

# Stable Homotopy Groups of Spheres [DRAFT]

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# 1 Introduction

- Define homotopy groups
- The Eilenberg-MacLane space is  $K(G, n)$ , and it has the property that

$$\pi_i(K(G, n)) = \begin{cases} \mathbb{Z} & i = n, \\ 0 & i \neq n. \end{cases}$$

They're unique up to weak homotopy equivalence (i.e. if you have another one  $X$ , there's a map between them which descends to an isomorphism on homotopy groups)

- Define suspension of a topological space
- Freudenthal's suspension theorem: if  $\pi_i(X) = 0$  for  $i \leq k$  (i.e.  $X$  is  $k$ -connected) then the map

$$\begin{aligned} \pi_n(X) &\rightarrow \pi_{n+1}(\Sigma X) \\ [\gamma : S^n \rightarrow X] &\mapsto [\Sigma \gamma : \Sigma S^n = S^{n+1} \rightarrow \Sigma X] \end{aligned}$$

is an isomorphism for  $n \leq 2k$  and surjective for  $n = 2k + 1$

- This implies  $\pi_{n+k}(S^n)$  depends only on  $k$  for  $n \geq k + 2$
- (Obviously be careful with basepoints above)
- Suppose  $X$  is  $k$ -connected. Then, for  $k \geq 0$ ,  $0 = \pi_k(X) \cong \pi_{k+1}(\Sigma X)$ , so whenever a space is  $k$ -connected its suspension is  $k + 1$ -connected.
- As you take suspensions, then, your successive bounds are  $n \leq 2k$ ,  $n + 1 \leq 2k + 2 \implies n \leq 2k + 1$ ,  $n \leq 2k + 2$ , etc ... so the sequence  $\pi_n(X) \rightarrow \pi_{n+1}(\Sigma X) \rightarrow \dots$  will eventually stabilise.
- Thus, if you take the colimit of that direct system, it'll just equal the stable value, with the higher legs just being the inverse isomorphisms.
- [10], Cor 1.9 [not 100% convinced of how this follows, but believing it for now]: if  $X$  is a CW complex of dimension  $d$  and  $Y$  a  $(k - 1)$ -connected space, then the suspension homomorphism  $[X, Y] \rightarrow [\Sigma X, \Sigma Y]$  is bijective if  $d < 2k - 1$  and surjective if  $d = 2k - 1$ .

Miscellaneous facts I might need later:

- Cohomology [possibly only of pointed CW complexes] is representable, and its representing object is the Eilenberg-MacLane space. i.e.  $H^n(-; G) \cong \text{Hom}(-, K(G, n))$ .
- There is an adjunction  $\Sigma \dashv \Omega$ , where  $\Omega$  is the loop functor.
- $\mathcal{A}_2$  is generated as an algebra by elements  $Sq^{2^k}$  ([5], Prop 4L.8).
- The map  $\mathcal{A}_2 \rightarrow \tilde{H}^*(K(\mathbb{Z}/2\mathbb{Z}, n); \mathbb{Z}/2\mathbb{Z}), Sq^I \mapsto Sq^I(\iota_n)$  is an isomorphism from the degree  $d$  part of  $\mathcal{A}_2$  onto  $H^{n+d}(K(\mathbb{Z}/2\mathbb{Z}, n); \mathbb{Z}/2\mathbb{Z})$  for  $d \geq n$ . In particular, the admissible monomials  $Sq^I$  form an additive basis for  $\mathcal{A}_2$ . Thus,  $\mathcal{A}_2$  is exactly the algebra of all  $\mathbb{Z}/2\mathbb{Z}$  cohomology operations that are stable, commuting with suspension ([6], Cor 5.38).
- "Stable homotopy groups are a homology theory" (whatever that means)
- Hurewicz theorem: for any path-connected space  $X$  and  $n > 0$  there exists a group homomorphism  $h_* : \pi_n(X) \rightarrow H_n(X)$ . For  $n = 1$  this induces an isomorphism  $\pi_1^{\text{ab}}(X) \cong H_1(X)$ . For  $n \geq 2$ , if  $X$  is  $(n - 1)$ -connected then  $\tilde{H}_i(X) = 0$  for all  $i < n$ , and the map  $h_* : \pi_n(X) \rightarrow H_n(X)$  is an isomorphism.

Algebraic background:

DEFINITION 1.0.1. Let  $M, N$  be modules over a ring  $R$ . A *free resolution*  $F$  of  $M$  is an exact sequence

$$\cdots \rightarrow F_2 \rightarrow F_1 \rightarrow F_0 \rightarrow M \rightarrow 0,$$

with each  $F_i$  a free  $R$ -module.

Applying  $\text{Hom}_R(-, N)$  gives us a chain complex

$$\cdots \leftarrow \text{Hom}_R(F_2, N) \leftarrow \text{Hom}_R(F_1, N) \leftarrow \text{Hom}_R(F_0, N) \leftarrow \text{Hom}_R(M, N) \leftarrow 0.$$

Dropping the term  $\text{Hom}_R(M, N)$  [why?] we get the sequence

$$\cdots \leftarrow \text{Hom}_R(F_2, N) \leftarrow \text{Hom}_R(F_1, N) \leftarrow \text{Hom}_R(F_0, N) \leftarrow 0,$$

and we define  $\text{Ext}_R^n(M, N)$  to be the  $n$ th homology group of this chain complex.

[these do not depend on the choice of free resolution of  $M$ ]

[10], [4], [5]

## 2 The Steenrod algebra

The following is from [5] 4L.

- There are maps  $Sq^i : H^n(-; \mathbb{Z}/2\mathbb{Z}) \rightarrow H^{n+i}(-; \mathbb{Z}/2\mathbb{Z})$  for each  $i$ , and they satisfy the following properties:
  1.  $Sq_X^i(f^*(\alpha)) = f^*(Sq_Y^i(\alpha))$  for  $f : X \rightarrow Y$  (i.e.  $Sq^i$  is a natural transformation).
  2.  $Sq_X^i(\alpha + \beta) = Sq_X^i(\alpha) + Sq_X^i(\beta)$  (i.e.  $Sq_X^i$  respects the group operation for all  $X$ ).
  3.  $Sq^i(\alpha \smile \beta) = \sum_{0 \leq j \leq i} (Sq^j(\alpha) \smile Sq^{i-j}(\beta))$  (the Cartan formula)
  4.  $Sq^i(\sigma(\alpha)) = \sigma(Sq^i(\alpha))$  where  $\sigma : H^n(X; \mathbb{Z}/2\mathbb{Z}) \rightarrow H^{n+1}(\Sigma X; \mathbb{Z}/2\mathbb{Z})$  is the “suspension isomorphism given by reduced cross product with a generator of  $H^1(S^1; \mathbb{Z}/2\mathbb{Z})$ ”
  5.  $Sq^i(\alpha) = \alpha^2$  if  $i = |\alpha|$  and  $Sq^i(\alpha) = 0$  if  $i > |\alpha|$ . [Hatcher doesn’t explain this notation at all, but I think he means by  $|\alpha|$  the degree of  $\alpha$  - this is what [2] says in C2]
  6.  $Sq^0 = \text{id}$ .
  7.  $Sq^1$  is the “ $\mathbb{Z}/2\mathbb{Z}$  Bockstein homomorphism  $\beta$  associated with the coefficient sequence  $0 \rightarrow \mathbb{Z}/2\mathbb{Z} \rightarrow \mathbb{Z}/4\mathbb{Z} \rightarrow \mathbb{Z}/2\mathbb{Z} \rightarrow 0$ ”.
- Define  $Sq := Sq^0 + Sq^1 + \cdots$ . Then  $Sq(\alpha \smile \beta) = Sq(\alpha) \smile Sq(\beta)$  (since  $(Sq(\alpha \smile \beta))_n = \sum_i Sq^i(\alpha) \smile Sq^{n-i}(\beta) = (Sq(\alpha) \smile Sq(\beta))_n$ ). Thus,  $Sq$  is a ring homomorphism.
- Adem relations:

$$Sq^a Sq^b = \sum_j \binom{b-j-1}{a-2j} Sq^{a+b-j} Sq^j \quad \text{if } a < 2b,$$

where  $\binom{m}{n}$  is zero if  $m$  or  $n$  is negative, or  $m < n$ , and  $\binom{m}{0} = 1$  for  $m \geq 0$ .

- The Steenrod algebra  $\mathcal{A}_2$  is the algebra over  $\mathbb{Z}/2\mathbb{Z}$  that is the quotient of the algebra of polynomials in the noncommuting variables  $Sq^1, Sq^2, \dots$  by the two-sided ideal generated by the Adem relations. Thus, for every space  $X$ ,  $H^*(X; \mathbb{Z}/2\mathbb{Z})$  is a module over  $\mathcal{A}_2$ , via  $\alpha \cdot f = f(\alpha)$ .
- $\mathcal{A}_2$  is graded, and its elements of degree  $k$  are those that map  $H^n(X; \mathbb{Z}/2\mathbb{Z})$  to  $H^{n+k}(X; \mathbb{Z}/2\mathbb{Z})$  for all  $n$ . [Presumably you've fixed a space  $X$  while you're doing all this?]

[1], [9], [10], [7], [5], [3]

### 3 Spectra may not be your friends, but I can introduce you

- [10]: There is a category  $\mathcal{H}$  of finite [because the corollary wanted f.d. CW complexes] based CW complexes, with  $\text{Hom}(X, Y) =: [X, Y]$  the set of homotopy classes of base-point preserving maps  $X \rightarrow Y$ .
- There is a category  $\mathbf{St}(\mathcal{H})$  of finite[?] based CW complexes, with  $\text{Hom}(X, Y) =: \{X, Y\}$  the set  $\text{colim}_i [\Sigma^i X, \Sigma^i Y]$  [it's just a colimit of sets, and  $\mathbf{Set}$  is cocomplete, so we should be fine. [10] says it's a group?]
- There is a functor  $\mathcal{H} \rightarrow \mathbf{St}(\mathcal{H})$ . [10] doesn't say what this is but it's presumably the one that is the identity on objects and sends  $[f : X \rightarrow Y] \in [\Sigma^0 X, \Sigma^0 Y]$  to whatever it gets sent to in  $\{X, Y\}$  using the universal property of the colimit. Uniqueness makes it functorial, etc.
- We have a fully faithful functor  $\mathbf{St}(\mathcal{H}) \rightarrow \mathbf{St}(\mathcal{H})$  given by the suspension on objects, and the unique isomorphism  $\{X, Y\} \rightarrow \{\Sigma X, \Sigma Y\}$  on maps (such an isomorphism exists, since both of those things are colimits for  $[\Sigma^i X, \Sigma^i Y]$  - one of the sequences is cut off at the beginning, but it doesn't matter because both reach the stable value (see above discussion and [10] 1.9), aka the colimit).
- It's not an equivalence, because not every object is isomorphic to a suspension (e.g. anything not connected, since suspensions always connected [?])
- We can formally adjoin desuspensions  $\Sigma^{-n} X$  for all  $n$  [does this mean just putting the objects there and defining  $\text{Hom}(Y, \Sigma^{-n} X) := \text{Hom}(\Sigma^n Y, X)$  and  $\text{Hom}(\Sigma^{-n} X, Y) := \text{Hom}(X, \Sigma^n Y)$ ?], but this category does not have weak colimits (i.e. colimits w/o uniqueness property). [why does it not, and why do we even want that?]
- We instead consider formal sequences of desuspensions  $X_0 \rightarrow \Sigma^{-1} X_1 \rightarrow \dots$ , or sequences  $(X_n)$  and maps  $\Sigma X_n \rightarrow X_{n+1}$ , i.e. spectra. [and this fixes the problem?]

Below follows [6], Section 5.2.

**DEFINITION 3.0.1.** A *spectrum* is a collection of pointed topological spaces  $\{X_n\}_{n \in \mathbb{N}}$ , together with basepoint-preserving maps  $\sigma_n : \Sigma X_n \rightarrow X_{n+1}$ .

**EXAMPLE 3.0.2.** Let  $X$  be a topological space. The *suspension spectrum* of  $X$  has  $X_n = \Sigma^n X$  and  $\sigma_n = \text{id} : \Sigma X_n \rightarrow X_{n+1}$ .

[Define EM spectrum]

**DEFINITION 3.0.3.** Let  $X = \{X_n\}$  be a spectrum. We define  $\pi_i(X) = \text{colim}_n \pi_{i+n}(X_n)$ ,

where the map  $\pi_{i+n}(X_n) \rightarrow \pi_{i+n+1}(X_{n+1})$  is given by the composition

$$\pi_{i+n}(X_n) \xrightarrow{\Sigma} \pi_{i+n+1}(\Sigma X_n) \xrightarrow{(\sigma_n)_*} \pi_{i+n+1}(X_{n+1}).$$

EXAMPLE 3.0.4. If  $X'$  is the suspension spectrum of a topological space  $X$ , then  $\pi_i(X')$  is the  $i$ th stable homotopy group of  $X$ .

DEFINITION 3.0.5. A CW spectrum is a spectrum  $X$  consisting of CW complexes  $X_n$  with the maps  $\Sigma X_n \hookrightarrow X_{n+1}$  inclusions of subcomplexes.

[Define cells and dimension of a CW spectrum]

DEFINITION 3.0.6. A spectrum  $X$  is *connective* if its cells have dimensions which are bounded below.

[The above is phrased exactly as in hatcher - presumably he means that there is some absolute bound below which no cell has dimension, rather than a bound dependent on each cell?]

DEFINITION 3.0.7. A CW spectrum is *finite* if it has only finitely many cells, and *of finite type* if it has only finitely many cells in each dimension.

[1], [9], [10], [7], [3], [6]

## 4 The Adams spectral sequence

[8], [1], [9], [3], [6]

## 5 $\text{Ext}_A^s(\mathbb{F}_2, \mathbb{F}_2)_t$

[1], [9], [10]

## 6 Methods of resolving ambiguities

[1], [9]

## References

- [1] J.F. Adams. *Stable Homotopy and Generalised Homology*. T<sub>E</sub>Xromancers, 2022.
- [2] J.F. Adams. *Stable Homotopy Theory*. Springer, 1969.
- [3] David Barnes and Constanze Roitzheim. *Foundations of Stable Homotopy Theory*. Cambridge University Press, 2020.
- [4] Maxine Calle. *The Freudenthal Suspension Theorem*. 2020. URL: <https://bpb-us-w2.wpmucdn.com/web.sas.upenn.edu/dist/0/713/files/2020/08/FSTnotes.pdf>.
- [5] Allen Hatcher. *Algebraic Topology*. 2001. URL: <https://pi.math.cornell.edu/~hatcher/AT/AT+.pdf>.
- [6] Allen Hatcher. *Spectral Sequences*. URL: <https://pi.math.cornell.edu/~hatcher/AT/ATch5.pdf> (visited on 01/02/2025).
- [7] H.R. Margolis. *Spectra and the Steenrod Algebra*. Elsevier Science Publishers B.V., 1983.
- [8] John McCleary. *A User's Guide to Spectral Sequences*. Cambridge University Press, 2001.

- [9] Douglas C. Ravenel. *Complex Cobordism and Stable Homotopy Groups of Spheres*. Academic Press. Inc, 1986.
- [10] John Rognes. *The Adams Spectral Sequence*. 2012. URL: <https://www.mn.uio.no/math/personer/vit/rognes/papers/notes.050612.pdf>.