Stable Homotopy Groups of Spheres [DRAFT]

Contents

1 Introduction

- Define homotopy groups
- Freudenthal's suspension theorem: if $\pi_i(X) = 0$ for $i \leq k$ (i.e. X is k-connected) then the map

$$\begin{array}{rcl} \pi_n(X) & \to & \pi_{n+1}(\Sigma X) \\ [\gamma : S^n \to X] & \mapsto & [\Sigma\gamma : \Sigma S^n = S^{n+1} \to \Sigma X] \end{array}$$

  is an isomorphism for $n \leq 2k$ and surjective for $n = 2k+1$
- This implies $\pi_{n+k}(S^n)$ depends only on k for $n \geq k+2$
- (Obviously be careful with basepoints above)
- Suppose X is k-connected. Then, for $k \geq 0$, $0 = \pi_k(X) \cong \pi_{k+1}(\Sigma X)$, so whenever a space is k-connected its suspension is $k+1$-connected.
- As you take suspensions, then, your successive bounds are $n \leq 2k$, $n+1 \leq 2k+2 \implies n \leq 2k+1$, $n \leq 2k+2$, etc ... so the sequence $\pi_n(X) \to \pi_{n+1}(\Sigma X) \to \cdots$ will eventually stabilise.
- Thus, if you take the colimit of that direct system, it'll just equal the stable value, with the higher legs just being the inverse isomorphisms.
- [11], Cor 1.9 [not 100% convinced of how this follows, but believing it for now]: if X is a CW complex of dimension d and Y a $(k-1)$-connected space, then the suspension homomorphism[1] $[X, Y] \to [\Sigma X, \Sigma Y]$ is bijective if $d < 2k-1$ and surjective if $d = 2k-1$.

Miscellaneous facts I might need later:

- Cohomology [possibly only of pointedCW complexes] is representable[2], and its representing object is the Eilenberg-MacLane space. i.e. $H^n(-; G) \cong \mathrm{Hom}(-, K(G, n))$.
- $\mathscr{A}_2$ is generated as an algebra by elements Sq^{2^k} ([5], Prop 4L.8).
- The map $\mathscr{A}_2 \to \tilde{H}^*(K(\mathbb{Z}/2\mathbb{Z}, n); \mathbb{Z}/2\mathbb{Z})$, $Sq^I \mapsto Sq^I(\iota_n)$ is an isomorphism from the degree d part of $\mathscr{A}_2$ onto $H^{n+d}(K(\mathbb{Z}/2\mathbb{Z}, n); \mathbb{Z}/2\mathbb{Z})$ for $d \geq n$. In particular, the admissible monomials Sq^I form an additive basis for $\mathscr{A}_2$. Thus, $\mathscr{A}_2$ is exactly the algebra of all $\mathbb{Z}/2\mathbb{Z}$ cohomology operations that are stable, commuting with suspension ([6], Cor 5.38).
- "Stable homotopy groups are a homology theory" (whatever that means)
- Hurewicz theorem: for any path-connected space X and $n > 0$ there exists a group homomorphism $h_* : \pi_n(X) \to H_n(X)$. For $n = 1$ this induces an isomorphism $\pi_1^{\mathrm{ab}}(X) \cong H_1(X)$. For $n \geq 2$, if X is $(n-1)$-connected then $\tilde{H}_i(X) = 0$ for all $i < n$, and the map $h_* : \pi_n(X) \to H_n(X)$ is an isomorphism.

[11], [4], [5]

[1] Hang on, the what?? X isn't the suspension of anything, why on earth would this be a group?

[2] As a set, or is this some sort of enriched thing? If it's enriched, is that over **Ab** or **Rng**? [5] says on p394 that there is a natural group structure on $\mathrm{Hom}(X, K(G, n))$ such that the natural isomorphism $\mathrm{Hom}(X, K(G, n)) \to H^n(X; G)$ is in fact an isomorphism of abelian groups. So, it's over **Ab**? N.B: There's a lot of talk about 'reduced cohomology theories', so a good next step would be to figure out what those are - if they only involve groups and not rings, maybe the cup product on H^* is not relevant here.

2 The Steenrod algebra

The following is from [5] 4L.

- There are maps $Sq^i : H^n(-;\mathbb{Z}/2\mathbb{Z}) \to H^{n+i}(-;\mathbb{Z}/2\mathbb{Z})$ for each i, and they satisfy the following properties:
  1. $Sq^i_X(f^*(\alpha)) = f^*(Sq^i_Y(\alpha))$ for $f : X \to Y$ (i.e. Sq^i is a natural transformation).
  2. $Sq^i_X(\alpha+\beta) = Sq^i_X(\alpha) + Sq^i_X(\beta)$ (i.e. Sq^i_X respects the group operation for all X).
  3. $Sq^i(\alpha \smile \beta) = \sum_{0\le j\le i} (Sq^j(\alpha) \smile Sq^{i-j}(\beta))$ (the Cartan formula)
  4. $Sq^i(\sigma(\alpha)) = \sigma(Sq^i(\alpha))$ where $\sigma : H^n(X;\mathbb{Z}/2\mathbb{Z}) \to H^{n+1}(\Sigma X;\mathbb{Z}/2\mathbb{Z})$ is the "suspension isomorphism given by reduced cross product with a generator of $H^1(S^1;\mathbb{Z}/2\mathbb{Z})$". [see: [5], p219. N.B: I think this "relative cross product" theory is required – you can argue that there is an isomorphism via MV, but this point says that it's this specific one. Maybe they're the same, but Hatcher doesn't say that anywhere and there could be many isomorphisms.]
  5. $Sq^i(\alpha) = \alpha^2$ if $i = \deg(\alpha)$ and $Sq^i(\alpha) = 0$ if $i > \deg(\alpha)$.
  6. $Sq^0 = \mathrm{id}$.
  7. Sq^1 is the "$\mathbb{Z}/2\mathbb{Z}$ Bockstein homomorphism β associated with the coefficient sequence $0 \to \mathbb{Z}/2\mathbb{Z} \to \mathbb{Z}/4\mathbb{Z} \to \mathbb{Z}/2\mathbb{Z} \to 0$".
- Define $Sq := Sq^0 + Sq^1 + \cdots$. Then $Sq(\alpha \smile \beta) = Sq(\alpha) \smile Sq(\beta)$ (since $(Sq(\alpha \smile \beta))_n = \sum_i Sq^i(\alpha) \smile Sq^{n-i}(\beta) = (Sq(\alpha) \smile Sq(\beta))_n$). Thus, Sq is a ring homomorphism.
- Adem relations:

$$Sq^a Sq^b = \sum_j \binom{b-j-1}{a-2j} Sq^{a+b-j} Sq^j \quad \text{if } a < 2b,$$

  where $\binom{m}{n}$ is zero if m or n is negative, or $m < n$, and $\binom{m}{0} = 1$ for $m \ge 0$.

Definition 2.0.1. The *Steenrod algebra* $\mathscr{A}_2$ is the algebra over $\mathbb{Z}/2\mathbb{Z}$ that is the quotient of the algebra of polynomials in the noncommuting variables $Sq^1, Sq^2, \ldots$ by the two-sided ideal generated by the Adem relations. Thus, for every space X, $H^*(X;\mathbb{Z}/2\mathbb{Z})$ is a module over $\mathscr{A}_2$.

- $\mathscr{A}_2$ is graded, and its elements of degree k are those that map $H^n(X;\mathbb{Z}/2\mathbb{Z})$ to $H^{n+k}(X,\mathbb{Z}/2\mathbb{Z})$ for all n. [Presumably you've fixed a space X while you're doing all this?]

Definition 2.0.2. Write Sq^I for the monomial $Sq^{i_1}Sq^{i_2}\cdots Sq^{i_n}$. Then Sq^I is *admissible* if $i_j \ge 2i_{j+1}$ for all $0 \le j < n$.

Note the admissible monomials are exactly those to which no Adem relations can be applied. Thus, $\mathscr{A}_2$ is generated as a $\mathbb{Z}/2\mathbb{Z}$ module by admissible monomials.

[1], [10], [11], [7], [5], [2]

3 Spectra

3.1 Categorical nonsense

- [11]: There is a category $\mathcal{H}$ of finite based CW complexes, with $\mathrm{Hom}(X,Y) =: [X,Y]$ the set of homotopy classes of base-point preserving maps $X \to Y$.

- There is a category $\mathbf{St}(\mathcal{H})$ of finite based CW complexes, with $\mathrm{Hom}(X,Y) =: \{X,Y\}$ the set $\mathrm{colim}_i[\Sigma^i X, \Sigma^i Y]$ [it's just a colimit of sets, and **Set** is cocomplete, so we should be fine. [11] says it's a group[3]] [Also, how do these guys compose?]

- There is a functor $\mathcal{H} \to \mathbf{St}(\mathcal{H})$. [11] doesn't say what this is but it's presumably the one that is the identity on objects and sends $[f : X \to Y] \in [\Sigma^0 X, \Sigma^0 Y]$ to whatever it gets sent to in $\{X,Y\}$ using the universal property of the colimit. Uniqueness makes it functorial, etc.

- We have a fully faithful functor $\mathbf{St}(\mathcal{H}) \to \mathbf{St}(\mathcal{H})$ given by the suspension on objects, and the unique isomorphism $\{X,Y\} \to \{\Sigma X, \Sigma Y\}$ on maps (such an isomorphism exists, since both of those things are colimits for $[\Sigma^i X, \Sigma^i Y]$ - one of the sequences is cut off at the beginning, but it doesn't matter because both reach the stable value (see above discussion and [11] 1.9), aka the colimit).

- It's not an equivalence, because not every object is isomorphic to a suspension (e.g. anything not connected, since suspensions always connected [?])

- We can formally adjoin desuspensions $\Sigma^{-n}X$ for all n [does this mean just putting the objects there and defining $\mathrm{Hom}(Y, \Sigma^{-n}X) := \mathrm{Hom}(\Sigma^n Y, X)$ and $\mathrm{Hom}(\Sigma^{-n}X, Y) := \mathrm{Hom}(X, \Sigma^n Y)$?], but this category does not have weak colimits (i.e. colimits w/o uniqueness property). [why does it not, and why do we even want that?]

- We instead consider formal sequences of desuspensions $X_0 \to \Sigma^{-1}X_1 \to \cdots$, or sequences (X_n) and maps $\Sigma X_n \to X_{n+1}$, i.e. spectra. [and this fixes the problem?]

3.2 Definitions and examples

Below follows [6], Section 5.2.

[Maybe I could also look at [5] p454 onwards?]

Definition 3.2.1. A *spectrum* is a collection of pointed topological spaces $\{X_n\}_{n\in\mathbb{N}}$, together with basepoint-preserving maps $\sigma_n : \Sigma X_n \to X_{n+1}$.

Example 3.2.2. Let X be a topological space. The *suspension spectrum* of X, denoted by $\Sigma^\infty X$, has $X_n = \Sigma^n X$ and $\sigma_n = \mathrm{id} : \Sigma X_n \to X_{n+1}$.

We write $\mathbb{S} := \Sigma^\infty S^0$, and call $\mathbb{S}$ the *sphere spectrum.*

Example 3.2.3. The *Eilenberg-MacLane spectrum* has X_n a CW complex $K(G,n)$ and $\sigma_n : \Sigma K(G,n) \to K(G,n+1)$ is the adjoint of the CW approximation $K(G,n) \to \Omega K(G,n+1)$.

Definition 3.2.4. Let $X = \{X_n\}$ be a spectrum. We define $\pi_i(X) = \mathrm{colim}_n\, \pi_{i+n}(X_n)$, where the map $\pi_{i+n}(X_n) \to \pi_{i+n+1}(X_{n+1})$ is given by the composition

$$\pi_{i+n}(X_n) \xrightarrow{\Sigma} \pi_{i+n+1}(\Sigma X_n) \xrightarrow{(\sigma_n)_*} \pi_{i+n+1}(X_{n+1}).$$

Example 3.2.5. If X is a topological space, then $\pi_i(\Sigma^\infty X) = \pi_i^S(X)$, the ith stable homotopy group of X.

[3] The colimit is equal to the stable value (which exists, by the corollary). After Σ^2, these guys are all groups, so the colimit also has a group structure inherited from whatever $[\Sigma^k X, \Sigma^k Y]$ it's equal to. N.B: Remarks about cocompleteness of **Set** are misleading because that doesn't actually matter - any sequence that stabilises in any category will have a filtered colimit equal to that stable value, you don't need any extra conditions.

Definition 3.2.6. A CW spectrum is a spectrum X consisting of CW complexes X_n with the maps $\Sigma X_n \hookrightarrow X_{n+1}$ inclusions of subcomplexes.

Definition 3.2.7. Let X be a CW spectrum. Then the *k-cells* of X are the equivalence classes of non-basepoint $k+n$-cells in X_n, where two cells are equivalent if one is an m-fold suspension of the other.

Definition 3.2.8. A CW spectrum X is *connective* if it has no cells below a given dimension. Further, X is *finite* if it has only finitely many cells, and *of finite type* if it has only finitely many cells in each dimension.

Example 3.2.9. If X is a finite (resp. finite type) CW complex, then Σ^∞ is a finite (resp. finite type) CW spectrum. In particular, $\mathbb{S}$ is a finite CW spectrum with a unique cell in dimension 1.

3.3 Homology and cohomology

[From Hatcher: "the inclusions $\Sigma X_n \hookrightarrow X_{n+1}$ induce inclusions $C_*(X_n; G) \hookrightarrow C_*(X_{n+1}; G)$ with a dimension shift to account for the suspension". Below is my vague explanation of what I understand this to mean.

$C_i(X_n; G)$ is the free abelian group on maps $\Delta^i \to X_n$. I claim $\Sigma\Delta^i \cong \Delta^{i+1}$. If this is true, it gives a map

$$\begin{gathered} C_i(X_n; G) \to C_{i+1}(\Sigma X_n; G) \\ f \mapsto \Sigma f. \end{gathered}$$

This is an injection, by Remark B.1.3. We also have an injection $C_{i+1}(\Sigma X_n; G) \to C_{i+1}(X_{n+1}; G)$ induced by the structure map σ_n, so we get an injection $C_i(X_n; G) \hookrightarrow C_{i+1}(X_{n+1}; G)$, which indeed has a dimension shift.

We then have that "the union $C_*(X; G)$ of this increasing sequence of chain complexes is then a chain complex having one G summand for each cell of X". I think that

$$C_n(X; G) = \bigcup_{i \in \mathbb{Z}} C_{i+n}(X_i; G),$$

so that there's a G summand for every $i+n$ cell of X_i up to treating suspensions of cells as equivalent to the cells themselves (since they include in), i.e. a G summand for every n-cell of X, since that's how we defined them.

We define H^* and H_* to be the cohomology and homology of this chain complex, respectively.]

Definition 3.3.1. Let $X = \{X_n\}$ be a CW spectrum. A *subspectrum* X' of X is a sequence of subcomplexes $\{X'_n \subseteq X_n\}$ satisfying $\Sigma X'_n \subseteq X'_{n+1}$. The subspectrum X' is *cofinal* if, for each n and each cell e^i_α of X_n, the cell $\Sigma^k e^i_\alpha$ belongs to X'_{n+k} for all sufficiently large k.

[If $\Sigma^k e^i_\alpha$ belongs to X'_{n+k} then $\Sigma^{k+1} e^i_\alpha$ belongs to $\Sigma X'_{n+k} \hookrightarrow X'_{n+k+1}$, so if it happens once it'll happen for all time after that. Thus, if X', X'' are cofinal spectra of X with $\Sigma^k e^i_\alpha$ a cell of X'_{n+k} and $\Sigma^l e^i_\alpha$ a cell of X''_{n+l} ($l \geq k$) then $\Sigma^l e^i_\alpha$ is a cell of X'_{n+l} and therefore of $X'_{n+l} \cap X''_{n+l}$. In other words, the intersection of two cofinal spectra is a cofinal spectrum.]

Definition 3.3.2. Let X, Y be CW spectra. A *strict map* $f : X \to Y$ is a sequence of cellular maps $f_n : X_n \to Y_n$ such that the diagram below commutes.

$$\begin{tikzcd} \Sigma X_n \arrow[r, "\sigma_n"] \arrow[d, "\sigma f_n"'] & X_{n+1} \arrow[d, "f_{n+1}"] \\ \Sigma Y_n \arrow[r, "\sigma_n"'] & Y_{n+1} \end{tikzcd}$$

[this induces maps $\pi_i(X) \to \pi_i(Y)$, $H^*(Y) \to H^*(X)$, $H_*(X) \to H_*(Y)$.]

Definition 3.3.3. A *map* of CW spectra $f : X \to Y$ is an equivalence class of strict maps $f' : X' \to Y$ with X' a subspectrum of X, where two strict maps $f' : X' \to Y$ and $f'' : X'' \to Y$ are equivalent if they agree on some common cofinal spectrum.

[this also induces maps $\pi_i(X) \to \pi_i(Y)$, $H^*(Y) \to H^*(X)$, $H_*(X) \to H_*(Y)$.]

[check composition is well defined]

[working definition of equivalence below:]

Definition 3.3.4. Two spectra X, Y are *equivalent* if there are maps $f : X \to Y$ and $g : Y \to X$ such that $fg = \mathrm{id}_Y$ and $gf = \mathrm{id}_X$.

[A spectrum is always equivalent to the suspension of some other spectrum]

Definition 3.3.5. A *homotopy* of maps between spectra is a map $X \times I \to Y$, where $X \times I$ is the spectrum with $(X \times I)_n = X_n \times_{\mathrm{red}} I$.

Note that $\Sigma(X_n \times_{\mathrm{red}} I) = \Sigma X_n \times_{\mathrm{red}} I$. The set of homotopy classes of maps $X \to Y$ is denoted by $[X, Y]$.

[Claim: (probably very obvious): $[\Sigma^\infty S^t, Z] = \pi_t(Z)$.]

[[1] says on p171 that "$[\Sigma X, Z]$ is obviously a group, because in ΣX we have a spare suspension coordinate out in front to manipulate. And for the same reason, $[\Sigma^2 X, Z]$ is an abelian group. But now we can give $[X, Y]$ the structure of an abelian group, because $[X, Y]$ is in 1-1 correspondence with $[\Sigma^2 X, \Sigma^2 Y]$ and we pull back the group structure on that. So now our sets of morphisms $[X, Y]$ are abelian groups, and it's easy to see that composition is bilinear".

Various claims:

- The stuff about normal CW complexes and their groups of maps (i.e. Remark B.1.5) translates to tell me the appropriate things about spectra and their groups of maps.
- After checking a lot of things, I can eventually conclude that $[X, Y]$ is an abelian group for spectra X, Y.]

[The suspension map $[X, Y] \to [\Sigma X, \Sigma Y]$ is an isomorphism of groups]

[Lemma B.3.3 and Theorem B.3.2 are both true for CW spectra too]

[Whitehead's theorem: a map between CW spectra that induces isomorphisms on all homotopy groups is a homotopy equivalence.]

[Prop: If a CW spectrum X is n-connected in the sense that $\pi_i(X) = 0$ for $i \leq n$, then X is homotopy equivalent to a CW spectrum with no cells of dimension $\leq n$]

3.4 Cofibration sequences

Definition 3.4.1. Let $X = \{X_n\}, Y = \{Y_n\}$ be spectra. Then their *wedge sum*is $X \vee Y := \{X_n \vee Y_n\}$. Note that Remark B.2.5 gives us an inclusion $\Sigma(X_n \vee Y_n) \hookrightarrow X_{n+1} \vee Y_{n+1}$.

[The rest of the section:

- Def: closed subspectrum
- Def: quotient of spectra
- Fact: for spectra X, Y and a subspectrum $A \subseteq X$ we have exact sequences[4]

$$\cdots \to [\Sigma X, Y] \to [\Sigma A, Y] \to [X/A, Y] \to [X, Y] \to [A, Y]$$

$$[Y, A] \to [Y, X] \to [Y, X/A] \to [Y, \Sigma A] \to [Y, \Sigma X] \to \cdots$$

]

[1], [10], [11], [7], [2], [6]

4 The Adams spectral sequence

4.1 Spectral sequences

[Maybe add some notes from [11]]

[Some notes from [9], C2 - just here as a placeholder/reference and I'll probably completely rewrite this bit.]

Definition 4.1.1. A *differential bigraded module* E over a ring R is a collection of R-modules $\{E^{p,q}\}$, $p, q \in \mathbb{Z}$, together with a map $d : E^{p,q} \to E^{p+s,q-s+1}$ for each p, q and some fixed $s \in \mathbb{Z}$, satisfying $d^2 = 0$.

We can take the homology of (E, d):

$$H^{p,q}(E^{*,*}, d) = \ker(d : E^{p,q} \to E^{p+s,q-s+1}) / \operatorname{im}(d : E^{p-s,q+s-1} \to E^{p,q}).$$

Definition 4.1.2. A *spectral sequence* (of *cohomological type*[5]) is a collection of differential bigraded R-modules $\{E_r^{*,*}, d_r\}, r \in \mathbb{N}$, with the differentials d_r of bidegree $(r, 1-r)$. These satisfy the further condition that for all p, q, r, $E_{r+1}^{p,q} \cong H^{p,q}(E_r^{*,*}, d_r)$.

We will sometimes write $d_r^{p,q}$ for the differential $d_r : E^{p,q} \to E^{p+r,q-s+1}$.

Consider the term $E_2^{*,*}$. Define

$$Z_2^{p,q} := \ker d_2^{p,q} \quad \text{and} \quad B_2^{p,q} := \operatorname{im} d_2^{p-2,q+1}.$$

The condition $d^2 = 0$ implies that $B_2^{p,q} \subseteq Z_2^{p,q} \subseteq E_2^{p,q}$, and by definition we have $E_3^{p,q} \cong Z_2^{p,q}/B_2^{p,q}$.

Now, write

$$Z_3^{p,q} := \ker d_3^{p,q} \quad \text{and} \quad B_3^{p,q} := \operatorname{im} d_3^{p-3,q+2}.$$

[4] Check if I actually need the first one.

[5] I'll have to rewrite this section because the Adams spectral sequence is not a cohomological or a homological spectral sequence I don't think - the grading is $d_r : E^{s,t} \to E^{s+r,t+r-1}$.

Since $Z_3^{p,q} \subseteq E_3^{p,q}$, it can be written as $\overline{Z}_3^{p,q}/B_2^{p,q}$ for some $\overline{Z}_3^{p,q} \subseteq Z_2^{p,q}$. Similarly, $B_3^{p,q} \cong \overline{B}_3^{p,q}/B_2^{p,q}$ for some $\overline{B}_3^{p,q} \subseteq Z_2^{p,q}$. Thus,

$$E_4^{p,q} \cong Z_3^{p,q}/B_3^{p,q} \cong \frac{\overline{Z}_2^{p,q}/B_2^{p,q}}{\overline{B}_3^{p,q}/B_2^{p,q}} \cong \overline{Z}_3^{p,q}/\overline{B}_3^{p,q}.$$

Iterating this process, we present the spectral sequence as an infinite tower of submodules of $E_2^{p,q}$:

$$B_2^{p,q} \subseteq \overline{B}_3^{p,q} \subseteq \cdots \subseteq \overline{B}_n^{p,q} \subseteq \cdots \subseteq \overline{Z}_n^{p,q} \subseteq \cdots \subseteq \overline{Z}_3^{p,q} \subseteq Z_2^{p,q} \subseteq E_2^{p,q},$$

with the property that $E_{n+1}^{p,q} \cong \overline{Z}_n^{p,q}/\overline{B}_n^{p,q}$. The differential $d_{n+1}^{p,q}$ can be taken as a map $\overline{Z}_n^{p,q}/\overline{B}_n^{p,q} \to \overline{Z}_n^{p,q}/\overline{B}_n^{p,q}$ with kernel $\overline{Z}_{n+1}^{p,q}/\overline{B}_n^{p,q}$ and image $\overline{B}_{n+1}^{p,q}$. The short exact sequence induced by d_{n+1},

$$0 \to \overline{Z}_{n+1}^{p,q}/\overline{B}_n^{p,q} \to \overline{Z}_n^{p,q}/\overline{B}_n^{p,q} \xrightarrow{d_{n+1}^{p,q}} \overline{B}_{n+1}^{p,q}/\overline{B}_n^{p,q} \to 0,$$

gives rise to isomorphisms $\overline{Z}_n^{p,q}/\overline{Z}_{n+1}^{p,q} \cong \overline{B}_{n+1}^{p,q}/\overline{B}_n^{p,q}$ for all n. Conversely, a tower of submodules of E_2, together with a set of isomorphisms, gives rise to a spectral sequence.

Definition 4.1.3. An element of $E_2^{p,q}$ *survives to the rth stage* if lies in $\overline{Z}_r^{p,q}$, having been in the kernel of the previous $r-2$ differentials, and is *bounded by the rth stage* if it lies in $\overline{B}_r^{p,q}$. The bigraded module $E_r^{*,*}$ is called the *E_r-term* of the spectral sequence.

We define

$$Z_\infty^{p,q} := \bigcap_n \overline{Z}_n^{p,q}, \quad B_\infty^{p,q} := \bigcup^n \overline{B}_n^{p,q}.$$

From the tower of inclusions, we see that $B_\infty^{p,q} \subseteq Z_\infty^{p,q}$, so we define $E_\infty^{p,q} := Z_\infty^{p,q}/B_\infty^{p,q}$.

Definition 4.1.4. A spectral sequence *collapses at the Nth term* if the differentials $d_r^{p,q} = 0$ for $r \geq N$.

From the short exact sequence

$$0 \to \overline{Z}_r^{p,q}/\overline{B}_{r-1}^{p,q} \to \overline{Z}_{r-1}^{p,q}/\overline{B}_{r-1}^{p,q} \xrightarrow{d_r^{p,q}} \overline{B}_r^{p,q}/\overline{B}_{r-1}^{p,q} \to 0,$$

the condition $d_r^{p,q}$ forces $\overline{Z}_r^{p,q} = \overline{Z}_{r-1}^{p,q}$ and $\overline{B}_r^{p,q} = \overline{B}_{r-1}^{p,q}$. The tower of submodules becomes

$$B_2^{p,q} \subseteq \overline{B}_3^{p,q} \subseteq \cdots \subseteq \overline{B}_{N-1}^{p,q} = B_N^{p,q} = \cdots = B_\infty^{p,q} \subseteq Z_\infty^{p,q} = \cdots = \overline{Z}_N^{p,q} = \overline{Z}_{N-1}^{p,q} \subseteq \cdots \subseteq \overline{Z}_3^{p,q} \subseteq Z_2^{p,q} \subseteq E_2^{p,q}.$$

Thus, $E_\infty^{p,q} = E_N^{p,q}$.

4.2 Exact couples

(Following [9], C2)

Definition 4.2.1. Let D, E be R-modules, and let $i : D \to D$, $j : D \to E$, $k : E \to D$ be module homomorphisms. We call $\mathcal{C} = \{D, E, i, j, k\}$ an *exact couple* if the diagram below is exact.

$$\begin{tikzcd} D \arrow[rr, "i"] & & D \arrow[dl, "j"] \\ & E \arrow[ul, "k"] & \end{tikzcd}$$

Let $d := jk$, and define the following:

$$\begin{aligned} E' &:= H(E,d) = \ker d / \operatorname{im} d \\ D' &:= i(D) = \ker j \\ i' &:= i|_{i(D)} : D' \to D' \\ j' &:= i(x) \mapsto j(x) + dE : D' \to E' \\ k' &:= (e + dE) \mapsto k(e) : E' \to D' \end{aligned}$$

We call $\mathcal{C}' = \{D', E'.i', j', k'\}$ the *derived couple* of $\mathcal{C}$.

Proposition 4.2.2 ([9], Prop 2.7). If $\mathcal{C} = \{D, E, i, j, k\}$ is an exact couple, then $\mathcal{C}'$ is also an exact couple.

Theorem 4.2.3 ([9], Thm 2.8). Suppose $D^{*,*} = \{D^{p,q}\}$ and $E^{*,*} = \{E^{p,q}\}$ are bigraded modules equipped with homomorphisms i of bidegree $(-1,1)$[6], j of bidegree $(0,0)$, and k of bidegree $(1,0)$, such that $\{D^{*,*}, E^{*,*}, i, j, k\}$ is an exact couple. Then these data determine a spectral sequence $\{E_r, d_r\}$ for $r \in \mathbb{Z}_+$ of cohomological type, with $E_r = (E^{*,*})^{(r-1)}$, the $(r-1)$st derived module of $E^{*,*}$ and $d_r = j^{(r)} \circ k^{(r)}$.

A bigraded exact couple may be displayed in the following diagram, known as a *staircase diagram*:

$$\begin{tikzcd}
 & \vdots \arrow[d, "i"] & & \vdots \arrow[d, "i"] & \\
\cdots \arrow[r, "k"] & D^{p+2,q-1} \arrow[r, "j"] \arrow[d, "i"] & E^{p+2,q-1} \arrow[r, "k"] & D^{p+3,q-1} \arrow[r, "j"] \arrow[d, "i"] & \cdots \\
\cdots \arrow[r, "k"] & D^{p+1,q} \arrow[r, "j"] \arrow[d, "i"] & E^{p+1,q} \arrow[r, "k"] & D^{p+2,q} \arrow[r, "j"] \arrow[d, "i"] & \cdots \\
\cdots \arrow[r, "k"] & D^{p,q+1} \arrow[r, "j"] \arrow[d, "i"] & E^{p,q+1} \arrow[r, "k"] & D^{p+1,q+1} \arrow[r, "j"] \arrow[d, "i"] & \cdots \\
 & \vdots & & \vdots &
\end{tikzcd}$$

[This is the same staircase diagram that Hatcher is talking about (set $D^{*,*} = \pi_*^S X_*$ and $E^{*,*} = \pi_* K_*$) - just modulo some tweak of i's bigrading because the bigrading is a bit weird in the Adams spectral sequence.]

4.3 The Adams spectral sequence

Things I need before I can set it up (according to Hatcher [6]):

Let X be a CW spectrum of finite type.

- Fact: $H^*(X)$ is finitely generated.
- Fact: $H^*(X)$ is an $\mathscr{A}_2$-module. [We know that's true for a topological space]

[6]This is the wrong index. For my purposes it should be $(-1,-1)$. So, the spectral sequence will not be of cohomological type, but hopefully the proof still goes through.

- Fact: We can pick generators α_i for $H^*(X)$ as an $\mathscr{A}_2$-module such that there are at most finitely many in each $H^n(X)$.
- Fact: There α_i determine a map $X \to K_0$, where K_0 is a wedge of EM spectra, and K_0 has finite type. [the first part probably follows from the fact that $H^m(X;G) \cong [X, K(G,m)]$ naturally in m and X.]
- Fact: We can replace that map with an inclusion [maybe via a mapping cylinder? What does Hatcher mean by replace - does he mean that the new map is homotopy equivalent to the old one?].
- Fact: A quotient of connective spectra of finite type is again a connective spectrum of finite type.
- Prop: [6], 5.46.
- Def: A cofibration. [see : [5], p398.]
- Fact: If Z is a connective spectrum of finite type, then $\pi_t(Z)$ is finitely generated.

[[6] Thm 5.47 says that there is a spectral sequence $\{E_r, d_r\}$ such that $E_2^{s,t} = \mathrm{Ext}_{\mathscr{A}_2}(H^*(\mathbb{S}), H^*(\mathbb{S}))$, and $\{E_r, d_r\} \implies \pi_*^S$ modulo any odd torsion.

At the bottom of p592 of [6], it's mentioned that for a spectrum X of finite type, $H^i(X) \cong \varprojlim_n H^{i+n}(X_n)$ as $\mathscr{A}_2$-modules. Thus, since $\mathbb{S}$ is of finite type (it is actually finite), we have

$$\begin{aligned} H^i(\mathbb{S}; \mathbb{Z}/2\mathbb{Z}) &= \varprojlim_n H^{i+n}(S^n; \mathbb{Z}/2\mathbb{Z}) \\ &= \begin{cases} \mathbb{Z}/2\mathbb{Z} & i = 0, \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

(since they're all just either 0 or $\mathbb{Z}/2\mathbb{Z}$, at least eventually). So, $H^*(\mathbb{S}) = \mathbb{Z}/2\mathbb{Z}$ in degree zero and nothing else.]

Theorem 4.3.1 ([6], Thm 5.47). *There is a spectral sequence $\{E_r, d_r\}$ such that $E_2^{s,t} = \mathrm{Ext}^{s,t}_{\mathscr{A}_2}(\mathbb{F}_2, \mathbb{F}_2)$ and $\{E_r, d_r\} \implies \pi_*^S$ modulo torsion of odd order.*

[9], [1], [10], [2], [6], [11], [3]

4.4 Multiplicative structure

[Rough sketch, details to fill in once I have the shape of things:

Fact: there is a multiplication
$$E_2^{s,t} \otimes E_2^{s',t'} \to E_2^{s+s',t+t'},$$
i.e.
$$F^{s,t}/F^{s+1,t+1} \otimes F^{s',t'}/F^{s'+1,t'+1} \to F^{s+s',t+t'}/F^{s+s'+1,t+t'+1}.$$

Using Lemma 5.1.1 (which comes later but whatever, I'll reorder things), we get $E_2^{s,t} = \mathrm{Hom}^t_{\mathscr{A}_2}(F_s, \mathbb{F}_2)$. So, if we have a minimal free resolution

$$\cdots \to F_2 \to F_1 \to F_0 \to \mathbb{F}_2$$

of $\mathbb{F}_2$ as an $\mathscr{A}_2$-module, and we have $f \in \mathrm{Hom}^t_{\mathscr{A}_2}(F_s, \mathbb{F}_2), g \in \mathrm{Hom}^{t'}_{\mathscr{A}_2}(F_{s'}, \mathbb{F}_2)$, we multiply them as follows:

First, we put these resolutions side by side.

$$\begin{tikzcd}
F_{s+s'} \arrow[r, dashed] \arrow[d] & F_{s'}[t] \arrow[r, "g"] \arrow[d] & \mathbb{F}_2[t+t'] \\
F_{s+s'-1} \arrow[r, dashed] \arrow[d] & F_{s'-1}[t] \arrow[d] & \\
\vdots \arrow[d] & \vdots \arrow[d] & \\
F_s \arrow[r, dashed] \arrow[d] \arrow[dr, "f"] & F_0[t] \arrow[d, two heads] & \\
F_{s-1} \arrow[d] & \mathbb{F}_2[t] & \\
\vdots \arrow[d] & & \\
F_0 \arrow[d] & & \\
\mathbb{F}_2 & &
\end{tikzcd}$$

Then we inductively fill in the dotted arrows from the bottom up. So you can fill the first one because the modules are all free so you just need to say where the generator goes (a priori maybe there are infinitely many generators but it's ok because these are degree t maps and there are only finitely many in each degree), and the right map is surjective, so just do it the way you would do it. Likewise for the higher ones.

This thing is hopefully well-defined.

A few other things:

1. We can check by other methods [elaborate] that $\pi_0^S = \mathbb{Z}$. Now, $E^{s,t}$ is actually supposed to converge to the associated graded of $\mathbb{Z}_2$ (i.e. the 2-adics) I think? But either way, we know that for our filtration all the quotients are $\mathbb{Z}/2\mathbb{Z}$. I claim then that the filtration must be $\cdots 4\mathbb{Z}_2 \subseteq 2\mathbb{Z}_2 \subseteq \mathbb{Z}_2$.

2. If that's true then $\iota = [1] \in \mathbb{Z}_2/2\mathbb{Z}_2$, and using the multiplication above we see ι is a unit. We also have $h_0 = [2] \in 2\mathbb{Z}_2/4\mathbb{Z}_2$ so $h_0 = [2] = [2[1]] = [2\iota]$, so h_0 acts on ι by multiplication by 2.

3. Now, for any $\kappa \in E_2^{s,t}$, $h_0 \cdot \kappa = (\iota h_0) \cdot \kappa = 2\kappa$.

The notation's a bit weird above, when I say 'multiplication by 2' I mean: take $\kappa \in E^{s,t} = F^{s,t}/F^{s+1,t+1}$. Then $2\kappa \in F^{s+1,t+1}$ since it's a bunch of $\mathbb{F}_2$'s. Take it's equivalence class to get an element of $F^{s+1,t+1}/F^{s+2,t+2}$. That's what I really mean by 2κ and it's in $E^{s+1,t+1}$.

All this is to say if I start multiplying higher things by h_0, that *is* multiplying by 2. So I can start resolving extensions this way.]

[11]

5 Calculating stable homotopy groups

5.1 The E_2 page

LEMMA 5.1.1 ([6], Lem 5.49). For a minimal free resolution

$$\cdots \to F_2 \to F_1 \to F_0 \to \mathbb{F}_2 \to 0$$

of $\mathbb{F}_2 = H^*(\mathbb{S})$ as an $\mathscr{A}_2$-module, we have $\mathrm{Ext}^{s,t}_{\mathscr{A}_2}(\mathbb{F}_2, \mathbb{F}_2) = \mathrm{Hom}^t_{\mathscr{A}_2}(F_s, \mathbb{F}_2)$, where $\mathrm{Hom}^t_{\mathscr{A}_2}(F_s, \mathbb{F}_2) \subseteq \mathrm{Hom}_{\mathscr{A}_2}(F_s, \mathbb{F}_2)$ consists of the morphisms which lower the degree by t.

[Now, since $\mathbb{F}_2$ just has stuff in degree 0, the only things that can be sent to $1 \in \mathbb{F}_2$ are the things in degree t, so for every generator for F_s in degree t, there's an $\mathbb{F}_2$'s worth of such homs.]

[1], [10], [11], [6].

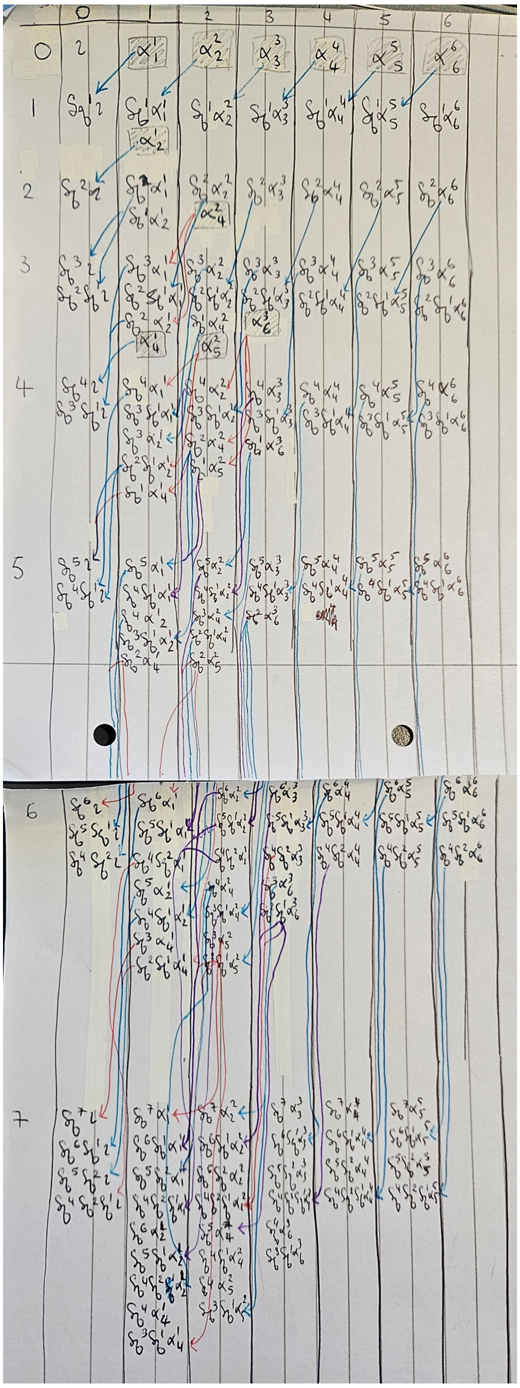

Figure 5.1: Calculating the first 5 rows

LEMMA 5.1.2 (?). There are no nontrivial differentials for $t - s \leq 13$.

[Should follow from the Leibniz rule, but we need the multiplicative structure first]

5.2 Resolving extensions

[Note: there is a multiplicative structure on the E_2 page of the spectral sequence, which Hatcher says is there and then proceeds to never ever mention again. So I'll have to find another source for this bit. It should correspond to the composition product on π_*^S, i.e. given $[f] \in \pi_i^S$, $[g] \in \pi_j^S$, we have $[f][g] = [g \circ \Sigma^j f] \in \pi_{i+j}^S$.

[11], in particular 6.1 and 6.2.

Also, [9] has some cryptic remarks on p424 and p407 about this but doesn't actually really explain it, and in particular doesn't explain how you could know that some stable homotopy group *isn't* cyclic (e.g. π_8^S).

Some slightly less cryptic comments are made in [10], in particular stemming from Lemma 3.1.3. If I just believe this lemma for the moment, assuming its a_0 is the same as my α_1^1, and further believe the calculations done in [11] in 6.2, then (I think!) this solves my problems.]

Proposition 5.2.1 ([11], Cor 6.5). We have the following relations:

$$\begin{aligned} \alpha_i^i &= (\alpha_1^1)^i \\ \alpha_4^2 &= (\alpha_2^1)^2 \\ \alpha_5^2 &= \alpha_1^1\alpha_4^1 \\ \alpha_6^3 &= (\alpha_1^1)^2\alpha_4^1 = (\alpha_2^1)^3. \end{aligned}$$

Proposition 5.2.2 ([10], Lem 3.1.3(b)). If $x \in \mathrm{Ext}^{s,t}(\mathbb{F}_2, \mathbb{F}_2)$ is a permanent cycle in the Adams spectral sequence represented by $\alpha \in \pi_*^S$, then $\alpha_1^1 x$ is a permanent cycle represented by 2α.

Theorem 5.2.3.

$$_{(2)}\pi_i^S = \begin{cases} \mathbb{Z}/2\mathbb{Z} & i = 1, 2 \\ \mathbb{Z}/8\mathbb{Z} & i = 3 \\ 0 & i = 4, 5. \end{cases}$$

5.3 Nontrivial differentials

[The point here is that all differentials interacting with $E^{s,t}$ for $t - s = 14$ are trivial, and thus computing the stable homotopy groups is purely mechanical, because everything that appears on this part of the E_2 page has to survive to E_∞. Thus, the 'ambiguity' at $t - s = 14$ is just the fact that this is the first time you need to actually compute differentials.

For $t - s < 14$ there are only actually the ones emanating from $E^{1,2}$ and the d_2 differential starting at $E^{2,10}$ to worry about, because all the others either enter or leave 0.]

[1], [10], [11].

A Algebra

A.1 Free resolutions

Definition 1.1.1. Let M, N be modules over a ring R. A *resolution* F of M is an exact sequence

$$\cdots \to F_2 \to F_1 \to F_0 \to M \to 0.$$

If in addition each F_i is a free R-module, then the resolution is called *free*.

Given a free resolution as above, applying $\mathrm{Hom}_R(-, N)$ gives us a chain complex

$$\cdots \leftarrow \mathrm{Hom}_R(F_2, N) \leftarrow \mathrm{Hom}_R(F_1, N) \leftarrow \mathrm{Hom}_R(F_0, N) \leftarrow \mathrm{Hom}_R(M, N) \leftarrow 0.$$

Dropping the term $\mathrm{Hom}_R(M, N)$ [why?] we get the sequence

$$\cdots \leftarrow \mathrm{Hom}_R(F_2, N) \leftarrow \mathrm{Hom}_R(F_1, N) \leftarrow \mathrm{Hom}_R(F_0, N) \leftarrow 0,$$

and we define $\mathrm{Ext}^n_R(M,N)$ to be the nth homology group of this chain complex.

[these do not depend on the choice of free resolution of M]

A free resolution is *minimal* if at each stage of its construction we choose the minimal number of free generators for F_i in each degree.

[The above definition is bad but I'm keeping it just for the moment.]

B Topology

All from [5] unless otherwise stated.

B.1 Suspension

DEFINITION 2.1.1. Let X be a topological space. The *suspension* SX is the space $(X \times I)/\sim$, where $(x,0) \sim (x',0)$ and $(x,1) \sim (x',1)$ for all $x, x' \in X$.

DEFINITION 2.1.2. Let X be a pointed topological space. The *reduced suspension* ΣX is the space $SX/\sim$, where $[x_0,t] \sim [x_0,t']$ for all $t,t' \in I$.

Given a map $f : X \to Y$, we can define $\Sigma f : \Sigma X \to \Sigma Y$ by $\Sigma f[(x,t)] = [(fx,t)]$. This makes Σ into a functor $\Sigma : \mathbf{Top} \to \mathbf{Top}$.

REMARK 2.1.3. Σ is faithful, since for any maps $f,g : X \to Y$, if $\Sigma f = \Sigma g$ then in particular $[(fx,\frac{1}{2})] = [(gx,\frac{1}{2})]$, so $fx = gx$.

[below is reconstructed from [8]]

Given pointed maps $f,g : \Sigma X \to Z$, define

$$\begin{aligned} f \star g : \Sigma X &\to Z \\ [x,t] &\mapsto \begin{cases} f[x,2t-1] & t \geq \frac{1}{2}, \\ g[x,2t] & t \leq \frac{1}{2}. \end{cases} \end{aligned}$$

This is well defined, since both f and g are basepoint-preserving.

REMARK 2.1.4. This defines a group structure on $[\Sigma X, Z]$, and thus $[\Sigma^i X, Z]$ is a group for all $i \geq 1$. For $i \geq 2$, these can be shown to be abelian, via the Eckmann-Hilton argument.

REMARK 2.1.5. The homotopy groups $\pi_i(Z)$ are a special case of the above construction, taking $X := S^{i-1}$.

- Loops; the adjunction $\Sigma \dashv \Omega$, where Ω is the loop functor.

[5], p395:

REMARK 2.1.6. It follows that $\pi_{n+1}(X) \cong \pi_n(\Omega X)$. In particular, $\Omega K(G,n)$ is a $K(G,n-1)$.

- [5] 2.1 Ex 20 and 2.2 Ex 32: $\tilde{H}_n(X) \cong \tilde{H}_{n+1}(SX)$, where S is the (non-reduced) suspension. (MV?)
- Hatcher also says on p219 that $\tilde{H}^n(X;R) \cong \tilde{H}^{n+k}(\Sigma^k X;R)$, where Σ is reduced suspension.

B.2 Other basic constructions

Definition 2.2.1. Let $(X, x_0), (Y, y_0)$ be pointed topological spaces, and consider their product $X \times Y$. The subspaces $X \times \{y_0\} \cong X$ and $\{x_0\} \times Y \cong Y$ intersect at exactly one point, (x_0, y_0), and so can be identified with the wedge $X \vee Y$. We thus define the *smash product* $X \wedge Y := (X \times Y)/(X \vee Y)$, with the canonical basepoint (x_0, y_0).

Example 2.2.2. We have $S^n \wedge S^m \cong S^{n+m}$. [is this obvious?]

Remark 2.2.3. Note that $\Sigma X \cong X \wedge S^1$.

Remark 2.2.4. Observe that $X \wedge (Y \wedge Z) \cong (X \wedge Y) \wedge Z$. Combining this with the remarks above, we see that $\Sigma^k X = X \wedge S^k$.

Remark 2.2.5. Note that $\Sigma(X \vee Y) \cong \Sigma X \vee \Sigma Y$.

- The Eilenberg-MacLane space is $K(G, n)$, and it has the property that

$$\pi_i(K(G,n)) = \begin{cases} G & i = n, \\ 0 & i \neq n. \end{cases}$$

  They're unique up to weak homotopy equivalence (i.e. if you have another one X, there's a map between them which descends to an isomorphism on homotopy groups). They can be taken to be CW complexes.

Definition 2.2.6. Let X, Y be topological spaces, where X has a basepoint x_0. Then the *reduced product* $X \times_{\text{red}} Y := (X \times Y)/(x_0 \times Y)$.

Definition 2.2.7. Let $f : X \to Y$ be a map. The *mapping cylinder* M_f is defined by $(X \times I) \sqcup Y/\sim$, where $(x, 1) \sim f(x)$ for all $x \in X$.

The mapping cylinder deformation retracts onto Y via $h : M_f \times I \to M_f$; $([x,t], s) \mapsto [x, t + s(1-t)]$. If $(X, x_0), (Y, y_0)$ are pointed spaces, the *reduced mapping cylinder* is the quotient $M_f/\sim$, where $[x_0, t] \sim [x_0, t']$ for all $t \in I$.

Definition 2.2.8. Let $f : X \to Y$ be a map. The *mapping cone*[7] C_f is defined to be $Y \sqcup_f CX := (Y \sqcup CX)/(f(x) \sim [x, 1])$.

B.3 Cell complexes

Definition 2.3.1. Let $f : X \to Y$ be a map between CW complexes. Then f is *cellular* if $f(X_{(n)}) \subseteq Y_{(n)}$ for all n, where $X_{(n)}$ is the n-skeleton of X.

Cellular approximation theorem:

Theorem 2.3.2 ([5], Thm 4.8). Let $f : X \to Y$ be a map of CW complexes. Then f is homotopic to a cellular map.

Lemma 2.3.3 ([5], Prop 0.16). Let $A \subseteq X$ be CW complexes. Then the pair (X, A) has the *homotopy extension property*; that is, for any map $f : X \to Y$ and homotopy $h : A \times I \to Y$ such that $h(a, 0) = f|_A$, there is a homotopy $\tilde{h} : X \times I \to Y$ extending h.

- The product of cell complexes is a cell complex (maybe only if one of them is finite?)
- The smash product of (pointed?) cell complexes is a cell complex (maybe only if one is them is finite?) [[5] says "the smash product $X \wedge Y$ is a cell complex if X and Y are cell

[7] Why does Hatcher not insist this guy is reduced, like he does with the mapping cylinders?

complexes with x_0 and y_0 0-cells, assuming that we give $X \times Y$ the cell-complex topology rather than the product topology in cases where these two topologies differ".]

- Quotient of a CW complex by a subcomplex is a CW complex, where the quotient map is cellular
- For a CW complex X, $SX \simeq \Sigma X$.
- The reduced suspension of a pointed cell complex (X, x_0) is another pointed cell complex ΣX with basepoint x_0 and an n-cell for each non-basepoint $n-1$ cell e_α^{n-1} of X.

DEFINITION 2.3.4. Let X is a topological space. A *CW approximation* to X is a CW complex Z equipped with a weak homotopy equivalence $f : Z \to X$.

THEOREM 2.3.5 ([5], Prop 4.13). Every space X has a CW approximation $f : Z \to X$.

- In particular, $\Omega K(G, n)$ has a CW approximation $Z \to \Omega K(G, n)$, and since $\Omega K(G, n)$ is a $K(G, n-1)$, so is Z.
- Something along the lines of 'compact $\leftrightarrow$ finite number of cells'.

C Notes to self

C.1 Vague problems and Questions

- What does Hatcher mean when he says two spectra are 'equivalent'?
- On p588 of [6], he says "every CW spectrum is equivalent to a suspension spectrum". Does he actually mean that, or does he mean 'equivalent to the suspension of a spectrum'? The former seems way too strong, although in fairness I still don't know what an equivalence of spectra actually *is*.
- Replacing maps by inclusions?
- I am definitely being told some lies about what the spectral sequence actually converges to. There's a strong implication/actual statement(!!) that at each i it's supposed to be a filtration of π_i^S modulo odd torsion, but I think this isn't true. I think it's actually the 2-completion[8] of π_i^S. Cor 2.20 of this nLab page says that that coincides with this for finite abelian groups, but for π_0^S it's supposed to be $\mathbb{Z}_2$ (i.e. the 2-adic integers), not $\mathbb{Z}$. I believe. Maybe get a source for this. Some people say it's the localisation at 2?? But I think that's also a lie.

C.2 To do

Now:

- The definition of Ext in A.1 has only one superscript, but the Ext in the Adams spectral sequence has a bigrading - figure out where the extra index came from and write it down so I don't forget it.
- Read about fibration and cofibration sequences of CW complexes.
- Find out what a (co)homology theory is.
- Find out what the deal with reduced cohomology and being stable under suspension is.

[8] Whatever that means.

Eventually:

- Be consistent with either cell complex or CW complex (eventually)
- Be consistent with $\mathbb{F}_2$ or $\mathbb{Z}/2\mathbb{Z}$ (don't use $\mathbb{Z}_2$, that's bad).
- Specialise the Adams spectral sequence (i.e. set $Y = \mathbb{S}$)
- Remember that you have to hand in the tex file, so for the love of god change anything stupid that's hidden in the pdf.

C.3 Other notes

- READ IF YOUR CALCULATIONS AREN'T WORKING: You are working modulo 2!!!
- If you have a bunch of maps between graded modules/algebras, they're graded homomorphisms. So they preserve degree.
- All (co)homology is supposed to be reduced.
- On p592 of [6], he says that "for spectra X of finite type [the more general] definition of an $\mathscr{A}_2$-module structure on $H^*(X)$ agrees with the definition using the usual $\mathscr{A}_2$-module structure on the cohomology of spaces and the identification of $H^*(X)$ with the inverse limit $\varprojlim H^{*+n}(X_n)$". Absolutely everything relevant to spheres in the construction of the Adams spectral sequence seems to only use spectra of finite type, but Hatcher says on p585 that we can't just take the inverse limit because it doesn't work for the 'more general spectra' used when constructing the Adams spectral sequence. I believe this is because the Adams spectral sequence is defined for X not of finite type, though Hatcher only constructs it for finite type guys.

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