Something True and Beautiful [DRAFT]

Contents

1 Introduction

Calculating the higher homotopy groups of spheres is an important and famously difficult problem in homotopy theory. However, the Freudenthal suspension theorem, given below, is the beginning of the field of *stable homotopy theory*, and in particular allows us to ask not about the homotopy groups of spheres in full generality, but to restrict to the colimits of these groups under suspension. This turns out to be a more approachable problem, and these 'stable' groups will be the subject of this essay.

Before stating the theorem below, we note that we will be working throughout with based spaces and maps, and reduced suspension; this ensures that the suspension of any based space has a canonical basepoint, and that the suspension of any pointed map is again a pointed map. We will, however, omit the basepoint from notation to reduce clutter.

THEOREM 1.0.1 ([15], Thm 1.8, Freudenthal suspension theorem). If $\pi_i(X) = 0$ for $i \leq k$ (i.e. X is k-connected) then the map

$$\begin{aligned} \pi_n(X) &\to \pi_{n+1}(\Sigma X) \\ [\gamma : S^n \to X] &\mapsto [\Sigma\gamma : \Sigma S^n = S^{n+1} \to \Sigma X] \end{aligned}$$

is an isomorphism for $n \leq 2k$ and surjective for $n = 2k+1$.

Now, let X be any topological space, and let $k \geq 0$ be such that X is k-connected. Then Theorem 1.0.1 implies that ΣX is $(k+1)$-connected, since $0 = \pi_i(X) \cong \pi_{i+1}(\Sigma X)$ for $i \leq k$. As we take suspensions of X, the successive bounds are $n \leq 2k$, $n \leq 2k+1$, $n \leq 2k+2$, and so on, so the sequence

$$\pi_n(X) \to \pi_{n+1}(\Sigma X) \to \pi_{n+2}(\Sigma^2 X) \to \cdots \tag{1}$$

will eventually stabilise. We thus define the *stable homotopy group* $\pi_n^s(X)$ to be the filtered colimit of the system (1), which is equal to its stable value.

We will denote the groups $\pi_k^s(S^0) \cong \pi_{n+k}^s(S^n)$ by π_k^s. The following classical theorem implies immediately that for $k > 0$, π_k^s is finite.

THEOREM 1.0.2 ([13], Thm 1.1.8). $\pi_{n+k}(S^n)$ is finite for $k > 0$ except when $n = 2m$, $k = 2m - 1$.

Our main computational tool for computing π_k^s, the Adams spectral sequence, will (in the form we present it here) give us information about the 2-completion of these groups, which for a finite abelian group A coincides with A modulo its odd torsion. The remainder of this essay will thus be dedicated to constructing this spectral sequence and using it to determine π_k^s modulo its odd torsion (in the case where $k \neq 0$) for $k \leq 15$.

2 The Steenrod algebra

In this section, we give a very brief introduction to the mod 2 Steenrod algebra $\mathscr{A}_2$, whose elements are characterised by the axioms below. It can be shown (see e.g. [7] p500) that $\mathscr{A}_2$ consists exactly of the stable $\mathbb{F}_2$ cohomology operations (i.e. the natural transformations $H^m(-;\mathbb{F}_2) \to H^n(-;\mathbb{F}_2)$ for fixed n, m).

It will turn out to be useful in computing $\pi_i^s(X) = \operatorname{colim}_k \pi_{i+k}(\Sigma^k X)$ to consider instead the group $\operatorname{Hom}(H^*(X), H^*(S^i))$ to which there is a natural map; the Steenrod algebra gives us a way of obtaining more structure on this group (and thus more information about $\pi_i^s(X)$), by restricting to $\mathrm{Hom}_{\mathscr{A}_2}(H^*(X), H^*(S^i))$ (and later considering the higher Ext groups).

PROPOSITION 2.0.1 ([7], p489). For all X and each n, there are maps $Sq^i : H^n(-;\mathbb{F}_2) \to H^{n+i}(-;\mathbb{F}_2)$ for each i, and they satisfy the following properties:

1. $Sq^i(f^*(\alpha)) = f^*(Sq^i(\alpha))$ for $f : X \to Y$ (i.e. Sq^i is a natural transformation).
2. $Sq^i(\alpha + \beta) = Sq^i(\alpha) + Sq^i(\beta)$ (i.e. Sq^i respects the group operation for all X).
3. $Sq^i(\alpha \smile \beta) = \sum_{0\le j\le i} (Sq^j(\alpha) \smile Sq^{i-j}(\beta))$ (the Cartan formula).
4. $Sq^i(\sigma(\alpha)) = \sigma(Sq^i(\alpha))$ where $\sigma : H^n(X;\mathbb{F}_2) \to \widetilde{H}^{n+1}(\Sigma X;\mathbb{F}_2)$ is the suspension isomorphism given by reduced cross product with a generator of $\widetilde{H}^1(S^1;\mathbb{F}_2)$.
5. $Sq^i(\alpha) = \alpha^2$ if $i = \deg(\alpha)$ and $Sq^i(\alpha) = 0$ if $i > \deg(\alpha)$.
6. $Sq^0 = \mathrm{id}$.

Define $Sq := Sq^0 + Sq^1 + \cdots$. Then $Sq(\alpha \smile \beta) = Sq(\alpha) \smile Sq(\beta)$ (since $(Sq(\alpha \smile \beta))_n = \sum_i Sq^i(\alpha) \smile Sq^{n-i}(\beta) = (Sq(\alpha) \smile Sq(\beta))_n$). Thus, Sq is a ring homomorphism.

The following proposition will be an important computational tool later.

PROPOSITION 2.0.2 ([7], p496). The Steenrod squares satisfy the following relations, known as the *Adem relations*:

$$Sq^a Sq^b = \sum_j \binom{b-j-1}{a-2j} Sq^{a+b-j} Sq^j \quad \text{if } a < 2b,$$

where $\binom{m}{n}$ is zero if m or n is negative, or $m < n$, and $\binom{m}{0} = 1$ for $m \ge 0$.

DEFINITION 2.0.3. The *Steenrod algebra* $\mathscr{A}_2$ is the algebra over $\mathbb{F}_2$ that is the quotient of the algebra of polynomials in the noncommuting variables $Sq^1, Sq^2, ...$ by the two-sided ideal generated by the Adem relations. Thus, for every space X, $H^*(X;\mathbb{F}_2)$ is a module over $\mathscr{A}_2$.

Note that $\mathscr{A}_2$ is graded, with elements of degree k those that map $H^n(X;\mathbb{F}_2)$ to $H^{n+k}(X,\mathbb{F}_2)$ for all n.

DEFINITION 2.0.4. Let $I = (i_1, ..., i_n)$, and write Sq^I for the monomial $Sq^{i_1}Sq^{i_2}\cdots Sq^{i_n}$. Then Sq^I is *admissible* if $i_j \ge 2i_{j+1}$ for all $0 \le j < n$.

The admissible monomials are exactly those to which no Adem relations can be applied. Thus, $\mathscr{A}_2$ is generated as an $\mathbb{F}_2$ module by admissible monomials.

Finally, recall that an Eilenberg-MacLane space $K(G, n)$ (for G an abelian group and $n \in \mathbb{Z}_{\ge 0}$) is a space for which $\pi_i(K(G, n)) = G$ if $i = n$ and 0 otherwise. The following result says the cohomology of an Eilenberg-MacLane space $K(\mathbb{F}_2, n)$ is free over $\mathscr{A}_2$ below dimension $2n$.

PROPOSITION 2.0.5 ([14], Cor 7.5.6). The homomorphism

$$\begin{aligned} \mathscr{A}_2[n] &\to \widetilde{H}^*(K(\mathbb{F}_2, n);\mathbb{F}_2) \\ Sq^I[n] &\mapsto Sq^I(u_n), \end{aligned}$$

where $[n]$ denotes a shift in degree by n, is an isomorphism in degrees $* \le 2n$.

Remark 2.0.6. The above result combined with the wedge axiom implies that for a fixed n, the cohomology of a wedge $\bigvee_i K(\mathbb{F}_2, n_i)$ (with each $n_i \geq n$ and only finitely many n_i in each dimension) is also free over $\mathscr{A}_2$ below dimension $2n$.

Thus, though no space X can have cohomology a free $\mathscr{A}_2$-module (since for any $\alpha \in H^*(X; \mathbb{F}_2)$ of degree n, $Sq^{n+1}(\alpha) = 0$ by Proposition 2.0.1 (5)), the cohomology of an Eilenberg-MacLane space is free in sufficiently low degree. This fact will be vital to the construction of the Adams spectral sequence in Section 4.4.

3 If spectra aren't your friends, I can introduce you

In this section, we introduce spectra, the stable analogue of CW complexes. These objects will turn out to have particularly nice properties; for example, in addition to many of the properties of CW complexes carrying over to spectra, we will see that the collection of maps between two spectra up to homotopy has a natural abelian group structure, and that the suspension map between these groups is an isomorphism. Sections 3.1-3.4 closely follow Section 5.2 of [8], developing some basic properties of spectra which will be used in 4.4, while in Section 3.5 we note some facts about p-completion which will be needed later.

3.1 Definitions and examples

Definition 3.1.1. A *spectrum* is a collection of pointed topological spaces $\{X_n\}_{n\in\mathbb{N}}$, together with basepoint-preserving maps $\sigma_n : \Sigma X_n \to X_{n+1}$.

Example 3.1.2. Let X be a topological space. The *suspension spectrum* of X, denoted by $\Sigma^\infty X$, has $X_n = \Sigma^n X$ and $\sigma_n = \mathrm{id} : \Sigma X_n \to X_{n+1}$.

We write $\mathbb{S}$ for the suspension spectrum $\Sigma^\infty S^0$, and call $\mathbb{S}$ the *sphere spectrum*. For $i > 0$, we write $\mathbb{S}^i$ for $\Sigma^\infty S^i$.

Example 3.1.3. An *Eilenberg-MacLane spectrum* $\mathbb{K}(G, m)$ has $(\mathbb{K}(G, m))_n$ a CW complex $K(G, m+n)$, and can be constructed inductively by attaching cells to $\Sigma K(G, m+n))$ to kill $\pi_i(\Sigma K(G, m+n))$ for $i > m+n+1$. By Theorem 1.0.1, $\pi_i(K(G, m+n)) \cong \pi_{i+1}(\Sigma K(G, m+n))$ for $i \leq 2m+2n-2$, so the cells attached can be taken to have dimension $\geq 2m+2n-1$. The maps σ_n are inclusions of subcomplexes.

Definition 3.1.4. Let $X = \{X_n\}$ be a spectrum. We define $\pi_i(X) = \mathrm{colim}_n\, \pi_{i+n}(X_n)$, where the map $\pi_{i+n}(X_n) \to \pi_{i+n+1}(X_{n+1})$ is given by the composition

$$\pi_{i+n}(X_n) \xrightarrow{\Sigma} \pi_{i+n+1}(\Sigma X_n) \xrightarrow{(\sigma_n)_*} \pi_{i+n+1}(X_{n+1}).$$

Example 3.1.5. If X is a topological space, then $\pi_i(\Sigma^\infty X) = \pi_i^s(X)$, the ith stable homotopy group of X.

Definition 3.1.6. A CW spectrum is a spectrum X consisting of CW complexes X_n with the maps $\Sigma X_n \hookrightarrow X_{n+1}$ inclusions of subcomplexes.

Definition 3.1.7. Let X be a CW spectrum. Then the *k-cells* of X are the equivalence classes of non-basepoint $(k+n)$-cells in X_n, where two cells are equivalent if one is an m-fold suspension of the other, for some $m > 0$.

We will say that a CW spectrum X is *connective* if it has no cells below a given dimension, *finite* if it has only finitely many cells, and *of finite type* if it has only finitely many cells in each dimension. The following two classes of examples will be particularly important for us.

EXAMPLE 3.1.8. If X is a finite (resp. finite type) CW complex, then $\Sigma^\infty X$ is a finite (resp. finite type) CW spectrum. In particular, $\mathbb{S}$ is a finite CW spectrum with a unique cell in dimension 0.

EXAMPLE 3.1.9. For each m, the Eilenberg-MacLane spectrum $\mathbb{K}(G,m)$ constructed in Example 3.1.3 has finite type. This follows from the fact that the dimensions of the cells added to $\Sigma K(G, n+m)$ are eventually all larger than $n+i$ for any i, so $\mathbb{K}(G,m)$ only has finitely many i-cells.

LEMMA 3.1.10. Let X be a connective spectrum of finite type. Then the groups $\pi_{i+n}(X_n)$ eventually stabilise; i.e. the maps $\pi_{i+n}(X_n) \xrightarrow{(\sigma_n)_* \circ \Sigma} \pi_{i+n+1}(X_{n+1})$ are isomorphisms for large enough n.

PROOF. Recall that whenever $(X_{n+1}, \Sigma X_n)$ are such that $X_{n+1} \setminus \Sigma X_n$ has no cells in dimension $\leq k$, the map $\pi_i(\Sigma X_n) \to \pi_i(X_{n+1})$ induced by the inclusion is an isomorphism for $i < k$ (this follows from cellular approximation and the long exact sequence of relative homotopy groups). Thus, if $(\sigma_n)_* : \pi_{i+n+1}(\Sigma X_n) \to \pi_{i+n+1}(X_{n+1})$ never stabilises, there must be infinitely many natural numbers N_j such that $(X_{N_j+1}, \Sigma X_{N_j})$ is not $(i+N_j+1)$-connected, and thus that $X_{N_j+1} \setminus \Sigma X_{N_j}$ has cells of dimension $\leq i + N_j + 2$. By connectivity, there is some fixed l such that these cells are of dimension $N_j + k + 1$ for $-l \leq k \leq i+1$. Thus, there must be some k such that infinitely many of the X_{N_j+1} have a $(k + N_j + 1)$-cell not included in ΣX_{N_j}. This then contradicts the assumption that X is of finite type, since it has infinitely many k-cells.

Thus, the maps $(\sigma_n)_* : \pi_{i+n+1}(\Sigma X_n) \to \pi_{i+n+1}(X_{n+1})$ are eventually isomorphisms. The argument above also shows that in the stable range, the maps $(\Sigma^j \sigma_n)_* : \pi_{i+n+1}(\Sigma^{j+1} X_n) \to \pi_{i+n+1}(\Sigma^j X_{n+1})$ are also isomorphisms for $j > 0$, since $\Sigma^j X_{n+1} \setminus \Sigma^{j+1} X_n$ has no cells in dimension $\leq k$ whenever $X_{n+1} \setminus \Sigma X_n$ doesn't.

Now, fix $m > 0$ such that $(\sigma_n)_*$ is an isomorphism for $n \geq m$. By Theorem Theorem 1.0.1, there exists some $m' > 0$ such that $\Sigma : \pi_{i+m+n'}(\Sigma^{n'} X_m) \to \pi_{i+m+n'+1}(\Sigma^{n'+1} X_m)$ is an isomorphism for all $n' \geq m'$. Consider the commutative diagram below.

$$\begin{tikzcd}
\pi_{i+m+n'}(\Sigma^{n'} X_m) \arrow[r, "(\Sigma^{n'-1}\sigma_m)_*"] \arrow[d, "\Sigma", "\cong"'] & \pi_{i+m+n'}(\Sigma^{n'-1} X_{m+1}) \arrow[r, "(\Sigma^{n'-2}\sigma_{m+1})_*"] \arrow[d, "\Sigma"] & \cdots \arrow[r, "(\sigma_{m+n'})_*"] & \pi_{i+m+n'}(X_{m+n'}) \arrow[d, "\Sigma"] \\
\pi_{i+m+n'+1}(\Sigma^{n'+1} X_m) \arrow[r, "(\Sigma^{n'}\sigma_m)_*"'] & \pi_{i+m+n'}(\Sigma^{n'} X_{m+1}) \arrow[r, "(\Sigma^{n'-1}\sigma_{m+1})_*"'] & \cdots \arrow[r, "(\Sigma\sigma_{m+n'-1})_*"'] & \pi_{i+m+n'+1}(\Sigma X_{m+n'})
\end{tikzcd}$$

The horizontal maps are all isomorphisms by the previous paragraphs, and the leftmost vertical map is an isomorphism by assumption, so the rightmost vertical map must also be an isomorphism for all $n' \geq m'$. Therefore, for sufficiently large n, the composite $\pi_{i+n}(X_n) \xrightarrow{\Sigma} \pi_{i+n+1}(\Sigma X_n) \xrightarrow{(\sigma_n)_*} \pi_{i+n+1}(X_{n+1})$ is an isomorphism, as required. $\square$

3.2 Homology and cohomology

We now move on to define $H^*(X)$ and $H_*(X)$ for any CW spectrum $X = \{X_n\}$.

Recall that $C_i^{\mathrm{cell}}(X_n;G)$ has a G-summand for every i-cell of X_n. We have an injection

$$\begin{aligned} C_i^{\mathrm{cell}}(X_n;G) &\to C_{i+1}^{\mathrm{cell}}(\Sigma X_n;G) \\ e_\alpha^i &\mapsto \Sigma e_\alpha^i, \end{aligned}$$

and an injection $C_{i+1}^{\mathrm{cell}}(\Sigma X_n;G) \to C_{i+1}^{\mathrm{cell}}(X_{n+1};G)$ induced by the structure map σ_n, so we get an injection $C_i^{\mathrm{cell}}(X_n;G) \hookrightarrow C_{i+1}^{\mathrm{cell}}(X_{n+1};G)$.

We define

$$C_n(X;G) := \bigcup_{i\in\mathbb{Z}} C_{i+n}^{\mathrm{cell}}(X_i;G).$$

Note that there is a G summand for every $i+n$ cell of X_i up to treating suspensions of cells as equivalent to the cells themselves, i.e. a G summand for every n-cell of X. We define $H^*(X;G)$ and $H_*(X;G)$ to be the cohomology and homology of this chain complex, respectively.

Lemma 3.2.1. Let X be a connective CW spectrum of finite type. Then $H_i(X;G)$, $H^i(X;G)$, and $\pi_i(X)$ are finitely generated for all i.

Proof. First, note that $H_i(X;G) = H_{i+n}(X_n;G)$ for sufficiently large n, since for large enough n, X_n contains all the cells of X of dimension $\leq i$ (which are the $(i+n)$-cells of X_n). Similarly, $H^i(X;G) = H^{i+n}(X_n;G)$ for sufficiently large n. Each $H_{i+n}(X_n;G)$ is finitely generated, since X_n has only finitely many cells in each dimension, and thus each $H^{i+n}(X_n;G)$ is also finitely generated (see [7] Cor 3.3). Therefore, $H_i(X;G)$ and $H^i(X;G)$ are finitely generated.

Now, $\pi_i(X) = \operatorname{colim}_n \pi_{i+n}(X_n)$, and the groups $\pi_{i+n}(X_n)$ stabilise by Lemma 3.1.10. The X_n must eventually be simply-connected, since X is connective. A simply-connected space has finitely generated homotopy groups if and only if it has finitely generated homology groups (see e.g. [7], Thm 5.7), and we have just seen that the $H_{i+n}(X_n;G)$ are finitely generated, so $\pi_i(X) = \pi_{i+n}(X_n)$ is finitely generated. $\square$

Example 3.2.2. Recall that $\mathbb{S}$ is a finite spectrum. We thus have

$$\begin{aligned} H^i(\mathbb{S};\mathbb{F}_2) &= \lim_n H^{i+n}(S^n;\mathbb{F}_2) \\ &= \begin{cases} \mathbb{F}_2 & i=0, \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

Next, we define the notions of subspectra and maps between spectra.

Definition 3.2.3. Let $X = \{X_n\}$ be a CW spectrum. A *subspectrum* X' of X is a sequence of subcomplexes $\{X'_n \subseteq X_n\}$ satisfying $\Sigma X'_n \subseteq X'_{n+1}$. The subspectrum X' is *cofinal* if, for each n and each cell e_α^i of X_n, the cell $\Sigma^k e_\alpha^i$ belongs to X'_{n+k} for all sufficiently large k.

Note that if $\Sigma^k e_\alpha^i$ belongs to X'_{n+k} then $\Sigma^{k+1} e_\alpha^i$ belongs to $\Sigma X'_{n+k} \subseteq X'_{n+k+1} \subseteq X'_{k+k+2} \subseteq \cdots$. Thus, if X', X'' are cofinal spectra of X with $\Sigma^k e_\alpha^i$ a cell of X'_{n+k} and $\Sigma^l e_\alpha^i$ a cell of X''_{n+l} (with $l \geq k$) then $\Sigma^l e_\alpha^i$ is a cell of X'_{n+l} and therefore of $X'_{n+l} \cap X''_{n+l}$. In other words, the intersection of two cofinal spectra is a cofinal spectrum.

Definition 3.2.4. Let X, Y be CW spectra. A *strict map* $f : X \to Y$ is a sequence of cellular maps $f_n : X_n \to Y_n$ such that the diagram below commutes.

$$\begin{tikzcd} \Sigma X_n \arrow[r, "\sigma_n"] \arrow[d, "\Sigma f_n"'] & X_{n+1} \arrow[d, "f_{n+1}"] \\ \Sigma Y_n \arrow[r, "\sigma_n"'] & Y_{n+1} \end{tikzcd}$$

Taking strict maps to be our notion of maps between spectra, however, turns out to be too strong a requirement. For instance, a strict map $\mathbb{S}^i \to \Sigma^\infty X$ would be given simply by a map $S^i \to X$, whereas if we want to know about the stable homotopy groups of X, we should also consider maps $S^{i+n} \to \Sigma^n X$ which cannot necessarily be desuspended. We will therefore relax the definition of maps between spectra to include maps that are 'defined eventually', in the following sense.

Definition 3.2.5. A *map of CW spectra* $f : X \to Y$ is an equivalence class of strict maps $f' : X' \to Y$ with X' a cofinal subspectrum of X, where two strict maps $f' : X' \to Y$ and $f'' : X'' \to Y$ are equivalent if they agree on some common cofinal subspectrum.

Given two maps $f : X \to Y$, $g : Y \to Z$ represented by $f' : X' \to Y$, $g' : Y' \to Z$ respectively, we compose as follows: let X'' be the subspectrum of X', where the cells of X''_n consist of the cells of X'_n mapped to Y'_n under f'_n. Then, for any cell e^i_α of X'_n, $f'_n(e^i_\alpha)$ is contained in a finite union of cells of Y_n (since the image of a compact set is compact), whose k-fold suspension lies in Y'_{n+k} for large enough k. Since f' is a strict map, $\Sigma^k f'_n(e^i_\alpha) = f'_{n+k}\Sigma^k e^i_\alpha$, so $\Sigma^k e^i_\alpha$ is a cell of X''_{n+k}. Thus, X'' is cofinal in X' and hence in X. We define $gf := [X'' \xrightarrow{f'|_{X''}} Y' \xrightarrow{g'} Z]$, which is well-defined since the intersection of cofinal subspectra is again a cofinal subspectrum.

Since any strict map $f' : X' \to Y$ can be taken to be cellular, a map $f : X \to Y$ induces a well-defined map $C_*(X) \to C_*(Y)$ (by cofinality), and thus maps on homology and cohomology.

Further, any map $\mathbb{S}^i \to X$ can be represented by a map $S^{i+n} \to X_n$, which has compact image and thus is contained in a finite subcomplex $\overline{X}_n \subseteq X_n$. Given any map $f : X \to Y$ represented by a strict map $f' : X' \to Y$, the kth suspension of the cells of $\overline{X}_n$ lie in X'_{n+k}, and thus f induces a map $\pi_*(X) \to \pi_*(Y)$.

Definition 3.2.6. Two spectra X, Y are *equivalent* if there are maps $f : X \to Y$ and $g : Y \to X$ such that $fg = \mathrm{id}_Y$ and $gf = \mathrm{id}_X$.

Note that a spectrum is equivalent to any of its cofinal subspectra. In particular, if $X = \{X_n\}$ is a spectrum, then $X' = \{\Sigma X_{n-1}\}$ is a cofinal subspectrum of X (where we take X_{-1} to be the basepoint of X_0). We define $\Sigma^{-1} X := \{X_{n-1}\}$, noting that $\Sigma\Sigma^{-1} X = \Sigma^{-1}\Sigma = X' \simeq X$. Thus, a spectrum is always equivalent to the suspension of some other spectrum.

Recall that for $i > 0$, we write $\mathbb{S}^i$ for the spectrum $\Sigma^\infty S^i = \Sigma^i \mathbb{S}$. We have just seen that spectra can always be desuspended, so we will extend this notation and write $\mathbb{S}^i := \Sigma^i \mathbb{S}$ for any nonzero $i \in \mathbb{Z}$.

Definition 3.2.7. A *homotopy* of maps between spectra is a map $X \times I \to Y$, where $X \times I$ is the spectrum with $(X \times I)_n = X_n \times_{\mathrm{red}} I := (X_n \times I)/(x_0 \times I)$.

Note that $\Sigma(X_n \times_{\mathrm{red}} I) = \Sigma X_n \times_{\mathrm{red}} I$. The set of homotopy classes of maps $X \to Y$ is denoted by $[X, Y]$.

Remark 3.2.8. For any CW spectra Z, $[\mathbb{S}^t, Z] = \pi_t(Z)$.

For any CW spectra X, Y, the set $[X, Y]$ can the structure of an abelian group, since X has be written as a double suspension $\Sigma^2 X'$, and each set $[\Sigma^2 X'_n, Y_n]$ has the structure of an abelian group.

Theorem 3.2.9. The suspension map $[X, Y] \to [\Sigma X, \Sigma Y]$ is an isomorphism of groups.

Proof. The suspension map is a homomorphism, since it is a homomorphism on maps between CW complexes. Thus, it suffices to show it is a bijection on maps between spectra.

Recall that $\Sigma^{-1}\Sigma X = \Sigma\Sigma^{-1}X \simeq X$. For any map $f : X \to Y$ given by strict maps $f_n : X'_n \to Y_n$, define $\Sigma^{-1}f : \Sigma^{-1}X \to \Sigma^{-1}Y$ by $\{f_{n-1} : X'_{n-1} \to Y_{n-1}\}$. Then $\Sigma\Sigma^{-1}f = \{\Sigma f_{n-1}\} = \{f_n|_{\Sigma X_{n-1}}\} = f$, and similarly $\Sigma^{-1}\Sigma f = f$. Thus, we have bijections $[X, Y] \cong [\Sigma\Sigma^{-1}X, \Sigma\Sigma^{-1}Y] \cong [\Sigma^{-1}\Sigma X, \Sigma^{-1}\Sigma Y]$, so Σ has a two-sided inverse. $\square$

3.3 Cofibration sequences

Definition 3.3.1. Let $X = \{X_n\}, Y = \{Y_n\}$ be spectra. Then their *wedge sum* is $X \vee Y := \{X_n \vee Y_n\}$. Note that we have an inclusion $\Sigma(X_n \vee Y_n) \hookrightarrow X_{n+1} \vee Y_{n+1}$.

Definition 3.3.2. Let $f : X \to Y$ be a map of CW spectra, and let $f' : X' \to Y$ be a representative for f, where $X' \subseteq X$ is cofinal. The *mapping cylinder* M_f has components $(M_f)_n = M_{f'_n}$, where $M_{f'_n}$ is the reduced mapping cylinder of f'_n. It is independent of the choice of X' up to equivalence.

Remark 3.3.3. Given any map $f : X \to Y$ of CW spectra, we have a deformation retraction of M_f onto Y. Since we will only be interested in spectra up to homotopy equivalence, by replacing Y by M_f we may assume any map $f : X \to Y$ is an inclusion.

Definition 3.3.4. Let X be a CW spectrum, $A \subseteq X$ a subspectrum. Then A is *closed* in X if for every cell e^n_α of X_n, if $\Sigma^k e^n_\alpha \in A_{n+k}$ then $e^n_\alpha \in A_n$.

Any subspectrum is cofinal in (and thus equivalent to) its closure. We define X/A to be the CW spectrum with $(X/A)_n = X_n/A'_n$, where $A' = \{A'_n\}$ is the closure of A. Note that a quotient of connective spectra of finite type is again a connective spectrum of finite type (since the quotient has fewer cells in each dimension than the original space).

The map $X \cup CA \to X/A$ is a homotopy equivalence of spectra, since each quotient $X_n \cup CA_n \to X_n/A_n$ is, so we have a cofibration sequence

$$A \hookrightarrow X \to X \cup CA \to \Sigma A \hookrightarrow \Sigma X \to \cdots.$$

Theorem 3.3.5. Let X, Y be spectra, and $A \subseteq X$ a subspectrum. Then there is an exact sequence

$$[Y, A] \to [Y, X] \to [Y, X/A] \to [Y, \Sigma A] \to [Y, \Sigma X] \to \cdots.$$

Proof. It suffices to show that

$$[Y, A] \to [Y, X] \to [Y, X/A]$$

is exact.

The composition $[Y, A] \to [Y, X] \to [Y, X/A]$ is clearly zero. Suppose $Y \xrightarrow{f} X \to X \cup CA$ is homotopic to the constant map. Then we have a map $h : CY \to X \cup CA$ making the solid diagram below commute.

$$\begin{tikzcd} Y \arrow[r, "\mathrm{id}"] \arrow[d, dashed] & Y \arrow[r, hook] \arrow[d, "f"] & CY \arrow[r] \arrow[d, "h"] & \Sigma Y \arrow[r, "-\mathrm{id}"] \arrow[d, dashed] & \Sigma Y \arrow[d, dashed, "\Sigma f"] \\ A \arrow[r, hook, "i"'] & X \arrow[r, hook] & X \cup CA \arrow[r] & \Sigma A \arrow[r, hook, "-\Sigma i"'] & \Sigma X \end{tikzcd} \tag{2}$$

We claim that we can fill in the two dotted maps on the right to make homotopy commutative squares. To see this, consider the diagram below,

$$\begin{tikzcd} Y \arrow[r, "\mathrm{id}"] \arrow[d, dashed] & Y \arrow[r, hook] \arrow[d, "f"] & Y \cup CY \arrow[r, hook] \arrow[d, "h"] & (Y \cup CY) \cup CY \arrow[r, hook] \arrow[d, "h \cup Cf"] & ((Y \cup CY) \cup CY) \cup C(Y \cup CY) \arrow[d, "(h \cup Cf) \cup Ch"] \\ A \arrow[r, hook, "i"'] & X \arrow[r, hook] & X \cup CA \arrow[r, hook] & (X \cup CA) \cup CX \arrow[r, hook] & ((X \cup CA) \cup CX) \cup C(X \cup CA) \end{tikzcd}$$

where $h \cup Cf$ is given by applying h to $Y \cup CY$ and Cf to CY (which is well-defined since the maps agree on the intersection $Y \times \{0\}$), and likewise for $(h \cup Cf) \cup Ch$. Now, the square below commutes, since the identification $((Y \cup CY) \cup \textcolor{red}{CY}) \cup C(Y \cup CY) \simeq \Sigma Y$ collapses everything except the factor in red (whose base is collapsed), and similarly for X.

$$\begin{tikzcd} ((Y \cup CY) \cup CY) \cup C(Y \cup CY) \arrow[r, two heads, "\simeq"] \arrow[d, "(h \cup Cf) \cup Ch"'] & \Sigma Y \arrow[d, "\Sigma f"] \\ ((X \cup CA) \cup CX) \cup C(X \cup CA) \arrow[r, two heads, "\simeq"'] & \Sigma X \end{tikzcd}$$

Now, let $p_Y : (Y \cup CY) \cup CY \to \Sigma Y$ and $p_A : (X \cup CA) \cup CX \to \Sigma A$ be the projections, with homotopy inverses h_Y and h_A respectively. We define $g : \Sigma Y \to \Sigma A$ by $g := p_A \circ (h \cup Cf) \circ h_Y$. This g makes the diagram in (2) commute up to homotopy, where the minus signs arise from the fact that opposite hemispheres of the spaces $((Y \cup CY) \cup CY) \cup C(Y \cup CY)$ and $(Y \cup CY) \cup CY$ are collapsed under the quotient map (and likewise for the bottom row).

By Theorem 3.2.9, we can take the map $g : \Sigma Y \to \Sigma A$ to be Σk for some $k : Y \to A$. Then $(\Sigma f) \circ (-\mathrm{id}) \simeq (-\Sigma i)(\Sigma k)$, so $\Sigma f \simeq \Sigma(ik)$, and thus $f \simeq ik$ as required. $\square$

Finally, we get the lemma below, which follows from the equivalent result for CW complexes.

Lemma 3.3.6. Let $A \overset{f}{\hookrightarrow} X \xrightarrow{i} C_f \xrightarrow{j} \Sigma A \to \cdots$ be a cofibration, where X, A are CW spectra of finite type. Then there is a long exact sequence

$$\cdots \leftarrow H^{n-1}(\Sigma A) \leftarrow H^n(X) \xleftarrow{i^*} H^n(C_f) \xleftarrow{j^*} H^n(\Sigma A) \leftarrow H^{n+1}(X) \leftarrow \cdots .$$

3.4 Eilenberg-MacLane spectra

In this section, we briefly record two important facts about Eilenberg-MacLane spectra which will be used later in the construction of the Adams spectral sequence. The first is the analogue of the representability of cohomology by Eilenberg-MacLane spaces for CW complexes, and can be proven similarly.

Theorem 3.4.1 ([8], Prop 5.45). There are natural isomorphisms $H^m(X; G) \cong [X, \mathbb{K}(G, m)]$ for all CW spectra.

Now, recall that giving a map into a product is equivalent to giving a map into each of its components. We have maps $F_i : [X, \bigvee_i \mathbb{K}(G, n_i)] \to [X, \mathbb{K}(G, n_i)]$ given by composition with the projections, giving a map $F : [X, \bigvee_i \mathbb{K}(G, n_i)] \to \prod_i [X, \mathbb{K}(G, n_i)]$.

Proposition 3.4.2 ([8], Prop 5.46). *The map $F : [X, \bigvee_i \mathbb{K}(G, n_i)] \to \prod_i [X, \mathbb{K}(G, n_i)]$ described above is an isomorphism if X is a connective spectrum of finite type and $n_i \to \infty$ as $i \to \infty$.*

3.5 p-completion of spectra

Definition 3.5.1 ([10], Def 10.1.1). Let A be an abelian group. Then its *p-adic completion* is the limit

$$A_p^\wedge = \lim_{\leftarrow n} (A/p^n A).$$

If $A = \mathbb{Z}$, we instead write $\mathbb{Z}_p := \mathbb{Z}_p^\wedge$ for the p-adic integers. There is a natural map $A \to A_p^\wedge$, whose component at n is reduction modulo $p^n A$.

When A is finitely generated, its p-adic completion is given by the map $A \to A \otimes \mathbb{Z}_p$; $a \mapsto a \otimes 1$.

Remark 3.5.2. Suppose A is finite, and write $|A| = np^r$ for $p \not| n$. Then $A_p^\wedge \cong A/T$, where $T \subseteq A$ is the subgroup generated by all torsion coprime to p, since $A/p^k A \cong A/T$ for all $k \geq r$.

Remark 3.5.3. If A is finite with order np^r for $p \not| n$, then $|A_p^\wedge| = p^r$, by Cauchy's theorem.

Definition 3.5.4 ([9], p129). Let X be a CW spectrum. Then a *p-completion* of X is a map $f : X \to X_p^\wedge$ such that for all i, $\pi_i f$ expresses $\pi_i(X_p^\wedge)$ as the p-completion of $\pi_i(X)$.

Throughout the remainder of this essay we will largely be concerned with connective spectra of finite type, for which we have the following results.

Theorem 3.5.5 ([9], Thm 9.1.1). *If X has finite type, then it has a p-completion unique up to equivalence.*

Theorem 3.5.6 ([9], Prop 9.2.22). *Let X be a connective spectrum of finite type, and let Y be p-complete. Then the map $[X_p^\wedge, Y] \to [X, Y]$ is an isomorphism.*

Equivalently, we have the following universal property: given any map $X \xrightarrow{f} Y$, there exists a unique (up to homotopy) map $X_p^\wedge \xrightarrow{\overline{f}} Y$ such that f factors as $X \to X_p^\wedge \xrightarrow{\overline{f}} Y$.

Note that this property holds for abelian groups A and B, with $B = B_p^\wedge$, where the component $\overline{f}_n : A_p^\wedge \to B/p^n B$ is defined by projecting onto $A/p^n A$ and composing with the map induced by f (uniqueness follows from the fact that group homomorphisms are continuous with respect to the p-adic topology, and the image of A is dense in $A_p^\wedge$). We have already seen that maps between spectra always form an abelian group, and the theorems above thus give another way in which (under certain hypotheses) spectra behave like abelian groups.

4 The Adams spectral sequence

We will now turn our attention to the main tool for computing stable homotopy groups of spheres - the Adams spectral sequence. Sections 4.1 and 4.2 give a very brief explanation of spectral sequences and one important way in which they arise, following Chapter 2 of [12]. Section 4.3 then covers some homological algebra which will be needed later, and is drawn mostly from [16]. In Section 4.4 we will finally construct the Adams spectral sequence, and in Section 4.5 we make some initial computations of the stable homotopy groups using the tools we have developed; both of these sections mostly follow [8]. Finally, Section 4.6 will explore further structure which can be put on the Adams spectral sequence, which will help narrow down possibilities for the stable homotopy groups; we will follow [15] and [14].

From this section onwards, all homology and cohomology will be taken with $\mathbb{F}_2$ coefficients, and we will thus ease notation by writing $H^*(X)$ (resp. $H_*(X)$) for $H^*(X;\mathbb{F}_2)$ (resp. $H_*(X;\mathbb{F}_2)$).

4.1 Spectral sequences

Definition 4.1.1. A *differential bigraded module* E over a ring R is a collection of R-modules $\{E^{p,q}\}$, $p,q \in \mathbb{Z}$, together with a map $d : E^{p,q} \to E^{p+s,q+r}$ for each p,q and some fixed $s,r \in \mathbb{Z}$, satisfying $d^2 = 0$.

We can take the homology of (E,d):

$$H^{p,q}(E^{*,*},d) = \ker(d : E^{p,q} \to E^{p+s,q+r}) / \operatorname{im}(d : E^{p-s,q-r} \to E^{p,q}).$$

In the definition of a spectral sequence below, we will specialise to a specific bigrading, the *Adams grading*, since it is the one we will encounter in Section 4.4. There are, however, different gradings which correspond to different types of spectral sequences, the most common of which are (co)homological spectral sequences.

Definition 4.1.2. A *spectral sequence* (of *Adams type*) is a collection of differential bigraded R-modules $\{E_r^{*,*}, d_r\}, r \in \mathbb{N}$, with the differentials d_r of bidegree $(r, r-1)$. These satisfy the further condition that for all p,q,r, $E_{r+1}^{p,q} \cong H^{p,q}(E_r^{*,*}, d_r)$.

Consider the term E_2. Define

$$Z_2 := \ker d_2 \quad \text{and} \quad B_2 := \operatorname{im} d_2.$$

The condition $d^2 = 0$ implies that $B_2 \subseteq Z_2 \subseteq E_2$, and by definition we have $E_3 \cong Z_2/B_2$.

Now, write

$$Z_3 := \ker d_3 \quad \text{and} \quad B_3 := \operatorname{im} d_3.$$

Since $Z_3 \subseteq E_3$, it can be written as $\overline{Z}_3/B_2$ for some $\overline{Z}_3 \subseteq Z_2$. Similarly, $B_3 \cong \overline{B}_3/B_2$ for some $\overline{B}_3 \subseteq Z_2$. Thus,

$$E_4 \cong Z_3/B_3 \cong \frac{\overline{Z}_2/B_2}{\overline{B}_3/B_2} \cong \overline{Z}_3/\overline{B}_3.$$

Iterating this process, we present the spectral sequence as an infinite tower of submodules of E_2:

$$B_2 \subseteq \overline{B}_3 \subseteq \cdots \subseteq \overline{B}_n \subseteq \cdots \subseteq \overline{Z}_n \subseteq \cdots \subseteq \overline{Z}_3 \subseteq Z_2 \subseteq E_2,$$

with the property that $E_{n+1} \cong \overline{Z}_n/\overline{B}_n$. The differential d_{n+1} can be taken as a map $\overline{Z}_n/\overline{B}_n \to \overline{Z}_n/\overline{B}_n$ with kernel $\overline{Z}_{n+1}/\overline{B}_n$ and image $\overline{B}_{n+1}/\overline{B}_n$. The short exact sequence induced by d_{n+1},

$$0 \to \overline{Z}_{n+1}/\overline{B}_n \to \overline{Z}_n/\overline{B}_n \xrightarrow{d_{n+1}} \overline{B}_{n+1}/\overline{B}_n \to 0,$$

gives rise to isomorphisms $\overline{Z}_n/\overline{Z}_{n+1} \cong \overline{B}_{n+1}/\overline{B}_n$ for all n. Conversely, a tower of submodules of E_2, together with a set of isomorphisms, gives rise to a spectral sequence.

Definition 4.1.3. An element of E_2 *survives to the rth stage* if lies in $\overline{Z}_r$, having been in the kernel of the previous $r-2$ differentials, and is *bounded by the rth stage* if it lies in $\overline{B}_r$. The bigraded module $E_r^{*,*}$ is called the *E_r-term* of the spectral sequence.

We define

$$Z_\infty := \bigcap_n \overline{Z}_n, \quad B_\infty := \bigcup_n \overline{B}_n.$$

From the tower of inclusions, we see that $B_\infty \subseteq Z_\infty$, so we define $E_\infty := Z_\infty/B_\infty$.

Definition 4.1.4. A spectral sequence *collapses at the Nth term* if the differentials $d_r = 0$ for $r \geq N$.

From the short exact sequence

$$0 \to \overline{Z}_r/\overline{B}_{r-1} \to \overline{Z}_{r-1}/\overline{B}_{r-1} \xrightarrow{d_r} \overline{B}_r/\overline{B}_{r-1} \to 0,$$

the condition $d_r = 0$ forces $\overline{Z}_r = \overline{Z}_{r-1}$ and $\overline{B}_r = \overline{B}_{r-1}$. The tower of submodules becomes

$$B_2 \subseteq \overline{B}_3 \subseteq \cdots \subseteq \overline{B}_{N-1} = \overline{B}_N = \cdots = B_\infty \subseteq Z_\infty = \cdots = \overline{Z}_N = \overline{Z}_{N-1} \subseteq \cdots \subseteq \overline{Z}_3 \subseteq Z_2 \subseteq E_2.$$

Thus, $E_\infty = E_N$.

Let M^* be a graded R-module, and suppose M^* has a filtration

$$\cdots \subseteq F^{n+1}M^* \subseteq F^n M^* \subseteq F^{n-1}M^* \subseteq \cdots \subseteq M^*.$$

We define the *associated graded* $E_0^{*,*}(M^*, F)$ of M to be the bigraded module whose degree (p,q) summand is

$$E_0^{p,q}(M) = F^p M^{p+q}/F^{p+1}M^{p+q}.$$

Definition 4.1.5. A spectral sequence $\{E_r^{*,*}\}$ *converges* to a graded R-module M^* if there is a filtration F on M^* such that the following conditions hold:

1. $\bigcup_n F^n M^* = M^*$.
2. $\bigcap_n F^n M^* = \{0\}$.
3. $E_\infty^{p,q} \cong E_0^{p,q}(M^*, F)$.

4.2 Exact couples

We have seen that certain towers of submodules give rise to spectral sequences; now, we describe a different structure from which a spectral sequence can be obtained, which will in fact be how the Adams spectral sequence arises.

Definition 4.2.1. Let D, E be R-modules, and let $i : D \to D$, $j : D \to E$, $k : E \to D$ be module homomorphisms. We call $\mathcal{C} = \{D, E, i, j, k\}$ an *exact couple* if the diagram below is exact.

$$\begin{tikzcd} D \arrow[rr, "i"] & & D \arrow[dl, "j"] \\ & E \arrow[ul, "k"] & \end{tikzcd}$$

Let $d := jk$, and define the following:

$$
\begin{aligned}
E' &:= H(E, d) = \ker d / \operatorname{im} d \\
D' &:= i(D) = \ker j \\
i' &:= i|_{i(D)} : D' \to D' \\
j' &:= i(x) \mapsto j(x) + dE : D' \to E' \\
k' &:= (e + dE) \mapsto k(e) : E' \to D'
\end{aligned}
$$

We call $\mathcal{C}' = \{D', E'.i', j', k'\}$ the *derived couple* of $\mathcal{C}$.

PROPOSITION 4.2.2 ([12], Prop 2.7). If $\mathcal{C} = \{D, E, i, j, k\}$ is an exact couple, then $\mathcal{C}'$ is also an exact couple.

The theorem below is proven in [12] when i has bidegree $(-1, 1)$ and the resulting spectral sequence has differentials of bidegree $(r, 1 - r)$, however an almost identical proof works in the case of the Adams grading.

THEOREM 4.2.3 ([12], Thm 2.8). Suppose $D^{*,*} = \{D^{p,q}\}$ and $E^{*,*} = \{E^{p,q}\}$ are bigraded modules equipped with homomorphisms i of bidegree $(-1, -1)$, j of bidegree $(0, 0)$, and k of bidegree $(1, 0)$, such that $\{D^{*,*}, E^{*,*}, i, j, k\}$ is an exact couple. Then these data determine a spectral sequence $\{E_r, d_r\}$ for $r \in \mathbb{Z}_+$ of Adams type, with $E_r = (E^{*,*})^{(r-1)}$, the $(r-1)$st derived module of $E^{*,*}$ and $d_r = j^{(r-1)} \circ k^{(r-1)}$.

Such a bigraded exact couple may be displayed in the diagram below, known as a *staircase diagram.*

$$
\begin{tikzcd}
& \vdots \arrow[d, "i"] & & \vdots \arrow[d, "i"] & \\
\cdots \arrow[r, "k"] & D^{p+1,q+1} \arrow[r, "j"] \arrow[d, "i"] & E^{p+1,q+1} \arrow[r, "k"] & D^{p+2,q+1} \arrow[r, "j"] \arrow[d, "i"] & \cdots \\
\cdots \arrow[r, "k"] & D^{p,q} \arrow[r, "j"] \arrow[d, "i"] & E^{p,q} \arrow[r, "k"] & D^{p+1,q} \arrow[r, "j"] \arrow[d, "i"] & \cdots \\
\cdots \arrow[r, "k"] & D^{p-1,q-1} \arrow[r, "j"] \arrow[d, "i"] & E^{p-1,q-1} \arrow[r, "k"] & D^{p,q-1} \arrow[r, "j"] \arrow[d, "i"] & \cdots \\
& \vdots & & \vdots &
\end{tikzcd}
$$

4.3 The Ext functor

Before constructing the Adams spectral sequence, we briefly recall the definition of the Ext functor and some of its basic properties, which will be of importance later. We mainly follow [16].

DEFINITION 4.3.1. Let M, N be modules over a ring R. A *projective resolution* P of M is an exact sequence,

$$\cdots \to P_2 \to P_1 \to P_0 \to M \to 0.$$

where each P_i is projective. If, in addition, each P_i is free, then the resolution is called *free.*

Dually, an *injective resolution* I of M is a exact sequence

$$0 \to M \to I_0 \to I_1 \to I_2 \to \cdots ,$$

where each I_i is injective.

The following result can be obtained from [16], Lemmas 2.2.5, 2.3.6, and Exercise 2.3.5.

LEMMA 4.3.2. Every R-module M has a projective resolution and an injective resolution.

Given a projective resolution as in Definition 4.3.1, applying $\mathrm{Hom}_R(-, N)$ gives us a chain complex

$$\cdots \leftarrow \mathrm{Hom}_R(P_2, N) \leftarrow \mathrm{Hom}_R(P_1, N) \leftarrow \mathrm{Hom}_R(P_0, N) \leftarrow \mathrm{Hom}_R(M, N) \leftarrow 0.$$

Dropping the term $\mathrm{Hom}_R(M, N)$, we get the chain complex

$$\cdots \leftarrow \mathrm{Hom}_R(P_2, N) \leftarrow \mathrm{Hom}_R(P_1, N) \leftarrow \mathrm{Hom}_R(P_0, N) \leftarrow 0,$$

which we denote by $\mathrm{Hom}_R(P_\bullet, N)$.

Dually, given an injective resolution as in Definition 4.3.1, we can form the chain complex

$$\cdots \leftarrow \mathrm{Hom}_R(N, I_2) \leftarrow \mathrm{Hom}_R(N, I_1) \leftarrow \mathrm{Hom}_R(N, I_0) \leftarrow 0,$$

denoted by $\mathrm{Hom}_R(N, I_\bullet)$.

The result below is a combination of [16], Lemma 2.4.1 and Theorem 2.7.6.

PROPOSITION 4.3.3. Let M, N be R-modules. For any projective resolution P and any injective resolution I of M, $H^*(\mathrm{Hom}_R(P_\bullet, N)) = H^*(\mathrm{Hom}_R(N, I_\bullet))$.

We define $\mathrm{Ext}^n_R(M, N) := H^n(\mathrm{Hom}_R(P_\bullet, N)) = H^n(\mathrm{Hom}_R(N, I_\bullet))$.

4.4 Setting up the Adams spectral sequence

Let X be a connective CW spectrum of finite type. Then $H^*(X)$ is an $\mathscr{A}_2$-module, since $H^i(X) \cong H^{i+n}(X_n)$ for sufficiently large n, so we can define $Sq^j : H^i(X) \to H^{i+j}(X)$ by evaluating $Sq^j : H^{i+n}(X_n) \to H^{i+j+n}(X_n)$ followed by enough suspensions. Note that we could also have first suspended $H^{i+n}(X_n)$ and $H^{i+j+n}(X_n)$ until they were both stable, and then evaluated Sq^j, but that these two $\mathscr{A}_2$-actions coincide since the Steenrod squares commute with suspension isomorphisms (Proposition 2.0.1 (4)).

We can pick generators α_i for $H^*(X)$ as an $\mathscr{A}_2$-module such that there are at most finitely many in each $H^n(X)$ (since each $H^n(X)$ is finitely generated by Lemma 3.2.1, and such a finite generating set would certainly also generate it as an $\mathscr{A}_2$-module). Each generator $\alpha_i \in H^{n_i}(X)$ corresponds to a map $X \to \mathbb{K}(\mathbb{F}_2, n_i)$ by Theorem 3.4.1, so putting these maps together gives an element of $\prod_i [X, \mathbb{K}(\mathbb{F}_2, n_i)]$. Now, $n_i \to \infty$ as $i \to \infty$ since there are only finitely many α_i in each $H^{n_i}(X)$, so Proposition 3.4.2 implies that we get an element of $[X, \bigvee_i \mathbb{K}(\mathbb{F}_2, n_i)]$. We write $K_0 := \bigvee_i \mathbb{K}(\mathbb{F}_2, n_i)$, and replace the map $X \to K_0$ by an inclusion (see Remark 3.3.3).

REMARK 4.4.1. K_0 has finite type, which can be seen as follows: first, recall from Example Example 3.1.9 that each spectrum $\mathbb{K}(\mathbb{F}_2, n_i)$ has finite type. Now, the j-cells of

$\bigvee_i \mathbb{K}(\mathbb{F}_2, n_i)$ consist of the $(j+k)$-cells of $\bigvee_i K(\mathbb{F}_2, n_i+k)$ for each k, up to equivalence under suspension. However, there are only finitely many n_i with $n_i \leq j$, and if $n_i > j$ the space $K(\mathbb{F}_2, n_i)$ can be taken to have no cells of dimension $\leq j$. Thus, the j-cells of $\bigvee_i \mathbb{K}(\mathbb{F}_2, n_i)$ are the j-cells of the finite wedge $\bigvee_{i, n_i \leq j} \mathbb{K}(\mathbb{F}_2, n_i)$, of which there are only finitely many (since a finite wedge of finite-type spectra has finite type).

Now, we set $X_1 = K_0/X$, and repeat the construction to get a diagram:

$$\begin{tikzcd}
X \arrow[r] & K_0 \arrow[rr] \arrow[dr] & & K_1 \arrow[rr] \arrow[dr] & & K_2 \arrow[rr] \arrow[dr] & & \cdots \\
& & K_0/X = X_1 \arrow[ur] & & K_1/X_1 = X_2 \arrow[ur] & & K_2/X_2 = X_3 \arrow[ur] &
\end{tikzcd}$$

Taking cohomology, we get a diagram

$$\begin{tikzcd}
0 & H^*(X) \arrow[l] & H^*(K_0) \arrow[l] & & H^*(K_1) \arrow[ll] \arrow[dl] & & H^*(K_2) \arrow[ll] \arrow[dl] & & \cdots \arrow[ll] \arrow[dl] \\
& & & H^*(X_1) \arrow[ul] \arrow[dl] & & H^*(X_2) \arrow[ul] \arrow[dl] & & H^*(X_3) \arrow[ul] \arrow[dl] & \\
& & 0 & & 0 \arrow[ul] & & 0 \arrow[ul] & & 0 \arrow[ul]
\end{tikzcd} \tag{3}$$

The induced map $H^*(X) \leftarrow H^*(K_0)$ is surjective by construction, and thus each map $H^*(X_i) \leftarrow H^*(K_i)$ is surjective.

Now, as with CW complexes, we have a long exact sequence

$$\cdots \leftarrow H^{n+1}(X_{s+1}) \leftarrow H^n(X_s) \leftarrow H^n(K_s) \leftarrow H^n(X_{s+1}) \leftarrow H^{n-1}(X_s) \leftarrow \cdots,$$

and surjectivity of the maps $H^*(X_s) \leftarrow H^*(K_s)$ implies that the boundary maps $H^{n+1}(X_{s+1}) \leftarrow H^n(X_s)$ are all zero (writing $X_0 := X$). We thus get short exact sequences

$$0 \leftarrow H^n(X_s) \leftarrow H^n(K_s) \leftarrow H^n(X_{s+1}) \leftarrow 0,$$

giving rise to a short exact sequence

$$0 \leftarrow H^*(X_s) \leftarrow H^*(K_s) \leftarrow H^*(X_{s+1}) \leftarrow 0,$$

for each s. This then implies that the top row of (3) is exact.

Each $H^*(K_s)$ is a free $\mathscr{A}_2$-module, by Remark 2.0.6 and the fact that K_s has finite type. Thus, the top row of (3) gives a free resolution of $H^*(X)$.

By Theorem 3.3.5, we obtain a long exact sequence

$$\cdots \to [\mathbb{S}^{t+1}, X_s] \to [\mathbb{S}^{t+1}, K_s] \to [\mathbb{S}^{t+1}, X_{s+1}] \to [\mathbb{S}^{t+1}, \Sigma X_s] \to [\mathbb{S}^{t+1}, \Sigma K_s] \to \cdots.$$

Using the isomorphism $[Y, Z] \cong [\Sigma Y, \Sigma Z]$, we get long exact sequences

$$\cdots \to \pi_{t+1} X_s \to \pi_{t+1} K_s \to \pi_{t+1} X_{s+1} \to \pi_t X_s \to \pi_t K_s \to \cdots,$$

which form the staircase diagram shown below.

$$\begin{tikzcd}
 & \vdots \arrow[d, "i"] & & \vdots \arrow[d, "i"] & & \vdots \arrow[d, "i"] & \\
\cdots \arrow[r, "k"] & \pi_{t+1}X_s \arrow[r, "j"] \arrow[d, "i"] & \pi_{t+1}K_s \arrow[r, "k"] & \pi_{t+1}X_{s+1} \arrow[r, "j"] \arrow[d, "i"] & \pi_{t+1}K_{s+1} \arrow[r, "k"] & \pi_{t+1}X_{s+2} \arrow[r, "j"] \arrow[d, "i"] & \cdots \\
\cdots \arrow[r, "k"] & \pi_t X_{s-1} \arrow[r, "j"] \arrow[d, "i"] & \pi_t K_{s-1} \arrow[r, "k"] & \pi_t X_s \arrow[r, "j"] \arrow[d, "i"] & \pi_t K_s \arrow[r, "k"] & \pi_t X_{s+1} \arrow[r, "j"] \arrow[d, "i"] & \cdots \\
\cdots \arrow[r, "k"] & \pi_{t-1}X_{s-2} \arrow[r, "j"] \arrow[d, "i"] & \pi_{t-1}K_{s-2} \arrow[r, "k"] & \pi_{t-1}X_{s-1} \arrow[r, "j"] \arrow[d, "i"] & \pi_{t-1}K_{s-1} \arrow[r, "k"] & \pi_{t-1}X_s \arrow[r, "j"] \arrow[d, "i"] & \cdots \\
 & \vdots & & \vdots & & \vdots &
\end{tikzcd} \tag{4}$$

This gives rise to a spectral sequence, by Theorem 4.2.3.

Now, since $K_s = \bigvee_i \mathbb{K}(\mathbb{F}_2, n_{s_i})$, Proposition Proposition 3.4.2 tells us that $[\mathbb{S}, K_s] \cong \prod_i [\mathbb{S}, \mathbb{K}(\mathbb{F}_2, n_{s_i})]$, which is naturally isomorphic to $\prod_i H^{n_{s_i}}(\mathbb{S}; \mathbb{F}_2)$. Thus, elements of $[\mathbb{S}, K_s]$ are tuples of elements of $H^*(\mathbb{S})$.

We have a map

$$\begin{array}{ccc} [\mathbb{S}, K_s] & \to & \mathrm{Hom}^0_{\mathscr{A}_2}(H^*(K_s), H^*(\mathbb{S})) \\ f & \mapsto & f^*, \end{array}$$

since f^* is an $\mathscr{A}_2$-module homomorphism by Proposition 2.0.1 (1), and the fact that $H^*(K_s)$ is free implies that it is an isomorphism.

We thus have

$$[\Sigma^t \mathbb{S}, K_s] = \mathrm{Hom}^0_{\mathscr{A}_2}(H^*(K_s), H^*(\Sigma^t \mathbb{S})) = \mathrm{Hom}^t_{\mathscr{A}_2}(H^*(K_s), H^*(\mathbb{S})),$$

where $\mathrm{Hom}^t_{\mathscr{A}_2}(H^*(K_s), H^*(\mathbb{S}))$ is the set of $\mathscr{A}_2$-module homomorphisms which lower the degree by t. In the case of CW complexes, we have $H^*(\Sigma^t X) \cong H^{*-t}(X)$. Since $\mathbb{S}$ has finite type, for i large enough we have $H^n(\Sigma^t \mathbb{S}) = H^{n+i}(\Sigma^t S^i) \cong H^{n+i-t}(S^i) = H^{n-t}(\mathbb{S})$.

Now, $E_1^{s,t} = \pi_t K_s = \mathrm{Hom}^t_{\mathscr{A}_2}(H^*(K_s), H^*(\mathbb{S}))$, since the staircase diagram comes from the exact couple

$$\begin{tikzcd}
\pi_* X_* \arrow[rr, "i"] & & \pi_* X_* \arrow[dl, "j"] \\
 & \pi_* K_* \arrow[ul, "k"] &
\end{tikzcd}$$

where $i : \pi_{t+1}X_{s+1} \to \pi_t X_s$, $j : \pi_{t+1}X_s \to \pi_{t+1}K_s$, and $k : \pi_{t+1}X_{s+1}$ are as in (4). The differential $d_1 : \pi_t(K_s) \to \pi_t K_{s+1}$ is induced by the map $K_s \to K_{s+1}$, since it is defined to be $j \circ k$.

Further, $E_2^{s,t} = H^{s,t}(E_1^{*,*}, d_1)$, so each $E^{*,t}$ is the homology of the chain complex

$$0 \to E_1^{0,t} \to E_1^{1,t} \to E_1^{2,t} \to \cdots,$$

which is by construction the chain complex below.

$$0 \to \mathrm{Hom}^t_{\mathscr{A}_2}(H^*(K_0), H^*(\mathbb{S})) \to \mathrm{Hom}^t_{\mathscr{A}_2}(H^*(K_1), H^*(\mathbb{S})) \to \cdots .$$

The homology of this is by definition $\mathrm{Ext}^{s,t}_{\mathscr{A}_2}(H^*(X), H^*(\mathbb{S}))$, so $E_2^{s,t} = \mathrm{Ext}^{s,t}_{\mathscr{A}_2}(H^*(X), H^*(\mathbb{S}))$.

THEOREM 4.4.2 ([15], Thm 4.29). The spectral sequence $\{E_r, d_r\}$ constructed above converges to $\pi_{t-s}(X)^\wedge_2$ in the sense of Definition Definition 4.1.5.

In particular, when $X = \mathbb{S}$, the Adams spectral sequence above converges to the stable homotopy groups $(\pi_*^s)^\wedge_2$. In other words, there are filtrations

$$\cdots \subseteq F^{s+1,t+1} \subseteq F^{s,t} \subseteq F^{s-1,t-1} \subseteq \cdots \subseteq (\pi^s_{t-s})^\wedge_2,$$

for $t - s \geq 0$, and we have isomorphisms $E_\infty^{s,t} \cong F^{s,t}/F^{s+1,t+1}$. Note that it is possible for a spectral sequence to converge to several different groups, since the E_∞ page only gives information about the associated graded of a group, and thus other arguments may be needed to determine the groups $(\pi^s_{t-s})^\wedge_2$ themselves.

4.5 First computations

Before we begin any calculations, we first prove a small lemma which will make computing the E_2 page much easier.

We will say that a free resolution

$$\cdots \xrightarrow{f_3} F_2 \xrightarrow{f_2} F_1 \xrightarrow{f_1} F_0 \xrightarrow{f_0} H^*(X)$$

is *minimal* if $\mathrm{im}\, f_i \subseteq \mathscr{A}_2^+ F_{i-1}$ for all i, where $\mathscr{A}_2^+ \subseteq \mathscr{A}_2$ is the irrelevant ideal.

LEMMA 4.5.1 ([8], Lem 5.49). For a minimal free resolution

$$\cdots \xrightarrow{f_3} F_2 \xrightarrow{f_2} F_1 \xrightarrow{f_1} F_0 \xrightarrow{f_1} H^*(X) \to 0$$

of $H^*(X)$ as an $\mathscr{A}_2$-module, we have $\mathrm{Ext}^{s,t}_{\mathscr{A}_2}(H^*(X), \mathbb{F}_2) = \mathrm{Hom}^t_{\mathscr{A}_2}(F_s, \mathbb{F}_2)$.

PROOF. Let $x \in F_i$. Since $f_{i-1}f_i = 0$, we have $f_i(x) \in \ker f_{i-1} = \mathrm{im}\, f_i \subseteq \mathscr{A}_2^+ F_{i-1}$. We can thus write $f_i(x) = \sum_j a_j x_{i-1,j}$ with $a_j \in \mathscr{A}_2^+$. Now, for $g \in \mathrm{Hom}_{\mathscr{A}_2}(F_{i-1}, \mathbb{F}_2)$, we have $gf_i(x) = \sum_j a_j g(x_{i-1}, j) = 0$, since a_j acts trivially on elements of $\mathbb{F}_2$.

Thus, the boundary maps in the complex

$$\cdots \xleftarrow{-\circ f_3} \mathrm{Hom}_{\mathscr{A}_2}(F_2, \mathbb{F}_2) \xleftarrow{-\circ f_2} \mathrm{Hom}_{\mathscr{A}_2}(F_1, \mathbb{F}_2) \xleftarrow{-\circ f_1} \mathrm{Hom}_{\mathscr{A}_2}(F_0, \mathbb{F}_2) \leftarrow 0$$

are all zero, so $\mathrm{Ext}^{s,t}_{\mathscr{A}_2}(H^*(X), \mathbb{F}_2) = \mathrm{Hom}^t_{\mathscr{A}_2}(F_s, \mathbb{F}_2)$. $\square$

Now, since $\mathbb{F}_2$ is concentrated in degree 0, the only elements of F_s which can be sent to $1 \in \mathbb{F}_2$ are the elements of degree t, so for every generator of F_s in degree t, there is an $\mathbb{F}_2$ summand in $\mathrm{Hom}^t_{\mathscr{A}_2}(F_s, \mathbb{F}_2)$.

Figure 4.1 shows part of a construction of a minimal free resolution of $H^*(\mathbb{S}) = \mathbb{F}_2$, where position $(t-s, s)$ consists of degree t elements of F_s. Instead of inducting on s and calculating column by column, we will instead induct on $t - s$, assuming the previous rows have been computed. Note that since we will add the minimum number of generators needed in each row, the addition of new generators in later rows will not affect the induction (because a new generator will not impact the kernel of any f_i).

For $t-s=0$, at position $(0,0)$, we need a generator $\iota \in F_0$ to map to $1 \in \mathbb{F}_2$, in order to make f_0 a surjection. The kernel of f_0 thus contains every multiple of ι by an element of $\mathscr{A}_2^+$, which by exactness should be contained in the image of f_1. Thus, we need a new generator α_1^1 at $(0,1)$ mapping to $Sq^1\iota$. The element $Sq^1\alpha_1^1 \in F_1$ is therefore in the kernel of f_1, since $Sq^1Sq^1 = 0$, so we need a generator α_2^2 at $(0,2)$ mapping to $Sq^1\alpha_1^1$. Now, it is clear that each position $(0,s)$ will require a new generator α_s^s, since each $Sq^1\alpha_{s-1}^{s-1}$ maps to $Sq^1Sq^1\alpha_{s-2}^{s-2} = 0$, so the first row is completely determined, and $\mathrm{Ext}_{\mathscr{A}_2}^{s,s}(\mathbb{F}_2, \mathbb{F}_2) = \mathbb{F}_2$.

When $t-s=1$, a generator α_2^1 is needed in position $(1,1)$ mapping to $Sq^2\iota$, since $f_1(Sq^1\alpha_1^1) = Sq^1f_1(\alpha_1^1) = Sq^1Sq^1\iota = 0$ but $Sq^2\iota \in \ker f_0$. No other generators are needed, since $Sq^1\alpha_2^1$ maps to $Sq^3\iota \neq 0$ and $Sq^2\alpha_s^s$ maps to $Sq^2Sq^1\alpha_{s-1}^{s-1} \neq 0$ for all $s > 1$.

For $t-s=2$, no generator is needed at $(2,1)$, since $f_1(Sq^2\alpha_1^1) = Sq^2Sq^1\iota \neq 0$ and $f_1(Sq^1\alpha_2^1) = Sq^3\iota \neq 0$. A generator α_4^2 is needed at $(2,2)$ to map to $S^3\alpha_1^1 + Sq^2\alpha_2^1$, since $f_1(Sq^3\alpha_1^1 + Sq^2\alpha_2^1) = 2Sq^3Sq^1\iota = 0$. No further generators are needed, as $Sq^1\alpha_4^2$ maps to $Sq^3\alpha_2^1 \neq 0$ and $Sq^3\alpha_s^s$ maps to $Sq^3Sq^1\alpha_{s-1}^{s-1} \neq 0$ for all $s > 1$.

When $t-s=3$, generators α_4^1, α_5^2, and α_6^3 are needed to map to $Sq^4\iota$, $Sq^4\alpha_1^1 + Sq^2Sq^1\alpha_2^1 + Sq^1\alpha_4^1$, and $Sq^4\alpha_2^2 + Sq^2\alpha_4^2 + Sq^1\alpha_5^2$ respectively, since the latter elements are in the kernel of their respective f_i's. No new generators are needed after $s=4$, since $Sq^1\alpha_6^3$ maps to $Sq^5\alpha_2^2 + Sq^3\alpha_4^2$, $Sq^4\alpha_s^s$ maps to $Sq^4Sq^1\alpha_{s-1}^{s-1}$, and although $Sq^3Sq^1\alpha_s^s$ maps to zero, it is hit by $Sq^3\alpha^{s+1,s+1}$.

Continuing in this fashion, the computations for $t-s=4,5$ are shown in Figure 4.1, though the Adem relations of Proposition 2.0.2 required to justify them are not. Note that although to compute each row, knowledge of maps involving the next two rows is required, the rows $t-s=6,7$ do not contain all the new generators needed.

Figure 4.2 shows the E_2 page of the Adams spectral sequence for $\mathbb{S}$ for $t-s \leq 5$. Recall that the Adams grading is $(r, r-1)$, and thus on Figure 4.2 the d_r differentials go one unit left and r units up. Now, from Figure 4.2, we see that $(\pi_1^s)_2^\wedge$ has order dividing 2, but a priori there could be a d_r differential mapping α_2^1 to α_{r+1}^{r+1}, in which case α_2^1 would not survive to the E_∞ page. On the other hand, any differential emanating from or entering position $(2,2)$ either enters or leaves 0, so α_4^2 must survive to the E_∞ page, and we see that $(\pi_2^s)_2^\wedge = \mathbb{Z}/2\mathbb{Z}$.

Similarly, we see that all of the generators in the column $t-s=3$ survive to E_∞ (since there are no possible nonzero differentials interacting with them), so $|(\pi_3^s)_2^\wedge| = 8$. However, we do not currently have the tools to determine the isomorphism class of $(\pi_3^s)_2^\wedge$, since any of the filtrations below are possible:

$$\{0\} \subseteq \mathbb{Z}/2\mathbb{Z} \subseteq \mathbb{Z}/4\mathbb{Z} \subseteq \mathbb{Z}/8\mathbb{Z},$$

$$\{0\} \subseteq \mathbb{Z}/2\mathbb{Z} \subseteq \mathbb{Z}/4\mathbb{Z} \subseteq \mathbb{Z}/4\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z},$$

$$\{0\} \subseteq \mathbb{Z}/2\mathbb{Z} \subseteq (\mathbb{Z}/2\mathbb{Z})^2 \subseteq \mathbb{Z}/4\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z},$$

$$\{0\} \subseteq \mathbb{Z}/2\mathbb{Z} \subseteq (\mathbb{Z}/2\mathbb{Z})^2 \subseteq (\mathbb{Z}/2\mathbb{Z})^3.$$

We will therefore spend some time describing a multiplication on the Adams spectral sequence which will allow us to resolve such ambiguities.

$t-s$ \ s	0	1	2	3	4
0	ι	α_1^1	α_2^2	α_3^3	α_4^4
1	$Sq^1\iota$	$Sq^1\alpha_1^1$ α_2^1	$Sq^1\alpha_2^2$	$Sq^1\alpha_3^3$	$Sq^1\alpha_4^4$
2	$Sq^2\iota$	$Sq^2\alpha_1^1$ $Sq^1\alpha_2^1$	$Sq^2\alpha_2^2$ α_4^2	$Sq^2\alpha_3^3$	$Sq^2\alpha_4^4$
3	$Sq^2Sq^1\iota$ $Sq^3\iota$	$Sq^2Sq^1\alpha_1^1$ $Sq^3\alpha_1^1$ $Sq^2\alpha_2^1$ α_4^1	$Sq^2Sq^1\alpha_2^2$ $Sq^3\alpha_2^2$ $Sq^1\alpha_4^2$ α_5^2	$Sq^2Sq^1\alpha_3^3$ $Sq^3\alpha_3^3$ α_6^3	$Sq^2Sq^1\alpha_4^4$ $Sq^3\alpha_4^4$
4	$Sq^3Sq^1\iota$ $Sq^4\iota$	$Sq^3Sq^1\alpha_1^1$ $Sq^4\alpha_1^1$ $Sq^2Sq^1\alpha_2^1$ $Sq^3\alpha_2^1$ $Sq^1\alpha_4^1$	$Sq^3Sq^1\alpha_2^2$ $Sq^4\alpha_2^2$ $Sq^2\alpha_4^2$ $Sq^1\alpha_5^2$	$Sq^3Sq^1\alpha_3^3$ $Sq^4\alpha_3^3$ $Sq^1\alpha_6^3$	$Sq^3Sq^1\alpha_4^4$ $Sq^4\alpha_4^4$
5	$Sq^4Sq^1\iota$ $Sq^5\iota$	$Sq^4Sq^1\alpha_1^1$ $Sq^5\alpha_1^1$ $Sq^3Sq^1\alpha_2^1$ $Sq^4\alpha_2^1$ $Sq^2\alpha_4^1$	$Sq^4Sq^1\alpha_2^2$ $Sq^5\alpha_2^2$ $Sq^2Sq^1\alpha_4^2$ $Sq^3\alpha_4^2$ $Sq^2\alpha_5^2$	$Sq^4Sq^1\alpha_3^3$ $Sq^5\alpha_3^3$ $Sq^2\alpha_6^3$	$Sq^4Sq^1\alpha_4^4$ $Sq^5\alpha_4^4$
6	$Sq^5Sq^1\iota$ $Sq^4Sq^2\iota$ $Sq^6\iota$	$Sq^5Sq^1\alpha_1^1$ $Sq^4Sq^2\alpha_1^1$ $Sq^6\alpha_1^1$ $Sq^4Sq^1\alpha_2^1$ $Sq^5\alpha_2^1$ $Sq^2Sq^1\alpha_4^1$ $Sq^3\alpha_4^1$	$Sq^5Sq^1\alpha_2^2$ $Sq^4Sq^2\alpha_2^2$ $Sq^6\alpha_2^2$ $Sq^3Sq^1\alpha_4^2$ $Sq^4\alpha_4^2$ $Sq^2Sq^1\alpha_5^2$ $Sq^3\alpha_5^2$	$Sq^5Sq^1\alpha_3^3$ $Sq^4Sq^2\alpha_3^3$ $Sq^6\alpha_3^3$ $Sq^2Sq^1\alpha_6^3$ $Sq^3\alpha_6^3$	$Sq^5Sq^1\alpha_4^4$ $Sq^4Sq^2\alpha_4^4$ $Sq^6\alpha_4^4$
7	$Sq^4Sq^2Sq^1\iota$ $Sq^6Sq^1\iota$ $Sq^5Sq^2\iota$ $Sq^7\iota$	$Sq^4Sq^2Sq^1\alpha_1^1$ $Sq^6Sq^1\alpha_1^1$ $Sq^5Sq^2\alpha_1^1$ $Sq^7\alpha_1^1$ $Sq^5Sq^1\alpha_2^1$ $Sq^4Sq^2\alpha_2^1$ $Sq^6\alpha_2^1$ $Sq^3Sq^1\alpha_4^1$ $Sq^4\alpha_4^1$	$Sq^4Sq^2Sq^1\alpha_2^2$ $Sq^6Sq^1\alpha_2^2$ $Sq^5Sq^2\alpha_2^2$ $Sq^7\alpha_2^2$ $Sq^4Sq^1\alpha_4^2$ $Sq^5\alpha_4^2$ $Sq^3Sq^1\alpha_5^2$ $Sq^4\alpha_5^2$	$Sq^4Sq^2Sq^1\alpha_3^3$ $Sq^6Sq^1\alpha_3^3$ $Sq^5Sq^2\alpha_3^3$ $Sq^7\alpha_3^3$ $Sq^3Sq^1\alpha_3^3$ $Sq^4\alpha_3^3$	$Sq^4Sq^2Sq^1\alpha_4^4$ $Sq^6Sq^1\alpha_4^4$ $Sq^5Sq^2\alpha_4^4$ $Sq^7\alpha_4^4$

Figure 4.1: A construction of a minimal free resolution of $H^*(\mathbb{S}) = \mathbb{F}_2$. Generators for $t-s \leq 5$ are highlighted in pink; further generators which may be needed in rows $t-s = 6, 7$ are not shown.

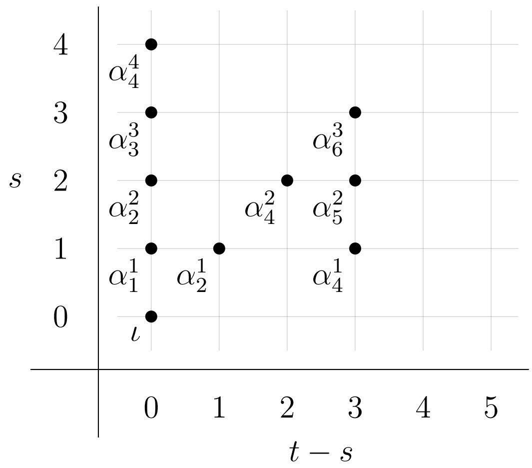


Figure 4.2: $\mathrm{Ext}^{s,t}_{\mathscr{A}_2}(\mathbb{F}_2, \mathbb{F}_2)$ for $t - s \leq 5$. Note that there is a generator α_s^s at $(0, s)$ for every $s \geq 0$, though only the first five are shown here.

4.6 Multiplicative structure

4.6.1 The Yoneda product

Definition 4.6.1 ([14], Def 11.8.1). For any algebra A and A-modules L, M, N, there is a product, the *Yoneda product*

$$\circ : \mathrm{Ext}^{s,t}_A(M, N) \otimes \mathrm{Ext}^{u,v}_A(L, M) \to \mathrm{Ext}^{s+u,t+v}_A(L, N),$$

defined as follows: let

$$\cdots \xrightarrow{f_3} F_2 \xrightarrow{f_2} F_1 \xrightarrow{f_1} F_0 \xrightarrow{f_0} L \to 0,$$

$$\cdots \xrightarrow{f'_3} F'_2 \xrightarrow{f'_2} F'_1 \xrightarrow{f'_1} F'_0 \xrightarrow{f'_0} M \to 0$$

be free resolutions for L and M. Then, given $[g] \in \mathrm{Ext}^{s,t}_A(M, N)$, $[h] \in \mathrm{Ext}^{u,v}_A(L, M)$, we inductively construct a chain map $h_\bullet : F_{u+\bullet} \to F'_\bullet[v]$, as shown in the diagram below (where square brackets denotes the shift in degree).

$$\begin{tikzcd}
F_{u+s} \arrow[r, "f_{u+s}"] \arrow[d, dashed, "h_s"'] & F_{u+s-1} \arrow[r, "f_{u+s-1}"] \arrow[d, dashed, "h_{s-1}"'] & \cdots \arrow[r, "f_{u+1}"] & F_u \arrow[r, "f_u"] \arrow[d, dashed, "h_0"'] \arrow[dr, "h"] & F_{u-1} \arrow[r, "f_{u-1}"] & \cdots \arrow[r, "f_1"] & F_0 \arrow[r, "f_0"] & L \\
F'_s[v] \arrow[r, "f'_s"'] \arrow[d, "g"'] & F'_{s-1}[v] \arrow[r, "f'_{s-1}"'] & \cdots \arrow[r, "f'_1"'] & F'_0[v] \arrow[r, two heads, "f'_0"'] & M[v] \\
N[v+t]
\end{tikzcd}$$

The map h_0 is defined as follows: let $\alpha \in F_u$ be a generator, and consider $h(\alpha) \in M[v]$. Since f'_0 is surjective, there exists some $\beta \in F'_0[v]$ such that $f'_0(\beta) = h(\alpha)$. We define $h_0(\alpha) = \beta$. Now, suppose the h_i have been constructed for $i < w$, and consider the diagram below.

$$\begin{tikzcd}
F_{u+w} \arrow[r, "f_{u+w}"] \arrow[d, dashed, "h_w"'] & F_{u+w-1} \arrow[r, "f_{u+w-1}"] \arrow[d, "h_{w-1}"] & F_{u+w-2} \arrow[d, "h_{w-2}"] \\
F'_w[v] \arrow[r, "f'_w"'] & F'_{w-1}[v] \arrow[r, "f'_{w-1}"'] & F'_{w-2}[v]
\end{tikzcd}$$

Let $\alpha \in F_{u+w}$ be a generator, and consider $f'_{w-1}h_{w-1}f_{u+w}(\alpha) \in F'_{w-2}[v]$. By induction, the right square commutes, so $f'_{w-1}h_{w-1}f_{u+w}(\alpha) = h_{w-2}f_{u+w-1}f_{u+w}(\alpha) = 0$, by exactness of the top row. Thus, $h_{w-1}f_{u+w}(\alpha) \in \ker f'_{w-1} = \operatorname{im} f'_w$. Write $h_{w-1}f_{u+w}(\alpha) = f'_w(\beta)$, and define $h_w(\alpha) = \beta$.

Now, consider the diagram below.

$$\begin{tikzcd}
F_{u+s+1} \arrow[r, "f_{u+s+1}"] \arrow[d, dashed, "h_{s+1}"'] & F_{u+s} \arrow[d, "h_s"] \\
F'_{s+1}[v] \arrow[r, "f'_{s+1}"'] & F'_s[v] \arrow[d, "g"] \\
 & N[v+t]
\end{tikzcd}$$

We have $gh_sf_{u+s+1} = gf'_{s+1}h_{s+1} = 0$, since $[g] \in \operatorname{Ext}_A^{s,t}(F'_s, N)$, so $[gh_s] \in \operatorname{Ext}_A^{u+s,v+t}$. We thus define $[g] \cdot [h] = [gh_s]$.

This definition is independent of the lifts chosen, which can be seen as follows. Suppose we have two chain maps $\{h_i\}, \{h'_i\}$; we will construct a chain homotopy between them. Define $k_0 : F_{u-1} \to F'_0[v]$ to be the zero map. By construction, $f'_0h_0 = f'_0h'_0 = h$, so $f'_0(h_0 - h'_0) = 0$. Thus, $\operatorname{im}(h_0 - h'_0) \subseteq \ker f'_0 = \operatorname{im} f'_1$, so $h_0 - h'_0 = f'_1k_1 = f'_1k_1 + k_0f_u$ for some map $k_1 : F_u \to F'_1[v]$. Now, suppose we have k_i, k_{i-1} such that $h_{i-1} - h'_{i-1} = f'_ik_i + k_{i-1}f_{u+i-1}$. Then $f'_ih_i = h_{i-1}f_{u+i}$ and $f'_ih'_i = h'_{i-1}f_{u+i}$, so $f'_i(h_i - h'_i) = (h_{i-1} - h'_{i-1})f_{u+i} = (f'_ik_i + k_{i-1}f_{u+i-1})f_{u+i} = f'_ik_if_{u+i}$, and thus we can construct k_{i+1} such that $h_i - h'_i = f'_{i+1}k_{i+1} + k_if_{u+i}$. Now, $g(h_s - h'_s) = g(f'_{s+1}k_{s+1} + k_sf_{u+s}) = gk_sf_{u+s}$, and therefore $[g(h_s - h'_s)] = [gk_sf_{u+s}] = [0]$.

Finally, if $h = lf_u$ for some $l : F_{u-1} \to M[v]$, with filling $\{l_i\}$, then $\{l_if_{u+i}\}$ is a filling for h, so $[g] \cdot [h] = [gl_sf_{u+s}] = [0]$. On the other hand, if $g = mf'_s$, then $[g] \cdot [h] = [gh_s] = [mf'_sh_s] = [mh_{s-1}f_{u+s}] = [0]$. Thus, the Yoneda product is well defined.

4.6.2 The composition product

Definition 4.6.2 ([15], p47). Let X, Y, Z be spectra. The *composition pairing* $\circ : [Y,Z]_* \otimes [X,Y]_* \to [X,Z]_*$ is defined as follows:

$$\begin{aligned} \circ :\ [Y,Z]_v \ \otimes \ [X,Y]_t \ &\to \ [X,Z]_{v+t} \\ [g : \Sigma^v Y \to Z] \otimes [f : \Sigma^t X \to Y] &\mapsto [g \circ \Sigma^v f : \Sigma^{v+t} X \to Z], \end{aligned}$$

where $[X,Y]_n = [\Sigma^n X, Y]$.

In particular, if $X = Y = Z = \mathbb{S}$, we have a product $\pi^s_v \otimes \pi^s_t \to \pi^s_{v+t}$.

Lemma 4.6.3. Let $f, g : S^n \to S^n$ be pointed maps such that $\deg f = \deg g$. Then $f \simeq g$.

Proof. We prove the contrapositive. Suppose $f \not\simeq g$. Then f, g represent two different elements in $\pi_nS^n \simeq \mathbb{Z}$, say $[f] = n \neq m = [g]$ for $n, m \in \mathbb{Z}$. The Hurewicz theorem then implies that for a fixed generator $u_n \in H^n(S^n)$, $f_*(u_n) \neq g_*(u_n) \in H^n(S^n)$, so $\deg f \neq \deg g$, as required. $\square$

Lemma 4.6.4 ([7], Prop 4.56). The composition product makes π^s_* into a graded commutative ring.

Proof. The identity map is clearly a two sided-identity for the composition product, and associativity follows from the fact that suspension respects composition. We now check graded commutativity.

Let $f : S^{i+k} \to S^k$, $g : S^{j+k} \to S^k$ represent elements of π_*^s; without loss of generality we may assume k is even. Note that under the identification $\Sigma^l S^{i+k} \cong S^{i+k} \wedge S^l$, the map $\Sigma^l f : \Sigma^l S^{i+k} \to \Sigma^l S^k$ corresponds to $f \wedge \mathrm{id} : S^{i+k} \wedge S^l \to S^k \wedge S^l$. Now, consider the commutative diagram below, where τ and σ swap the two factors.

$$\begin{tikzcd} S^k \wedge S^{j+k} \arrow[r, "\mathrm{id}\wedge g"] \arrow[d, "\sigma"'] & S^k \wedge S^k \arrow[d, "\tau"] \\ S^{j+k} \wedge S^k \arrow[r, "g\wedge \mathrm{id}"'] & S^k \wedge S^k \end{tikzcd}$$

The map σ is a composition of $k(j+k)$ transpositions $S^1 \wedge S^1 \to S^1 \wedge S^1$, each of which has degree -1 (since such a transposition is homotopic to a reflection), so σ has degree $(-1)^{k(j+k)} = 1$. Similarly, $\deg \tau = 1$, so by Lemma 4.6.3 we see that σ and τ are both homotopic to the identity. Thus, $f \wedge g = (\mathrm{id} \wedge g) \circ (f \wedge \mathrm{id}) \simeq (g \wedge \mathrm{id}) \circ (f \wedge \mathrm{id})$. Since $(g \wedge \mathrm{id})(f \wedge \mathrm{id})$ and $g \cdot f$ represent the same element in π_*^s, we have $f \wedge g \simeq g \cdot f$, and by the same argument $g \wedge f \simeq f \cdot g$. It now suffices to show that $f \wedge g \simeq (-1)^{ij} g \wedge f$.

Consider the commutative diagram below.

$$\begin{tikzcd} S^{i+k} \wedge S^{j+k} \arrow[r, "f\wedge g"] \arrow[d, "\sigma"'] & S^k \wedge S^k \arrow[d, "\tau"] \\ S^{j+k} \wedge S^{i+k} \arrow[r, "g\wedge f"'] & S^k \wedge S^k \end{tikzcd}$$

We have $\deg \sigma = (-1)^{(i+k)(j+k)} = (-1)^{ij}$ and $\deg \tau = (-1)^{k^2} = 1$, so $f \wedge g \simeq (-1)^{ij} g \wedge f$, as required.

Finally, for $f' : S^{i+k} \to S^k, h : S^{l+k} \to S^k$, we have $(f+f') \cdot h = (f+f') \circ \Sigma^i h = f \cdot h + g \cdot h$, and $h \cdot (f+g) = (-1)^{il}(f+g) \cdot h = (-1)^{il} f \cdot h + (-1)^{il} g \cdot h = h \cdot f + h \cdot g$, so the distributivity laws also follow. $\square$

Lemma 4.6.5. *There is a unique ring structure on $(\pi_*^s)_2^\wedge$ which makes the completion map $c : \pi_*^s \to (\pi_*^s)_2^\wedge$ into a ring homomorphism.*

Proof. We show uniqueness first. Let $f \in (\pi_i^s)_2^\wedge$, $g \in (\pi_j^s)_2^\wedge$. If $i, j \geq 1$, then the completion map is surjective, so $f = c(\tilde{f}), g = c(\tilde{g})$ for some $\tilde{f} \in \pi_i^s, \tilde{g} \in \pi_j^s$. Then $fg = c(\tilde{f})c(\tilde{g}) = c(\tilde{f}\tilde{g})$.

If $i = 0, j \geq 1$, then let $\hat{f} \in \pi_0^s$ be a lift of $q(f) \in \pi_0^s / 2^r \pi_0^s$, where 2^r is the highest power of 2 dividing the order of π_j^s. Then $f \equiv c(\hat{f}) \mod 2^r$, so $f = c(\hat{f}) + 2^r w$. We have $fg = fc(\tilde{g}) = (c(\hat{f}) + 2^r w)c(\tilde{g}) = c(\hat{f})c(\tilde{g}) + 2^r(wc(\tilde{g})) = c(\hat{f}\tilde{g}) \in (\pi_j^s)_2^\wedge$.

Finally, if $i = j = 0$, we claim that any two multiplications on $\mathbb{Z}_2$ which agree on $\mathbb{Z}$ must agree on all of $\mathbb{Z}_2$, and thus the multiplication is given by the usual product on $\mathbb{Z}_2$.

Suppose not; let $\star, \cdot$ be two products on $\mathbb{Z}_2$, agreeing on $\mathbb{Z}$, with $f \star g \neq f \cdot g$. Then there is some k such that $f \star g \not\equiv f \cdot g \mod k$. Pick integers n, m such that $n \equiv f \mod k$ and $m \equiv g \mod k$. Then, modulo k, $f \cdot g \equiv n \cdot m = n \star m \equiv f \star g$, giving a contradiction.

Now, for $i, j \geq 1$, the multiplication above is well defined, since if $\tilde{f}' = \tilde{f} + t$, with $nt = 0$ for odd n, then $c(\tilde{f}'\tilde{g}) = c(\tilde{f}\tilde{g} + t\tilde{g}) = c(\tilde{f}\tilde{g})$ (since multiplication by n is an isomorphism in a group of order 2^r). Likewise, if $\tilde{g}' = \tilde{g} + t$, then $c(\tilde{f}\tilde{g}') = c(\tilde{f}\tilde{g})$. If $i = 0, j \geq 1$ (or vice versa), then picking a different representative for g does not change the product, by the previous argument. If $\hat{f}'$ is a different lift of $q(f)$, we have $\hat{f} \equiv \hat{f}' \mod 2^r$, so $c(\hat{f}'\tilde{g}) = c(\hat{f}\tilde{g} + 2^r u\tilde{g}) = c(\hat{f}\tilde{g}) + 2^r c(u\tilde{g}) = c(\hat{f}\tilde{g})$ (for some $u \in \mathbb{Z}$). The usual product on $\mathbb{Z}_2$ is of course well-defined. Finally, associativity, distributivity, and unitality are inherited from π_*^s. $\square$

Given spectra X, Y, Z, we can define a pairing $\circ : [Y, Z_2^\wedge]_* \otimes [X, Y_2^\wedge]_* \to [X, Z_2^\wedge]$ as follows: let $f \in [Y, Z_2^\wedge]_s, g \in [X, Y_2^\wedge]_t$. By Theorem 3.5.6, there exists a unique (up to homotopy) map $\overline{f} : (\Sigma^s Y)_2^\wedge \to Z_2^\wedge$ such that f factors through $\overline{f}$. Now, note that $(\Sigma^s Y)_2^\wedge \simeq \Sigma^s Y_2^\wedge$, since $\pi_i(\Sigma^s Y) = \pi_{i-s}(Y)$. We can thus define the pairing of f and g to be $\overline{f} \circ \Sigma^s g$, as shown below.

$$\begin{tikzcd}
 & \Sigma^s Y \arrow[d] \arrow[dr, "f"] & \\
\Sigma^{s+t} X \arrow[r, "\Sigma^s g"'] & \Sigma^s Y_2^\wedge \arrow[r, dashed, "\overline{f}"'] & Z_2^\wedge
\end{tikzcd}$$

Lemma 4.6.6. The completion map $c_* : \pi_*^s \to \pi_*(\mathbb{S}_2^\wedge)$ is a ring homomorphism. In particular, by Lemma 4.6.5, the composition product on $\pi_*(\mathbb{S}_2^\wedge)$ coincides with the product on $(\pi_*^s)_2^\wedge$ inherited from π_*^s, so the two groups are also isomorphic as rings.

Proof. Let $f : \mathbb{S}^i \to \mathbb{S}$, $g : \mathbb{S}^j \to \mathbb{S}$ be elements of π_i^s and π_j^s respectively. Then $c_*(f)c_*(g) = (cf)(cg)$ is given by factorising $cg = \overline{cg}c$ and composing to get $\overline{cg}c\Sigma^j f$. We thus have the commutative diagram below.

$$\begin{tikzcd}
\mathbb{S}^{i+j} \arrow[r, "\Sigma^j f"] & \mathbb{S}^j \arrow[r, "g"] \arrow[d, "c"] & \mathbb{S} \arrow[d, "c"] \\
 & \Sigma^j \mathbb{S}_2^\wedge \arrow[r, "\overline{cg}"'] & \mathbb{S}_2^\wedge
\end{tikzcd}$$

The upper path is exactly $c_*(fg)$, so $c_*(f)c_*(g) = c_*(fg)$. Further, the completion map clearly preserves the identity, so it is a ring homomorphism. $\square$

4.6.3 Multiplication on the Adams spectral sequence

Definition 4.6.7 ([15], Def 5.5). Let $\{'E_r\}, \{''E_r\}, \{E_r\}$ be three spectral sequences. A *pairing* of these spectral sequences is a sequence of homomorphisms

$$\phi_r : {}'E_r^{*,*} \otimes {}''E_r^{*,*} \to E_r^{*,*},$$

such that the Leibniz rule $d_r\phi_r(x \otimes y) = \phi_r(d_r(x) \otimes y) + (-1)^{\deg x}\phi_r(x \otimes d_r(y))$ holds, and

$$\phi_{r+1}([x] \otimes [y]) = [\phi_r(x \otimes y)], \tag{5}$$

where $[x] \in {}'E_{r+1}^{*,*}$ is the homology class of a d_r-cycle $x \in {}'E_r^{*,*}$, and similarly for y.

A spectral sequence pairing $\{\phi_r\}$ induces a pairing

$$\phi_\infty : {}'E_\infty^{*,*} \otimes {}''E_\infty^{*,*} \to E_\infty^{*,*}.$$

Theorem 4.6.8 ([15], Thm 5.8). Let X, Y, Z be spectra, with Y, Z connective and of finite type. There is a pairing of spectral sequences

$$E_r^{*,*}(Y,Z) \otimes E_r^{*,*}(X,Y) \to E_r^{*,*}(X,Z)$$

which agrees for $r = 2$ with the Yoneda pairing

$$\mathrm{Ext}^{*,*}_{\mathscr{A}_2}(H^*(Z), H^*(Y)) \otimes \mathrm{Ext}^{*,*}_{\mathscr{A}_2}(H^*(Y), H^*(X)) \to \mathrm{Ext}^{*,*}_{\mathscr{A}_2}(H^*(Z), H^*(X))$$

and which converges to the composition pairing

$$[Y, Z_2^\wedge]_* \otimes [X, Y_2^\wedge]_* \to [X, Z_2^\wedge]_*.$$

The pairing is associative and unital.

Remark 4.6.9. Condition (5) of Definition 4.6.7 ensures that if a product is computed on the E_2 page, and both terms survive to the E_r page for some $r > 2$, then the computation is still valid on that page.

5 Calculating stable homotopy groups

In this section, we will calculate the groups $(\pi_{t-s}^s)_2^\wedge$ for $t - s \leq 15$. We saw in sections 4.5 and 4.6.1 that computing $\mathrm{Ext}^{s,t}_{\mathscr{A}_2}(H^*(X), \mathbb{F}_2)$ and Yoneda products is entirely algorithmic; for $t - s \leq 5$ we will show how to use the Yoneda product to resolve by hand the ambiguities mentioned in Section 4.5, and for $t - s > 5$ we will use the Adams spectral sequence calculator (see [2]) to compute $\mathrm{Ext}^{s,t}_{\mathscr{A}_2}(\mathbb{F}_2, \mathbb{F}_2)$. We will see in sections 5.1 and 5.2 that the groups $(\pi_{t-s}^s)_2^\wedge$ can be completely determined this way for $t - s \leq 13$. Section 5.3 will be dedicated to proving that certain differentials at $t - s = 15$ are nontrivial, and will culminate in the computation of $(\pi_{14}^s)_2^\wedge$ and $(\pi_{15}^s)_2^\wedge$. We follow [15] and [14] throughout.

5.1 Resolving extensions

Proposition 5.1.1 ([15], Cor 6.5). We have the following relations:

$$\begin{aligned}
\alpha_i^i &= (\alpha_1^1)^i \\
\alpha_4^2 &= (\alpha_2^1)^2 \\
\alpha_5^2 &= \alpha_1^1 \alpha_4^1 \\
\alpha_6^3 &= (\alpha_1^1)^2 \alpha_4^1 = (\alpha_2^1)^3.
\end{aligned}$$

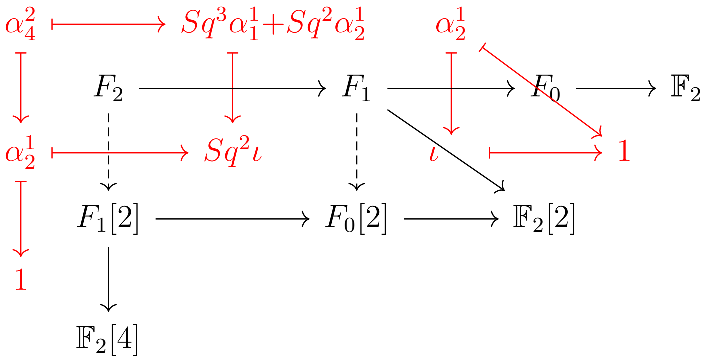
Proof. We show the first two relations; the final two are obtained similarly.

Consider the diagram below, where $F_\bullet$ is the free resolution in Figure 4.1.

$$\begin{tikzcd}
\alpha_s^s \arrow[rr, mapsto] \arrow[dd, mapsto] & & Sq^1\alpha_{s-1}^{s-1} \arrow[dd, mapsto] & & \alpha_{s-1}^{s-1} \arrow[dd, mapsto] \arrow[ddrr, mapsto] & & & \\
 & F_s \arrow[rr] \arrow[dd, dashed] & & F_{s-1} \arrow[rr] \arrow[dd, dashed] \arrow[ddrr] & & \cdots \arrow[r] & F_0 \arrow[r] & \mathbb{F}_2 \\
\alpha_1^1 \arrow[rr, mapsto] \arrow[dd, mapsto] & & Sq^1\iota & & \iota \arrow[rr, mapsto] & & 1 & \\
 & F_1[s-1] \arrow[rr] \arrow[dd] & & F_0[s-1] \arrow[rr] & & \mathbb{F}_2[s-1] & & \\
1 & & & & & & & \\
 & \mathbb{F}_2[s] & & & & & &
\end{tikzcd}$$

Since $\alpha_{s-1}^{s-1} \in F_{s-1}$ is the only generator of degree $s-1$, to write down a lift $F_{s-1} \to F_0[s-1]$ it suffices to say where α_{s-1}^{s-1} is sent. In order for the right triangle to commute, we must send α_{s-1}^{s-1} to ι. Now, to write down a lift $F_s \to F_1[s-1]$, it again suffices to write down the image of α_s^s. In order for the left square to commute, we must send α_s^s to α_1^1. The composite map $F_s \to \mathbb{F}_2[2]$ is the unique map sending α_s^s to 1, so $\alpha_1^1 \cdot \alpha_{s-1}^{s-1} = \alpha_s^s$ for all $s > 0$. Thus, $\alpha_s^s = (\alpha_1^1)^s$.

Similarly, the calculation below shows that $\alpha_2^1 \cdot \alpha_2^1 = \alpha_4^2$.



□

From now on, we will write h_i for the generator $\alpha_{2^i}^1 \in \mathrm{Ext}_{\mathscr{A}_2}^{1,2^i}(\mathbb{F}_2, \mathbb{F}_2)$.

Proposition 5.1.2. Suppose $\alpha \in (\pi_i^s)_2^\wedge$ represents $a \in E_\infty$. Then 2α represents $h_0 a$. In other words, multiplication by h_0 is induced by multiplication by 2.

Proof. Recall that $\pi_0^s = \mathbb{Z}$, since $\pi_1 S^1 = \mathbb{Z}$ and $n = 1 \leq 2 = 2(1)$, so this lies in the stable region. Now, $E_r^{s,s}(\mathbb{S})$ converges to some filtration of $(\pi_0^s)_2^\wedge = \mathbb{Z}_2$ whose quotients are all $\mathbb{Z}/2\mathbb{Z}$. The filtration must therefore be

$$\cdots \subseteq 4\mathbb{Z}_2 \subseteq 2\mathbb{Z}_2 \subseteq \mathbb{Z}_2,$$

since finite index subgroups of $\mathbb{Z}_p$ are of the form $p^k\mathbb{Z}_p$.

Thus, $\iota = [1] \in \mathbb{Z}_2/2\mathbb{Z}_2$, and by computing the Yoneda product we see that ι is a unit. We also have $h_0 = [2] \in 2\mathbb{Z}_2/4\mathbb{Z}_2$ so $[2] = h_0 = h_0 \cdot \iota$, and hence h_0 acts on ι by multiplication by 2.

Now, suppose $(\pi_{t-s}^s)_2^\wedge = G$ has a filtration

$$\cdots \subseteq F^{s+1,t+1}G \subseteq F^{s,t}G \subseteq F^{s-1,t-1}G \subseteq \cdots \subseteq G,$$

with $E_\infty^{s,t}(\mathbb{S}) \cong F^{s,t}/F^{s+1,t+1}$. For any $a \in E_\infty^{s,t}(\mathbb{S})$ represented by $\alpha \in F^{s,t}G$, $h_0 \cdot a = (h_0 \cdot \iota) \cdot a \in E_\infty^{s+1,t+1}(\mathbb{S})$ is represented by $2\alpha \in F^{s+1,t+1}$. □

Lemma 5.1.3. There are no nontrivial differentials for $t - s \leq 5$.

Proof. First, note from Figure 4.2 that the only possible nontrivial differentials in this range are the differentials $d_r : E_r^{1,2}(\mathbb{S}) \to E_r^{1+r,1+r}(\mathbb{S})$. Now, $0 = d_r(h_0 h_1) = d_r(h_0)h_1 + h_0 d_r(h_1) = h_0 d_r(h_1)$, so $d_r(h_1) = 0$. Since $E_r^{1,2}(\mathbb{S})$ is generated by h_1, we must have $d_r = 0$□

Theorem 5.1.4.

$$(\pi_i^s)_2^\wedge = \begin{cases} \mathbb{Z}/2\mathbb{Z} & i = 1,2 \\ \mathbb{Z}/8\mathbb{Z} & i = 3 \\ 0 & i = 4,5. \end{cases}$$

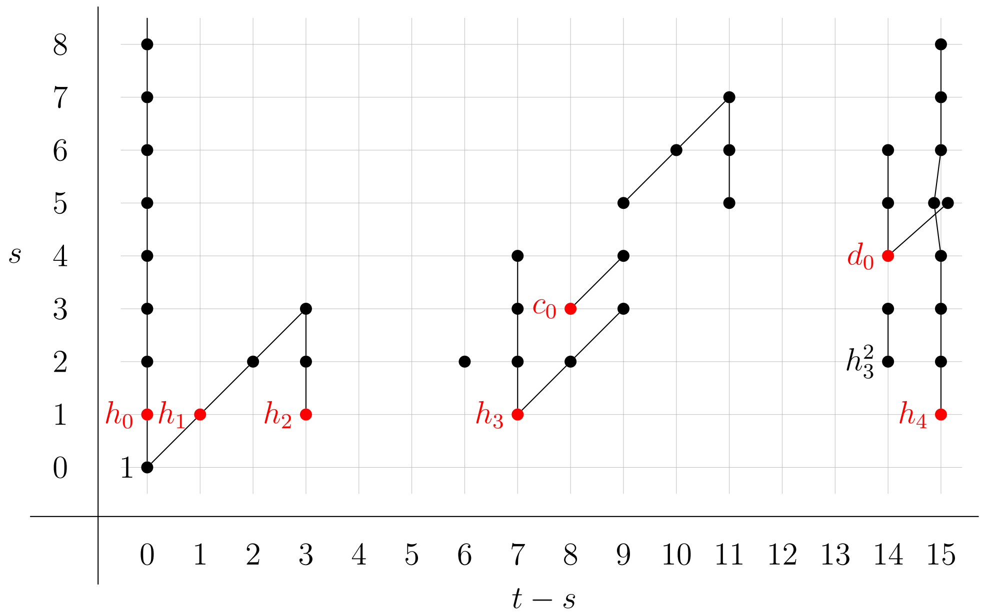


Figure 5.1: $\mathrm{Ext}^{s,t}_{\mathscr{A}_2}(\mathbb{F}_2, \mathbb{F}_2)$ for $t-s \leq 15$, calculated using [2]. The vertical and diagonal lines indicate multiplication by h_0 and h_1 respectively. Some of the algebra generators are shown in red, with their standard names.

Proof. We have already shown that $(\pi_4^s)_2^\wedge = (\pi_5^s)_2^\wedge = 0$ and that $(\pi_2^s)_2^\wedge = \mathbb{Z}/2\mathbb{Z}$. Further, Lemma 5.1.3 implies that $(\pi_1^s)_2^\wedge = \mathbb{Z}/2\mathbb{Z}$, since h_1 survives to E_∞. Now, we have some filtration

$$\{0\} \subseteq F^{3,6} \subseteq F^{2,5} \subseteq F^{1,4} = (\pi_3^s)_2^\wedge,$$

where $F^{3,6}$, $F^{2,5}/F^{3,6}, (\pi_3^s)_2^\wedge/F^{2,5} \cong \mathbb{Z}/2\mathbb{Z}$. Let $h_2 \in E^{1,4} = (\pi_3^s)_2^\wedge/F^{2,5}$ be represented by $\nu \in (\pi_3^s)_2^\wedge$. Then $0 \neq h_0^2 \cdot h_2 \in F^{3,6}$ is represented by 4ν, so $4\nu \neq 0$ in $(\pi_3^s)_2^\wedge$. Thus, $(\pi_3^s)_2^\wedge \cong \mathbb{Z}/8\mathbb{Z}$. □

5.2 The E_2 page for $t-s \leq 15$

Figure 5.1 shows the E_2 page for $\mathbb{S}$ in the range $t-s \leq 15$, and was calculated using [2]. Lines indicating multiplication by h_0 and h_1 are also shown, and from this we will be able to compute $(\pi_{t-s}^s)_2^\wedge$ for $t-s \leq 13$.

Lemma 5.2.1. There are no nontrivial differentials for $t-s \leq 13$.

Proof. We have shown in Lemma 5.1.3 that there are no nontrivial differentials for $t-s \leq 5$; for degree reasons, the only remaining possibility is that $d_2 : E_2^{2,10}(\mathbb{S}) \to E_2^{4,11}(\mathbb{S})$ is nonzero.

From Figure 5.1, we see that $E_2^{2,10}(\mathbb{S})$ is generated by h_1h_3, and $d_2(h_1h_3) = d_2(h_1)h_3 + h_1d_2(h_3) = 0 + 0 = 0$ (the first factor is zero by Lemma 5.1.3, and the second is an element of a trivial group). □

Theorem 5.2.2.

$$(\pi_i^s)_2^\wedge = \begin{cases} \mathbb{Z}/2\mathbb{Z} & i = 6, 10, \\ \mathbb{Z}/16\mathbb{Z} & i = 7, \\ (\mathbb{Z}/2\mathbb{Z})^2 & i = 8, \\ (\mathbb{Z}/2\mathbb{Z})^3 & i = 9, \\ \mathbb{Z}/8\mathbb{Z} & i = 11, \\ 0 & i = 12, 13. \end{cases}$$

Proof. The cases where $i = 6, 10$, $i = 7, 11$, and $i = 12, 13$ are proven analogously to the cases $i = 1, 2$, $i = 3$, and $i = 4, 5$ respectively. We thus show the result for $i = 8, 9$.

When $i = 8$, we have a filtration

$$\{0\} \subseteq F^{3,11} \subseteq F^{2,10} = (\pi_8^s)_2^\wedge,$$

where each quotient is isomorphic to $\mathbb{Z}/2\mathbb{Z}$. If $(\pi_8^s)_2^\wedge \cong \mathbb{Z}/4\mathbb{Z}$, then $h_1 h_3 \in (\pi_8^s)_2^\wedge / F^{3,11}$ is represented by $\alpha \in (\pi_8^s)_2^\wedge$ of order 4. However, $0 = h_0 h_1 h_3 \in F^{3,11}$, so $2\alpha = 0$, contradicting the assumption that it had order 4. Thus $(\pi_8^s)_2^\wedge \cong (\mathbb{Z}/2\mathbb{Z})^2$.

For $i = 9$, we again have a filtration

$$\{0\} \subseteq F^{5,14} \subseteq F^{4,13} \subseteq F^{3,12} = (\pi_9^s)_2^\wedge,$$

with quotients isomorphic to $\mathbb{Z}/2\mathbb{Z}$. From Figure 5.1 we see that $(\pi_9^s)_2^\wedge$ cannot be isomorphic to $\mathbb{Z}/8\mathbb{Z}$, since for any representative α of $h_1^2 h_3$, $0 = h_0^2 h_1^2 h_3 \in F^{5,14}$ is equal to 4α. However, we cannot immediately discount the possibility that $(\pi_9^s)_2^\wedge \cong \mathbb{Z}/4\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$; indeed, the filtration

$$\{0\} \subseteq \mathbb{Z}/2\mathbb{Z} = \langle (2,0) \rangle \subseteq (\mathbb{Z}/2\mathbb{Z})^2 = \langle (2,0), (0,1) \rangle \subseteq \mathbb{Z}/4\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} = \langle (1,0), (0,1) \rangle$$

would also give rise to a tower of three nodes unconnected by vertical lines on the E_2 page. This is because a generator $a \in (\mathbb{Z}/4\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z})/(\mathbb{Z}/2\mathbb{Z})^2 \cong \mathbb{Z}/2\mathbb{Z}$ can be represented by $(1,0) \in \mathbb{Z}/4\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$, so $[(2,0)] \in (\mathbb{Z}/2\mathbb{Z})^2/(\mathbb{Z}/2\mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z}$ is a representative for $h_0 \cdot a$. However, the final copy of $\mathbb{Z}/2\mathbb{Z}$ in the filtration is generated by $(2,0)$, so $[(2,0)] = 0$ in $(\mathbb{Z}/2\mathbb{Z})^2/(\mathbb{Z}/2\mathbb{Z})$, meaning that $h_0 \cdot a = 0$. Also, any generator β of $(\mathbb{Z}/2\mathbb{Z})^2$ doubles to zero, so any element $b \in (\mathbb{Z}/2\mathbb{Z})^2/(\mathbb{Z}/2\mathbb{Z})$ satisfies $h_0 \cdot b = 0$.

The situation described above is known as a 'hidden extension', since it cannot be read off from the spectral sequence diagram. However, we can often determine by other methods whether or not they arise.

In our case, this does not in fact occur; let $h_3 \in E_2^{1,8}$ be represented by $\sigma \in (\pi_7^s)_2^\wedge$, and let $h_1 \in E_2^{1,2}$ be represented by $\eta \in (\pi_1^s)_2^\wedge$. Then $h_1 h_3$ is represented by $\eta\sigma$, and $2(\eta\sigma) = (2\eta)\sigma = 0$, since $(\pi_1^s)_2^\wedge \cong \mathbb{Z}/2\mathbb{Z}$. Thus, $(\pi_9^s)_2^\wedge \cong (\mathbb{Z}/2\mathbb{Z})^3$. □

5.3 Differentials at $14 \leq t - s \leq 15$

For $t - s < 14$, the computation of $(\pi_{t-s}^s)_2^\wedge$ involves only the E_2 page of the spectral sequence, since there are no nontrivial differentials in this range. However, the first nonzero differential will appear in the range $14 \leq t - s \leq 15$, and in fact there are many differentials after this point, though we will only fully compute those in this range. In general, the

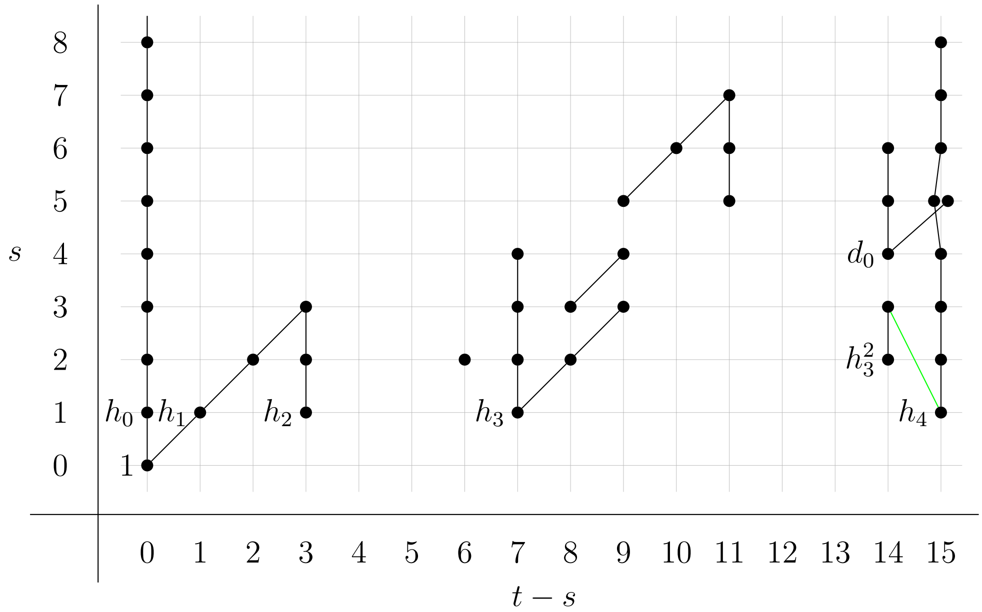


Figure 5.2: The E_2 page of the Adams spectral sequence for $\mathbb{S}$, in the range $t-s \leq 15$; the unique d_2 differential is shown in green.

problem of computing differentials is much harder than determining $\mathrm{Ext}^{s,t}_{\mathscr{A}_2}(\mathbb{F}_2, \mathbb{F}_2)$, and is not algorithmic.

THEOREM 5.3.1 ([14], Thm 11.10.2). $d_2(h_4) = h_0 h_3^2 \neq 0$.

PROOF. We have shown that h_0 detects $2 \in (\pi_*^s)_2^\wedge$ (i.e. 2 is a representative for h_0). Let $\sigma \in (\pi_7^s)_2^\wedge$ be a representative for h_3. Then $2\sigma^2$ is a representative for $h_0 h_3^2$. By graded commutativity of $(\pi_*^s)_2^\wedge$, $\sigma^2 = -\sigma^2$, so $2\sigma^2 = 0$, and thus $h_0 h_3^2 = 0$ in $E_\infty^{3,17}(\mathbb{S})$ (since $2\sigma^2 = 0$ is a representative for it). Therefore, $h_0 h_3^2$ is the boundary of a differential, so we must have $d_2(h_4) = h_0 h_3^2$. $\square$

Note that the d_2 differentials at $E_2^{2,17}(\mathbb{S}), E_2^{3,18}(\mathbb{S}), E_2^{4,19}(\mathbb{S})$ are all trivial, since $d_2(h_0^n h_4) = h_0^n d_2(h_4) = h_0^{n-1}(h_0 h_3^2) = 0$.

There are two possible d_3 differentials for $t-s \leq 15$ (emanating from $E_3^{2,17}$ and $E_3^{3,18}$), and in fact it will turn out that both are nontrivial. The method of proof will be by comparison to the Adams spectral sequence of a different spectrum, so we will first state a result comparing the Adams spectral sequences of two spectra with a map between them.

THEOREM 5.3.2 ([15], Cor 4.17). Let $f : Y \to Z$ be a map of connective spectra of finite type. Then there is a map

$$f_* : \{E_r(Y), d_r\}_r \to \{E_r(Z), d_r\}_r$$

of Adams spectral sequences, given at the E_2-level by the homomorphism

$$(f^*)^* : \mathrm{Ext}^{s,t}_{\mathscr{A}_2}(H^*(Y), \mathbb{F}_2) \to \mathrm{Ext}^{s,t}_{\mathscr{A}_2}(H^*(Z), \mathbb{F}_2)$$

induced by the $\mathscr{A}_2$-module homomorphism $f^* : H^*(Z) \to H^*(Y)$, converging to the homomorphism

$$f_* : \pi_*(Y) \to \pi_*(Z).$$

Let X be a connective spectrum of finite type. Then $\mathrm{Ext}^{*,*}_{\mathscr{A}_2}(H^*(X), \mathbb{F}_2)$ is an $(\mathrm{Ext}^{*,*}_{\mathscr{A}_2}(\mathbb{F}_2, \mathbb{F}_2))$-module, with action given by specialising the Yoneda product of Definition 4.6.1 to $M = N = \mathbb{F}_2$, $L = H^*(X)$.

REMARK 5.3.3. For any map $f : \mathbb{S} \to X$, where X is a connective spectrum of finite type, the induced map

$$(f^*)^* : \mathrm{Ext}^{s,t}_{\mathscr{A}_2}(\mathbb{F}_2, \mathbb{F}_2) \to \mathrm{Ext}^{s,t}_{\mathscr{A}_2}(H^*(X), \mathbb{F}_2)$$

satisfies $(f^*)^*(\alpha\beta) = \alpha \cdot (f^*)^*(\beta)$ for any $\alpha, \beta \in \mathrm{Ext}^{s,t}_{\mathscr{A}_2}(\mathbb{F}_2, \mathbb{F}_2)$. This follows from the definition of the Yoneda product, since both $(f^*)^*(\alpha\beta)$ and $\alpha \cdot (f^*)^*(\beta)$ arise filling in the same diagram with dotted lifts, forming the same chain maps up to chain homotopy.

LEMMA 5.3.4 ([14], Table 14.1 (9)). $d_2(f_0), d_2(e_0) \neq 0$.

PROOF. Let i, j, k, l be as shown in Figure 5.3. One can calculate (using e.g. [5]) that $h_4 i = 0$ and $h_0 h_3^2 i \neq 0$. Now, $d_2(i)$ is nontrivial, since $h_4 i = 0$ and $h_0 h_3^2 i \neq 0$, so $0 = d_2(h_4 i) = h_0 h_3^2 i + h_4 d_2(i)$ implies that $d_2(i) \neq 0$. Further, $d_2(j) \neq 0$ since $h_0 d_2(j) = d_2(h_0 j) = d_2(h_2 i) = h_2 d_2(i) \neq 0$. An almost identical argument shows that $d_2(k), d_2(l) \neq 0$, and thus $d_2(h_0 l) = h_0 d_2(l) \neq 0$. Finally, we have $h_0 l = d_0 f_0$, so $0 \neq d_2(h_0 l) = d_2(d_0 f_0) = d_0 d_2(f_0)$.

Now, $d_2(f_0) \neq 0$, so looking at Figure 5.3 we see that $0 \neq h_0 d_2(f_0) = d_2(h_0 f_0) = d_2(h_1 e_0) = h_1 d_2(e_0)$, and thus $d_2(e_0) \neq 0$. □

In the following proof, we write $\sigma \in \pi_7^s$ for the stable class of the octonionic Hopf fibration $S^{15} \to S^8$, which can be shown to detected by h_3 (recall that we previously denoted by σ an arbitrary generator of π_7^s).

LEMMA 5.3.5 ([14], Table 14.9 (4)). Consider the cofibration

$$\mathbb{S}^7 \xrightarrow{\sigma} \mathbb{S} \xrightarrow{i} C_\sigma \xrightarrow{j} \mathbb{S}^8 \to \cdots .$$

Let $a \in E_2^{3,18}(C_\sigma)$ be the generator shown in Figure 5.4. Then $d_2(a) \neq 0$.

PROOF. Let $\hat{i} = (i^*)^* : \mathrm{Ext}^{s,t}_{\mathscr{A}_2}(\mathbb{F}_2, \mathbb{F}_2) \to \mathrm{Ext}^{s,t}_{\mathscr{A}_2}(C_\sigma, \mathbb{F}_2)$ be the map induced by $i^* : H^*(C_\sigma) \to H^*(\mathbb{S})$. We first show that $d_2(h_2 \cdot a) = d_2(\hat{i}(f_0))$. By Lemma 3.3.6, we have a long exact sequence

$$\cdots \leftarrow H^{n-1}(\mathbb{S}^8) \leftarrow H^n(\mathbb{S}) \xleftarrow{i^*} H^n(C_\sigma) \xleftarrow{j^*} H^n(\mathbb{S}^8) \leftarrow H^{n+1}(\mathbb{S}) \leftarrow \cdots .$$

However, any map $H^n(\mathbb{S}) \to H^{n-1}(\mathbb{S}^8)$ must be zero, so we get short exact sequences

$$0 \leftarrow H^n(\mathbb{S}) \xleftarrow{i^*} H^n(C_\sigma) \xleftarrow{j^*} H^n(\mathbb{S}^8) \leftarrow 0.$$

Taking a direct sum gives a short exact sequence

$$0 \leftarrow \mathbb{F}_2 \xleftarrow{i^*} H^*(C_\sigma) \xleftarrow{j^*} \mathbb{F}_2[8] \leftarrow 0,$$

and from this we get a short exact sequence of chain complexes

$$0 \to \mathrm{Hom}(\mathbb{F}_2, I_\bullet) \xrightarrow{i^*} \mathrm{Hom}(H^*(C_\sigma), I_\bullet) \xrightarrow{j^*} \mathrm{Hom}(\mathbb{F}_2[8], I_\bullet) \to 0,$$

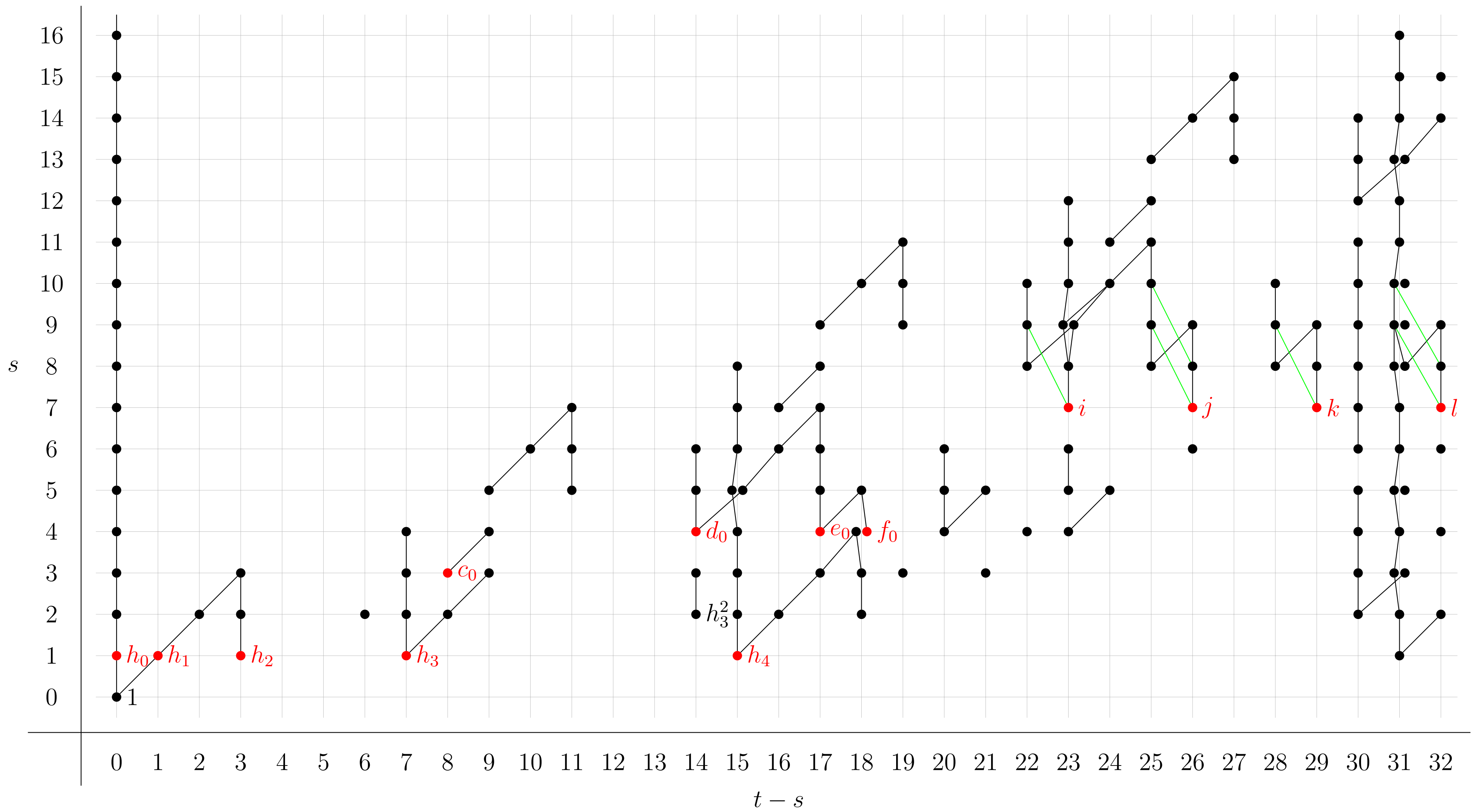


Figure 5.3: $\mathrm{Ext}^{s,t}_{\mathscr{A}_2}(\mathbb{F}_2, \mathbb{F}_2)$ for $t - s \leq 32$. The vertical and diagonal lines indicate multiplication by h_0 and h_1 respectively. Some of the algebra generators are shown in red, with naming conventions as in [14]. The d_2 differentials referenced in the proof of Lemma 5.3.4 are shown in green.

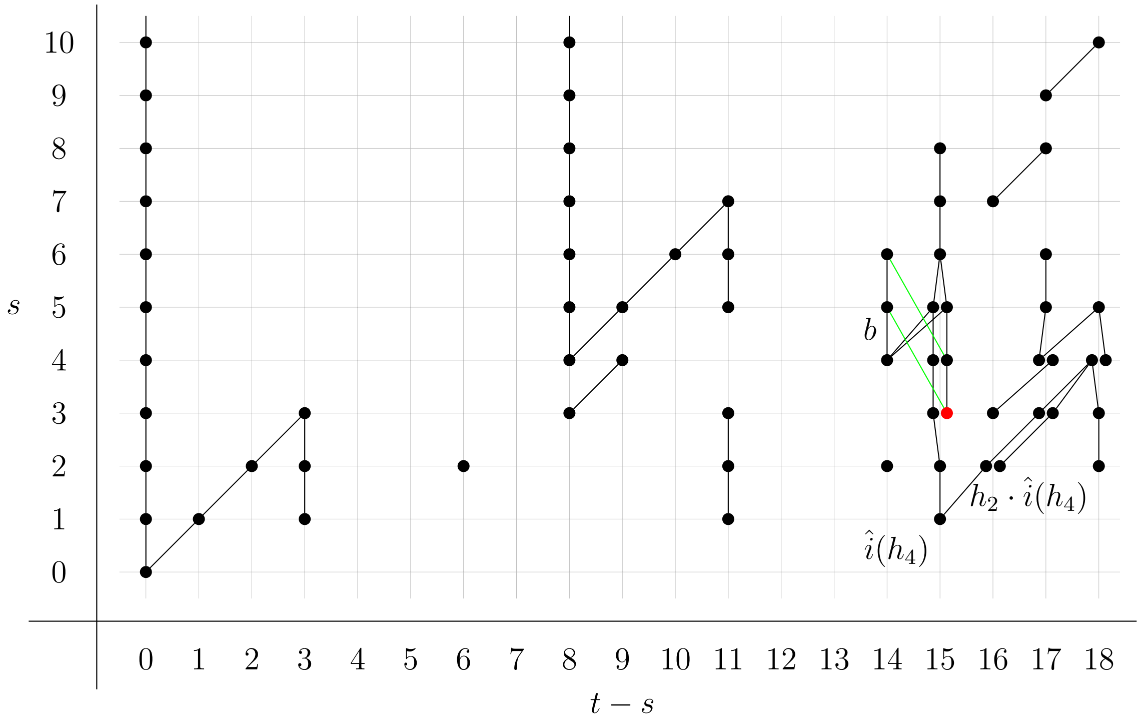


Figure 5.4: The E_2 page of the Adams spectral sequence for C_σ, in the range $t-s \leq 18$, with the generator a shown in red, and two of the differentials shown in green.

for any injective resolution I, and thus the long exact sequence below.

$$\cdots \longrightarrow \mathrm{Ext}^{s,t}_{\mathscr{A}_2}(\mathbb{F}_2,\mathbb{F}_2) \xrightarrow{\hat{i}} \mathrm{Ext}^{s,t}_{\mathscr{A}_2}(H^*(C_\sigma),\mathbb{F}_2) \xrightarrow{\hat{j}} \mathrm{Ext}^{s,t-8}_{\mathscr{A}_2}(\mathbb{F}_2,\mathbb{F}_2) \longrightarrow$$
$$\longrightarrow \mathrm{Ext}^{s+1,t}_{\mathscr{A}_2}(\mathbb{F}_2,\mathbb{F}_2) \xrightarrow{\hat{i}} \mathrm{Ext}^{s+1,t}_{\mathscr{A}_2}(H^*(C_\sigma),\mathbb{F}_2) \xrightarrow{\hat{j}} \mathrm{Ext}^{s+1,t-8}_{\mathscr{A}_2}(\mathbb{F}_2,\mathbb{F}_2) \longrightarrow \cdots$$

Now, $f_0 \in \mathrm{Ext}^{4,22}_{\mathscr{A}_2}(\mathbb{F}_2,\mathbb{F}_2)$; we consider the exact sequence

$$\mathrm{Ext}^{3,14}_{\mathscr{A}_2}(\mathbb{F}_2,\mathbb{F}_2) \to \mathrm{Ext}^{4,22}_{\mathscr{A}_2}(\mathbb{F}_2,\mathbb{F}_2) \xrightarrow{\hat{i}} \mathrm{Ext}^{4,22}_{\mathscr{A}_2}(H^*(C_\sigma),\mathbb{F}_2).$$

Figure 5.1 shows us that $\mathrm{Ext}^{3,14}_{\mathscr{A}_2}(\mathbb{F}_2,\mathbb{F}_2) = 0$, so $\hat{i}$ is injective at this point, and thus $\hat{i}(f_0) \neq 0$. Similarly, $\mathrm{Ext}^{4,15}_{\mathscr{A}_2}(\mathbb{F}_2,\mathbb{F}_2) = 0$ and $\mathrm{Ext}^{1,12}_{\mathscr{A}_2}(\mathbb{F}_2,\mathbb{F}_2) = 0$, so $\hat{i}(h_0 f_0), \hat{i}(h_2 h_4) \neq 0$. Since $E_2(C_\sigma)$ is an $E_2(\mathbb{S})$-module and $\hat{i}$ respects the $E_2(\mathbb{S})$ action (by Remark 5.3.3), $\hat{i}(h_0 f_0) = h_0 \cdot \hat{i}(f_0) \neq 0$, so $\hat{i}(f_0)$ is equal to either $h_2 \cdot a$ or $h_2 \cdot a + h_0^2 \cdot \hat{i}(h_2 h_4)$. Now, $d_2(h_2 h_4) = 0$, since otherwise it would be equal to e_0, and we would have $d_2^2(h_2 h_4) \neq 0$, contradicting the fact that d_2 is a differential. Thus, by linearity of d_2, we have $d_2(\hat{i}(f_0)) = d_2(h_2 \cdot a)$ (since $d_2(h_0^2 \cdot \hat{i}(h_2 h_4)) = d_2(\hat{i}(h_0^2 h_2 h_4)) = \hat{i}(h_0^2 d_2(h_2 h_4)) = 0$).

Finally, $\hat{i}(h_0^2 e_0) \neq 0$, since $\mathrm{Ext}^{5,15}_{\mathscr{A}_2}(\mathbb{F}_2,\mathbb{F}_2) = 0$ (using the long exact sequence in Ext again). Thus, $d_2(h_2 \cdot a) = d_2(\hat{i}(f_0)) = \hat{i}(d_2(f_0)) = \hat{i}(h_0^2 e_0) \neq 0$. Therefore, $d_2(a) \neq 0$. □

THEOREM 5.3.6 ([14], Table 14.2 (10)). *$d_3(h_0 h_4) = h_0 d_0$ in $E_3(\mathbb{S})$.*

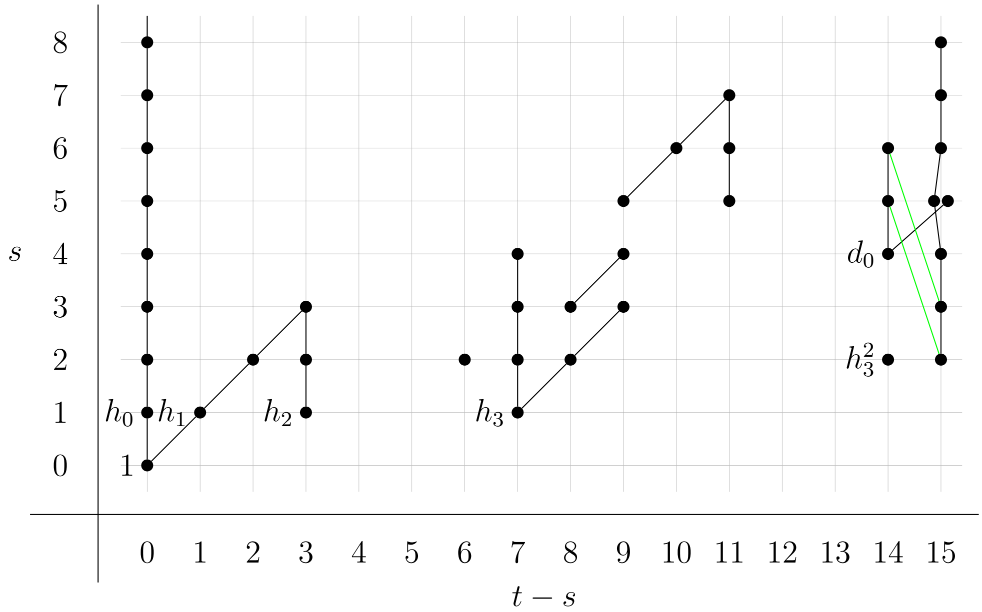


Figure 5.5: The E_3 page of the Adams spectral sequence for $\mathbb{S}$, in the range $t-s \leq 15$; the differentials are shown in green.

PROOF. From the cofibration

$$\mathbb{S}^7 \xhookrightarrow{\sigma} \mathbb{S} \xrightarrow{i} C_\sigma \xrightarrow{j} \mathbb{S}^8 \hookrightarrow \mathbb{S}^1 \to \cdots,$$

we get an exact sequence

$$\pi_7^s \xrightarrow{\sigma_*} \pi_{14}^s \xrightarrow{i_*} \pi_{14}(C_\sigma) \xrightarrow{j_*} \pi_6^s \to \pi_{13}^s,$$

by Theorem 3.3.5. Since these stable homotopy groups are all finite[1], this induces an exact sequence

$$(\pi_7^s)_2^\wedge \xrightarrow{\sigma_*} (\pi_{14}^s)_2^\wedge \xrightarrow{i_*} \pi_{14}(C_\sigma)_2^\wedge \xrightarrow{j_*} (\pi_6^s)_2^\wedge \xrightarrow{\sigma_*} (\pi_{13}^s)_2^\wedge = 0.$$

In $E_2(C_\sigma)$ we have $d_2(a) = b$, where b is the generator shown in Figure 5.4 (by Lemma 5.3.5), so $\pi_{14}(C_\sigma)_2^\wedge$ has order dividing four. Let $\nu \in (\pi_3^s)_2^\wedge$ be a representative for h_2. Then $(\pi_6^s)_2^\wedge = \mathbb{Z}/2\mathbb{Z}\langle \nu^2 \rangle$, and $\nu^2\sigma = 0$. By exactness, we see that j_* is surjective, so $(\pi_6^s)_2^\wedge \cong \pi_{14}(C_\sigma)_2^\wedge / \ker j_* = \pi_{14}(C_\sigma)_2^\wedge / \operatorname{im} i_*$. We know $\pi_{14}(C_\sigma)_2^\wedge$ has order dividing 4 and $(\pi_6^s)_2^\wedge$ has order 2, so $\operatorname{im} i_*$ has order dividing 2.

Now, $(\pi_7^s)_2^\wedge = \mathbb{Z}/16\mathbb{Z}\langle \sigma \rangle$, and $2\sigma^2 = 0$ by graded commutativity, so the first isomorphism theorem implies that $(\pi_{14}^s)_2^\wedge$ has order dividing four. Thus, $h_0 d_0$ and $h_0^2 d_0$ must be boundaries, and $d_3(h_0 h_4) = h_0 d_0$ is the only possibility. □

THEOREM 5.3.7.

$$(\pi_i^s)_2^\wedge = \begin{cases} (\mathbb{Z}/2\mathbb{Z})^2 & i = 14, \\ \mathbb{Z}/32\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} & i = 15. \end{cases}$$

PROOF. From Figure 5.6, we see that $(\pi_{15}^s)_2^\wedge = \mathbb{Z}/32\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$, and that $|(\pi_{14}^s)_2^\wedge| = 2$. Now, if h_3 is represented by $\sigma \in (\pi_7^s)_2^\wedge$, then h_3^2 is represented by σ^2. By graded commutativity, $\sigma^2 = -\sigma^2$, so $2\sigma^2 = 0$. Thus, since h_3^2 is represented by a generator of $(\pi_{14}^s)_2^\wedge$, we have $(\pi_{14}^s)_2^\wedge \cong (\mathbb{Z}/2\mathbb{Z})^2$. □

[1] A priori $\pi_{14}(C_\sigma)$ is only finitely generated, but from Figure 5.4 we see that its 2-completion is finite, so the group itself must be finite.

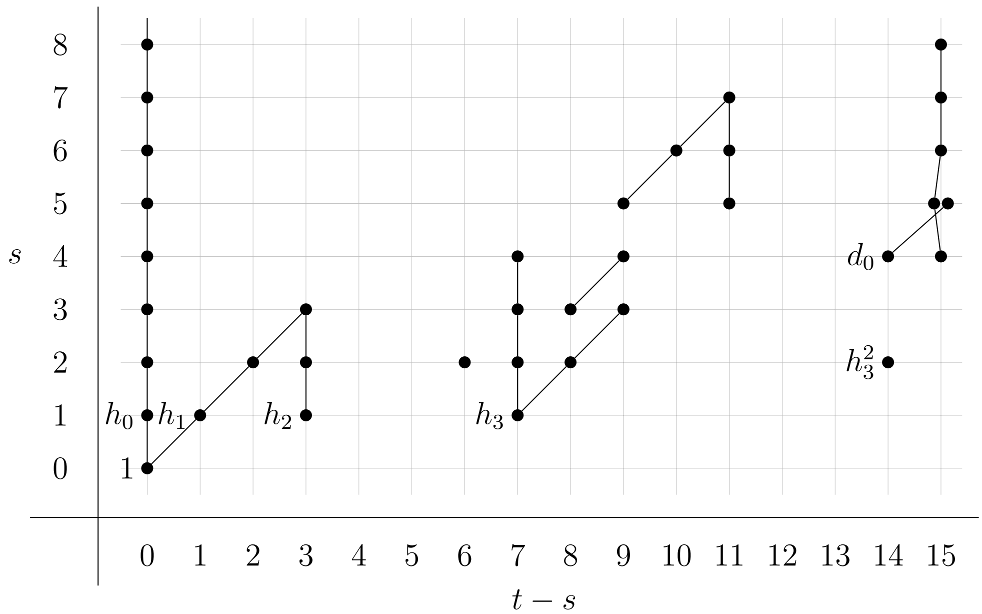


Figure 5.6: The E_4 page of the Adams spectral sequence for $\mathbb{S}$, in the range $t-s \leq 15$. There are no possible higher differentials, so this coincides with the E_∞ page for $t-s \leq 15$.

A Topology

All from [7] unless otherwise stated.

A.1 Suspension

DEFINITION 1.1.1. Let X be a topological space. The *suspension* SX is the space $(X \times I)/\sim$, where $(x, 0) \sim (x', 0)$ and $(x, 1) \sim (x', 1)$ for all $x, x' \in X$.

DEFINITION 1.1.2. Let X be a pointed topological space. The *reduced suspension* ΣX is the space $SX/\sim$, where $[x_0, t] \sim [x_0, t']$ for all $t, t' \in I$.

Given a map $f : X \to Y$, we can define $\Sigma f : \Sigma X \to \Sigma Y$ by $\Sigma f[(x, t)] = [(fx, t)]$. This makes Σ into a functor $\Sigma : \mathbf{Top} \to \mathbf{Top}$.

REMARK 1.1.3. Σ is faithful, since for any maps $f, g : X \to Y$, if $\Sigma f = \Sigma g$ then in particular $[(fx, \frac{1}{2})] = [(gx, \frac{1}{2})]$, so $fx = gx$.

[below is reconstructed from [11]]

Given pointed maps $f, g : \Sigma X \to Z$, define

$$\begin{aligned} f \star g : \Sigma X &\to Z \\ [x, t] &\mapsto \begin{cases} f[x, 2t-1] & t \geq \frac{1}{2}, \\ g[x, 2t] & t \leq \frac{1}{2}. \end{cases} \end{aligned}$$

This is well defined, since both f and g are basepoint-preserving.

REMARK 1.1.4. This defines a group structure on $[\Sigma X, Z]$, and thus $[\Sigma^i X, Z]$ is a group for all $i \geq 1$. For $i \geq 2$, these can be shown to be abelian, via the Eckmann-Hilton argument. The suspension map $[\Sigma X, Y] \to [\Sigma^2 X, \Sigma Y]$ is a homomorphism.[2]

REMARK 1.1.5. The homotopy groups $\pi_i(Z)$ are a special case of the above construction, taking $X := S^{i-1}$.

- Loops; the adjunction $\Sigma \dashv \Omega$, where Ω is the loop functor.

[7], p395:

REMARK 1.1.6. It follows that $\pi_{n+1}(X) \cong \pi_n(\Omega X)$. In particular, $\Omega K(G, n)$ is a $K(G, n-1)$.

- [7] 2.1 Ex 20 and 2.2 Ex 32: $\widetilde{H}_n(X) \cong \widetilde{H}_{n+1}(SX)$, where S is the (non-reduced) suspension. (MV?)
- Hatcher also says on p219 that $\widetilde{H}^n(X; R) \cong \widetilde{H}^{n+k}(\Sigma^k X; R)$, where Σ is reduced suspension.

A.2 Other basic constructions

DEFINITION 1.2.1. Let $(X, x_0), (Y, y_0)$ be pointed topological spaces, and consider their product $X \times Y$. The subspaces $X \times \{y_0\} \cong X$ and $\{x_0\} \times Y \cong Y$ intersect at exactly one

[2] Probably follows from the result for $\pi_*(Y)$ and induction on the cells of X, but I'll check this.

point, (x_0, y_0), and so can be identified with the wedge $X \vee Y$. We thus define the *smash product* $X \wedge Y := (X \times Y)/(X \vee Y)$, with the canonical basepoint (x_0, y_0).

EXAMPLE 1.2.2. We have $S^n \wedge S^m \cong S^{n+m}$. [is this obvious?]

REMARK 1.2.3. Note that $\Sigma X \cong X \wedge S^1$.

REMARK 1.2.4. Observe that $X \wedge (Y \wedge Z) \cong (X \wedge Y) \wedge Z$. Combining this with the remarks above, we see that $\Sigma^k X \cong X \wedge S^k$.

REMARK 1.2.5. Note that $\Sigma(X \vee Y) \cong \Sigma X \vee \Sigma Y$.

- An Eilenberg-MacLane space is $K(G, n)$, and it has the property that

$$\pi_i(K(G, n)) = \begin{cases} G & i = n, \\ 0 & i \neq n. \end{cases}$$

  They're unique up to weak homotopy equivalence (i.e. if you have another one X, there's a map between them which descends to an isomorphism on homotopy groups). They can be taken to be CW complexes.

DEFINITION 1.2.6. Let X, Y be topological spaces, where X has a basepoint x_0. Then the *reduced product* $X \times_{\text{red}} Y := (X \times Y)/(x_0 \times Y)$.

DEFINITION 1.2.7. Let $f : X \to Y$ be a map. The *mapping cylinder* M_f is defined by $((X \times I) \sqcup Y)/\sim$, where $(x, 1) \sim f(x)$ for all $x \in X$. If $(X, x_0), (Y, y_0)$ are pointed spaces, the *reduced mapping cylinder* is the quotient $M_f/\sim$, where $[x_0, t] \sim [x_0, t']$ for all $t \in I$.

REMARK 1.2.8. The mapping cylinder deformation retracts onto Y via $h : M_f \times I \to M_f$; $([x, t], s) \mapsto [x, t + s(1 - t)]$.

DEFINITION 1.2.9. Let $f : X \to Y$ be a map. The *mapping cone*[3] C_f is defined to be $Y \sqcup_f CX := (Y \sqcup CX)/(f(x) \sim [x, 1])$.

Relative Künneth Theorem:

THEOREM 1.2.10 ([7]). For CW pairs $(X, A), (Y, B)$, the cross product homomorphism $H^*(X, A; R) \otimes_R H^*(Y, B; R) \to H^*(X \times Y, A \times Y \cup X \times B; R)$ is an isomorphism of rings if $H^k(Y, B)$ is a finitely generated free R-module for each k.

In particular, for pointed spaces $(X, x_0), (Y, y_0)$, we have an isomorphism

$$\bigoplus_{i+j=n} H^i(X, x_0; R) \otimes_R H^j(Y, y_0; R) \to H^n(X \times Y, X \vee Y; R).$$

Or, in other words,

$$\bigoplus_{i+j=n} \widetilde{H}^i(X; R) \otimes_R \widetilde{H}^j(Y; R) \to \widetilde{H}^n(X \wedge Y; R).$$

Setting $Y = S^1$, we get an isomorphism

$$\widetilde{H}^{n-1}(X; R) \to \widetilde{H}^n(\Sigma X; R).$$

[3]Why does Hatcher not insist this guy is reduced, like he does with the mapping cylinders?

A.3 Cell complexes

Definition 1.3.1. Let X be a cell complex, $A \subseteq X$ a subcomplex. Then the quotient X/A has a cell complex structure, with cells the cells of $X \setminus A$ along with a basepoint (the image of A in X).

Definition 1.3.2. Let $f : X \to Y$ be a map between CW complexes. Then f is *cellular* if $f(X_{(n)}) \subseteq Y_{(n)}$ for all n, where $X_{(n)}$ is the n-skeleton of X.

Cellular approximation theorem:

Theorem 1.3.3 ([7], Thm 4.8). Let $f : X \to Y$ be a map of CW complexes. Then f is homotopic to a cellular map.

Lemma 1.3.4 ([7], Prop 0.16). Let $A \subseteq X$ be CW complexes. Then the pair (X, A) has the *homotopy extension property*; that is, for any map $f : X \to Y$ and homotopy $h : A \times I \to Y$ such that $h(a, 0) = f|_A$, there is a homotopy $\widetilde{h} : X \times I \to Y$ extending h.

- The product of cell complexes is a cell complex (maybe only if one of them is finite?)
- The smash product of (pointed?) cell complexes is a cell complex (maybe only if one is them is finite?) [[7] says "the smash product $X \wedge Y$ is a cell complex if X and Y are cell complexes with x_0 and y_0 0-cells, assuming that we give $X \times Y$ the cell-complex topology rather than the product topology in cases where these two topologies differ".]
- For a CW complex X, $SX \simeq \Sigma X$.
- The reduced suspension of a pointed cell complex (X, x_0) is another pointed cell complex ΣX with basepoint x_0 and an n-cell for each non-basepoint $n-1$ cell e_α^{n-1} of X.

Definition 1.3.5. Let X is a topological space. A *CW approximation* to X is a CW complex Z equipped with a weak homotopy equivalence $f : Z \to X$.

Theorem 1.3.6 ([7], Prop 4.13). Every space X has a CW approximation $f : Z \to X$.

- In particular, $\Omega K(G, n)$ has a CW approximation $Z \to \Omega K(G, n)$, and since $\Omega K(G, n)$ is a $K(G, n-1)$, so is Z.

Any finite CW complex is compact.

Proposition 1.3.7 ([7], A.1). A compact subspace of a CW complex is contained in a finite subcomplex.

B Notes to self

B.1 Vague problems and questions....

B.1.1 ...that probably don't matter

- On p588 of [8], he says "every CW spectrum is equivalent to a suspension spectrum". Does he actually mean that, or does he mean 'equivalent to the suspension of a spectrum'? The former seems way too strong, although in fairness I still don't know what an equivalence of spectra actually *is*.
- On p586 of [8], Hatcher says "If X is of finite type then for each i there is an n such that X_n contains all the i-cells of X. It follows that $H_i(X; G) = H_i(X_n; G)$ for all sufficiently

large n, and the same is true for cohomology." But from the way he set up H_* and H^* earlier, shouldn't this be $H_i(X;G) = H_{i+n}(X_n;G)$? Because $H_i(X;G) = \lim_{\rightarrow} H_{i+n}(X_n)$, and he talks about things stabilising in the next sentence, so shouldn't the stable point be at some H_{i+n}?

- I write $\mathscr{A}_2$ where Hatcher writes $\mathscr{A}$. We mean the same thing, right...?

B.1.2 ...that probably do matter

- The Leibniz rule is $d_r(xy) = d_r(x)y \pm x d_r(y)$ (can't remember the sign). But anything I'm using that rule on is some generator of an $\mathbb{F}_2$, right? So the sign shouldn't matter. But then, shouldn't the Yoneda product be graded commutative (and thus commutative, because again, in the target signs don't matter)? So why does [15] have some comment (in Cor 6.5) about how the Yoneda product is commutative "in [some] range"??

- On p592 of [8], he says that "for spectra X of finite type [the more general] definition of an $\mathscr{A}_2$-module structure on $H^*(X)$ agrees with the definition using the usual $\mathscr{A}_2$-module structure on the cohomology of spaces and the identification of $H^*(X)$ with the inverse limit $\lim_{\leftarrow} H^{*+n}(X_n)$". Um? Sure, we have that each $H^{i+n}(X_n)$ stabilises eventually, but is Hatcher saying $H^{*+n}(X_n)$ stabilises? Like, as an $\mathscr{A}_2$-module? And if not, what's going on here? Because inverse limits don't commute with infinite direct sums - they're not biproducts anymore, they're coproducts and there's no reason limits should commute with them.

- There's something weird going on with products. So, things are ok in **Top**, because we have the ordinary product of two spaces, which is a categorical product. But with CW complexes, supposedly sometimes the product topology differs from the 'cell complex topology'? But, regardless, we're supposed to be working with pointed things - so in $\mathbf{Top}_*$, the pointed coproduct is the wedge sum, and the pointed product is just the normal product $X \times Y$ with the basepoint (x_0, y_0) (it's not the smash product). But what about in spectra? No one ever seems to talk about products of spectra, but for example a collection of maps $X \to \mathbb{K}(G, n_i)$ should correspond to a single map $X \to \prod_i \mathbb{K}(G, n_i)$, whatever that last object is.

  The plot thickens. From the nLab: "[some smash product] is non-canonically equivalent to a product of EM-spectra (hence a wedge sum of EM-spectra in the finite case)". ???????

- I'm not happy with Theorem 4.4.2...

B.2 To do

Now:

Eventually:

- Be consistent with either cell complex or CW complex.
- Be consistent with $\mathbb{F}_2$ or $\mathbb{Z}/2\mathbb{Z}$ (don't use $\mathbb{Z}_2$, that's really bad).
- Specialise the Adams spectral sequence (i.e. set $Y = \mathbb{S}$).

- Remember that you have to hand in the tex file, so for the love of god change anything stupid that's hidden in the pdf.
- Sometimes I say π_*^s or ${}_{(2)}\pi_*^s$ (localised at 2?) instead of its completion at 2 or whatever. So make sure it's correct.
- Stick to a convention on suspension/cone/homotopy numbering. I.e. Does a homotopy start at 0 or 1? Does a suspension go from -1 to 1 with the space in the middle at 0, or 0 to 1 with the space at 1/2? Do cones go from 0 to 1, and if so, make sure when they include into suspensions they do so consistently.
- Have any sort of consistency in using or not using brackets (e.g. $\pi_t X_s$ v.s. $\pi_t(X_s)$).
- When I say 'spectrum' at any point after defining CW spectra I mean 'CW spectrum'. And I basically always mean 'connective CW spectrum of finite type' too.
- Be consistent with the composition product (i.e. does $f \otimes g$ get sent to $f \circ \Sigma^i g$ or $g \circ \Sigma^j f$?)

B.3 Other notes

- READ IF YOUR CALCULATIONS AREN'T WORKING: You are working modulo 2!!!
- If you have a bunch of maps between graded modules/algebras, they're graded homomorphisms. So they preserve degree.
- All (co)homology is supposed to be reduced.
- Signs don't matter with the Leibniz rule either!! You are working modulo 2!!!!!!!!
- Remember, once you know that $d_2(h_4) = h_0 h_3^2$, you know h_4 *doesn't survive to the third page*. So, for example, $d_3(h_0 h_4) \neq h_0 d_3(h_4)$ because h_4 doesn't exist anymore. That's why $d_3(h_0 h_4)$ can be nonzero.
- As previously mentioned, we are working modulo 2!! What this also implies is that if anything is hit by any sort of differential, or has any nonzero differential coming out of it, it's completely killed by the next page. Because the summands are just a bunch of $\mathbb{F}_2$'s (so you don't need to worry about 'how much' of something is killed, it all is).
- Sometimes Hatcher says that you can replace any map of CW complexes by an inclusion. I think the point here is that if you have a map $f : X \to Y$, Remark A.2.8 says that M_f deformation retracts onto Y. So if you only care about X and Y up to homotopy equivalence, you can replace Y by M_f and then X definitely includes into M_f.
- Where it's ambiguous, I'm marking things I definitely need by ! and things I think I may not need by ?.
- In literature, $A_p^\wedge$ is the p-adic completion of A. Sometimes I'll write this as ${}_pA$ because of some stupid notational decisions I made earlier.
- The 'abutment' of a spectral sequence apparently means the thing it converges to (i.e. if E_∞ computes the associated graded of some H^*, the abutment of $\{E\}$ is H^* (not its associated graded)).

- [14] has some n_m notation where n_m is supposed to be the mth generator in row n. This is a bit arbitrary when there are two generators in the same row and column; I don't know how he counts them, but he's using the `ext` program, whereas I'm using sseq. Unless there's some Canonical Ordering, there's no reason why these different programs written by different people would use the same convention. In particular, even though [14] says $\overline{\overline{h_0^2 h_3}} = 3_4$, I'm pretty sure it is the one on the right (i.e. the one I would label 3_5).

Sources I've used: [13], [15], [9], [8], [7], [14], [10], [12], [16]

Sources I probably won't use: [6], [1], [3], [4], [11] (I think the construction I need is in Hatcher)

References

[1] J. F. Adams. *Stable Homotopy and Generalised Homology*. TEXromancers, 2022.

[2] *Adams spectral sequence calculator*. URL: https://spectralsequences.github.io/sseq/ (visited on 29/04/2025).

[3] David Barnes and Constanze Roitzheim. *Foundations of Stable Homotopy Theory*. Cambridge University Press, 2020.

[4] R. R. Bruner. *An Adams Spectral Sequence Primer*. 2009. URL: http://www.rrb.wayne.edu/papers/adams.pdf (visited on 08/02/2025).

[5] Robert R. Bruner. *Ext*. URL: https://rrb.wayne.edu/papers/index.html (visited on 29/04/2025).

[6] Maxine Calle. *The Freudenthal Suspension Theorem*. 2020. URL: https://bpb-us-w2.wpmucdn.com/web.sas.upenn.edu/dist/0/713/files/2020/08/FSTnotes.pdf (visited on 08/02/2025).

[7] Allen Hatcher. *Algebraic Topology*. 2001. URL: https://pi.math.cornell.edu/~hatcher/AT/AT+.pdf (visited on 01/02/2025).

[8] Allen Hatcher. *Spectral Sequences*. URL: https://pi.math.cornell.edu/~hatcher/AT/ATch5.pdf (visited on 01/02/2025).

[9] H. R. Margolis. *Spectra and the Steenrod Algebra*. Elsevier Science Publishers B. V., 1983.

[10] J. P. May and K. Ponto. *More Concise Algebraic Topology*. 2010. URL: https://www.maths.ed.ac.uk/~v1ranick/papers/mayponto.pdf (visited on 01/04/2025).

[11] Aaron Mazel-Gee. *An introduction to spectra*. 2011. URL: https://etale.site/writing/an-introduction-to-spectra.pdf (visited on 19/02/2025).

[12] John McCleary. *A User's Guide to Spectral Sequences*. Cambridge University Press, 2001.

[13] Douglas C. Ravenel. *Complex Cobordism and Stable Homotopy Groups of Spheres*. Academic Press. Inc, 1986.

[14] John Rognes. *Spectral Sequences*. 2010. URL: https://www.uio.no/studier/emner/matnat/math/MAT9580/v21/dokumenter/spseq.pdf (visited on 13/03/2025).

[15] John Rognes. *The Adams Spectral Sequence*. 2012. URL: https://www.mn.uio.no/math/personer/vit/rognes/papers/notes.050612.pdf (visited on 08/02/2025).

[16] Charles A. Weibel. *An Introduction to Homological Algebra*. Cambridge University Press, 1994.