

# Stable Homotopy Groups of Spheres

# 1 Introduction

Calculating the higher homotopy groups of spheres is an important and famously difficult problem in homotopy theory. However, the Freudenthal suspension theorem, given below, is the beginning of the field of *stable homotopy theory*, and in particular allows us to ask not about the homotopy groups of spheres in full generality, but to restrict to the colimits of these groups under suspension. This turns out to be a more approachable problem, and these ‘stable’ groups will be the subject of this essay.

Before stating the theorem below, we note that we will be working throughout with based spaces and maps, and reduced suspension; this ensures that the suspension of any based space has a canonical basepoint, and that the suspension of any pointed map is again a pointed map. We will, however, omit the basepoint from notation to reduce clutter.

**Theorem 1.0.1** ([10], Thm 1.8, Freudenthal suspension theorem). *If  $\pi_i(X) = 0$  for  $i \leq k$  (i.e.  $X$  is  $k$ -connected) then the map*

$$\begin{aligned} \pi_n(X) &\rightarrow \pi_{n+1}(\Sigma X) \\ [\gamma : S^n \rightarrow X] &\mapsto [\Sigma\gamma : \Sigma S^n = S^{n+1} \rightarrow \Sigma X] \end{aligned}$$

*is an isomorphism for  $n \leq 2k$  and surjective for  $n = 2k + 1$ .*

Now, let  $X$  be any topological space, and let  $k \geq 0$  be such that  $X$  is  $k$ -connected. Then Theorem 1.0.1 implies that  $\Sigma X$  is  $(k + 1)$ -connected, since  $0 = \pi_i(X) \cong \pi_{i+1}(\Sigma X)$  for  $i \leq k$ . As we take suspensions of  $X$ , the successive bounds are  $n \leq 2k$ ,  $n \leq 2k + 1$ ,  $n \leq 2k + 2$ , and so on, so the sequence

$$\pi_n(X) \rightarrow \pi_{n+1}(\Sigma X) \rightarrow \pi_{n+2}(\Sigma^2 X) \rightarrow \dots \quad (1.0.1)$$

will eventually stabilise. We thus define the *stable homotopy group*  $\pi_n^s(X)$  to be the filtered colimit of the system (1.0.1), which is equal to its stable value.

We will denote the groups  $\pi_k^s(S^0) \cong \pi_{n+k}^s(S^n)$  by  $\pi_k^s$ . The following classical theorem implies immediately that for  $k > 0$ ,  $\pi_k^s$  is finite.

**Theorem 1.0.2** ([8], Thm 1.1.8).  *$\pi_{n+k}^s(S^n)$  is finite for  $k > 0$  except when  $n = 2m$ ,  $k = 2m - 1$ .*

Our main computational tool for computing  $\pi_k^s$ , the Adams spectral sequence, will (in the form we present it here) give us information about the 2-completion of these groups, which for a finite abelian group  $A$  coincides with  $A$  modulo its odd torsion. The remainder of this essay will thus be dedicated to constructing this spectral sequence and using it to determine  $\pi_k^s$  modulo its odd torsion (in the case where  $k \neq 0$ ) for  $k \leq 15$ .

# 2 The Steenrod algebra

In this section, we give a very brief introduction to the mod 2 Steenrod algebra  $\mathcal{A}_2$ , whose elements are characterised by the axioms below. It can be shown (see e.g. [3] p500) that  $\mathcal{A}_2$  consists exactly of the stable  $\mathbb{F}_2$  cohomology operations (i.e. the natural transformations  $H^m(-; \mathbb{F}_2) \rightarrow H^n(-; \mathbb{F}_2)$  for fixed  $n, m$ ).

It will turn out to be useful in computing  $\pi_i^s(X) = \text{colim}_k \pi_{i+k}(\Sigma^k X)$  to consider instead the group  $\text{Hom}(H^*(X), H^*(S^i))$  to which there is a natural map; the Steenrod algebra gives us a way of obtaining more structure on this group (and thus more information about  $\pi_i^s(X)$ ), by restricting to  $\text{Hom}_{\mathcal{A}_2}(H^*(X), H^*(S^i))$  (and later considering the higher Ext groups).

**Proposition 2.0.1** ([3], p489). *For all  $X$  and each  $n$ , there are maps  $Sq^i : H^n(X; \mathbb{F}_2) \rightarrow H^{n+i}(X; \mathbb{F}_2)$  for each  $i$ , and they satisfy the following properties:*

1.  $Sq^i(f^*(\alpha)) = f^*(Sq^i(\alpha))$  for  $f : X \rightarrow Y$  (i.e.  $Sq^i$  is a natural transformation).
2.  $Sq^i(\alpha + \beta) = Sq^i(\alpha) + Sq^i(\beta)$  (i.e.  $Sq^i$  respects the group operation).
3.  $Sq^i(\alpha \smile \beta) = \sum_{0 \leq j \leq i} (Sq^j(\alpha) \smile Sq^{i-j}(\beta))$  (the Cartan formula).
4.  $Sq^i(\sigma(\alpha)) = \sigma(Sq^i(\alpha))$  where  $\sigma : H^n(X; \mathbb{F}_2) \rightarrow \tilde{H}^{n+1}(\Sigma X; \mathbb{F}_2)$  is the suspension isomorphism given by reduced cross product with a generator of  $\tilde{H}^1(S^1; \mathbb{F}_2)$ .
5.  $Sq^i(\alpha) = \alpha^2$  if  $i = \deg(\alpha)$  and  $Sq^i(\alpha) = 0$  if  $i > \deg(\alpha)$ .
6.  $Sq^0 = id$ .

Define  $Sq := Sq^0 + Sq^1 + \dots$ . Then  $Sq(\alpha \smile \beta) = Sq(\alpha) \smile Sq(\beta)$  (since  $(Sq(\alpha \smile \beta))_n = \sum_i Sq^i(\alpha) \smile Sq^{n-i}(\beta) = (Sq(\alpha) \smile Sq(\beta))_n$ ). Thus,  $Sq$  is a ring homomorphism.

The following proposition will be an important computational tool later.

**Proposition 2.0.2** ([3], p496). *The Steenrod squares satisfy the following relations, known as the Adem relations:*

$$Sq^a Sq^b = \sum_j \binom{b-j-1}{a-2j} Sq^{a+b-j} Sq^j \quad \text{if } a < 2b,$$

where  $\binom{m}{n}$  is zero if  $m$  or  $n$  is negative, or  $m < n$ , and  $\binom{m}{0} = 1$  for  $m \geq 0$ .

**Definition 2.0.3.** The Steenrod algebra  $\mathcal{A}_2$  is the algebra over  $\mathbb{F}_2$  that is the quotient of the algebra of polynomials in the noncommuting variables  $Sq^1, Sq^2, \dots$  by the two-sided ideal generated by the Adem relations. Thus, for every space  $X$ ,  $H^*(X; \mathbb{F}_2)$  is a module over  $\mathcal{A}_2$ .

Note that  $\mathcal{A}_2$  is graded, with elements of degree  $k$  those that map  $H^n(X; \mathbb{F}_2)$  to  $H^{n+k}(X; \mathbb{F}_2)$  for all  $n$ .

**Definition 2.0.4.** Let  $I = (i_1, \dots, i_n)$ , and write  $Sq^I$  for the monomial  $Sq^{i_1} Sq^{i_2} \dots Sq^{i_n}$ . Then  $Sq^I$  is *admissible* if  $i_j \geq 2i_{j+1}$  for all  $0 \leq j < n$ .

The admissible monomials are exactly those to which no Adem relations can be applied. Thus,  $\mathcal{A}_2$  is generated as an  $\mathbb{F}_2$  module by admissible monomials.

Finally, recall that an Eilenberg-MacLane space  $K(G, n)$  (for  $G$  an abelian group and  $n \in \mathbb{Z}_{\geq 0}$ ) is a space for which  $\pi_i(K(G, n)) = G$  if  $i = n$  and 0 otherwise. The following result says the cohomology of an Eilenberg-MacLane space  $K(\mathbb{F}_2, n)$  is free over  $\mathcal{A}_2$  below dimension  $2n$ .

**Proposition 2.0.5** ([9], Cor 7.5.6). *The homomorphism*

$$\begin{aligned} \mathcal{A}_2[n] &\rightarrow \tilde{H}^*(K(\mathbb{F}_2, n); \mathbb{F}_2) \\ Sq^I[n] &\mapsto Sq^I(u_n), \end{aligned}$$

where  $[n]$  denotes a shift in degree by  $n$ , is an isomorphism in degrees  $* \leq 2n$ .

**Remark 2.0.6.** The above result combined with the wedge axiom implies that for a fixed  $n$ , the cohomology of a wedge  $\bigvee_i K(\mathbb{F}_2, n_i)$  (with each  $n_i \geq n$  and only finitely many  $n_i$  in each dimension) is also free over  $\mathcal{A}_2$  below dimension  $2n$ .

Thus, though no space  $X$  can have cohomology a free  $\mathcal{A}_2$ -module (since for any  $\alpha \in H^*(X; \mathbb{F}_2)$  of degree  $n$ ,  $Sq^{n+1}(\alpha) = 0$  by Proposition 2.0.1 (5)), the cohomology of an Eilenberg-MacLane space is free in sufficiently low degree. This fact will be vital to the construction of the Adams spectral sequence in Section 4.4.

### 3 Spectra

In this section, we introduce spectra, the stable analogue of CW complexes. These objects will turn out to have particularly nice properties; for example, in addition to many of the properties of CW complexes carrying over to spectra, we will see that the collection of maps between two spectra up to homotopy has a natural abelian group structure, and that the suspension map between these groups is an isomorphism. Sections 3.1-3.4 closely follow Section 5.2 of [4], developing some basic properties of spectra which will be used in 4.4, while in Section 3.5 we note some facts about  $p$ -completion which will be needed later.

#### 3.1 Definitions and examples

**Definition 3.1.1.** A *spectrum* is a collection of pointed topological spaces  $\{X_n\}_{n \in \mathbb{N}}$ , together with basepoint-preserving maps  $\sigma_n : \Sigma X_n \rightarrow X_{n+1}$ .

**Example 3.1.2.** Let  $X$  be a topological space. The *suspension spectrum* of  $X$ , denoted by  $\Sigma^\infty X$ , has  $X_n = \Sigma^n X$  and  $\sigma_n = \text{id} : \Sigma X_n \rightarrow X_{n+1}$ .

We write  $\mathbb{S}$  for the suspension spectrum  $\Sigma^\infty S^0$ , and call  $\mathbb{S}$  the *sphere spectrum*. For  $i > 0$ , we write  $\mathbb{S}^i$  for  $\Sigma^\infty S^i$ .

**Example 3.1.3.** An *Eilenberg-MacLane spectrum*  $\mathbb{K}(G, m)$  has  $(\mathbb{K}(G, m))_n$  a CW complex  $K(G, m+n)$ , and can be constructed inductively by attaching cells to  $\Sigma K(G, m+n)$  to kill  $\pi_i(\Sigma K(G, m+n))$  for  $i > m+n+1$ . By Theorem 1.0.1,  $\pi_i(K(G, m+n)) \cong \pi_{i+1}(\Sigma K(G, m+n))$  for  $i \leq 2m+2n-2$ , so the cells attached can be taken to have dimension  $\geq 2m+2n-1$ . The maps  $\sigma_n$  are inclusions of subcomplexes.

**Definition 3.1.4.** Let  $X = \{X_n\}$  be a spectrum. We define  $\pi_i(X) = \text{colim}_n \pi_{i+n}(X_n)$ , where the map  $\pi_{i+n}(X_n) \rightarrow \pi_{i+n+1}(X_{n+1})$  is given by the composition

$$\pi_{i+n}(X_n) \xrightarrow{\Sigma} \pi_{i+n+1}(\Sigma X_n) \xrightarrow{(\sigma_n)_*} \pi_{i+n+1}(X_{n+1}).$$

**Example 3.1.5.** If  $X$  is a topological space, then  $\pi_i(\Sigma^\infty X) = \pi_i^s(X)$ , the  $i$ th stable homotopy group of  $X$ .

**Definition 3.1.6.** A CW spectrum is a spectrum  $X$  consisting of CW complexes  $X_n$  with the maps  $\Sigma X_n \hookrightarrow X_{n+1}$  inclusions of subcomplexes.

**Definition 3.1.7.** Let  $X$  be a CW spectrum. Then the  $k$ -cells of  $X$  are the equivalence classes of non-basepoint  $(k+n)$ -cells in  $X_n$ , where two cells are equivalent if one is an  $m$ -fold suspension of the other, for some  $m > 0$ .

We will say that a CW spectrum  $X$  is *connective* if it has no cells below a given dimension, *finite* if it has only finitely many cells, and *of finite type* if it has only finitely many cells in each dimension. The following two classes of examples will be particularly important for us.

**Example 3.1.8.** If  $X$  is a finite (resp. finite type) CW complex, then  $\Sigma^\infty X$  is a finite (resp. finite type) CW spectrum. In particular,  $\mathbb{S}$  is a finite CW spectrum with a unique cell in dimension 0.

**Example 3.1.9.** For each  $m$ , the Eilenberg-MacLane spectrum  $\mathbb{K}(G, m)$  constructed in Example 3.1.3 has finite type. This follows from the fact that the dimensions of the cells added to  $\Sigma K(G, n + m)$  are eventually all larger than  $n + i$  for any  $i$ , so  $\mathbb{K}(G, m)$  only has finitely many  $i$ -cells.

**Lemma 3.1.10.** *Let  $X$  be a connective spectrum of finite type. Then the groups  $\pi_{i+n}(X_n)$  eventually stabilise; i.e. the maps  $\pi_{i+n}(X_n) \xrightarrow{(\sigma_n)_* \circ \Sigma} \pi_{i+n+1}(X_{n+1})$  are isomorphisms for large enough  $n$ .*

*Proof.* Recall that whenever  $(X_{n+1}, \Sigma X_n)$  are such that  $X_{n+1} \setminus \Sigma X_n$  has no cells in dimension  $\leq k$ , the map  $\pi_i(\Sigma X_n) \rightarrow \pi_i(X_{n+1})$  induced by the inclusion is an isomorphism for  $i < k$  (this follows from cellular approximation and the long exact sequence of relative homotopy groups). Thus, if  $(\sigma_n)_* : \pi_{i+n+1}(\Sigma X_n) \rightarrow \pi_{i+n+1}(X_{n+1})$  never stabilises, there must be infinitely many natural numbers  $N_j$  such that  $(X_{N_j+1}, \Sigma X_{N_j})$  is not  $(i + N_j + 1)$ -connected, and thus that  $X_{N_j+1} \setminus \Sigma X_{N_j}$  has cells of dimension  $\leq i + N_j + 2$ . By connectivity, there is some fixed  $l$  such that these cells are of dimension  $N_j + k + 1$  for  $-l \leq k \leq i + 1$ . Thus, there must be some  $k$  such that infinitely many of the  $X_{N_j+1}$  have a  $(k + N_j + 1)$ -cell not included in  $\Sigma X_{N_j}$ . This then contradicts the assumption that  $X$  is of finite type, since it has infinitely many  $k$ -cells.

Thus, the maps  $(\sigma_n)_* : \pi_{i+n+1}(\Sigma X_n) \rightarrow \pi_{i+n+1}(X_{n+1})$  are eventually isomorphisms. The argument above also shows that in the stable range, the maps  $(\Sigma^j \sigma_n)_* : \pi_{i+n+1}(\Sigma^{j+1} X_n) \rightarrow \pi_{i+n+1}(\Sigma^j X_{n+1})$  are also isomorphisms for  $j > 0$ , since  $\Sigma^j X_{n+1} \setminus \Sigma^{j+1} X_n$  has no cells in dimension  $\leq k$  whenever  $X_{n+1} \setminus \Sigma X_n$  doesn't.

Now, fix  $m > 0$  such that  $(\sigma_n)_*$  is an isomorphism for  $n \geq m$ . By Theorem 1.0.1, there exists some  $m' > 0$  such that  $\Sigma : \pi_{i+m+n'}(\Sigma^{n'} X_m) \rightarrow \pi_{i+m+n'+1}(\Sigma^{n'+1} X_m)$  is an isomorphism for all  $n' \geq m'$ . Consider the commutative diagram below.

$$\begin{array}{ccccccc}
\pi_{i+m+n'}(\Sigma^{n'} X_m) & \xrightarrow{(\Sigma^{n'-1} \sigma_m)_*} & \pi_{i+m+n'}(\Sigma^{n'-1} X_{m+1}) & \xrightarrow{(\Sigma^{n'-2} \sigma_{m+1})_*} \cdots & \xrightarrow{(\sigma_{m+n'})_*} & \pi_{i+m+n'}(X_{m+n'}) \\
\cong \downarrow \Sigma & & \downarrow \Sigma & & & \downarrow \Sigma \\
\pi_{i+m+n'+1}(\Sigma^{n'+1} X_m) & \xrightarrow{(\Sigma^{n'} \sigma_m)_*} & \pi_{i+m+n'}(\Sigma^{n'} X_{m+1}) & \xrightarrow{(\Sigma^{n'-1} \sigma_{m+1})_*} \cdots & \xrightarrow{(\Sigma \sigma_{m+n'-1})_*} & \pi_{i+m+n'+1}(\Sigma X_{m+n'})
\end{array}$$

The horizontal maps are all isomorphisms by the previous paragraphs, and the leftmost vertical map is an isomorphism by assumption, so the rightmost vertical map must also be an isomorphism for all  $n' \geq m'$ . Therefore, for sufficiently large  $n$ , the composite  $\pi_{i+n}(X_n) \xrightarrow{\Sigma} \pi_{i+n+1}(\Sigma X_n) \xrightarrow{(\sigma_n)_*} \pi_{i+n+1}(X_{n+1})$  is an isomorphism, as required.  $\square$

## 3.2 Homology and cohomology

We now move on to define  $H^*(X)$  and  $H_*(X)$  for any CW spectrum  $X = \{X_n\}$ .

Recall that  $C_i^{\text{cell}}(X_n; G)$  has a  $G$ -summand for every  $i$ -cell of  $X_n$ . We have an injection

$$\begin{aligned}
C_i^{\text{cell}}(X_n; G) &\rightarrow C_{i+1}^{\text{cell}}(\Sigma X_n; G) \\
e_\alpha^i &\mapsto \Sigma e_\alpha^i,
\end{aligned}$$

and an injection  $C_{i+1}^{\text{cell}}(\Sigma X_n; G) \rightarrow C_{i+1}^{\text{cell}}(X_{n+1}; G)$  induced by the structure map  $\sigma_n$ , so we get an injection  $C_i^{\text{cell}}(X_n; G) \hookrightarrow C_{i+1}^{\text{cell}}(X_{n+1}; G)$ .

We define

$$C_n(X; G) := \bigcup_{i \in \mathbb{Z}} C_{i+n}^{\text{cell}}(X_i; G).$$

Note that there is a  $G$  summand for every  $i+n$  cell of  $X_i$  up to treating suspensions of cells as equivalent to the cells themselves, i.e. a  $G$  summand for every  $n$ -cell of  $X$ . We define  $H^*(X; G)$  and  $H_*(X; G)$  to be the cohomology and homology of this chain complex, respectively.

**Lemma 3.2.1.** *Let  $X$  be a connective CW spectrum of finite type. Then  $H_i(X; G)$ ,  $H^i(X; G)$ , and  $\pi_i(X)$  are finitely generated for all  $i$ .*

*Proof.* First, note that  $H_i(X; G) = H_{i+n}(X_n; G)$  for sufficiently large  $n$ , since for large enough  $n$ ,  $X_n$  contains all the cells of  $X$  of dimension  $\leq i$  (which are the  $(i+n)$ -cells of  $X_n$ ). Similarly,  $H^i(X; G) = H^{i+n}(X_n; G)$  for sufficiently large  $n$ . Each  $H_{i+n}(X_n; G)$  is finitely generated, since  $X_n$  has only finitely many cells in each dimension, and thus each  $H^{i+n}(X_n; G)$  is also finitely generated (see [3] Cor 3.3). Therefore,  $H_i(X; G)$  and  $H^i(X; G)$  are finitely generated.

Now,  $\pi_i(X) = \text{colim}_n \pi_{i+n}(X_n)$ , and the groups  $\pi_{i+n}(X_n)$  stabilise by Lemma 3.1.10. The  $X_n$  must eventually be simply-connected, since  $X$  is connective. A simply-connected space has finitely generated homotopy groups if and only if it has finitely generated homology groups (see e.g. [3], Thm 5.7), and we have just seen that the  $H_{i+n}(X_n; G)$  are finitely generated, so  $\pi_i(X) = \pi_{i+n}(X_n)$  is finitely generated.  $\square$

**Example 3.2.2.** Recall that  $\mathbb{S}$  is a finite spectrum. We thus have

$$\begin{aligned} H^i(\mathbb{S}; \mathbb{F}_2) &= \lim_n H^{i+n}(S^n; \mathbb{F}_2) \\ &= \begin{cases} \mathbb{F}_2 & i = 0, \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

Next, we define the notions of subspectra and maps between spectra.

**Definition 3.2.3.** Let  $X = \{X_n\}$  be a CW spectrum. A *subspectrum*  $X'$  of  $X$  is a sequence of subcomplexes  $\{X'_n \subseteq X_n\}$  satisfying  $\Sigma X'_n \subseteq X'_{n+1}$ . The subspectrum  $X'$  is *cofinal* if, for each  $n$  and each cell  $e_\alpha^i$  of  $X_n$ , the cell  $\Sigma^k e_\alpha^i$  belongs to  $X'_{n+k}$  for all sufficiently large  $k$ .

Note that if  $\Sigma^k e_\alpha^i$  belongs to  $X'_{n+k}$  then  $\Sigma^{k+1} e_\alpha^i$  belongs to  $\Sigma X'_{n+k} \subseteq X'_{n+k+1} \subseteq X'_{k+k+2} \subseteq \dots$ . Thus, if  $X'$ ,  $X''$  are cofinal spectra of  $X$  with  $\Sigma^k e_\alpha^i$  a cell of  $X'_{n+k}$  and  $\Sigma^l e_\alpha^i$  a cell of  $X''_{n+l}$  (with  $l \geq k$ ) then  $\Sigma^l e_\alpha^i$  is a cell of  $X'_{n+l}$  and therefore of  $X'_{n+l} \cap X''_{n+l}$ . In other words, the intersection of two cofinal spectra is a cofinal spectrum.

**Definition 3.2.4.** Let  $X, Y$  be CW spectra. A *strict map*  $f : X \rightarrow Y$  is a sequence of cellular maps  $f_n : X_n \rightarrow Y_n$  such that the diagram below commutes.

$$\begin{array}{ccc} \Sigma X_n & \xrightarrow{\sigma_n} & X_{n+1} \\ \Sigma f_n \downarrow & & \downarrow f_{n+1} \\ \Sigma Y_n & \xrightarrow{\sigma_n} & Y_{n+1} \end{array}$$

Taking strict maps to be our notion of maps between spectra, however, turns out to be too strong a requirement. For instance, a strict map  $\mathbb{S}^i \rightarrow \Sigma^\infty X$  would be given simply by a map  $S^i \rightarrow X$ , whereas if we want to know about the stable homotopy groups of  $X$ , we should also consider maps

$S^{i+n} \rightarrow \Sigma^n X$  which cannot necessarily be desuspended. We will therefore relax the definition of maps between spectra to include maps that are ‘defined eventually’, in the following sense.

**Definition 3.2.5.** A map of CW spectra  $f : X \rightarrow Y$  is an equivalence class of strict maps  $f' : X' \rightarrow Y$  with  $X'$  a cofinal subspectrum of  $X$ , where two strict maps  $f' : X' \rightarrow Y$  and  $f'' : X'' \rightarrow Y$  are equivalent if they agree on some common cofinal subspectrum.

Given two maps  $f : X \rightarrow Y$ ,  $g : Y \rightarrow Z$  represented by  $f' : X' \rightarrow Y$ ,  $g' : Y' \rightarrow Z$  respectively, we compose as follows: let  $X''$  be the subspectrum of  $X'$ , where the cells of  $X''$  consist of the cells of  $X'_n$  mapped to  $Y'_n$  under  $f'_n$ . Then, for any cell  $e_\alpha^i$  of  $X'_n$ ,  $f'_n(e_\alpha^i)$  is contained in a finite union of cells of  $Y_n$  (since the image of a compact set is compact), whose  $k$ -fold suspension lies in  $Y'_{n+k}$  for large enough  $k$ . Since  $f'$  is a strict map,  $\Sigma^k f'_n(e_\alpha^i) = f'_{n+k} \Sigma^k e_\alpha^i$ , so  $\Sigma^k e_\alpha^i$  is a cell of  $X''_{n+k}$ . Thus,  $X''$  is cofinal in  $X'$  and hence in  $X$ . We define  $gf := [X'' \xrightarrow{f'|_{X''}} Y' \xrightarrow{g'} Z]$ , which is well-defined since the intersection of cofinal subspectra is again a cofinal subspectrum.

Since any strict map  $f' : X' \rightarrow Y$  can be taken to be cellular, a map  $f : X \rightarrow Y$  induces a well-defined map  $C_*(X) \rightarrow C_*(Y)$  (by cofinality), and thus maps on homology and cohomology.

Further, any map  $S^i \rightarrow X$  can be represented by a map  $S^{i+n} \rightarrow X_n$ , which has compact image and thus is contained in a finite subcomplex  $\overline{X}_n \subseteq X_n$ . Given any map  $f : X \rightarrow Y$  represented by a strict map  $f' : X' \rightarrow Y$ , the  $k$ th suspension of the cells of  $\overline{X}_n$  lie in  $X'_{n+k}$ , and thus  $f$  induces a map  $\pi_*(X) \rightarrow \pi_*(Y)$ .

**Definition 3.2.6.** Two spectra  $X, Y$  are *equivalent* if there are maps  $f : X \rightarrow Y$  and  $g : Y \rightarrow X$  such that  $fg = \text{id}_Y$  and  $gf = \text{id}_X$ .

Note that a spectrum is equivalent to any of its cofinal subspectra. In particular, if  $X = \{X_n\}$  is a spectrum, then  $X' = \{\Sigma X_{n-1}\}$  is a cofinal subspectrum of  $X$  (where we take  $X_{-1}$  to be the basepoint of  $X_0$ ). We define  $\Sigma^{-1}X := \{X_{n-1}\}$ , noting that  $\Sigma \Sigma^{-1}X = \Sigma^{-1}\Sigma = X' \simeq X$ . Thus, a spectrum is always equivalent to the suspension of some other spectrum.

Recall that for  $i > 0$ , we write  $S^i$  for the spectrum  $\Sigma^\infty S^i = \Sigma^i S$ . We have just seen that spectra can always be desuspended, so we will extend this notation and write  $S^i := \Sigma^i S$  for any nonzero  $i \in \mathbb{Z}$ .

**Definition 3.2.7.** A *homotopy* of maps between spectra is a map  $X \times I \rightarrow Y$ , where  $X \times I$  is the spectrum with  $(X \times I)_n = X_n \times_{\text{red}} I := (X_n \times I)/(x_0 \times I)$ .

Note that  $\Sigma(X_n \times_{\text{red}} I) = \Sigma X_n \times_{\text{red}} I$ . The set of homotopy classes of maps  $X \rightarrow Y$  is denoted by  $[X, Y]$ .

**Remark 3.2.8.** For any CW spectra  $Z$ ,  $[S^t, Z] = \pi_t(Z)$ .

For any CW spectra  $X, Y$ , the set  $[X, Y]$  can have the structure of an abelian group, since  $X$  can be written as a double suspension  $\Sigma^2 X'$ , and each set  $[\Sigma^2 X'_n, Y_n]$  has the structure of an abelian group.

**Theorem 3.2.9.** The suspension map  $[X, Y] \rightarrow [\Sigma X, \Sigma Y]$  is an isomorphism of groups.

*Proof.* The suspension map is a homomorphism, since it is a homomorphism on maps between CW complexes. Thus, it suffices to show it is a bijection on maps between spectra.

Recall that  $\Sigma^{-1}\Sigma X = \Sigma \Sigma^{-1}X \simeq X$ . For any map  $f : X \rightarrow Y$  given by strict maps  $f_n : X'_n \rightarrow Y_n$ , define  $\Sigma^{-1}f : \Sigma^{-1}X \rightarrow \Sigma^{-1}Y$  by  $\{f_{n-1} : X'_{n-1} \rightarrow Y_{n-1}\}$ . Then  $\Sigma \Sigma^{-1}f = \{\Sigma f_{n-1}\} = \{f_n|_{\Sigma X'_{n-1}}\} = f$ , and similarly  $\Sigma^{-1}\Sigma f = f$ . Thus, we have bijections  $[X, Y] \cong [\Sigma \Sigma^{-1}X, \Sigma \Sigma^{-1}Y] \cong [\Sigma^{-1}\Sigma X, \Sigma^{-1}\Sigma Y]$ , so  $\Sigma$  has a two-sided inverse.  $\square$

### 3.3 Cofibration sequences

**Definition 3.3.1.** Let  $X = \{X_n\}, Y = \{Y_n\}$  be spectra. Then their *wedge sum* is  $X \vee Y := \{X_n \vee Y_n\}$ . Note that we have an inclusion  $\Sigma(X_n \vee Y_n) \hookrightarrow X_{n+1} \vee Y_{n+1}$ .

**Definition 3.3.2.** Let  $f : X \rightarrow Y$  be a map of CW spectra, and let  $f' : X' \rightarrow Y$  be a representative for  $f$ , where  $X' \subseteq X$  is cofinal. The *mapping cylinder*  $M_f$  has components  $(M_f)_n = M_{f'_n}$ , where  $M_{f'_n}$  is the reduced mapping cylinder of  $f'_n$ . It is independent of the choice of  $X'$  up to equivalence.

**Remark 3.3.3.** Given any map  $f : X \rightarrow Y$  of CW spectra, we have a deformation retraction of  $M_f$  onto  $Y$ . Since we will only be interested in spectra up to homotopy equivalence, by replacing  $Y$  by  $M_f$  we may assume any map  $f : X \rightarrow Y$  is an inclusion.

**Definition 3.3.4.** Let  $X$  be a CW spectrum,  $A \subseteq X$  a subspectrum. Then  $A$  is *closed* in  $X$  if for every cell  $e_\alpha^n$  of  $X_n$ , if  $\Sigma^k e_\alpha^n \in A_{n+k}$  then  $e_\alpha^n \in A_n$ .

Any subspectrum is cofinal in (and thus equivalent to) its closure. We define  $X/A$  to be the CW spectrum with  $(X/A)_n = X_n/A'_n$ , where  $A' = \{A'_n\}$  is the closure of  $A$ . Note that a quotient of connective spectra of finite type is again a connective spectrum of finite type (since the quotient has fewer cells in each dimension than the original space).

The map  $X \cup CA \rightarrow X/A$  is a homotopy equivalence of spectra, since each quotient  $X_n \cup CA_n \rightarrow X_n/A_n$  is, so we have a cofibration sequence

$$A \hookrightarrow X \rightarrow X \cup CA \rightarrow \Sigma A \hookrightarrow \Sigma X \rightarrow \dots$$

**Theorem 3.3.5.** Let  $X, Y$  be spectra, and  $A \subseteq X$  a subspectrum. Then there is an exact sequence

$$[Y, A] \rightarrow [Y, X] \rightarrow [Y, X/A] \rightarrow [Y, \Sigma A] \rightarrow [Y, \Sigma X] \rightarrow \dots$$

*Proof.* It suffices to show that

$$[Y, A] \rightarrow [Y, X] \rightarrow [Y, X/A]$$

is exact.

The composition  $[Y, A] \rightarrow [Y, X] \rightarrow [Y, X/A]$  is clearly zero. Suppose  $Y \xrightarrow{f} X \rightarrow X \cup CA$  is homotopic to the constant map. Then we have a map  $h : CY \rightarrow X \cup CA$  making the solid diagram below commute.

$$\begin{array}{ccccccc} Y & \xrightarrow{\text{id}} & Y & \hookrightarrow & CY & \longrightarrow & \Sigma Y \xrightarrow{-\text{id}} \Sigma Y \\ \downarrow & & \downarrow f & & \downarrow h & & \downarrow \Sigma f \\ A & \xrightarrow{i} & X & \hookrightarrow & X \cup CA & \longrightarrow & \Sigma A \xrightarrow{-\Sigma i} \Sigma X \end{array} \quad (3.3.1)$$

We claim that we can fill in the two dotted maps on the right to make homotopy commutative squares. To see this, consider the diagram below,

$$\begin{array}{ccccccc} Y & \xrightarrow{\text{id}} & Y & \hookrightarrow & Y \cup CY & \hookrightarrow & (Y \cup CY) \cup CY \hookrightarrow ((Y \cup CY) \cup CY) \cup C(Y \cup CY) \\ \downarrow & & \downarrow f & & \downarrow h & & \downarrow h \cup Cf \\ A & \xrightarrow{i} & X & \hookrightarrow & X \cup CA & \hookrightarrow & (X \cup CA) \cup CX \hookrightarrow ((X \cup CA) \cup CX) \cup C(X \cup CA) \end{array}$$

where  $h \cup Cf$  is given by applying  $h$  to  $Y \cup CY$  and  $Cf$  to  $CY$  (which is well-defined since the maps agree on the intersection  $Y \times \{0\}$ ), and likewise for  $(h \cup Cf) \cup Ch$ . Now, the square below commutes,



since the identification  $((Y \cup CY) \cup \textcolor{red}{CY}) \cup C(Y \cup CY) \simeq \Sigma Y$  collapses everything except the factor in red (whose base is collapsed), and similarly for  $X$ .

$$\begin{array}{ccc} ((Y \cup CY) \cup CY) \cup C(Y \cup CY) & \xrightarrow{\simeq} & \Sigma Y \\ (h \cup Cf) \cup Ch \downarrow & & \downarrow \Sigma f \\ ((X \cup CA) \cup CX) \cup C(X \cup CA) & \xrightarrow{\simeq} & \Sigma X \end{array}$$

Now, let  $p_Y : (Y \cup CY) \cup CY \rightarrow \Sigma Y$  and  $p_A : (X \cup CA) \cup CX \rightarrow \Sigma A$  be the projections, with homotopy inverses  $h_Y$  and  $h_A$  respectively. We define  $g : \Sigma Y \rightarrow \Sigma A$  by  $g := p_A \circ (h \cup Cf) \circ h_Y$ . This  $g$  makes the diagram in (3.3.1) commute up to homotopy, where the minus signs arise from the fact that opposite hemispheres of the spaces  $((Y \cup CY) \cup CY) \cup C(Y \cup CY)$  and  $(Y \cup CY) \cup CY$  are collapsed under the quotient map (and likewise for the bottom row).

By Theorem 3.2.9, we can take the map  $g : \Sigma Y \rightarrow \Sigma A$  to be  $\Sigma k$  for some  $k : Y \rightarrow A$ . Then  $(\Sigma f) \circ (-\text{id}) \simeq (-\Sigma i)(\Sigma k)$ , so  $\Sigma f \simeq \Sigma(ik)$ , and thus  $f \simeq ik$  as required.  $\square$

Finally, we get the lemma below, which follows from the equivalent result for CW complexes.

**Lemma 3.3.6.** *Let  $A \xrightarrow{f} X \xrightarrow{i} C_f \xrightarrow{j} \Sigma A \rightarrow \dots$  be a cofibration, where  $X, A$  are CW spectra of finite type. Then there is a long exact sequence*

$$\dots \leftarrow H^{n-1}(\Sigma A) \leftarrow H^n(X) \xleftarrow{i^*} H^n(C_f) \xleftarrow{j^*} H^n(\Sigma A) \leftarrow H^{n+1}(X) \leftarrow \dots$$

### 3.4 Eilenberg-MacLane spectra

In this section, we briefly record two important facts about Eilenberg-MacLane spectra which will be used later in the construction of the Adams spectral sequence. The first is the analogue of the representability of cohomology by Eilenberg-MacLane spaces for CW complexes, and can be proven similarly.

**Theorem 3.4.1** ([4], Prop 5.45). *There are natural isomorphisms  $H^m(X; G) \cong [X, \mathbb{K}(G, m)]$  for all CW spectra.*

Now, recall that giving a map into a product is equivalent to giving a map into each of its components. We have maps  $F_i : [X, \bigvee_i \mathbb{K}(G, n_i)] \rightarrow [X, \mathbb{K}(G, n_i)]$  given by composition with the projections, giving a map  $F : [X, \bigvee_i \mathbb{K}(G, n_i)] \rightarrow \prod_i [X, \mathbb{K}(G, n_i)]$ .

**Proposition 3.4.2** ([4], Prop 5.46). *The map  $F : [X, \bigvee_i \mathbb{K}(G, n_i)] \rightarrow \prod_i [X, \mathbb{K}(G, n_i)]$  described above is an isomorphism if  $X$  is a connective spectrum of finite type and  $n_i \rightarrow \infty$  as  $i \rightarrow \infty$ .*

### 3.5 $p$ -completion of spectra

**Definition 3.5.1** ([6], Def 10.1.1). Let  $A$  be an abelian group. Then its  $p$ -adic completion is the limit

$$A_p^\wedge = \varprojlim_n (A/p^n A).$$

If  $A = \mathbb{Z}$ , we instead write  $\mathbb{Z}_p := \mathbb{Z}_p^\wedge$  for the  $p$ -adic integers. There is a natural map  $A \rightarrow A_p^\wedge$ , whose component at  $n$  is reduction modulo  $p^n A$ .

When  $A$  is finitely generated, its  $p$ -adic completion is given by the map  $A \rightarrow A \otimes \mathbb{Z}_p$ ;  $a \mapsto a \otimes 1$ .

**Remark 3.5.2.** Suppose  $A$  is finite, and write  $|A| = np^r$  for  $p \nmid n$ . Then  $A_p^\wedge \cong A/T$ , where  $T \subseteq A$  is the subgroup generated by all torsion coprime to  $p$ , since  $A/p^k A \cong A/T$  for all  $k \geq r$ .

**Remark 3.5.3.** If  $A$  is finite with order  $np^r$  for  $p \nmid n$ , then  $|A_p^\wedge| = p^r$ , by Cauchy's theorem.

**Definition 3.5.4** ([5], p129). Let  $X$  be a CW spectrum. Then a  $p$ -completion of  $X$  is a map  $f : X \rightarrow X_p^\wedge$  such that for all  $i$ ,  $\pi_i f$  expresses  $\pi_i(X_p^\wedge)$  as the  $p$ -completion of  $\pi_i(X)$ .

Throughout the remainder of this essay we will largely be concerned with connective spectra of finite type, for which we have the following results.

**Theorem 3.5.5** ([5], Thm 9.1.1). *If  $X$  has finite type, then it has a  $p$ -completion unique up to equivalence.*

**Theorem 3.5.6** ([5], Prop 9.2.22). *Let  $X$  be a connective spectrum of finite type, and let  $Y$  be  $p$ -complete. Then the map  $[X_p^\wedge, Y] \rightarrow [X, Y]$  is an isomorphism.*

Equivalently, we have the following universal property: given any map  $X \xrightarrow{f} Y$ , there exists a unique (up to homotopy) map  $X_p^\wedge \xrightarrow{\bar{f}} Y$  such that  $f$  factors as  $X \rightarrow X_p^\wedge \xrightarrow{\bar{f}} Y$ .

Note that this property holds for abelian groups  $A$  and  $B$ , with  $B = B_p^\wedge$ , where the component  $\bar{f}_n : A_p^\wedge \rightarrow B/p^n B$  is defined by projecting onto  $A/p^n A$  and composing with the map induced by  $f$  (uniqueness follows from the fact that group homomorphisms are continuous with respect to the  $p$ -adic topology, and the image of  $A$  is dense in  $A_p^\wedge$ ). We have already seen that maps between spectra always form an abelian group, and the theorems above thus give another way in which (under certain hypotheses) spectra behave like abelian groups.

## 4 The Adams spectral sequence

We will now turn our attention to the main tool for computing stable homotopy groups of spheres - the Adams spectral sequence. Sections 4.1 and 4.2 give a very brief explanation of spectral sequences and one important way in which they arise, following Chapter 2 of [7]. Section 4.3 then covers some homological algebra which will be needed later, and is drawn mostly from [11]. In Section 4.4 we will finally construct the Adams spectral sequence, and in Section 4.5 we make some initial computations of the stable homotopy groups using the tools we have developed; both of these sections mostly follow [4]. Finally, Section 4.6 will explore further structure which can be put on the Adams spectral sequence, which will help narrow down possibilities for the stable homotopy groups; we will follow [10] and [9].

From this section onwards, all homology and cohomology will be taken with  $\mathbb{F}_2$  coefficients, and we will thus ease notation by writing  $H^*(X)$  (resp.  $H_*(X)$ ) for  $H^*(X; \mathbb{F}_2)$  (resp.  $H_*(X; \mathbb{F}_2)$ ).

### 4.1 Spectral sequences

**Definition 4.1.1.** A *differential bigraded module*  $E$  over a ring  $R$  is a collection of  $R$ -modules  $\{E^{p,q}\}$ ,  $p, q \in \mathbb{Z}$ , together with a map  $d : E^{p,q} \rightarrow E^{p+s, q+r}$  for each  $p, q$  and some fixed  $s, r \in \mathbb{Z}$ , satisfying  $d^2 = 0$ .

We can take the homology of  $(E, d)$ :

$$H^{p,q}(E^{*,*}, d) = \ker(d : E^{p,q} \rightarrow E^{p+s, q+r}) / \text{im}(d : E^{p-s, q-r} \rightarrow E^{p,q}).$$

In the definition of a spectral sequence below, we will specialise to a specific bigrading, the *Adams grading*, since it is the one we will encounter in Section 4.4. There are, however, different gradings which correspond to different types of spectral sequences, the most common of which are (co)homological spectral sequences.

**Definition 4.1.2.** A *spectral sequence* (of *Adams type*) is a collection of differential bigraded  $R$ -modules  $\{E_r^{*,*}, d_r\}$ ,  $r \in \mathbb{N}$ , with the differentials  $d_r$  of bidegree  $(r, r-1)$ . These satisfy the further condition that for all  $p, q, r$ ,  $E_{r+1}^{p,q} \cong H^{p,q}(E_r^{*,*}, d_r)$ .

Consider the term  $E_2$ . Define

$$Z_2 := \ker d_2 \quad \text{and} \quad B_2 := \text{im } d_2.$$

The condition  $d^2 = 0$  implies that  $B_2 \subseteq Z_2 \subseteq E_2$ , and by definition we have  $E_3 \cong Z_2/B_2$ .

Now, write

$$Z_3 := \ker d_3 \quad \text{and} \quad B_3 := \text{im } d_3.$$

Since  $Z_3 \subseteq E_3$ , it can be written as  $\overline{Z}_3/B_2$  for some  $\overline{Z}_3 \subseteq Z_2$ . Similarly,  $B_3 \cong \overline{B}_3/B_2$  for some  $\overline{B}_3 \subseteq Z_2$ . Thus,

$$E_4 \cong Z_3/B_3 \cong \frac{\overline{Z}_2/B_2}{\overline{B}_3/B_2} \cong \overline{Z}_3/\overline{B}_3.$$

Iterating this process, we present the spectral sequence as an infinite tower of submodules of  $E_2$ :

$$B_2 \subseteq \overline{B}_3 \subseteq \cdots \subseteq \overline{B}_n \subseteq \cdots \subseteq \overline{Z}_n \subseteq \cdots \subseteq \overline{Z}_3 \subseteq Z_2 \subseteq E_2,$$

with the property that  $E_{n+1} \cong \overline{Z}_n/\overline{B}_n$ . The differential  $d_{n+1}$  can be taken as a map  $\overline{Z}_n/\overline{B}_n \rightarrow \overline{Z}_n/\overline{B}_n$  with kernel  $\overline{Z}_{n+1}/\overline{B}_n$  and image  $\overline{B}_{n+1}/\overline{B}_n$ . The short exact sequence induced by  $d_{n+1}$ ,

$$0 \rightarrow \overline{Z}_{n+1}/\overline{B}_n \rightarrow \overline{Z}_n/\overline{B}_n \xrightarrow{d_{n+1}} \overline{B}_{n+1}/\overline{B}_n \rightarrow 0,$$

gives rise to isomorphisms  $\overline{Z}_n/\overline{Z}_{n+1} \cong \overline{B}_{n+1}/\overline{B}_n$  for all  $n$ . Conversely, a tower of submodules of  $E_2$ , together with a set of isomorphisms, gives rise to a spectral sequence.

**Definition 4.1.3.** An element of  $E_2$  *survives to the  $r$ th stage* if lies in  $\overline{Z}_r$ , having been in the kernel of the previous  $r-2$  differentials, and is *bounded by the  $r$ th stage* if it lies in  $\overline{B}_r$ . The bigraded module  $E_r^{*,*}$  is called the  $E_r$ -*term* of the spectral sequence.

We define

$$Z_\infty := \bigcap_n \overline{Z}_n, \quad B_\infty := \bigcup_n \overline{B}_n.$$

From the tower of inclusions, we see that  $B_\infty \subseteq Z_\infty$ , so we define  $E_\infty := Z_\infty/B_\infty$ .

**Definition 4.1.4.** A spectral sequence *collapses at the  $N$ th term* if the differentials  $d_r = 0$  for  $r \geq N$ .

From the short exact sequence

$$0 \rightarrow \overline{Z}_r/\overline{B}_{r-1} \rightarrow \overline{Z}_{r-1}/\overline{B}_{r-1} \xrightarrow{d_r} \overline{B}_r/\overline{B}_{r-1} \rightarrow 0,$$

the condition  $d_r = 0$  forces  $\overline{Z}_r = \overline{Z}_{r-1}$  and  $\overline{B}_r = \overline{B}_{r-1}$ . The tower of submodules becomes

$$B_2 \subseteq \overline{B}_3 \subseteq \cdots \subseteq \overline{B}_{N-1} = \overline{B}_N = \cdots = B_\infty \subseteq Z_\infty = \cdots = \overline{Z}_N = \overline{Z}_{N-1} \subseteq \cdots \subseteq \overline{Z}_3 \subseteq Z_2 \subseteq E_2.$$

Thus,  $E_\infty = E_N$ .

Let  $M^*$  be a graded  $R$ -module, and suppose  $M^*$  has a filtration

$$\dots \subseteq F^{n+1}M^* \subseteq F^n M^* \subseteq F^{n-1}M^* \subseteq \dots \subseteq M^*.$$

We define the *associated graded*  $E_0^{*,*}(M^*, F)$  of  $M$  to be the bigraded module whose degree  $(p, q)$  summand is

$$E_0^{p,q}(M) = F^p M^{p-q} / F^{p+1} M^{p-q}.$$

**Definition 4.1.5.** A spectral sequence  $\{E_r^{*,*}\}$  *converges* to a graded  $R$ -module  $M^*$  if there is a filtration  $F$  on  $M^*$  such that the following conditions hold:

1.  $\bigcup_n F^n M^* = M^*$ .
2.  $\bigcap_n F^n M^* = \{0\}$ .
3.  $E_\infty^{p,q} \cong E_0^{p,q}(M^*, F)$ .

## 4.2 Exact couples

We have seen that certain towers of submodules give rise to spectral sequences; now, we describe a different structure from which a spectral sequence can be obtained, which will in fact be how the Adams spectral sequence arises.

**Definition 4.2.1.** Let  $D, E$  be  $R$ -modules, and let  $i : D \rightarrow D$ ,  $j : D \rightarrow E$ ,  $k : E \rightarrow D$  be module homomorphisms. We call  $\mathcal{C} = \{D, E, i, j, k\}$  an *exact couple* if the diagram below is exact.

$$\begin{array}{ccc} D & \xrightarrow{i} & D \\ & \swarrow k & \searrow j \\ & E & \end{array}$$

Let  $d := jk$ , and define the following:

$$\begin{aligned} E' &:= H(E, d) = \ker d / \operatorname{im} d \\ D' &:= i(D) = \ker j \\ i' &:= i|_{D'} : D' \rightarrow D' \\ j' &:= i(x) \mapsto j(x) + dE : D' \rightarrow E' \\ k' &:= (e + dE) \mapsto k(e) : E' \rightarrow D' \end{aligned}$$

We call  $\mathcal{C}' = \{D', E', i', j', k'\}$  the *derived couple* of  $\mathcal{C}$ .

**Proposition 4.2.2** ([7], Prop 2.7). *If  $\mathcal{C} = \{D, E, i, j, k\}$  is an exact couple, then  $\mathcal{C}'$  is also an exact couple.*

The theorem below is proven in [7] when  $i$  has bidegree  $(-1, 1)$  and the resulting spectral sequence has differentials of bidegree  $(r, 1 - r)$ , however an almost identical proof works in the case of the Adams grading.

**Theorem 4.2.3** ([7], Thm 2.8). *Suppose  $D^{*,*} = \{D^{p,q}\}$  and  $E^{*,*} = \{E^{p,q}\}$  are bigraded modules equipped with homomorphisms  $i$  of bidegree  $(-1, -1)$ ,  $j$  of bidegree  $(0, 0)$ , and  $k$  of bidegree  $(1, 0)$ , such that  $\{D^{*,*}, E^{*,*}, i, j, k\}$  is an exact couple. Then these data determine a spectral sequence  $\{E_r, d_r\}$  for  $r \in \mathbb{Z}_+$  of Adams type, with  $E_r = (E^{*,*})^{(r-1)}$ , the  $(r-1)$ st derived module of  $E^{*,*}$  and  $d_r = j^{(r-1)} \circ k^{(r-1)}$ .*

Such a bigraded exact couple may be displayed in the diagram below, known as a *staircase diagram*.

$$\begin{array}{ccccccc}
& & \vdots & & \vdots & & \\
& & \downarrow i & & \downarrow i & & \\
\cdots & \xrightarrow{k} & D^{p+1,q+1} & \xrightarrow{j} & E^{p+1,q+1} & \xrightarrow{k} & D^{p+2,q+1} \xrightarrow{j} \cdots \\
& & \downarrow i & & \downarrow i & & \\
\cdots & \xrightarrow{k} & D^{p,q} & \xrightarrow{j} & E^{p,q} & \xrightarrow{k} & D^{p+1,q} \xrightarrow{j} \cdots \\
& & \downarrow i & & \downarrow i & & \\
\cdots & \xrightarrow{k} & D^{p-1,q-1} & \xrightarrow{j} & E^{p-1,q-1} & \xrightarrow{k} & D^{p,q-1} \xrightarrow{j} \cdots \\
& & \downarrow i & & \downarrow i & & \\
& & \vdots & & \vdots & & 
\end{array}$$

### 4.3 The Ext functor

Before constructing the Adams spectral sequence, we briefly recall the definition of the Ext functor and some of its basic properties, which will be of importance later. We mainly follow [11].

**Definition 4.3.1.** Let  $M, N$  be modules over a ring  $R$ . A *projective resolution*  $P$  of  $M$  is an exact sequence,

$$\cdots \rightarrow P_2 \rightarrow P_1 \rightarrow P_0 \rightarrow M \rightarrow 0.$$

where each  $P_i$  is projective. If, in addition, each  $P_i$  is free, then the resolution is called *free*.

Dually, an *injective resolution*  $I$  of  $M$  is an exact sequence

$$0 \rightarrow M \rightarrow I_0 \rightarrow I_1 \rightarrow I_2 \rightarrow \cdots,$$

where each  $I_i$  is injective.

The following result can be obtained from [11], Lemmas 2.2.5, 2.3.6, and Exercise 2.3.5.

**Lemma 4.3.2.** Every  $R$ -module  $M$  has a projective resolution and an injective resolution.

Given a projective resolution as in Definition 4.3.1, applying  $\text{Hom}_R(-, N)$  gives us a chain complex

$$\cdots \leftarrow \text{Hom}_R(P_2, N) \leftarrow \text{Hom}_R(P_1, N) \leftarrow \text{Hom}_R(P_0, N) \leftarrow \text{Hom}_R(M, N) \leftarrow 0.$$

Dropping the term  $\text{Hom}_R(M, N)$ , we get the chain complex

$$\cdots \leftarrow \text{Hom}_R(P_2, N) \leftarrow \text{Hom}_R(P_1, N) \leftarrow \text{Hom}_R(P_0, N) \leftarrow 0,$$

which we denote by  $\text{Hom}_R(P_\bullet, N)$ .

Dually, given an injective resolution as in Definition 4.3.1, we can form the chain complex

$$\cdots \leftarrow \text{Hom}_R(N, I_2) \leftarrow \text{Hom}_R(N, I_1) \leftarrow \text{Hom}_R(N, I_0) \leftarrow 0,$$

denoted by  $\text{Hom}_R(N, I_\bullet)$ .

The result below is a combination of [11], Lemma 2.4.1 and Theorem 2.7.6.

**Proposition 4.3.3.** Let  $M, N$  be  $R$ -modules. For any projective resolution  $P$  and any injective resolution  $I$  of  $M$ ,  $H^*(\text{Hom}_R(P_\bullet, N)) = H^*(\text{Hom}_R(N, I_\bullet))$ .

We define  $\text{Ext}_R^n(M, N) := H^n(\text{Hom}_R(P_\bullet, N)) = H^n(\text{Hom}_R(N, I_\bullet))$ .

#### 4.4 Setting up the Adams spectral sequence

Let  $X$  be a connective CW spectrum of finite type. Then  $H^*(X)$  is an  $\mathcal{A}_2$ -module, since  $H^i(X) \cong H^{i+n}(X_n)$  for sufficiently large  $n$ , so we can define  $Sq^j : H^i(X) \rightarrow H^{i+j}(X)$  by evaluating  $Sq^j : H^{i+n}(X_n) \rightarrow H^{i+j+n}(X_n)$  followed by enough suspensions. Note that we could also have first suspended  $H^{i+n}(X_n)$  and  $H^{i+j+n}(X_n)$  until they were both stable, and then evaluated  $Sq^j$ , but that these two  $\mathcal{A}_2$ -actions coincide since the Steenrod squares commute with suspension isomorphisms (Proposition 2.0.1 (4)).

We can pick generators  $\alpha_i$  for  $H^*(X)$  as an  $\mathcal{A}_2$ -module such that there are at most finitely many in each  $H^n(X)$  (since each  $H^n(X)$  is finitely generated by Lemma 3.2.1, and such a finite generating set would certainly also generate it as an  $\mathcal{A}_2$ -module). Each generator  $\alpha_i \in H^{n_i}(X)$  corresponds to a map  $X \rightarrow \mathbb{K}(\mathbb{F}_2, n_i)$  by Theorem 3.4.1, so putting these maps together gives an element of  $\prod_i [X, \mathbb{K}(\mathbb{F}_2, n_i)]$ . Now,  $n_i \rightarrow \infty$  as  $i \rightarrow \infty$  since there are only finitely many  $\alpha_i$  in each  $H^{n_i}(X)$ , so Proposition 3.4.2 implies that we get an element of  $[X, \bigvee_i \mathbb{K}(\mathbb{F}_2, n_i)]$ . We write  $K_0 := \bigvee_i \mathbb{K}(\mathbb{F}_2, n_i)$ , and replace the map  $X \rightarrow K_0$  by an inclusion (see Remark 3.3.3).

**Remark 4.4.1.**  $K_0$  has finite type, which can be seen as follows: first, recall from Example 3.1.9 that each spectrum  $\mathbb{K}(\mathbb{F}_2, n_i)$  has finite type. Now, the  $j$ -cells of  $\bigvee_i \mathbb{K}(\mathbb{F}_2, n_i)$  consist of the  $(j+k)$ -cells of  $\mathbb{K}(\mathbb{F}_2, n_i+k)$  for each  $k$ , up to equivalence under suspension. However, there are only finitely many  $n_i$  with  $n_i \leq j$ , and if  $n_i > j$  the space  $\mathbb{K}(\mathbb{F}_2, n_i)$  can be taken to have no cells of dimension  $\leq j$ . Thus, the  $j$ -cells of  $\bigvee_i \mathbb{K}(\mathbb{F}_2, n_i)$  are the  $j$ -cells of the finite wedge  $\bigvee_{i, n_i \leq j} \mathbb{K}(\mathbb{F}_2, n_i)$ , of which there are only finitely many (since a finite wedge of finite-type spectra has finite type).

Now, we set  $X_1 = K_0/X$ , and repeat the construction to get a diagram:

$$\begin{array}{ccccccc} X & \longrightarrow & K_0 & \longrightarrow & K_1 & \longrightarrow & K_2 \longrightarrow \cdots \\ & & \searrow & & \swarrow & & \searrow \\ & & K_0/X = X_1 & & K_1/X_1 = X_2 & & K_2/X_2 = X_3 \end{array}$$

Taking cohomology, we get a diagram

$$\begin{array}{ccccccc} 0 \leftarrow H^*(X) \leftarrow H^*(K_0) & \longleftarrow & H^*(K_1) & \longleftarrow & H^*(K_2) & \longleftarrow & \cdots \\ & \swarrow & \swarrow & \swarrow & \swarrow & \swarrow & \\ & H^*(X_1) & & H^*(X_2) & & H^*(X_3) & \\ & \swarrow & \swarrow & \swarrow & \swarrow & \swarrow & \\ 0 & & 0 & & 0 & & 0 \end{array} \quad (4.4.1)$$

The induced map  $H^*(X) \leftarrow H^*(K_0)$  is surjective by construction, and thus each map  $H^*(X_i) \leftarrow H^*(K_i)$  is surjective.

Now, as with CW complexes, we have a long exact sequence

$$\cdots \leftarrow H^{n+1}(X_{s+1}) \leftarrow H^n(X_s) \leftarrow H^n(K_s) \leftarrow H^n(X_{s+1}) \leftarrow H^{n-1}(X_s) \leftarrow \cdots,$$

and surjectivity of the maps  $H^*(X_s) \leftarrow H^*(K_s)$  implies that the boundary maps  $H^{n+1}(X_{s+1}) \leftarrow H^n(X_s)$  are all zero (writing  $X_0 := X$ ). We thus get short exact sequences

$$0 \leftarrow H^n(X_s) \leftarrow H^n(K_s) \leftarrow H^n(X_{s+1}) \leftarrow 0,$$

giving rise to a short exact sequence

$$0 \leftarrow H^*(X_s) \leftarrow H^*(K_s) \leftarrow H^*(X_{s+1}) \leftarrow 0,$$

for each  $s$ . This then implies that the top row of (4.4.1) is exact.

Each  $H^*(K_s)$  is a free  $\mathcal{A}_2$ -module, by Remark 2.0.6 and the fact that  $K_s$  has finite type. Thus, the top row of (4.4.1) gives a free resolution of  $H^*(X)$ .

By Theorem 3.3.5, we obtain a long exact sequence

$$\cdots \rightarrow [\mathbb{S}^{t+1}, X_s] \rightarrow [\mathbb{S}^{t+1}, K_s] \rightarrow [\mathbb{S}^{t+1}, X_{s+1}] \rightarrow [\mathbb{S}^{t+1}, \Sigma X_s] \rightarrow [\mathbb{S}^{t+1}, \Sigma K_s] \rightarrow \cdots.$$

Using the isomorphism  $[Y, Z] \cong [\Sigma Y, \Sigma Z]$ , we get long exact sequences

$$\cdots \rightarrow \pi_{t+1}X_s \rightarrow \pi_{t+1}K_s \rightarrow \pi_{t+1}X_{s+1} \rightarrow \pi_tX_s \rightarrow \pi_tK_s \rightarrow \cdots,$$

which form the staircase diagram shown below.

$$\begin{array}{ccccccccccc} & \vdots & & \vdots & & \vdots & & & & & \\ & \downarrow i & & \downarrow i & & \downarrow i & & & & & \\ \cdots & \xrightarrow{k} & \pi_{t+1}X_s & \xrightarrow{j} & \pi_{t+1}K_s & \xrightarrow{k} & \pi_{t+1}X_{s+1} & \xrightarrow{j} & \pi_{t+1}K_{s+1} & \xrightarrow{k} & \pi_{t+1}X_{s+2} \xrightarrow{j} \cdots \\ & \downarrow i & & \downarrow i & & \downarrow i & & & & & \\ \cdots & \xrightarrow{k} & \pi_tX_{s-1} & \xrightarrow{j} & \pi_tK_{s-1} & \xrightarrow{k} & \pi_tX_s & \xrightarrow{j} & \pi_tK_s & \xrightarrow{k} & \pi_tX_{s+1} \xrightarrow{j} \cdots \\ & \downarrow i & & \downarrow i & & \downarrow i & & & & & \\ \cdots & \xrightarrow{k} & \pi_{t-1}X_{s-2} & \xrightarrow{j} & \pi_{t-1}K_{s-2} & \xrightarrow{k} & \pi_{t-1}X_{s-1} & \xrightarrow{j} & \pi_{t-1}K_{s-1} & \xrightarrow{k} & \pi_{t-1}X_s \xrightarrow{j} \cdots \\ & \downarrow i & & \downarrow i & & \downarrow i & & & & & \\ & \vdots & & \vdots & & \vdots & & & & & \end{array} \quad (4.4.2)$$

This gives rise to a spectral sequence, by Theorem 4.2.3.

Now, since  $K_s = \bigvee_i \mathbb{K}(\mathbb{F}_2, n_{s_i})$ , Proposition 3.4.2 tells us that  $[\mathbb{S}, K_s] \cong \prod_i [\mathbb{S}, \mathbb{K}(\mathbb{F}_2, n_{s_i})]$ , which is naturally isomorphic to  $\prod_i H^{n_{s_i}}(\mathbb{S}; \mathbb{F}_2)$ . Thus, elements of  $[\mathbb{S}, K_s]$  are tuples of elements of  $H^*(\mathbb{S})$ .

We have a map  $[\mathbb{S}, K_s] \rightarrow \text{Hom}_{\mathcal{A}_2}^0(H^*(K_s), H^*(\mathbb{S}))$  sending  $f \in [\mathbb{S}, K_s]$  to  $f^*$ , since  $f^*$  is an  $\mathcal{A}_2$ -module homomorphism by Proposition 2.0.1 (1). The fact that  $H^*(K_s)$  is free implies that it is an isomorphism.

We thus have

$$[\Sigma^t \mathbb{S}, K_s] = \text{Hom}_{\mathcal{A}_2}^0(H^*(K_s), H^*(\Sigma^t \mathbb{S})) = \text{Hom}_{\mathcal{A}_2}^t(H^*(K_s), H^*(\mathbb{S})),$$

where  $\text{Hom}_{\mathcal{A}_2}^t(H^*(K_s), H^*(\mathbb{S}))$  is the set of  $\mathcal{A}_2$ -module homomorphisms which lower the degree by  $t$ . In the case of CW complexes, we have  $H^*(\Sigma^t X) \cong H^{*-t}(X)$ . Since  $\mathbb{S}$  has finite type, for  $i$  large enough we have  $H^n(\Sigma^t \mathbb{S}) = H^{n+i}(\Sigma^t S^i) \cong H^{n+i-t}(S^i) = H^{n-t}(\mathbb{S})$ .

Now,  $E_1^{s,t} = \pi_t K_s = \text{Hom}_{\mathcal{A}_2}^t(H^*(K_s), H^*(\mathbb{S}))$ , since the staircase diagram comes from the exact couple

$$\begin{array}{ccc} \pi_* X_* & \xrightarrow{i} & \pi_* X_* \\ & \nwarrow k \quad \swarrow j & \\ & \pi_* K_* & \end{array}$$

where  $i : \pi_{t+1}X_{s+1} \rightarrow \pi_tX_s$ ,  $j : \pi_{t+1}X_s \rightarrow \pi_{t+1}K_s$ , and  $k : \pi_{t+1}X_{s+1} \rightarrow \pi_tK_s$  are as in (4.4.2). The differential  $d_1 : \pi_t(K_s) \rightarrow \pi_tK_{s+1}$  is induced by the map  $K_s \rightarrow K_{s+1}$ , since it is defined to be  $j \circ k$ .

Further,  $E_2^{s,t} = H^{s,t}(E_1^{*,*}, d_1)$ , so each  $E^{*,t}$  is the homology of the chain complex

$$0 \rightarrow E_1^{0,t} \rightarrow E_1^{1,t} \rightarrow E_1^{2,t} \rightarrow \cdots,$$

which is by construction the chain complex below.

$$0 \rightarrow \text{Hom}_{\mathcal{A}_2}^t(H^*(K_0), H^*(\mathbb{S})) \rightarrow \text{Hom}_{\mathcal{A}_2}^t(H^*(K_1), H^*(\mathbb{S})) \rightarrow \cdots.$$

The homology of this is by definition  $\text{Ext}_{\mathcal{A}_2}^{s,t}(H^*(X), H^*(\mathbb{S}))$ , so  $E_2^{s,t} = \text{Ext}_{\mathcal{A}_2}^{s,t}(H^*(X), H^*(\mathbb{S}))$ .

**Theorem 4.4.2.** *The spectral sequence  $\{E_r, d_r\}$  constructed above converges to  $\pi_{t-s}(X)_2^\wedge$ , in the sense of Definition 4.1.5.*

The version of the theorem above stated in [4] (Thm 4.29) says instead that  $\{E_r, d_r\}$  converges to  $\pi_{t-s}^s$  modulo its odd torsion, however, these two statements are equivalent. First, recall that  $\pi_{t-s}^s$  modulo its odd torsion is isomorphic to  $(\pi_{t-s}^s)_2^\wedge$  whenever  $\pi_{t-s}^s$  is finite. Now, all the terms  $E_\infty^{s,t}$  are finitely generated  $\mathbb{F}_2$ -vector spaces, so each term of the filtration must be a finite index subgroup of whichever group the spectral sequence converges to. The finite index subgroups of  $\mathbb{Z}_2$  are of the form  $2^k\mathbb{Z}_2$  and the finite index subgroups of  $\mathbb{Z}$  (giving quotients of order a power of 2) are of the form  $2^k\mathbb{Z}$ . The associated graded thus cannot tell the difference between a  $\mathbb{Z}$  summand and a  $\mathbb{Z}_2$  summand in the group the spectral sequence converges to, so the two theorems are equivalent.

## 4.5 First computations

Before we begin any calculations, we first prove a small lemma which will make computing the  $E_2$  page much easier.

We will say that a free resolution

$$\cdots \xrightarrow{f_3} F_2 \xrightarrow{f_2} F_1 \xrightarrow{f_1} F_0 \xrightarrow{f_0} H^*(X)$$

is *minimal* if  $\text{im } f_i \subseteq \mathcal{A}_2^+ F_{i-1}$  for all  $i$ , where  $\mathcal{A}_2^+ \subseteq \mathcal{A}_2$  is the irrelevant ideal.

**Lemma 4.5.1** ([4], Lem 5.49). *For a minimal free resolution*

$$\cdots \xrightarrow{f_3} F_2 \xrightarrow{f_2} F_1 \xrightarrow{f_1} F_0 \xrightarrow{f_0} H^*(X) \rightarrow 0$$

*of  $H^*(X)$  as an  $\mathcal{A}_2$ -module, we have  $\text{Ext}_{\mathcal{A}_2}^{s,t}(H^*(X), \mathbb{F}_2) = \text{Hom}_{\mathcal{A}_2}^t(F_s, \mathbb{F}_2)$ .*

*Proof.* Let  $x \in F_i$ . Since  $f_{i-1}f_i = 0$ , we have  $f_i(x) \in \ker f_{i-1} = \text{im } f_i \subseteq \mathcal{A}_2^+ F_{i-1}$ . We can thus write  $f_i(x) = \sum_j a_j x_{i-1,j}$  with  $a_j \in \mathcal{A}_2^+$ . Now, for  $g \in \text{Hom}_{\mathcal{A}_2}(F_{i-1}, \mathbb{F}_2)$ , we have  $gf_i(x) = \sum_j a_j g(x_{i-1,j}) = 0$ , since  $a_j$  acts trivially on elements of  $\mathbb{F}_2$ .

Thus, the boundary maps in the complex

$$\cdots \xleftarrow{-\circ f_3} \text{Hom}_{\mathcal{A}_2}(F_2, \mathbb{F}_2) \xleftarrow{-\circ f_2} \text{Hom}_{\mathcal{A}_2}(F_1, \mathbb{F}_2) \xleftarrow{-\circ f_1} \text{Hom}_{\mathcal{A}_2}(F_0, \mathbb{F}_2) \leftarrow 0$$

are all zero, so  $\text{Ext}_{\mathcal{A}_2}^{s,t}(H^*(X), \mathbb{F}_2) = \text{Hom}_{\mathcal{A}_2}^t(F_s, \mathbb{F}_2)$ . □

Now, since  $\mathbb{F}_2$  is concentrated in degree 0, the only elements of  $F_s$  which can be sent to  $1 \in \mathbb{F}_2$  are the elements of degree  $t$ , so for every generator of  $F_s$  in degree  $t$ , there is an  $\mathbb{F}_2$  summand in  $\text{Hom}_{\mathcal{A}_2}^t(F_s, \mathbb{F}_2)$ .



Figure 4.1 shows part of a construction of a minimal free resolution of  $H^*(\mathbb{S}) = \mathbb{F}_2$ , where position  $(t-s, s)$  consists of degree  $t$  elements of  $F_s$ . Instead of inducting on  $s$  and calculating column by column, we will instead induct on  $t-s$ , assuming the previous rows have been computed. Note that since we will add the minimum number of generators needed in each row, the addition of new generators in later rows will not affect the induction (because a new generator will not impact the kernel of any  $f_i$ ).

For  $t-s=0$ , at position  $(0,0)$ , we need a generator  $\iota \in F_0$  to map to  $1 \in \mathbb{F}_2$ , in order to make  $f_0$  a surjection. The kernel of  $f_0$  thus contains every multiple of  $\iota$  by an element of  $\mathcal{A}_2^+$ , which by exactness should be contained in the image of  $f_1$ . Thus, we need a new generator  $\alpha_1^1$  at  $(0,1)$  mapping to  $Sq^1\iota$ . The element  $Sq^1\alpha_1^1 \in F_1$  is therefore in the kernel of  $f_1$ , since  $Sq^1Sq^1 = 0$ , so we need a generator  $\alpha_2^2$  at  $(0,2)$  mapping to  $Sq^1\alpha_1^1$ . Now, it is clear that each position  $(0,s)$  will require a new generator  $\alpha_s^s$ , since each  $Sq^1\alpha_{s-1}^{s-1}$  maps to  $Sq^1Sq^1\alpha_{s-2}^{s-2} = 0$ , so the first row is completely determined, and  $\text{Ext}_{\mathcal{A}_2}^{s,s}(\mathbb{F}_2, \mathbb{F}_2) = \mathbb{F}_2$ .

When  $t-s=1$ , a generator  $\alpha_2^1$  is needed in position  $(1,1)$  mapping to  $Sq^2\iota$ , since  $f_1(Sq^1\alpha_1^1) = Sq^1f_1(\alpha_1^1) = Sq^1Sq^1\iota = 0$  but  $Sq^2\iota \in \ker f_0$ . No other generators are needed, since  $Sq^1\alpha_2^1$  maps to  $Sq^3\iota \neq 0$  and  $Sq^2\alpha_s^s$  maps to  $Sq^2Sq^1\alpha_{s-1}^{s-1} \neq 0$  for all  $s > 1$ .

For  $t-s=2$ , no generator is needed at  $(2,1)$ , since  $f_1(Sq^2\alpha_1^1) = Sq^2Sq^1\iota \neq 0$  and  $f_1(Sq^1\alpha_2^1) = Sq^3\iota \neq 0$ . A generator  $\alpha_4^2$  is needed at  $(2,2)$  to map to  $Sq^3\alpha_1^1 + Sq^2\alpha_2^1$ , since  $f_1(Sq^3\alpha_1^1 + Sq^2\alpha_2^1) = 2Sq^3Sq^1\iota = 0$ . No further generators are needed, as  $Sq^1\alpha_4^2$  maps to  $Sq^3\alpha_2^2 \neq 0$  and  $Sq^3\alpha_s^s$  maps to  $Sq^3Sq^1\alpha_{s-1}^{s-1} \neq 0$  for all  $s > 1$ .

When  $t-s=3$ , generators  $\alpha_4^1$ ,  $\alpha_5^2$ , and  $\alpha_6^3$  are needed to map to  $Sq^4\iota$ ,  $Sq^4\alpha_1^1 + Sq^2Sq^1\alpha_2^1 + Sq^1\alpha_4^1$ , and  $Sq^4\alpha_2^2 + Sq^2\alpha_4^2 + Sq^1\alpha_5^2$  respectively, since the latter elements are in the kernel of their respective  $f_i$ 's. No new generators are needed after  $s=4$ , since  $Sq^1\alpha_6^3$  maps to  $Sq^5\alpha_2^2 + Sq^3\alpha_4^2$ ,  $Sq^4\alpha_s^s$  maps to  $Sq^4Sq^1\alpha_{s-1}^{s-1}$ , and although  $Sq^3Sq^1\alpha_s^s$  maps to zero, it is hit by  $Sq^3\alpha_{s+1}^{s+1}$ .

Continuing in this fashion, the computations for  $t-s=4, 5$  are shown in Figure 4.1, though the Adem relations of Proposition 2.0.2 required to justify them are not. Note that although to compute each row, knowledge of maps involving the next two rows is required, the rows  $t-s=6, 7$  do not contain all the new generators needed.

Figure 4.2 shows the  $E_2$  page of the Adams spectral sequence for  $\mathbb{S}$  for  $t-s \leq 5$ . Recall that the Adams grading is  $(r, r-1)$ , and thus on Figure 4.2 the  $d_r$  differentials go one unit left and  $r$  units up. Now, from Figure 4.2, we see that  $(\pi_1^s)_2^\wedge$  has order dividing 2, but a priori there could be a  $d_r$  differential mapping  $\alpha_2^1$  to  $\alpha_{r+1}^{r+1}$ , in which case  $\alpha_2^1$  would not survive to the  $E_\infty$  page. On the other hand, any differential emanating from or entering position  $(2,2)$  either enters or leaves 0, so  $\alpha_4^2$  must survive to the  $E_\infty$  page, and we see that  $(\pi_2^s)_2^\wedge = \mathbb{Z}/2\mathbb{Z}$ .

Similarly, we see that all of the generators in the column  $t-s=3$  survive to  $E_\infty$  (since there are no possible nonzero differentials interacting with them), so  $|(\pi_3^s)_2^\wedge| = 8$ . However, we do not currently have the tools to determine the isomorphism class of  $(\pi_3^s)_2^\wedge$ , since any of the filtrations below are possible:

$$\begin{aligned} \{0\} &\subseteq \mathbb{Z}/2\mathbb{Z} \subseteq \mathbb{Z}/4\mathbb{Z} \subseteq \mathbb{Z}/8\mathbb{Z}, \\ \{0\} &\subseteq \mathbb{Z}/2\mathbb{Z} \subseteq \mathbb{Z}/4\mathbb{Z} \subseteq \mathbb{Z}/4\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}, \\ \{0\} &\subseteq \mathbb{Z}/2\mathbb{Z} \subseteq (\mathbb{Z}/2\mathbb{Z})^2 \subseteq \mathbb{Z}/4\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}, \\ \{0\} &\subseteq \mathbb{Z}/2\mathbb{Z} \subseteq (\mathbb{Z}/2\mathbb{Z})^2 \subseteq (\mathbb{Z}/2\mathbb{Z})^3. \end{aligned}$$

We will therefore spend some time describing a multiplication on the Adams spectral sequence which will allow us to resolve such ambiguities.

$\begin{smallmatrix} s \\ t-s \end{smallmatrix}$	0	1	2	3	4
0	$\iota$	$\alpha_1^1$	$\alpha_2^2$	$\alpha_3^3$	$\alpha_4^4$
1	$Sq^1 \iota$	$Sq^1 \alpha_1^1$	$Sq^1 \alpha_2^2$	$Sq^1 \alpha_3^3$	$Sq^1 \alpha_4^4$
2	$Sq^2 \iota$	$\alpha_2^1$ $Sq^2 \alpha_1^1$ $Sq^1 \alpha_2^2$	$Sq^2 \alpha_2^2$	$Sq^2 \alpha_3^3$	$Sq^2 \alpha_4^4$
3	$Sq^2 Sq^1 \iota$ $Sq^3 \iota$	$Sq^2 Sq^1 \alpha_1^1$ $Sq^3 \alpha_1^1$ $Sq^2 \alpha_2^1$	$\alpha_4^2$ $Sq^2 Sq^1 \alpha_2^2$ $Sq^3 \alpha_2^2$ $Sq^1 \alpha_4^2$	$Sq^2 Sq^1 \alpha_3^3$ $Sq^3 \alpha_3^3$	$Sq^2 Sq^1 \alpha_4^4$ $Sq^3 \alpha_4^4$
4	$Sq^3 Sq^1 \iota$ $Sq^4 \iota$	$\alpha_4^1$ $Sq^3 Sq^1 \alpha_1^1$ $Sq^4 \alpha_1^1$ $Sq^2 Sq^1 \alpha_2^1$ $Sq^3 \alpha_2^1$ $Sq^1 \alpha_4^1$	$\alpha_5^2$ $Sq^3 Sq^1 \alpha_2^2$ $Sq^4 \alpha_2^2$ $Sq^2 \alpha_4^2$ $Sq^1 \alpha_5^2$	$\alpha_6^3$ $Sq^3 Sq^1 \alpha_3^3$ $Sq^4 \alpha_3^3$ $Sq^1 \alpha_6^3$	$Sq^3 Sq^1 \alpha_4^4$ $Sq^4 \alpha_4^4$
5	$Sq^4 Sq^1 \iota$ $Sq^5 \iota$	$Sq^4 Sq^1 \alpha_1^1$ $Sq^5 \alpha_1^1$ $Sq^3 Sq^1 \alpha_2^1$ $Sq^4 \alpha_2^1$ $Sq^2 \alpha_4^1$	$Sq^4 Sq^1 \alpha_2^2$ $Sq^5 \alpha_2^2$ $Sq^2 Sq^1 \alpha_4^2$ $Sq^3 \alpha_4^2$ $Sq^2 \alpha_5^2$	$Sq^4 Sq^1 \alpha_3^3$ $Sq^5 \alpha_3^3$ $Sq^2 \alpha_6^3$	$Sq^4 Sq^1 \alpha_4^4$ $Sq^5 \alpha_4^4$
6	$Sq^5 Sq^1 \iota$ $Sq^4 Sq^2 \iota$ $Sq^6 \iota$	$Sq^5 Sq^1 \alpha_1^1$ $Sq^4 Sq^2 \alpha_1^1$ $Sq^6 \alpha_1^1$ $Sq^4 Sq^1 \alpha_2^1$ $Sq^5 \alpha_2^1$ $Sq^2 Sq^1 \alpha_4^1$ $Sq^3 \alpha_4^1$	$Sq^5 Sq^1 \alpha_2^2$ $Sq^4 Sq^2 \alpha_2^2$ $Sq^6 \alpha_2^2$ $Sq^3 Sq^1 \alpha_4^2$ $Sq^4 \alpha_4^2$ $Sq^2 Sq^1 \alpha_5^2$ $Sq^3 \alpha_5^2$	$Sq^5 Sq^1 \alpha_3^3$ $Sq^4 Sq^2 \alpha_3^3$ $Sq^6 \alpha_3^3$ $Sq^2 Sq^1 \alpha_6^3$ $Sq^3 \alpha_6^3$	$Sq^5 Sq^1 \alpha_4^4$ $Sq^4 Sq^2 \alpha_4^4$ $Sq^6 \alpha_4^4$
7	$Sq^4 Sq^2 Sq^1 \iota$ $Sq^6 Sq^1 \iota$ $Sq^5 Sq^2 \iota$ $Sq^7 \iota$	$Sq^4 Sq^2 Sq^1 \alpha_1^1$ $Sq^6 Sq^1 \alpha_1^1$ $Sq^5 Sq^2 \alpha_1^1$ $Sq^7 \alpha_1^1$ $Sq^5 Sq^1 \alpha_2^1$ $Sq^4 Sq^2 \alpha_2^1$ $Sq^6 \alpha_2^1$ $Sq^3 Sq^1 \alpha_4^1$ $Sq^4 \alpha_4^1$	$Sq^4 Sq^2 Sq^1 \alpha_2^2$ $Sq^6 Sq^1 \alpha_2^2$ $Sq^5 Sq^2 \alpha_2^2$ $Sq^7 \alpha_2^2$ $Sq^4 Sq^1 \alpha_4^2$ $Sq^5 \alpha_4^2$ $Sq^3 Sq^1 \alpha_5^2$ $Sq^4 \alpha_5^2$	$Sq^4 Sq^2 Sq^1 \alpha_3^3$ $Sq^6 Sq^1 \alpha_3^3$ $Sq^5 Sq^2 \alpha_3^3$ $Sq^7 \alpha_3^3$ $Sq^3 Sq^1 \alpha_6^3$ $Sq^4 \alpha_6^3$	$Sq^4 Sq^2 Sq^1 \alpha_4^4$ $Sq^6 Sq^1 \alpha_4^4$ $Sq^5 Sq^2 \alpha_4^4$ $Sq^7 \alpha_4^4$

Figure 4.1: A construction of a minimal free resolution of  $H^*(\mathbb{S}) = \mathbb{F}_2$ . Generators for  $t - s \leq 5$  are highlighted in pink; further generators which may be needed in rows  $t - s = 6, 7$  are not shown.

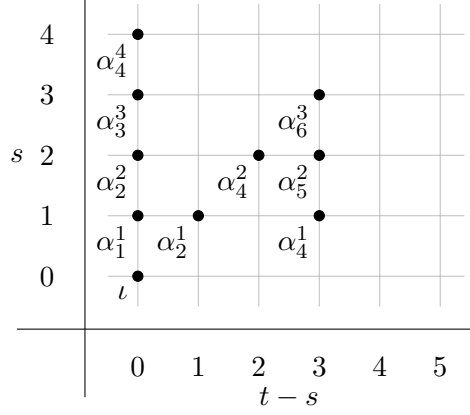


Figure 4.2:  $\text{Ext}_{\mathcal{A}_2}^{s,t}(\mathbb{F}_2, \mathbb{F}_2)$  for  $t - s \leq 5$ . Note that there is a generator  $\alpha_s^s$  at  $(0, s)$  for every  $s \geq 0$ , though only the first five are shown here.

## 4.6 Multiplicative structure

### 4.6.1 The Yoneda product

**Definition 4.6.1** ([9], Def 11.8.1). For any algebra  $A$  and  $A$ -modules  $L, M, N$ , there is a product, the *Yoneda product*

$$\circ : \text{Ext}_A^{s,t}(M, N) \otimes \text{Ext}_A^{u,v}(L, M) \rightarrow \text{Ext}_A^{s+u, t+v}(L, N),$$

defined as follows: let

$$\begin{aligned} \dots &\xrightarrow{f_3} F_2 \xrightarrow{f_2} F_1 \xrightarrow{f_1} F_0 \xrightarrow{f_0} L \rightarrow 0, \\ \dots &\xrightarrow{f'_3} F'_2 \xrightarrow{f'_2} F'_1 \xrightarrow{f'_1} F'_0 \xrightarrow{f'_0} M \rightarrow 0 \end{aligned}$$

be free resolutions for  $L$  and  $M$ . Then, given  $[g] \in \text{Ext}_A^{s,t}(M, N)$ ,  $[h] \in \text{Ext}_A^{u,v}(L, M)$ , we inductively construct a chain map  $h_\bullet : F_{u+\bullet} \rightarrow F'_\bullet[v]$ , as shown in the diagram below (where square brackets denotes the shift in degree).

$$\begin{array}{ccccccccccc} F_{u+s} & \xrightarrow{f_{u+s}} & F_{u+s-1} & \xrightarrow{f_{u+s-1}} & \dots & \xrightarrow{f_{u+1}} & F_u & \xrightarrow{f_u} & F_{u-1} & \xrightarrow{f_{u-1}} & \dots & \xrightarrow{f_1} & F_0 & \xrightarrow{f_0} & L \\ \downarrow h_s & & \downarrow h_{s-1} & & & & \downarrow h_0 & \searrow h & & & & & & & \\ F'_s[v] & \xrightarrow{f'_s} & F'_{s-1}[v] & \xrightarrow{f'_{s-1}} & \dots & \xrightarrow{f'_1} & F'_0[v] & \xrightarrow{f'_0} & M[v] & & & & & & \\ \downarrow g & & & & & & & & & & & & & & \\ N[v+t] & & & & & & & & & & & & & & \end{array}$$

The map  $h_0$  is defined as follows: let  $\alpha \in F_u$  be a generator, and consider  $h(\alpha) \in M[v]$ . Since  $f'_0$  is surjective, there exists some  $\beta \in F'_0[v]$  such that  $f'_0(\beta) = h(\alpha)$ . We define  $h_0(\alpha) = \beta$ . Now, suppose the  $h_i$  have been constructed for  $i < w$ , and consider the diagram below.

$$\begin{array}{ccccc} F_{u+w} & \xrightarrow{f_{u+w}} & F_{u+w-1} & \xrightarrow{f_{u+w-1}} & F_{u+w-2} \\ \downarrow h_w & & \downarrow h_{w-1} & & \downarrow h_{w-2} \\ F'_w[v] & \xrightarrow{f'_w} & F'_{w-1}[v] & \xrightarrow{f'_{w-1}} & F'_{w-2}[v] \end{array}$$

Let  $\alpha \in F_{u+w}$  be a generator, and consider  $f'_{w-1}h_{w-1}f_{u+w}(\alpha) \in F'_{w-2}[v]$ . By induction, the right square commutes, so  $f'_{w-1}h_{w-1}f_{u+w}(\alpha) = h_{w-2}f_{u+w-1}f_{u+w}(\alpha) = 0$ , by exactness of the top row. Thus,  $h_{w-1}f_{u+w}(\alpha) \in \ker f'_{w-1} = \text{im } f'_w$ . Write  $h_{w-1}f_{u+w}(\alpha) = f'_w(\beta)$ , and define  $h_w(\alpha) = \beta$ .

Now, consider the diagram below.

$$\begin{array}{ccc} F_{u+s+1} & \xrightarrow{f_{u+s+1}} & F_{u+s} \\ \downarrow h_{s+1} & & \downarrow h_s \\ F'_{s+1}[v] & \xrightarrow{f'_{s+1}} & F'_s[v] \\ & & \downarrow g \\ & & N[v+t] \end{array}$$

We have  $gh_s f_{u+s+1} = gf'_{s+1}h_{s+1} = 0$ , since  $[g] \in \text{Ext}_A^{s,t}(F'_s, N)$ , so  $[gh_s] \in \text{Ext}_A^{u+s, v+t}$ . We thus define  $[g] \cdot [h] = [gh_s]$ .

This definition is independent of the lifts chosen, which can be seen as follows. Suppose we have two chain maps  $\{h_i\}, \{h'_i\}$ ; we will construct a chain homotopy between them. Define  $k_0 : F_{u-1} \rightarrow F'_0[v]$  to be the zero map. By construction,  $f'_0 h_0 = f'_0 h'_0 = h$ , so  $f'_0(h_0 - h'_0) = 0$ . Thus,  $\text{im}(h_0 - h'_0) \subseteq \ker f'_0 = \text{im } f'_1$ , so  $h_0 - h'_0 = f'_1 k_1 = f'_1 k_1 + k_0 f_u$  for some map  $k_1 : F_u \rightarrow F'_1[v]$ . Now, suppose we have  $k_i, k_{i-1}$  such that  $h_{i-1} - h'_{i-1} = f'_i k_i + k_{i-1} f_{u+i-1}$ . Then  $f'_i h_i = h_{i-1} f_{u+i}$  and  $f'_i h'_i = h'_{i-1} f_{u+i}$ , so  $f'_i(h_i - h'_i) = (h_{i-1} - h'_{i-1})f_{u+i} = (f'_i k_i + k_{i-1} f_{u+i-1})f_{u+i} = f'_i k_i f_{u+i}$ , and thus we can construct  $k_{i+1}$  such that  $h_i - h'_i = f'_{i+1} k_{i+1} + k_i f_{u+i}$ . Now,  $g(h_s - h'_s) = g(f'_{s+1} k_{s+1} + k_s f_{u+s}) = gk_s f_{u+s}$ , and therefore  $[g(h_s - h'_s)] = [gk_s f_{u+s}] = [0]$ .

Finally, if  $h = l f_u$  for some  $l : F_{u-1} \rightarrow M[v]$ , with filling  $\{l_i\}$ , then  $\{l_i f_{u+i}\}$  is a filling for  $h$ , so  $[g] \cdot [h] = [gl_s f_{u+s}] = [0]$ . On the other hand, if  $g = m f'_s$ , then  $[g] \cdot [h] = [gh_s] = [m f'_s h_s] = [m h_{s-1} f_{u+s}] = [0]$ . Thus, the Yoneda product is well defined.

#### 4.6.2 The composition product

**Definition 4.6.2** ([10], p47). Let  $X, Y, Z$  be spectra. The *composition pairing*  $\circ : [Y, Z]_* \otimes [X, Y]_* \rightarrow [X, Z]_*$  is defined as follows:

$$\begin{aligned} \circ : [Y, Z]_v \otimes [X, Y]_t &\rightarrow [X, Z]_{v+t} \\ [g : \Sigma^v Y \rightarrow Z] \otimes [f : \Sigma^t X \rightarrow Y] &\mapsto [g \circ \Sigma^v f : \Sigma^{v+t} X \rightarrow Z], \end{aligned}$$

where  $[X, Y]_n = [\Sigma^n X, Y]$ .

In particular, if  $X = Y = Z = \mathbb{S}$ , we have a product  $\pi_v^s \otimes \pi_t^s \rightarrow \pi_{v+t}^s$ .

**Lemma 4.6.3.** *Let  $f, g : S^n \rightarrow S^n$  be pointed maps such that  $\deg f = \deg g$ . Then  $f \simeq g$ .*

*Proof.* We prove the contrapositive. Suppose  $f \not\simeq g$ . Then  $f, g$  represent two different elements in  $\pi_n S^n \simeq \mathbb{Z}$ , say  $[f] = n \neq m = [g]$  for  $n, m \in \mathbb{Z}$ . The Hurewicz theorem then implies that for a fixed generator  $u_n \in H^n(S^n)$ ,  $f_*(u_n) \neq g_*(u_n) \in H^n(S^n)$ , so  $\deg f \neq \deg g$ , as required.  $\square$

**Lemma 4.6.4** ([3], Prop 4.56). *The composition product makes  $\pi_*^s$  into a graded commutative ring.*

*Proof.* The identity map is clearly a two sided-identity for the composition product, and associativity follows from the fact that suspension respects composition. We now check graded commutativity.

Let  $f : S^{i+k} \rightarrow S^k$ ,  $g : S^{j+k} \rightarrow S^k$  represent elements of  $\pi_*^s$ ; without loss of generality we may assume  $k$  is even. Note that under the identification  $\Sigma^l S^{i+k} \cong S^{i+k} \wedge S^l$ , the map  $\Sigma^l f : \Sigma^l S^{i+k} \rightarrow \Sigma^l S^k$  corresponds to  $f \wedge \text{id} : S^{i+k} \wedge S^l \rightarrow S^k \wedge S^l$ . Now, consider the commutative diagram below, where  $\tau$  and  $\sigma$  swap the two factors.

$$\begin{array}{ccc} S^k \wedge S^{j+k} & \xrightarrow{\text{id} \wedge g} & S^k \wedge S^k \\ \sigma \downarrow & & \downarrow \tau \\ S^{j+k} \wedge S^k & \xrightarrow{g \wedge \text{id}} & S^k \wedge S^k \end{array}$$

The map  $\sigma$  is a composition of  $k(j+k)$  transpositions  $S^1 \wedge S^1 \rightarrow S^1 \wedge S^1$ , each of which has degree  $-1$  (since such a transposition is homotopic to a reflection), so  $\sigma$  has degree  $(-1)^{k(j+k)} = 1$ . Similarly,  $\deg \tau = 1$ , so by Lemma 4.6.3 we see that  $\sigma$  and  $\tau$  are both homotopic to the identity. Thus,  $f \wedge g = (\text{id} \wedge g) \circ (f \wedge \text{id}) \simeq (g \wedge \text{id}) \circ (f \wedge \text{id})$ . Since  $(g \wedge \text{id})(f \wedge \text{id})$  and  $g \cdot f$  represent the same element in  $\pi_*^s$ , we have  $f \wedge g \simeq g \cdot f$ , and by the same argument  $g \wedge f \simeq f \cdot g$ . It now suffices to show that  $f \wedge g \simeq (-1)^{ij} g \wedge f$ .

Consider the commutative diagram below.

$$\begin{array}{ccc} S^{i+k} \wedge S^{j+k} & \xrightarrow{f \wedge g} & S^k \wedge S^k \\ \sigma \downarrow & & \downarrow \tau \\ S^{j+k} \wedge S^{i+k} & \xrightarrow{g \wedge f} & S^k \wedge S^k \end{array}$$

We have  $\deg \sigma = (-1)^{(i+k)(j+k)} = (-1)^{ij}$  and  $\deg \tau = (-1)^{k^2} = 1$ , so  $f \wedge g \simeq (-1)^{ij} g \wedge f$ , as required.

Finally, for  $f' : S^{i+k} \rightarrow S^k$ ,  $h : S^{l+k} \rightarrow S^k$ , we have  $(f + f') \cdot h = (f + f') \circ \Sigma^l h = f \cdot h + g \cdot h$ , and  $h \cdot (f + g) = (-1)^{il}(f + g) \cdot h = (-1)^{il}f \cdot h + (-1)^{il}g \cdot h = h \cdot f + h \cdot g$ , so the distributivity laws also follow.  $\square$

**Lemma 4.6.5.** *There is a unique ring structure on  $(\pi_*^s)_2^\wedge$  which makes the completion map  $c : \pi_*^s \rightarrow (\pi_*^s)_2^\wedge$  into a ring homomorphism.*

*Proof.* We show uniqueness first. Let  $f \in (\pi_i^s)_2^\wedge$ ,  $g \in (\pi_j^s)_2^\wedge$ . If  $i, j \geq 1$ , then the completion map is surjective, so  $f = c(\tilde{f})$ ,  $g = c(\tilde{g})$  for some  $\tilde{f} \in \pi_i^s$ ,  $\tilde{g} \in \pi_j^s$ . Then  $fg = c(\tilde{f})c(\tilde{g}) = c(\tilde{f}\tilde{g})$ .

If  $i = 0, j \geq 1$ , then let  $\hat{f} \in \pi_0^s$  be a lift of  $q(f) \in \pi_0^s/2^r\pi_0^s$ , where  $2^r$  is the highest power of 2 dividing the order of  $\pi_j^s$ . Then  $f \equiv c(\hat{f}) \pmod{2^r}$ , so  $f = c(\hat{f}) + 2^r w$ . We have  $fg = f c(\tilde{g}) = (c(\hat{f}) + 2^r w)c(\tilde{g}) = c(\hat{f})c(\tilde{g}) + 2^r(wc(\tilde{g})) = c(\hat{f}\tilde{g}) \in (\pi_j^s)_2^\wedge$ .

Finally, if  $i = j = 0$ , we claim that any two multiplications on  $\mathbb{Z}_2$  which agree on  $\mathbb{Z}$  must agree on all of  $\mathbb{Z}_2$ , and thus the multiplication is given by the usual product on  $\mathbb{Z}_2$ .

Suppose not; let  $\star, \cdot$  be two products on  $\mathbb{Z}_2$ , agreeing on  $\mathbb{Z}$ , with  $f \star g \neq f \cdot g$ . Then there is some  $k$  such that  $f \star g \not\equiv f \cdot g \pmod{k}$ . Pick integers  $n, m$  such that  $n \equiv f \pmod{k}$  and  $m \equiv g \pmod{k}$ . Then, modulo  $k$ ,  $f \cdot g \equiv n \cdot m = n \star m \equiv f \star g$ , giving a contradiction.

Now, for  $i, j \geq 1$ , the multiplication above is well defined, since if  $\tilde{f}' = \tilde{f} + t$ , with  $nt = 0$  for odd  $n$ , then  $c(\tilde{f}'\tilde{g}) = c(\tilde{f}\tilde{g} + t\tilde{g}) = c(\tilde{f}\tilde{g})$  (since multiplication by  $n$  is an isomorphism in a group of order  $2^r$ ). Likewise, if  $\tilde{g}' = \tilde{g} + t$ , then  $c(\tilde{f}\tilde{g}') = c(\tilde{f}\tilde{g})$ . If  $i = 0, j \geq 1$  (or vice versa), then picking a different representative for  $g$  does not change the product, by the previous argument. If  $\hat{f}'$  is a different lift of  $q(f)$ , we have  $\hat{f} \equiv \hat{f}' \pmod{2^r}$ , so  $c(\hat{f}'\tilde{g}) = c(\hat{f}\tilde{g} + 2^r u\tilde{g}) = c(\hat{f}\tilde{g}) + 2^r c(u\tilde{g}) = c(\hat{f}\tilde{g})$  (for some  $u \in \mathbb{Z}$ ).

The usual product on  $\mathbb{Z}_2$  is of course well-defined. Finally, associativity, distributivity, and unitality are inherited from  $\pi_*^s$ .  $\square$

Given spectra  $X, Y, Z$ , we can define a pairing  $\circ : [Y, Z_2^\wedge]_* \otimes [X, Y_2^\wedge]_* \rightarrow [X, Z_2^\wedge]$  as follows: let  $f \in [Y, Z_2^\wedge]_s, g \in [X, Y_2^\wedge]_t$ . By Theorem 3.5.6, there exists a unique (up to homotopy) map  $\bar{f} : (\Sigma^s Y)_2^\wedge \rightarrow Z_2^\wedge$  such that  $f$  factors through  $\bar{f}$ . Now, note that  $(\Sigma^s Y)_2^\wedge \simeq \Sigma^s Y_2^\wedge$ , since  $\pi_i(\Sigma^s Y) = \pi_{i-s}(Y)$ . We can thus define the pairing of  $f$  and  $g$  to be  $\bar{f} \circ \Sigma^s g$ , as shown below.

$$\begin{array}{ccccc} & & \Sigma^s Y & & \\ & & \downarrow & \searrow f & \\ \Sigma^{s+t} X & \xrightarrow{\Sigma^s g} & \Sigma^s Y_2^\wedge & \xrightarrow{\bar{f}} & Z_2^\wedge \end{array}$$

**Lemma 4.6.6.** *The completion map  $c_* : \pi_*^s \rightarrow \pi_*(\mathbb{S}_2^\wedge)$  is a ring homomorphism. In particular, by Lemma 4.6.5, the composition product on  $\pi_*(\mathbb{S}_2^\wedge)$  coincides with the product on  $(\pi_*^s)_2^\wedge$  inherited from  $\pi_*^s$ , so the two groups are also isomorphic as rings.*

*Proof.* Let  $f : \mathbb{S}^i \rightarrow \mathbb{S}, g : \mathbb{S}^j \rightarrow \mathbb{S}$  be elements of  $\pi_i^s$  and  $\pi_j^s$  respectively. Then  $c_*(f)c_*(g) = (cf)(cg)$  is given by factorising  $cg = \bar{c}gc$  and composing to get  $\bar{c}gc\Sigma^j f$ . We thus have the commutative diagram below.

$$\begin{array}{ccccc} \mathbb{S}^{i+j} & \xrightarrow{\Sigma^j f} & \mathbb{S}^j & \xrightarrow{g} & \mathbb{S} \\ & & \downarrow c & & \downarrow c \\ & & \Sigma^j \mathbb{S}_2^\wedge & \xrightarrow{\bar{c}g} & \mathbb{S}_2^\wedge \end{array}$$

The upper path is exactly  $c_*(fg)$ , so  $c_*(f)c_*(g) = c_*(fg)$ . Further, the completion map clearly preserves the identity, so it is a ring homomorphism.  $\square$

### 4.6.3 Multiplication on the Adams spectral sequence

**Definition 4.6.7** ([10], Def 5.5). Let  $\{^l E_r\}, \{'' E_r\}, \{E_r\}$  be three spectral sequences. A *pairing* of these spectral sequences is a sequence of homomorphisms

$$\phi_r : ^l E_r^{*,*} \otimes '' E_r^{*,*} \rightarrow E_r^{*,*},$$

such that the Leibniz rule  $d_r \phi_r(x \otimes y) = \phi_r(d_r(x) \otimes y) + (-1)^{\deg x} \phi_r(x \otimes d_r(y))$  holds, and

$$\phi_{r+1}([x] \otimes [y]) = [\phi_r(x \otimes y)], \quad (4.6.1)$$

where  $[x] \in ^l E_{r+1}^{*,*}$  is the homology class of a  $d_r$ -cycle  $x \in ^l E_r^{*,*}$ , and similarly for  $y$ .

A spectral sequence pairing  $\{\phi_r\}$  induces a pairing

$$\phi_\infty : ^l E_\infty^{*,*} \otimes '' E_\infty^{*,*} \rightarrow E_\infty^{*,*}.$$

**Theorem 4.6.8** ([10], Thm 5.8). *Let  $X, Y, Z$  be spectra, with  $Y, Z$  connective and of finite type. There is a pairing of spectral sequences*

$$E_r^{*,*}(Y, Z) \otimes E_r^{*,*}(X, Y) \rightarrow E_r^{*,*}(X, Z)$$

which agrees for  $r = 2$  with the Yoneda pairing

$$\mathrm{Ext}_{\mathcal{A}_2}^{*,*}(H^*(Z), H^*(Y)) \otimes \mathrm{Ext}_{\mathcal{A}_2}^{*,*}(H^*(Y), H^*(X)) \rightarrow \mathrm{Ext}_{\mathcal{A}_2}^{*,*}(H^*(Z), H^*(X))$$

and which converges to the composition pairing

$$[Y, Z_2^\wedge]_* \otimes [X, Y_2^\wedge]_* \rightarrow [X, Z_2^\wedge]_*.$$

The pairing is associative and unital.

**Remark 4.6.9.** Condition (4.6.1) of Definition 4.6.7 ensures that if a product is computed on the  $E_2$  page, and both terms survive to the  $E_r$  page for some  $r > 2$ , then the computation is still valid on that page.

## 5 Calculating stable homotopy groups

In this section, we will calculate the groups  $(\pi_{t-s}^s)_2^\wedge$  for  $t - s \leq 15$ . We saw in sections 4.5 and 4.6.1 that computing  $\text{Ext}_{\mathcal{A}_2}^{s,t}(H^*(X), \mathbb{F}_2)$  and Yoneda products is entirely algorithmic; for  $t - s \leq 5$  we will show how to use the Yoneda product to resolve by hand the ambiguities mentioned in Section 4.5, and for  $t - s > 5$  we will use the Adams spectral sequence calculator (see [1]) to compute  $\text{Ext}_{\mathcal{A}_2}^{s,t}(\mathbb{F}_2, \mathbb{F}_2)$ . We will see in sections 5.1 and 5.2 that the groups  $(\pi_{t-s}^s)_2^\wedge$  can be completely determined this way for  $t - s \leq 13$ . Section 5.3 will be dedicated to proving that certain differentials at  $t - s = 15$  are nontrivial, and will culminate in the computation of  $(\pi_{14}^s)_2^\wedge$  and  $(\pi_{15}^s)_2^\wedge$ . We follow [10] and [9] throughout.

### 5.1 Resolving extensions

**Proposition 5.1.1** ([10], Cor 6.5). *We have the following relations:*

$$\begin{aligned} \alpha_i^i &= (\alpha_1^1)^i \\ \alpha_4^2 &= (\alpha_2^1)^2 \\ \alpha_5^2 &= \alpha_1^1 \alpha_4^1 \\ \alpha_6^3 &= (\alpha_1^1)^2 \alpha_4^1 = (\alpha_2^1)^3. \end{aligned}$$

*Proof.* We show the first two relations; the final two are obtained similarly.

Consider the diagram below, where  $F_\bullet$  is the free resolution in Figure 4.1.

$$\begin{array}{ccccccc} \alpha_s^s & \xrightarrow{\quad} & Sq^1 \alpha_{s-1}^{s-1} & & \alpha_{s-1}^{s-1} & \searrow & \\ \downarrow & & \downarrow & & \downarrow & & \\ F_s & \xrightarrow{\quad} & F_{s-1} & \xrightarrow{\quad} & \cdots & \xrightarrow{\quad} & F_0 \rightarrow \mathbb{F}_2 \\ \downarrow & & \downarrow & & \downarrow & & \\ \alpha_1^1 & \xrightarrow{\quad} & Sq^1 \iota & & \iota & \xrightarrow{\quad} & 1 \\ \downarrow & & \downarrow & & \downarrow & & \\ F_1[s-1] & \xrightarrow{\quad} & F_0[s-1] & \xrightarrow{\quad} & \mathbb{F}_2[s-1] & & \\ \downarrow & & \downarrow & & \downarrow & & \\ 1 & & \mathbb{F}_2[s] & & & & \end{array}$$

Since  $\alpha_{s-1}^{s-1} \in F_{s-1}$  is the only generator of degree  $s-1$ , to write down a lift  $F_{s-1} \rightarrow F_0[s-1]$  it suffices to say where  $\alpha_{s-1}^{s-1}$  is sent. In order for the right triangle to commute, we must send  $\alpha_{s-1}^{s-1}$  to  $\iota$ . Now, to write down a lift  $F_s \rightarrow F_1[s-1]$ , it again suffices to write down the image of  $\alpha_s^s$ . In order for the left square to commute, we must send  $\alpha_s^s$  to  $\alpha_1^1$ . The composite map  $F_s \rightarrow \mathbb{F}_2[2]$  is the unique map sending  $\alpha_s^s$  to 1, so  $\alpha_1^1 \cdot \alpha_{s-1}^{s-1} = \alpha_s^s$  for all  $s > 0$ . Thus,  $\alpha_s^s = (\alpha_1^1)^s$ .

Similarly, the calculation below shows that  $\alpha_2^1 \cdot \alpha_2^1 = \alpha_4^2$ .

$$\begin{array}{ccccccc}
\alpha_4^2 & \xrightarrow{\quad} & Sq^3\alpha_1^1 + Sq^2\alpha_2^1 & & \alpha_2^1 & & \\
\downarrow \alpha_2^1 & & \downarrow Sq^2\iota & & \downarrow \iota & \searrow & \\
F_2 & \xrightarrow{\quad} & F_1 & \xrightarrow{\quad} & F_0 & \xrightarrow{\quad} & \mathbb{F}_2 \\
\downarrow \alpha_2^1 & & \downarrow \iota & & \downarrow \iota & & \\
F_1[2] & \xrightarrow{\quad} & F_0[2] & \xrightarrow{\quad} & \mathbb{F}_2[2] & & \\
\downarrow 1 & & & & & & \\
\mathbb{F}_2[4] & & & & & & 
\end{array}$$

□

From now on, we will write  $h_i$  for the generator  $\alpha_{2^i}^1 \in \text{Ext}_{\mathcal{A}_2}^{1,2^i}(\mathbb{F}_2, \mathbb{F}_2)$ .

**Proposition 5.1.2.** *Suppose  $\alpha \in (\pi_i^s)_2^\wedge$  represents  $a \in E_\infty$ . Then  $2\alpha$  represents  $h_0 a$ . In other words, multiplication by  $h_0$  is induced by multiplication by 2.*

*Proof.* Recall that  $\pi_0^s = \mathbb{Z}$ , since  $\pi_1 S^1 = \mathbb{Z}$  and  $n = 1 \leq 2 = 2(1)$ , so this lies in the stable region. Now,  $E_r^{s,s}(\mathbb{S})$  converges to some filtration of  $(\pi_0^s)_2^\wedge = \mathbb{Z}_2$  whose quotients are all  $\mathbb{Z}/2\mathbb{Z}$ . The filtration must therefore be

$$\dots \subseteq 4\mathbb{Z}_2 \subseteq 2\mathbb{Z}_2 \subseteq \mathbb{Z}_2,$$

since finite index subgroups of  $\mathbb{Z}_p$  are of the form  $p^k \mathbb{Z}_p$ .

Thus,  $\iota = [1] \in \mathbb{Z}_2/2\mathbb{Z}_2$ , and by computing the Yoneda product we see that  $\iota$  is a unit. We also have  $h_0 = [2] \in 2\mathbb{Z}_2/4\mathbb{Z}_2$  so  $[2] = h_0 = h_0 \cdot \iota$ , and hence  $h_0$  acts on  $\iota$  by multiplication by 2.

Now, suppose  $(\pi_{t-s}^s)_2^\wedge = G$  has a filtration

$$\dots \subseteq F^{s+1,t+1}G \subseteq F^{s,t}G \subseteq F^{s-1,t-1}G \subseteq \dots \subseteq G,$$

with  $E_\infty^{s,t}(\mathbb{S}) \cong F^{s,t}/F^{s+1,t+1}$ . For any  $a \in E_\infty^{s,t}(\mathbb{S})$  represented by  $\alpha \in F^{s,t}G$ ,  $h_0 \cdot a = (h_0 \cdot \iota) \cdot a \in E_\infty^{s+1,t+1}(\mathbb{S})$  is represented by  $2\alpha \in F^{s+1,t+1}$ . □

**Lemma 5.1.3.** *There are no nontrivial differentials for  $t - s \leq 5$ .*

*Proof.* First, note from Figure 4.2 that the only possible nontrivial differentials in this range are the differentials  $d_r : E_r^{1,2}(\mathbb{S}) \rightarrow E_r^{1+r,1+r}(\mathbb{S})$ . Now,  $0 = d_r(h_0 h_1) = d_r(h_0)h_1 + h_0 d_r(h_1) = h_0 d_r(h_1)$ , so  $d_r(h_1) = 0$ . Since  $E_r^{1,2}(\mathbb{S})$  is generated by  $h_1$ , we must have  $d_r = 0$ . □

**Theorem 5.1.4.**

$$(\pi_i^s)_2^\wedge = \begin{cases} \mathbb{Z}/2\mathbb{Z} & i = 1, 2 \\ \mathbb{Z}/8\mathbb{Z} & i = 3 \\ 0 & i = 4, 5. \end{cases}$$

*Proof.* We have already shown that  $(\pi_4^s)_2^\wedge = (\pi_5^s)_2^\wedge = 0$  and that  $(\pi_2^s)_2^\wedge = \mathbb{Z}/2\mathbb{Z}$ . Further, Lemma 5.1.3 implies that  $(\pi_1^s)_2^\wedge = \mathbb{Z}/2\mathbb{Z}$ , since  $h_1$  survives to  $E_\infty$ . Now, we have some filtration

$$\{0\} \subseteq F^{3,6} \subseteq F^{2,5} \subseteq F^{1,4} = (\pi_3^s)_2^\wedge,$$

where  $F^{3,6}, F^{2,5}/F^{3,6}, (\pi_3^s)_2^\wedge/F^{2,5} \cong \mathbb{Z}/2\mathbb{Z}$ . Let  $h_2 \in E^{1,4} = (\pi_3^s)_2^\wedge/F^{2,5}$  be represented by  $\nu \in (\pi_3^s)_2^\wedge$ . Then  $0 \neq h_0^2 \cdot h_2 \in F^{3,6}$  is represented by  $4\nu$ , so  $4\nu \neq 0$  in  $(\pi_3^s)_2^\wedge$ . Thus,  $(\pi_3^s)_2^\wedge \cong \mathbb{Z}/8\mathbb{Z}$ . □



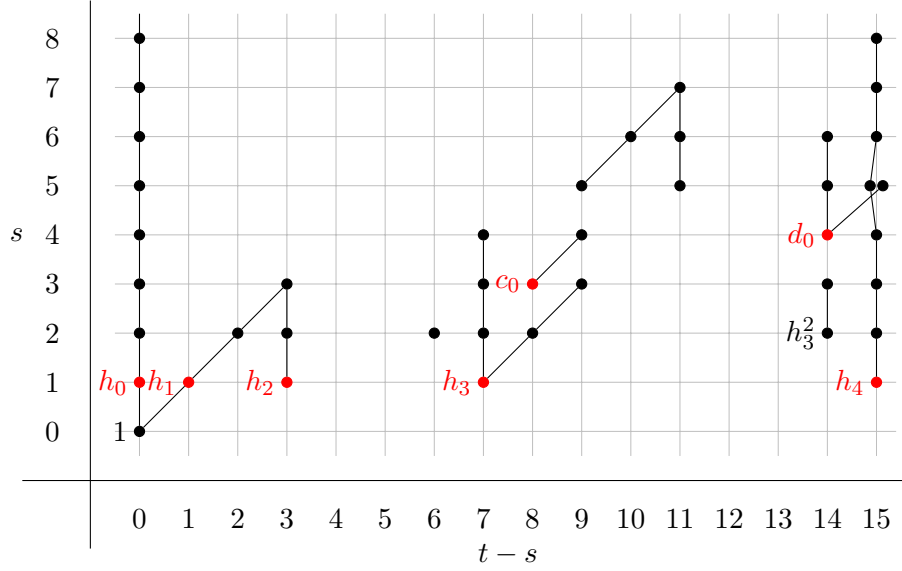


Figure 5.1:  $\text{Ext}_{\mathcal{A}_2}^{s,t}(\mathbb{F}_2, \mathbb{F}_2)$  for  $t - s \leq 15$ , calculated using [1]. The vertical and diagonal lines indicate multiplication by  $h_0$  and  $h_1$  respectively. Some of the algebra generators are shown in red, with their standard names.

## 5.2 The $E_2$ page for $t - s \leq 15$

Figure 5.1 shows the  $E_2$  page for  $\mathbb{S}$  in the range  $t - s \leq 15$ , and was calculated using [1]. Lines indicating multiplication by  $h_0$  and  $h_1$  are also shown, and from this we will be able to compute  $(\pi_{t-s}^s)_2^\wedge$  for  $t - s \leq 13$ .

**Lemma 5.2.1.** *There are no nontrivial differentials for  $t - s \leq 13$ .*

*Proof.* We have shown in Lemma 5.1.3 that there are no nontrivial differentials for  $t - s \leq 5$ ; for degree reasons, the only remaining possibility is that  $d_2 : E_2^{2,10}(\mathbb{S}) \rightarrow E_2^{4,11}(\mathbb{S})$  is nonzero.

From Figure 5.1, we see that  $E_2^{2,10}(\mathbb{S})$  is generated by  $h_1 h_3$ , and  $d_2(h_1 h_3) = d_2(h_1) h_3 + h_1 d_2(h_3) = 0 + 0 = 0$  (the first factor is zero by Lemma 5.1.3, and the second is an element of a trivial group).  $\square$

**Theorem 5.2.2.**

$$(\pi_i^s)_2^\wedge = \begin{cases} \mathbb{Z}/2\mathbb{Z} & i = 6, 10, \\ \mathbb{Z}/16\mathbb{Z} & i = 7, \\ (\mathbb{Z}/2\mathbb{Z})^2 & i = 8, \\ (\mathbb{Z}/2\mathbb{Z})^3 & i = 9, \\ \mathbb{Z}/8\mathbb{Z} & i = 11, \\ 0 & i = 12, 13. \end{cases}$$

*Proof.* The cases where  $i = 6, 10$ ,  $i = 7, 11$ , and  $i = 12, 13$  are proven analogously to the cases  $i = 1, 2$ ,  $i = 3$ , and  $i = 4, 5$  respectively. We thus show the result for  $i = 8, 9$ .

When  $i = 8$ , we have a filtration

$$\{0\} \subseteq F^{3,11} \subseteq F^{2,10} = (\pi_8^s)_2^\wedge,$$

where each quotient is isomorphic to  $\mathbb{Z}/2\mathbb{Z}$ . If  $(\pi_8^s)_2^\wedge \cong \mathbb{Z}/4\mathbb{Z}$ , then  $h_1h_3 \in (\pi_8^s)_2^\wedge/F^{3,11}$  is represented by  $\alpha \in (\pi_8^s)_2^\wedge$  of order 4. However,  $0 = h_0h_1h_3 \in F^{3,11}$ , so  $2\alpha = 0$ , contradicting the assumption that it had order 4. Thus  $(\pi_8^s)_2^\wedge \cong (\mathbb{Z}/2\mathbb{Z})^2$ .

For  $i = 9$ , we again have a filtration

$$\{0\} \subseteq F^{5,14} \subseteq F^{4,13} \subseteq F^{3,12} = (\pi_9^s)_2^\wedge,$$

with quotients isomorphic to  $\mathbb{Z}/2\mathbb{Z}$ . From Figure 5.1 we see that  $(\pi_9^s)_2^\wedge$  cannot be isomorphic to  $\mathbb{Z}/8\mathbb{Z}$ , since for any representative  $\alpha$  of  $h_1^2h_3$ ,  $0 = h_0^2h_1^2h_3 \in F^{5,14}$  is equal to  $4\alpha$ . However, we cannot immediately discount the possibility that  $(\pi_9^s)_2^\wedge \cong \mathbb{Z}/4\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$ ; indeed, the filtration

$$\{0\} \subseteq \mathbb{Z}/2\mathbb{Z} = \langle (2, 0) \rangle \subseteq (\mathbb{Z}/2\mathbb{Z})^2 = \langle (2, 0), (0, 1) \rangle \subseteq \mathbb{Z}/4\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} = \langle (1, 0), (0, 1) \rangle$$

would also give rise to a tower of three nodes unconnected by vertical lines on the  $E_2$  page. This is because a generator  $a \in (\mathbb{Z}/4\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z})/(\mathbb{Z}/2\mathbb{Z})^2 \cong \mathbb{Z}/2\mathbb{Z}$  can be represented by  $(1, 0) \in \mathbb{Z}/4\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$ , so  $[(2, 0)] \in (\mathbb{Z}/2\mathbb{Z})^2/(\mathbb{Z}/2\mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z}$  is a representative for  $h_0 \cdot a$ . However, the final copy of  $\mathbb{Z}/2\mathbb{Z}$  in the filtration is generated by  $(2, 0)$ , so  $[(2, 0)] = 0$  in  $(\mathbb{Z}/2\mathbb{Z})^2/(\mathbb{Z}/2\mathbb{Z})$ , meaning that  $h_0 \cdot a = 0$ . Also, any generator  $\beta$  of  $(\mathbb{Z}/2\mathbb{Z})^2$  doubles to zero, so any element  $b \in (\mathbb{Z}/2\mathbb{Z})^2/(\mathbb{Z}/2\mathbb{Z})$  satisfies  $h_0 \cdot b = 0$ .

The situation described above is known as a ‘hidden extension’, since it cannot be read off from the spectral sequence diagram. However, we can often determine by other methods whether or not they arise.

In our case, this does not in fact occur; let  $h_3 \in E_2^{1,8}$  be represented by  $\sigma \in (\pi_7^s)_2^\wedge$ , and let  $h_1 \in E_2^{1,2}$  be represented by  $\eta \in (\pi_1^s)_2^\wedge$ . Then  $h_1h_3$  is represented by  $\eta\sigma$ , and  $2(\eta\sigma) = (2\eta)\sigma = 0$ , since  $(\pi_1^s)_2^\wedge \cong \mathbb{Z}/2\mathbb{Z}$ . Thus,  $(\pi_9^s)_2^\wedge \cong (\mathbb{Z}/2\mathbb{Z})^3$ .  $\square$

### 5.3 Differentials at $14 \leq t - s \leq 15$

For  $t - s < 14$ , the computation of  $(\pi_{t-s}^s)_2^\wedge$  involves only the  $E_2$  page of the spectral sequence, since there are no nontrivial differentials in this range. However, the first nonzero differential will appear in the range  $14 \leq t - s \leq 15$ , and in fact there are many differentials after this point, though we will only fully compute those in this range. In general, the problem of computing differentials is much harder than determining  $\text{Ext}_{\mathcal{A}_2}^{s,t}(\mathbb{F}_2, \mathbb{F}_2)$ , and is not algorithmic.

**Theorem 5.3.1** ([9], Thm 11.10.2).  $d_2(h_4) = h_0h_3^2 \neq 0$ .

*Proof.* We have shown that  $h_0$  detects  $2 \in (\pi_7^s)_2^\wedge$  (i.e. 2 is a representative for  $h_0$ ). Let  $\sigma \in (\pi_7^s)_2^\wedge$  be a representative for  $h_3$ . Then  $2\sigma^2$  is a representative for  $h_0h_3^2$ . By graded commutativity of  $(\pi_*^s)_2^\wedge$ ,  $\sigma^2 = -\sigma^2$ , so  $2\sigma^2 = 0$ , and thus  $h_0h_3^2 = 0$  in  $E_\infty^{3,17}(\mathbb{S})$  (since  $2\sigma^2 = 0$  is a representative for it). Therefore,  $h_0h_3^2$  is the boundary of a differential, so we must have  $d_2(h_4) = h_0h_3^2$ .  $\square$

Note that the  $d_2$  differentials at  $E_2^{2,17}(\mathbb{S})$ ,  $E_2^{3,18}(\mathbb{S})$ ,  $E_2^{4,19}(\mathbb{S})$  are all trivial, since  $d_2(h_0^n h_4) = h_0^n d_2(h_4) = h_0^{n-1}(h_0h_3^2) = 0$ .

There are two possible  $d_3$  differentials for  $t - s \leq 15$  (emanating from  $E_3^{2,17}$  and  $E_3^{3,18}$ ), and in fact it will turn out that both are nontrivial. The method of proof will be by comparison to the Adams spectral sequence of a different spectrum, so we will first state a result comparing the Adams spectral sequences of two spectra with a map between them.

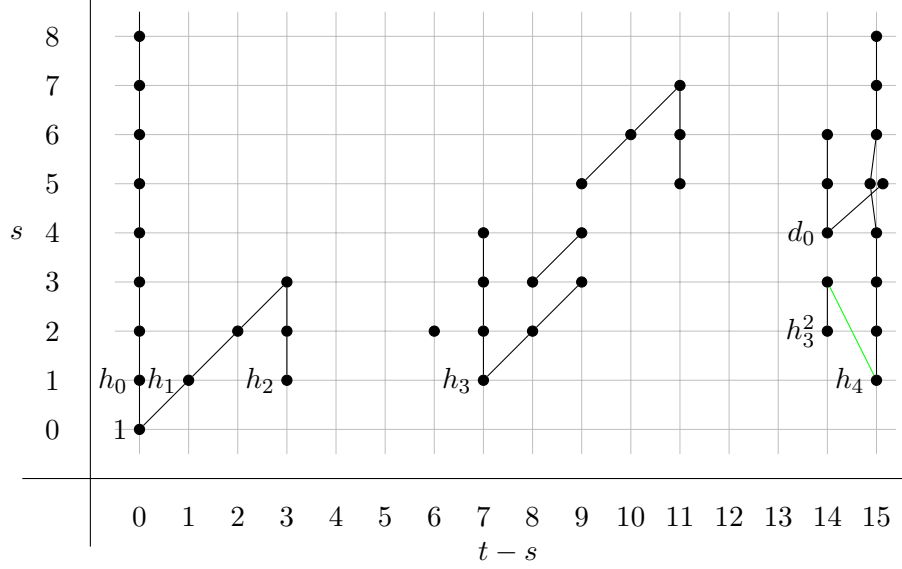


Figure 5.2: The  $E_2$  page of the Adams spectral sequence for  $\mathbb{S}$ , in the range  $t - s \leq 15$ ; the unique  $d_2$  differential is shown in green.

**Theorem 5.3.2** ([10], Cor 4.17). *Let  $f : Y \rightarrow Z$  be a map of connective spectra of finite type. Then there is a map*

$$f_* : \{E_r(Y), d_r\}_r \rightarrow \{E_r(Z), d_r\}_r$$

*of Adams spectral sequences, given at the  $E_2$ -level by the homomorphism*

$$(f^*)^* : \text{Ext}_{\mathcal{A}_2}^{s,t}(H^*(Y), \mathbb{F}_2) \rightarrow \text{Ext}_{\mathcal{A}_2}^{s,t}(H^*(Z), \mathbb{F}_2)$$

*induced by the  $\mathcal{A}_2$ -module homomorphism  $f^* : H^*(Z) \rightarrow H^*(Y)$ , converging to the homomorphism*

$$f_* : \pi_*(Y) \rightarrow \pi_*(Z).$$

Let  $X$  be a connective spectrum of finite type. Then  $\text{Ext}_{\mathcal{A}_2}^{*,*}(H^*(X), \mathbb{F}_2)$  is an  $(\text{Ext}_{\mathcal{A}_2}^{*,*}(\mathbb{F}_2, \mathbb{F}_2))$ -module, with action given by specialising the Yoneda product of Definition 4.6.1 to  $M = N = \mathbb{F}_2$ ,  $L = H^*(X)$ .

**Remark 5.3.3.** For any map  $f : \mathbb{S} \rightarrow X$ , where  $X$  is a connective spectrum of finite type, the induced map

$$(f^*)^* : \text{Ext}_{\mathcal{A}_2}^{s,t}(\mathbb{F}_2, \mathbb{F}_2) \rightarrow \text{Ext}_{\mathcal{A}_2}^{s,t}(H^*(X), \mathbb{F}_2)$$

satisfies  $(f^*)^*(\alpha\beta) = \alpha \cdot (f^*)^*(\beta)$  for any  $\alpha, \beta \in \text{Ext}_{\mathcal{A}_2}^{s,t}(\mathbb{F}_2, \mathbb{F}_2)$ . This follows from the definition of the Yoneda product, since both  $(f^*)^*(\alpha\beta)$  and  $\alpha \cdot (f^*)^*(\beta)$  arise filling in the same diagram with dotted lifts, forming the same chain maps up to chain homotopy.

**Lemma 5.3.4** ([9], Table 14.1 (9)).  $d_2(f_0), d_2(e_0) \neq 0$ .

*Proof.* Let  $i, j, k, l$  be as shown in Figure 5.3. One can calculate (using e.g. [2]) that  $h_4i = 0$  and  $h_0h_3^2i \neq 0$ . Now,  $d_2(i)$  is nontrivial, since  $h_4i = 0$  and  $h_0h_3^2i \neq 0$ , so  $0 = d_2(h_4i) = h_0h_3^2i + h_4d_2(i)$  implies that  $d_2(i) \neq 0$ . Further,  $d_2(j) \neq 0$  since  $h_0d_2(j) = d_2(h_0j) = d_2(h_2i) = h_2d_2(i) \neq 0$ . An almost identical argument shows that  $d_2(k), d_2(l) \neq 0$ , and thus  $d_2(h_0l) = h_0d_2(l) \neq 0$ . Finally, we have  $h_0l = d_0f_0$ , so  $0 \neq d_2(h_0l) = d_2(d_0f_0) = d_0d_2(f_0)$ .

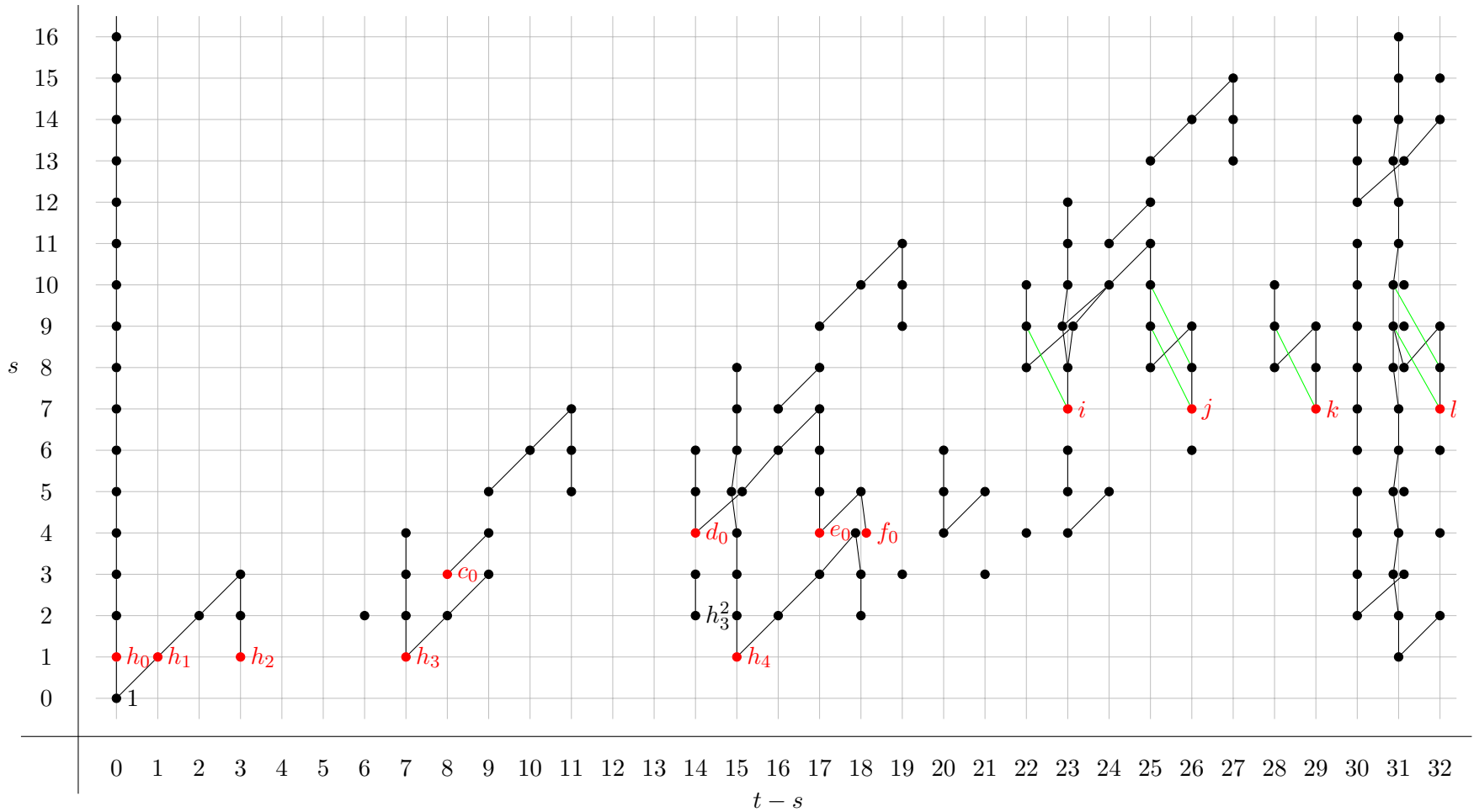


Figure 5.3:  $\text{Ext}_{\mathcal{A}_2}^{s,t}(\mathbb{F}_2, \mathbb{F}_2)$  for  $t - s \leq 32$ . The vertical and diagonal lines indicate multiplication by  $h_0$  and  $h_1$  respectively. Some of the algebra generators are shown in red, with naming conventions as in [9]. The  $d_2$  differentials referenced in the proof of Lemma 5.3.4 are shown in green.

Now,  $d_2(f_0) \neq 0$ , so looking at Figure 5.3 we see that  $0 \neq h_0 d_2(f_0) = d_2(h_0 f_0) = d_2(h_1 e_0) = h_1 d_2(e_0)$ , and thus  $d_2(e_0) \neq 0$ .  $\square$

In the following proof, we write  $\sigma \in \pi_7^s$  for the stable class of the octonionic Hopf fibration  $S^{15} \rightarrow S^8$ , which can be shown to detected by  $h_3$  (recall that we previously denoted by  $\sigma$  an arbitrary generator of  $\pi_7^s$ ).

**Lemma 5.3.5** ([9], Table 14.9 (4)). *Consider the cofibration*

$$\mathbb{S}^7 \xrightarrow{\sigma} \mathbb{S} \xrightarrow{i} C_\sigma \xrightarrow{j} \mathbb{S}^8 \rightarrow \dots$$

Let  $a \in E_2^{3,18}(C_\sigma)$  be the generator shown in Figure 5.4. Then  $d_2(a) \neq 0$ .

*Proof.* Let  $\hat{i} = (i^*)^* : \text{Ext}_{\mathcal{A}_2}^{s,t}(\mathbb{F}_2, \mathbb{F}_2) \rightarrow \text{Ext}_{\mathcal{A}_2}^{s,t}(C_\sigma, \mathbb{F}_2)$  be the map induced by  $i^* : H^*(C_\sigma) \rightarrow H^*(\mathbb{S})$ . We first show that  $d_2(h_2 \cdot a) = d_2(\hat{i}(f_0))$ . By Lemma 3.3.6, we have a long exact sequence

$$\dots \leftarrow H^{n-1}(\mathbb{S}^8) \leftarrow H^n(\mathbb{S}) \xleftarrow{i^*} H^n(C_\sigma) \xleftarrow{j^*} H^n(\mathbb{S}^8) \leftarrow H^{n+1}(\mathbb{S}) \leftarrow \dots$$

However, any map  $H^n(\mathbb{S}) \rightarrow H^{n-1}(\mathbb{S}^8)$  must be zero, so we get short exact sequences

$$0 \leftarrow H^n(\mathbb{S}) \xleftarrow{i^*} H^n(C_\sigma) \xleftarrow{j^*} H^n(\mathbb{S}^8) \leftarrow 0.$$

Taking a direct sum gives a short exact sequence

$$0 \leftarrow \mathbb{F}_2 \xleftarrow{i^*} H^*(C_\sigma) \xleftarrow{j^*} \mathbb{F}_2[8] \leftarrow 0,$$

and from this we get a short exact sequence of chain complexes

$$0 \rightarrow \text{Hom}(\mathbb{F}_2, I_\bullet) \xrightarrow{i^*} \text{Hom}(H^*(C_\sigma), I_\bullet) \xrightarrow{j^*} \text{Hom}(\mathbb{F}_2[8], I_\bullet) \rightarrow 0,$$

for any injective resolution  $I$ , and thus the long exact sequence below.

$$\begin{array}{ccccccc} \dots & \longrightarrow & \text{Ext}_{\mathcal{A}_2}^{s,t}(\mathbb{F}_2, \mathbb{F}_2) & \xrightarrow{\hat{i}} & \text{Ext}_{\mathcal{A}_2}^{s,t}(H^*(C_\sigma), \mathbb{F}_2) & \xrightarrow{\hat{j}} & \text{Ext}_{\mathcal{A}_2}^{s,t-8}(\mathbb{F}_2, \mathbb{F}_2) \\ & & & & & & \downarrow \\ & & & & & & \text{Ext}_{\mathcal{A}_2}^{s+1,t-8}(\mathbb{F}_2, \mathbb{F}_2) \\ & & & & & & \downarrow \\ & & & & & & \text{Ext}_{\mathcal{A}_2}^{s+1,t}(H^*(C_\sigma), \mathbb{F}_2) \\ & & & & & & \downarrow \\ & & & & & & \text{Ext}_{\mathcal{A}_2}^{s+1,t}(\mathbb{F}_2, \mathbb{F}_2) \longrightarrow \dots \end{array}$$

Now,  $f_0 \in \text{Ext}_{\mathcal{A}_2}^{4,22}(\mathbb{F}_2, \mathbb{F}_2)$ ; we consider the exact sequence

$$\text{Ext}_{\mathcal{A}_2}^{3,14}(\mathbb{F}_2, \mathbb{F}_2) \rightarrow \text{Ext}_{\mathcal{A}_2}^{4,22}(\mathbb{F}_2, \mathbb{F}_2) \xrightarrow{\hat{i}} \text{Ext}_{\mathcal{A}_2}^{4,22}(H^*(C_\sigma), \mathbb{F}_2).$$

Figure 5.1 shows us that  $\text{Ext}_{\mathcal{A}_2}^{3,14}(\mathbb{F}_2, \mathbb{F}_2) = 0$ , so  $\hat{i}$  is injective at this point, and thus  $\hat{i}(f_0) \neq 0$ . Similarly,  $\text{Ext}_{\mathcal{A}_2}^{4,15}(\mathbb{F}_2, \mathbb{F}_2) = 0$  and  $\text{Ext}_{\mathcal{A}_2}^{1,12}(\mathbb{F}_2, \mathbb{F}_2) = 0$ , so  $\hat{i}(h_0 f_0), \hat{i}(h_2 h_4) \neq 0$ . Since  $E_2(C_\sigma)$  is an  $E_2(\mathbb{S})$ -module and  $\hat{i}$  respects the  $E_2(\mathbb{S})$  action (by Remark 5.3.3),  $\hat{i}(h_0 f_0) = h_0 \cdot \hat{i}(f_0) \neq 0$ , so  $\hat{i}(f_0)$  is equal to either  $h_2 \cdot a$  or  $h_2 \cdot a + h_0^2 \cdot \hat{i}(h_2 h_4)$ . Now,  $d_2(h_2 h_4) = 0$ , since otherwise it would be equal to  $e_0$ , and we would have  $d_2^2(h_2 h_4) \neq 0$ , contradicting the fact that  $d_2$  is a differential. Thus, by linearity of  $d_2$ , we have  $d_2(\hat{i}(f_0)) = d_2(h_2 \cdot a)$  (since  $d_2(h_0^2 \cdot \hat{i}(h_2 h_4)) = d_2(\hat{i}(h_0^2 h_2 h_4)) = \hat{i}(h_0^2 d_2(h_2 h_4)) = 0$ ).

Finally,  $\hat{i}(h_0^2 e_0) \neq 0$ , since  $\text{Ext}_{\mathcal{A}_2}^{5,15}(\mathbb{F}_2, \mathbb{F}_2) = 0$  (using the long exact sequence in Ext again). Thus,  $d_2(h_2 \cdot a) = d_2(\hat{i}(f_0)) = \hat{i}(d_2(f_0)) = \hat{i}(h_0^2 e_0) \neq 0$ . Therefore,  $d_2(a) \neq 0$ .  $\square$

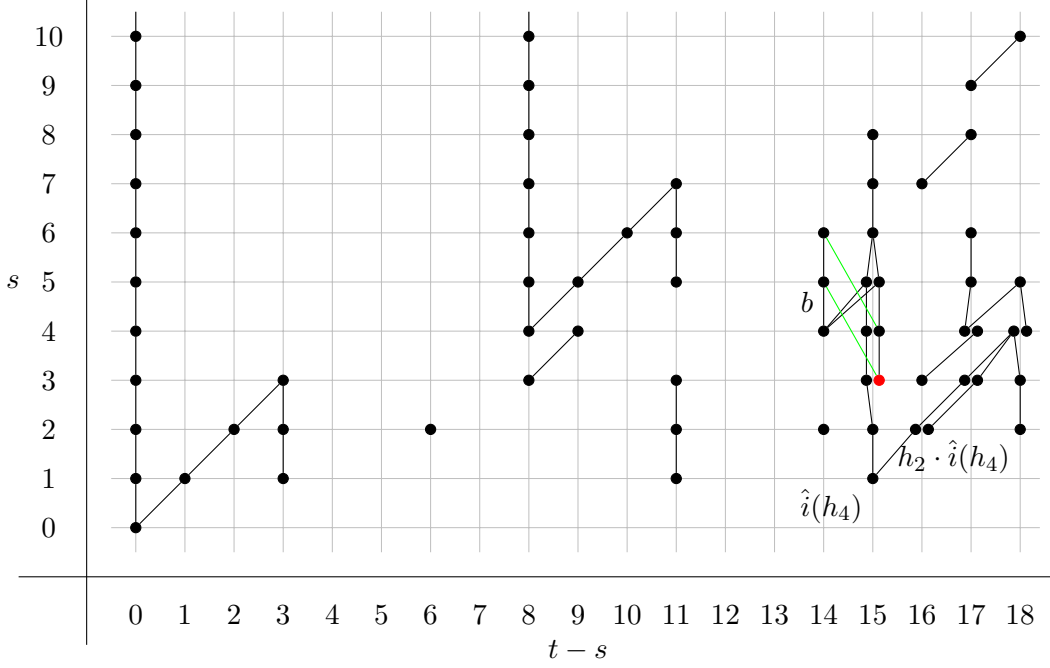


Figure 5.4: The  $E_2$  page of the Adams spectral sequence for  $C_\sigma$ , in the range  $t - s \leq 18$ , with the generator  $a$  shown in red, and two of the differentials shown in green.

**Theorem 5.3.6** ([9], Table 14.2 (10)).  $d_3(h_0h_4) = h_0d_0$  in  $E_3(\mathbb{S})$ .

*Proof.* From the cofibration

$$\mathbb{S}^7 \xrightarrow{\sigma} \mathbb{S} \xrightarrow{i} C_\sigma \xrightarrow{j} \mathbb{S}^8 \hookrightarrow \mathbb{S}^1 \rightarrow \dots,$$

we get an exact sequence

$$\pi_7^s \xrightarrow{\sigma_*} \pi_{14}^s \xrightarrow{i_*} \pi_{14}(C_\sigma) \xrightarrow{j_*} \pi_6^s \rightarrow \pi_{13}^s,$$

by Theorem 3.3.5. Since these stable homotopy groups are all finite<sup>1</sup>, this induces an exact sequence

$$(\pi_7^s)_2^\wedge \xrightarrow{\sigma_*} (\pi_{14}^s)_2^\wedge \xrightarrow{i_*} \pi_{14}(C_\sigma)_2^\wedge \xrightarrow{j_*} (\pi_6^s)_2^\wedge \xrightarrow{\sigma_*} (\pi_{13}^s)_2^\wedge = 0.$$

In  $E_2(C_\sigma)$  we have  $d_2(a) = b$ , where  $b$  is the generator shown in Figure 5.4 (by Lemma 5.3.5), so  $\pi_{14}(C_\sigma)_2^\wedge$  has order dividing four. Let  $\nu \in (\pi_3^s)_2^\wedge$  be a representative for  $h_2$ . Then  $(\pi_6^s)_2^\wedge = \mathbb{Z}/2\mathbb{Z}\langle\nu^2\rangle$ , and  $\nu^2\sigma = 0$ . By exactness, we see that  $j_*$  is surjective, so  $(\pi_6^s)_2^\wedge \cong \pi_{14}(C_\sigma)_2^\wedge / \ker j_* = \pi_{14}(C_\sigma)_2^\wedge / \text{im } i_*$ . We know  $\pi_{14}(C_\sigma)_2^\wedge$  has order dividing 4 and  $(\pi_6^s)_2^\wedge$  has order 2, so  $\text{im } i_*$  has order dividing 2.

Now,  $(\pi_7^s)_2^\wedge = \mathbb{Z}/16\mathbb{Z}\langle\sigma\rangle$ , and  $2\sigma^2 = 0$  by graded commutativity, so the first isomorphism theorem implies that  $(\pi_{14}^s)_2^\wedge$  has order dividing four. Thus,  $h_0d_0$  and  $h_0^2d_0$  must be boundaries, and  $d_3(h_0h_4) = h_0d_0$  is the only possibility.  $\square$

**Theorem 5.3.7.**

$$(\pi_i^s)_2^\wedge = \begin{cases} (\mathbb{Z}/2\mathbb{Z})^2 & i = 14, \\ \mathbb{Z}/32\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} & i = 15. \end{cases}$$

<sup>1</sup>A priori  $\pi_{14}(C_\sigma)$  is only finitely generated, but from Figure 5.4 we see that its 2-completion is finite, so the group itself must be finite.

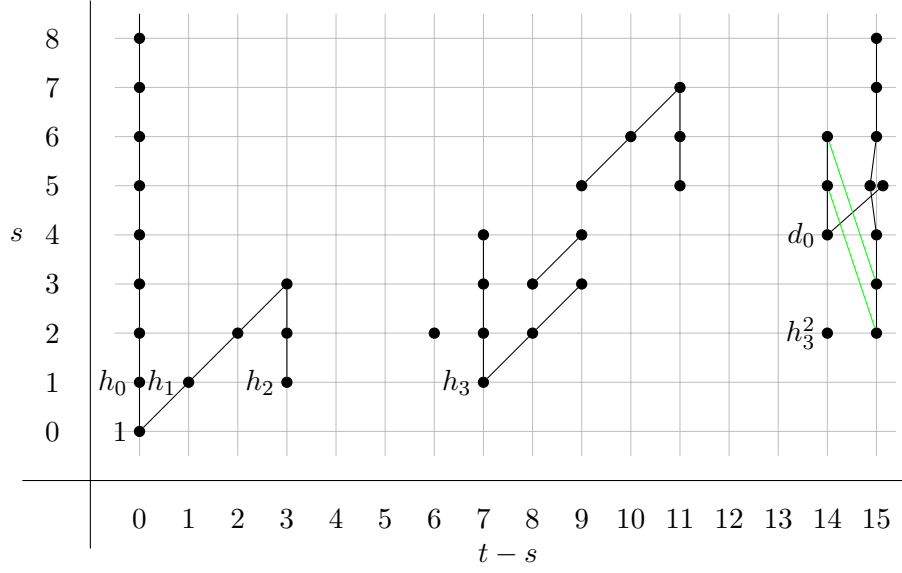


Figure 5.5: The  $E_3$  page of the Adams spectral sequence for  $\mathbb{S}$ , in the range  $t - s \leq 15$ ; the differentials are shown in green.

*Proof.* From Figure 5.6, we see that  $(\pi_{15}^s)_2^\wedge = \mathbb{Z}/32\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$ , and that  $|(\pi_{14}^s)_2^\wedge| = 2$ . Now, if  $h_3$  is represented by  $\sigma \in (\pi_7^s)_2^\wedge$ , then  $h_3^2$  is represented by  $\sigma^2$ . By graded commutativity,  $\sigma^2 = -\sigma^2$ , so  $2\sigma^2 = 0$ . Thus, since  $h_3^2$  is represented by a generator of  $(\pi_{14}^s)_2^\wedge$ , we have  $(\pi_{14}^s)_2^\wedge \cong (\mathbb{Z}/2\mathbb{Z})^2$ .  $\square$

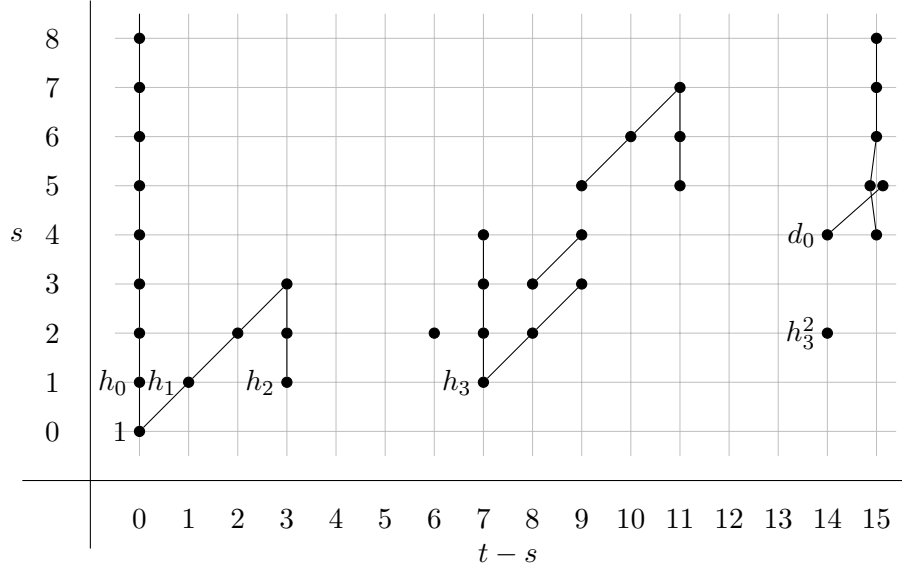


Figure 5.6: The  $E_4$  page of the Adams spectral sequence for  $\mathbb{S}$ , in the range  $t - s \leq 15$ . There are no possible higher differentials, so this coincides with the  $E_\infty$  page for  $t - s \leq 15$ .

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