2 ...that probably do matter

- I am definitely being told some lies about what the spectral sequence actually converges to. There's a strong implication/actual statement(!!) that at each i it's supposed to be a filtration of π_i^S modulo odd torsion, but I think this isn't true. I think it's actually the 2-completion of π_i^S. That coincides with the p-primary part for finite abelian groups, but for π_0^S it's supposed to be $\mathbb{Z}_2$ (i.e. the 2-adic integers), not $\mathbb{Z}$. I believe. Maybe get a source for this. Some people say it's the localisation at 2?? But I think that's also a lie.

- The Leibniz rule is $d_r(xy) = d_r(x)y \pm xd_r(y)$ (can't remember the sign). But anything I'm using that rule on is some generator of an $\mathbb{F}_2$, right? So the sign shouldn't matter. But then, shouldn't the Yoneda product be graded commutative (and thus commutative, because again, in the target signs don't matter)? So why does [13] have some comment (in Cor 6.5) about how the Yoneda product is commutative "in [some] range"??

- On p592 of [6], he says that "for spectra X of finite type [the more general] definition of an $\mathscr{A}_2$-module structure on $H^*(X)$ agrees with the definition using the usual $\mathscr{A}_2$-module structure on the cohomology of spaces and the identification of $H^*(X)$ with the inverse limit $\varprojlim H^{*+n}(X_n)$". Um? Sure, we have that each $H^{i+n}(X_n)$ stabilises eventually, but is Hatcher saying $H^{*+n}(X_n)$ stabilises? Like, as an $\mathscr{A}_2$-module? And if not, what's going on here? Because inverse limits don't commute with infinite direct sums - they're not biproducts anymore, they're coproducts and there's no reason limits should commute with them.

- There's something weird going on with products. So, things are ok in **Top**, because we have the ordinary product of two spaces, which is a categorical product. But with CW complexes, supposedly sometimes the product topology differs from the 'cell complex topology'? But, regardless, we're supposed to be working with pointed things - so in **Top**$_*$, the pointed coproduct is the wedge sum, and the pointed product is just the normal product $X \times Y$ with the basepoint (x_0, y_0) (it's not the smash product). But what about in spectra? No one ever seems to talk about products of spectra, but for example a collection of maps $X \to \mathbb{K}(G, n_i)$ should correspond to a single map $X \to \prod_i \mathbb{K}(G, n_i)$, whatever that last object is.

  The plot thickens. From the nLab: "[some smash product] is non-canonically equivalent to a product of EM-spectra (hence a wedge sum of EM-spectra in the finite case)". ???????

- I'm a bit suspicious of the proof of Theorem 3.2.9, because the proof is more complicated in [6]. Maybe raise this.

B.2 To do

Now:

- Rewrite Sections 4.1 and 4.2 and fix the grading.
- Figure out the whole wedge/product of EM spectra nonsense and why K_0 has finite type.
- Figure out why applying H^* to the sequence $K_0 \to K_1 \to \cdots$ makes it exact.
- Show that the $H^*(K_s)$'s are free $\mathscr{A}_2$-modules.
- Figure out how to state Theorem 4.4.2 without lying about 2-completion.
- Prove that π_*^S is a graded commutative ring.
- Finish the proof of Theorem 3.3.5.

Eventually:

- Be consistent with either cell complex or CW complex.
- Be consistent with $\mathbb{F}_2$ or $\mathbb{Z}/2\mathbb{Z}$ (don't use $\mathbb{Z}_2$, that's really bad).
- Specialise the Adams spectral sequence (i.e. set $Y = \mathbb{S}$).
- Remember that you have to hand in the tex file, so for the love of god change anything stupid that's hidden in the pdf.
- Sometimes I say π_*^S or ${}_{(2)}\pi_*^S$ (localised at 2?) instead of its completion at 2 or whatever. So make sure it's correct.
- Stick to a convention on suspension/cone/homotopy numbering. I.e. Does a homotopy start at 0 or 1? Does a suspension go from -1 to 1 with the space in the middle at 0, or 0 to 1 with the space at 1/2? Do cones go from 0 to 1, and if so, make sure when they include into suspensions they do so consistently.
- Have any sort of consistency in using or not using brackets (e.g. $\pi_t X_s$ v.s. $\pi_t(X_s)$).
- When I say 'spectrum' at any point after defining CW spectra I mean 'CW spectrum'. And I basically always mean 'connective CW spectrum of finite type' too.
- Connect 1 and h_1 (if possible without messing up the labels).

B.3 Other notes

- READ IF YOUR CALCULATIONS AREN'T WORKING: You are working modulo 2!!!
- If you have a bunch of maps between graded modules/algebras, they're graded homomorphisms. So they preserve degree.
- All (co)homology is supposed to be reduced.
- Signs don't matter with the Leibniz rule either!! You are working modulo 2!!!!!!!!

- Remember, once you know that $d_2(h_4) = h_0 h_3^2$, you know h_4 *doesn't survive to the third page*. So, for example, $d_3(h_0 h_4) \neq h_0 d_3(h_4)$ because h_4 doesn't exist anymore. That's why $d_3(h_0 h_4)$ can be nonzero.
- As previously mentioned, we are working modulo 2!! What this also implies is that if anything is hit by any sort of differential, or has any nonzero differential coming out of it, it's completely killed by the next page. Because the summands are just a bunch of $\mathbb{F}_2$'s (so you don't need to worry about 'how much' of something is killed, it all is).
- Sometimes Hatcher says that you can replace any map of CW complexes by an inclusion. I think the point here is that if you have a map $f : X \to Y$, Remark A.2.8 says that M_f deformation retracts onto Y. So if you only care about X and Y up to homotopy equivalence, you can replace Y by M_f and then X definitely includes into M_f.
- Where it's ambiguous, I'm marking things I definitely need by ! and things I think I may not need by ?.
- In literature, $A_p^\wedge$ is the p-adic completion of A. Sometimes I'll write this as ${}_pA$ because of some stupid notational decisions I made earlier.
- The 'abutment' of a spectral sequence apparently means the thing it converges to (i.e. if E_∞ computes the associated graded of some H^*, the abutment of $\{E\}$ is H^* (not its associated graded)).
- [12] has some n_m notation where n_m is supposed to be the mth generator in row n. This is a bit arbitrary when there are two generators in the same row and column; I don't know how he counts them, but he's using the `ext` program, whereas I'm using sseq. Unless there's some Canonical Ordering, there's no reason why these different programs written by different people would use the same convention. In particular, even though [12] says $\overline{\overline{h_0^2 h_3}} = 3_4$, I'm pretty sure it is the one on the right (i.e. the one I would label 3_5).

Sources I've used: [11], [13], [7], [6], [5], [12], [8], [10], [14]

Sources I probably won't use: [4], [1], [2], [3], [9] (I think the construction I need is in Hatcher)

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