

# Stable Homotopy Groups of Spheres [DRAFT]

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Key:

To do (likely straightforward)

To do (likely difficult)

Problems

## 1 Introduction

- Define homotopy groups

THEOREM 1.0.1 ([11], Thm 1.1.4, Freudenthal suspension theorem). If  $\pi_i(X) = 0$  for  $i \leq k$  (i.e.  $X$  is  $k$ -connected) then the map

$$\begin{aligned} \pi_n(X) &\rightarrow \pi_{n+1}(\Sigma X) \\ [\gamma : S^n \rightarrow X] &\mapsto [\Sigma \gamma : \Sigma S^n = S^{n+1} \rightarrow \Sigma X] \end{aligned}$$

is an isomorphism for  $n \leq 2k$  and surjective for  $n = 2k + 1$

- This implies  $\pi_{n+k}(S^n)$  depends only on  $k$  for  $n \geq k + 2$
- (Obviously be careful with basepoints above)
- Suppose  $X$  is  $k$ -connected. Then, for  $k \geq 0$ ,  $0 = \pi_k(X) \cong \pi_{k+1}(\Sigma X)$ , so whenever a space is  $k$ -connected its suspension is  $k + 1$ -connected.
- As you take suspensions, then, your successive bounds are  $n \leq 2k$ ,  $n + 1 \leq 2k + 2 \implies n \leq 2k + 1$ ,  $n \leq 2k + 2$ , etc ... so the sequence  $\pi_n(X) \rightarrow \pi_{n+1}(\Sigma X) \rightarrow \dots$  will eventually stabilise.
- Thus, if you take the colimit of that direct system, it'll just equal the stable value, with the higher legs just being the inverse isomorphisms.
- ?[13], Cor 1.9 [not 100% convinced of how this follows, but believing it for now]: if  $X$  is a CW complex of dimension  $d$  and  $Y$  a  $(k - 1)$ -connected space, then the suspension homomorphism<sup>1</sup>  $[X, Y] \rightarrow [\Sigma X, \Sigma Y]$  is bijective if  $d < 2k - 1$  and surjective if  $d = 2k - 1$ .
- [11], Thm 1.1.8:  $\pi_{n+k}(S^n)$  is finite for  $k > 0$  except when  $n = 2m$ ,  $k = 2m - 1$ .
- Immediate corollary:  $\pi_i^S$  is finite for  $i > 0$ .

Miscellaneous facts I might need later:

- Cohomology [possibly only of pointed CW complexes] is representable<sup>2</sup>, and its representing object is the Eilenberg-MacLane space. i.e.  $H^n(-; G) \cong \text{Hom}(-, K(G, n))$ .
- $\mathcal{A}_2$  is generated as an algebra by elements  $Sq^{2^k}$  ([5], Prop 4L.8).
- The map  $\mathcal{A}_2 \rightarrow \tilde{H}^*(K(\mathbb{Z}/2\mathbb{Z}, n); \mathbb{Z}/2\mathbb{Z})$ ,  $Sq^I \mapsto Sq^I(\iota_n)$  is an isomorphism from the degree  $d$  part of  $\mathcal{A}_2$  onto  $H^{n+d}(K(\mathbb{Z}/2\mathbb{Z}, n); \mathbb{Z}/2\mathbb{Z})$  for  $d \geq n$ . In particular, the admissible monomials  $Sq^I$  form an additive basis for  $\mathcal{A}_2$ . Thus,  $\mathcal{A}_2$  is exactly the algebra of all  $\mathbb{Z}/2\mathbb{Z}$  cohomology operations that are stable, commuting with suspension ([6], Cor 5.38).

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<sup>1</sup>Hang on, the what??  $X$  isn't the suspension of anything, why on earth would this be a group?

<sup>2</sup>As a set, or is this some sort of enriched thing? If it's enriched, is that over **Ab** or **Rng**? [5] says on p394 that there is a natural group structure on  $\text{Hom}(X, K(G, n))$  such that the natural isomorphism  $\text{Hom}(X, K(G, n)) \rightarrow H^n(X; G)$  is in fact an isomorphism of abelian groups. So, it's over **Ab**? N.B: There's a lot of talk about 'reduced cohomology theories', so a good next step would be to figure out what those are - if they only involve groups and not rings, maybe the cup product on  $H^*$  is not relevant here.

- ?Hurewicz theorem: for any path-connected space  $X$  and  $n > 0$  there exists a group homomorphism  $h_* : \pi_n(X) \rightarrow H_n(X)$ . For  $n = 1$  this induces an isomorphism  $\pi_1^{\text{ab}}(X) \cong H_1(X)$ . For  $n \geq 2$ , if  $X$  is  $(n-1)$ -connected then  $\tilde{H}_i(X) = 0$  for all  $i < n$ , and the map  $h_* : \pi_n(X) \rightarrow H_n(X)$  is an isomorphism.

- ?Axioms for a homology theory  $h_*$ :

1. If  $f \simeq g : X \rightarrow Y$ , then  $f_* = g_* : h_n(X) \rightarrow h_n(Y)$ .
2. For each  $A \subseteq X$ , there is a long exact sequence

$$\cdots \xleftarrow{\partial} h_n(X/A) \xleftarrow{q_*} h_n(X) \xleftarrow{i_*} h_n(A) \xleftarrow{\partial} h_{n+1}(X/A) \xleftarrow{q_*} \cdots$$

3. For a wedge sum  $X = \bigvee_{\alpha} X_{\alpha}$  with inclusions  $i_{\alpha} : X_{\alpha} \hookrightarrow X$ , the coproduct map  $\bigoplus_{\alpha} (i_{\alpha})_* : \bigoplus_{\alpha} h_n(X_{\alpha}) \rightarrow h_n(X)$  is an isomorphism for each  $n$ .

?Axioms for a (reduced) cohomology theory  $h^*$ :

1. If  $f \simeq g : X \rightarrow Y$ , then  $f^* = g^* : h^n(Y) \rightarrow h^n(X)$ .
2. For each  $A \subseteq X$ , there is a long exact sequence

$$\cdots \xrightarrow{\delta} h^n(X/A) \xrightarrow{q^*} h^n(X) \xrightarrow{i^*} h^n(A) \xrightarrow{\delta} h^{n+1}(X/A) \xrightarrow{q^*} \cdots$$

3. For a wedge sum  $X = \bigvee_{\alpha} X_{\alpha}$  with inclusions  $i_{\alpha} : X_{\alpha} \hookrightarrow X$ , the product map  $(i_{\alpha}^*) : h^n(X) \rightarrow \prod_{\alpha} h^n(X_{\alpha})$  is an isomorphism for each  $n$ .

- ?Stable homotopy groups are a homology theory.

[13], [5]

## 2 The Steenrod algebra

The following is from [5] 4L.

- There are maps  $Sq^i : H^n(-; \mathbb{Z}/2\mathbb{Z}) \rightarrow H^{n+i}(-; \mathbb{Z}/2\mathbb{Z})$  for each  $i$ , and they satisfy the following properties:
  1.  $Sq_X^i(f^*(\alpha)) = f^*(Sq_Y^i(\alpha))$  for  $f : X \rightarrow Y$  (i.e.  $Sq^i$  is a natural transformation).
  2.  $Sq_X^i(\alpha + \beta) = Sq_X^i(\alpha) + Sq_X^i(\beta)$  (i.e.  $Sq_X^i$  respects the group operation for all  $X$ ).
  3.  $Sq^i(\alpha \smile \beta) = \sum_{0 \leq j \leq i} (Sq^j(\alpha) \smile Sq^{i-j}(\beta))$  (the Cartan formula)
  4.  $Sq^i(\sigma(\alpha)) = \sigma(Sq^i(\alpha))$  where  $\sigma : H^n(X; \mathbb{Z}/2\mathbb{Z}) \rightarrow H^{n+1}(\Sigma X; \mathbb{Z}/2\mathbb{Z})$  is the suspension isomorphism given by reduced cross product with a generator of  $H^1(S^1; \mathbb{Z}/2\mathbb{Z})$ . [Something is a little strange here, I think maybe it doesn't matter which suspension isomorphism you pick, though it's not immediately clear why. It's just that the wikipedia page that  $\sigma$  is the connecting homomorphism of the long exact sequence in cohomology. [12] just says it's the suspension isomorphism :|. Something interesting: on p219 of [5], it says "the suspension isomorphisms  $\tilde{H}^n(X; R) \cong \tilde{H}^{n+k}(\Sigma^k X; R)$  derivable by elementary exact sequence arguments can also be obtained via cross product with a generator of  $\tilde{H}^*(S^k; R)$ ". Does he mean the *same* isomorphism, or just the fact that there is (at least) one?].
  5.  $Sq^i(\alpha) = \alpha^2$  if  $i = \deg(\alpha)$  and  $Sq^i(\alpha) = 0$  if  $i > \deg(\alpha)$ .

6.  $Sq^0 = \text{id}$ .

- Define  $Sq := Sq^0 + Sq^1 + \dots$ . Then  $Sq(\alpha \smile \beta) = Sq(\alpha) \smile Sq(\beta)$  (since  $(Sq(\alpha \smile \beta))_n = \sum_i Sq^i(\alpha) \smile Sq^{n-i}(\beta) = (Sq(\alpha) \smile Sq(\beta))_n$ ). Thus,  $Sq$  is a ring homomorphism.
- Adem relations:

$$Sq^a Sq^b = \sum_j \binom{b-j-1}{a-2j} Sq^{a+b-j} Sq^j \quad \text{if } a < 2b,$$

where  $\binom{m}{n}$  is zero if  $m$  or  $n$  is negative, or  $m < n$ , and  $\binom{m}{0} = 1$  for  $m \geq 0$ .

DEFINITION 2.0.1. The *Steenrod algebra*  $\mathcal{A}_2$  is the algebra over  $\mathbb{Z}/2\mathbb{Z}$  that is the quotient of the algebra of polynomials in the noncommuting variables  $Sq^1, Sq^2, \dots$  by the two-sided ideal generated by the Adem relations. Thus, for every space  $X$ ,  $H^*(X; \mathbb{Z}/2\mathbb{Z})$  is a module over  $\mathcal{A}_2$ .

- $\mathcal{A}_2$  is graded, and its elements of degree  $k$  are those that map  $H^n(X; \mathbb{Z}/2\mathbb{Z})$  to  $H^{n+k}(X, \mathbb{Z}/2\mathbb{Z})$  for all  $n$ . [Presumably you've fixed a space  $X$  while you're doing all this?]

DEFINITION 2.0.2. Write  $Sq^I$  for the monomial  $Sq^{i_1} Sq^{i_2} \dots Sq^{i_n}$ . Then  $Sq^I$  is *admissible* if  $i_j \geq 2i_{j+1}$  for all  $0 \leq j < n$ .

Note the admissible monomials are exactly those to which no Adem relations can be applied. Thus,  $\mathcal{A}_2$  is generated as a  $\mathbb{Z}/2\mathbb{Z}$  module by admissible monomials.

[5]

## 3 Spectra

### 3.1 ?Categorical nonsense

- [13]: There is a category  $\mathcal{H}$  of finite based CW complexes, with  $\text{Hom}(X, Y) =: [X, Y]$  the set of homotopy classes of base-point preserving maps  $X \rightarrow Y$ .
- There is a category  $\mathbf{St}(\mathcal{H})$  of finite based CW complexes, with  $\text{Hom}(X, Y) =: \{X, Y\}$  the set  $\text{colim}_i [\Sigma^i X, \Sigma^i Y]$  [it's just a colimit of sets, and  $\mathbf{Set}$  is cocomplete, so we should be fine. [13] says it's a group<sup>3</sup>] [Also, how do these guys compose?]
- There is a functor  $\mathcal{H} \rightarrow \mathbf{St}(\mathcal{H})$ . [13] doesn't say what this is but it's presumably the one that is the identity on objects and sends  $[f : X \rightarrow Y] \in [\Sigma^0 X, \Sigma^0 Y]$  to whatever it gets sent to in  $\{X, Y\}$  using the universal property of the colimit. Uniqueness makes it functorial, etc.
- We have a fully faithful functor  $\mathbf{St}(\mathcal{H}) \rightarrow \mathbf{St}(\mathcal{H})$  given by the suspension on objects, and the unique isomorphism  $\{X, Y\} \rightarrow \{\Sigma X, \Sigma Y\}$  on maps (such an isomorphism exists, since both of those things are colimits for  $[\Sigma^i X, \Sigma^i Y]$  - one of the sequences is cut off at the beginning, but it doesn't matter because both reach the stable value (see above discussion and [13] 1.9), aka the colimit).
- It's not an equivalence, because not every object is isomorphic to a suspension (e.g. anything not connected, since suspensions always connected)

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<sup>3</sup>The colimit is equal to the stable value (which exists, by the corollary). After  $\Sigma^2$ , these guys are all groups, so the colimit also has a group structure inherited from whatever  $[\Sigma^k X, \Sigma^k Y]$  it's equal to. N.B: Remarks about cocompleteness of  $\mathbf{Set}$  are misleading because that doesn't actually matter - any sequence that stabilises in any category will have a filtered colimit equal to that stable value, you don't need any extra conditions.

- We can formally adjoin desuspensions  $\Sigma^{-n}X$  for all  $n$  [does this mean just putting the objects there and defining  $\text{Hom}(Y, \Sigma^{-n}X) := \text{Hom}(\Sigma^n Y, X)$  and  $\text{Hom}(\Sigma^{-n}X, Y) := \text{Hom}(X, \Sigma^n Y)$ ?], but this category does not have weak colimits (i.e. colimits w/o uniqueness property). [why does it not, and why do we even want that?]
- We instead consider formal sequences of desuspensions  $X_0 \rightarrow \Sigma^{-1}X_1 \rightarrow \dots$ , or sequences  $(X_n)$  and maps  $\Sigma X_n \rightarrow X_{n+1}$ , i.e. spectra. [and this fixes the problem?]

### 3.2 Definitions and examples

[Some intro, below follows [6], Section 5.2. Some intuition on spectra and why we're doing this.]

**DEFINITION 3.2.1.** A *spectrum* is a collection of pointed topological spaces  $\{X_n\}_{n \in \mathbb{N}}$ , together with basepoint-preserving maps  $\sigma_n : \Sigma X_n \rightarrow X_{n+1}$ .

**EXAMPLE 3.2.2.** Let  $X$  be a topological space. The *suspension spectrum* of  $X$ , denoted by  $\Sigma^\infty X$ , has  $X_n = \Sigma^n X$  and  $\sigma_n = \text{id} : \Sigma X_n \rightarrow X_{n+1}$ .

We write  $\mathbb{S}$  for the suspension spectrum  $\Sigma^\infty S^0$ , and call  $\mathbb{S}$  the *sphere spectrum*. For  $i > 0$ , we write  $\mathbb{S}^i$  for  $\Sigma^\infty S^i$ .

**EXAMPLE 3.2.3.** The *Eilenberg-MacLane spectrum*  $\mathbb{K}(G, m)$  has  $(\mathbb{K}(G, m))_n$  a CW complex  $K(G, m+n)$  and  $\sigma_n : \Sigma K(G, m+n) \rightarrow K(G, m+n+1)$  is the adjoint of the CW approximation  $K(G, m+n) \rightarrow \Omega K(G, m+n+1)$ .

**DEFINITION 3.2.4.** Let  $X = \{X_n\}$  be a spectrum. We define  $\pi_i(X) = \text{colim}_n \pi_{i+n}(X_n)$ , where the map  $\pi_{i+n}(X_n) \rightarrow \pi_{i+n+1}(X_{n+1})$  is given by the composition

$$\pi_{i+n}(X_n) \xrightarrow{\Sigma} \pi_{i+n+1}(\Sigma X_n) \xrightarrow{(\sigma_n)_*} \pi_{i+n+1}(X_{n+1}).$$

**EXAMPLE 3.2.5.** If  $X$  is a topological space, then  $\pi_i(\Sigma^\infty X) = \pi_i^S(X)$ , the  $i$ th stable homotopy group of  $X$ .

**DEFINITION 3.2.6.** A CW spectrum is a spectrum  $X$  consisting of CW complexes  $X_n$  with the maps  $\Sigma X_n \hookrightarrow X_{n+1}$  inclusions of subcomplexes.

**DEFINITION 3.2.7.** Let  $X$  be a CW spectrum. Then the  $k$ -cells of  $X$  are the equivalence classes of non-basepoint  $(k+n)$ -cells in  $X_n$ , where two cells are equivalent if one is an  $m$ -fold suspension of the other, for some  $m > 0$ .

**DEFINITION 3.2.8.** A CW spectrum  $X$  is *connective* if it has no cells below a given dimension, *finite* if it has only finitely many cells, and *of finite type* if it has only finitely many cells in each dimension.

**EXAMPLE 3.2.9.** If  $X$  is a finite (resp. finite type) CW complex, then  $\Sigma^\infty X$  is a finite (resp. finite type) CW spectrum. In particular,  $\mathbb{S}$  is a finite CW spectrum with a unique cell in dimension 1.

**LEMMA 3.2.10.** Let  $X$  be a connective spectrum of finite type. Then the groups  $\pi_{i+n}(X_n)$  eventually stabilise; i.e. the maps  $\pi_{i+n}(X_n) \xrightarrow{(\sigma_n)_* \circ \Sigma} \pi_{i+n+1}(X_{n+1})$  are isomorphisms for large enough  $n$ .

**PROOF.** First, note that the maps  $\pi_{i+n}(X_n) \xrightarrow{\Sigma} \pi_{i+n+1}(\Sigma X_n)$  are eventually isomorphisms by [Theorem 1.0.1](#).

Recall that whenever  $(X_{n+1}, \Sigma X_n)$  are such that  $X_{n+1} \setminus X_n$  has no cells in dimension  $\leq k$ , the map  $\pi_i(\Sigma X_n) \rightarrow \pi_i(X_{n+1})$  induced by the inclusion is an isomorphism ([5], Cor 4.12 and a long exact sequence). Thus, if  $(\sigma_n)_* : \pi_{i+n+1}(\Sigma X_n) \rightarrow \pi_{i+n+1}$  never stabilises, there must be infinitely many natural numbers  $N_j$  such that  $(X_{N_j+1}, \Sigma X_{N_j})$  is not  $(i + N_j + 1)$ -connected, and thus that  $X_{N_j+1} \setminus \Sigma X_{N_j}$  has cells of dimension  $\leq i + N_j + 2$ . By connectivity, there is some fixed  $l$  such that these cells are of dimension  $N_j + k + 1$  for  $-l \leq k \leq i + 1$ . Thus, there must be some  $k$  such that infinitely many of the  $X_{N_j+1}$  have a  $(k + N_j + 1)$ -cell not included in  $\Sigma X_{N_j}$ . This then contradicts the assumption that  $X$  is of finite type, since it has infinitely many  $k$ -cells.

Thus, the maps  $(\sigma_n)_* : \pi_{i+n+1}(\Sigma X_n) \rightarrow \pi_{i+n+1}$  are also eventually isomorphisms, so the groups  $\pi_{i+n}(X_n)$  do stabilise.  $\square$

### 3.3 Homology and cohomology

Recall that  $C_i^{\text{cell}}(X_n; G)$  is the free abelian group on the  $i$ -cells of  $X_n$ . We have an injection

$$\begin{aligned} C_i^{\text{cell}}(X_n; G) &\rightarrow C_{i+1}^{\text{cell}}(\Sigma X_n; G) \\ e_\alpha^i &\mapsto \Sigma e_\alpha^i, \end{aligned}$$

and an injection  $C_{i+1}^{\text{cell}}(\Sigma X_n; G) \rightarrow C_{i+1}^{\text{cell}}(X_{n+1}; G)$  induced by the structure map  $\sigma_n$ , so we get an injection  $C_i^{\text{cell}}(X_n; G) \hookrightarrow C_{i+1}^{\text{cell}}(X_{n+1}; G)$ .

We define

$$C_n(X; G) := \bigcup_{i \in \mathbb{Z}} C_{i+n}^{\text{cell}}(X_i; G).$$

Note that there is a  $G$  summand for every  $i + n$  cell of  $X_i$  up to treating suspensions of cells as equivalent to the cells themselves, i.e. a  $G$  summand for every  $n$ -cell of  $X$ . We define  $H^*(X; G)$  and  $H_*(X; G)$  to be the cohomology and homology of this chain complex, respectively.

[Important example:

At the bottom of p592 of [6], it's mentioned that for a spectrum  $X$  of finite type,  $H^i(X) \cong \lim_{\leftarrow n} H^{i+n}(X_n)$  as  $\mathcal{A}_2$ -modules<sup>4</sup>. Thus, since  $\mathbb{S}$  is of finite type (it is actually finite), we have

$$\begin{aligned} H^i(\mathbb{S}; \mathbb{Z}/2\mathbb{Z}) &= \lim_{\leftarrow n} H^{i+n}(\mathbb{S}^n; \mathbb{Z}/2\mathbb{Z}) \\ &= \begin{cases} \mathbb{Z}/2\mathbb{Z} & i = 0, \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

(since they're all just either 0 or  $\mathbb{Z}/2\mathbb{Z}$ , at least eventually). So,  $H^*(\mathbb{S}) = \mathbb{Z}/2\mathbb{Z}$  in degree zero and nothing else.]

LEMMA 3.3.1. Let  $X$  be a connective CW spectrum of finite type. Then  $H_i(X)$ ,  $H^i(X)$ , and  $\pi_i(X)$  are finitely generated for all  $i$ .

PROOF. First, note that  $H_i(X) = H_{i+n}(X_n)$  for sufficiently large  $n$ , since for large enough  $n$ ,  $X_n$  contains all the cells of dimension  $\leq i$ . Similarly,  $H^i = H^{i+n}(X_n)$  for sufficiently large  $n$ . Each  $H_{i+n}(X_n)$  is finitely generated, since  $X_n$  has only finitely many cells in each dimension, and thus each  $H^{i+n}(X_n)$  is also finitely generated ([5] Cor 3.3). Thus,  $H_i(X)$ ,  $H^i(X)$  are finitely generated.

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<sup>4</sup>Maybe I need to move this somewhere else. I haven't defined the  $\mathcal{A}_2$ -action yet...

Now,  $\pi_i(X) = \operatorname{colim}_n \pi_{i+n}(X_n)$ , and the groups  $\pi_{i+n}(X_n)$  stabilise by [Lemma 3.2.10](#). The  $X_n$  must eventually be simply-connected, since  $X$  is connective. A simply-connected space has finitely generated homotopy groups if and only if it has finitely generated homology groups (see e.g. [\[5\]](#), Thm 5.7), and we have just seen that the  $H_{i+n}(X_n)$  are finitely generated, so  $\pi_i(X) = \pi_{i+n}(X_n)$  is finitely generated.  $\square$

**DEFINITION 3.3.2.** Let  $X = \{X_n\}$  be a CW spectrum. A *subspectrum*  $X'$  of  $X$  is a sequence of subcomplexes  $\{X'_n \subseteq X_n\}$  satisfying  $\Sigma X'_n \subseteq X'_{n+1}$ . The subspectrum  $X'$  is *cofinal* if, for each  $n$  and each cell  $e_\alpha^i$  of  $X_n$ , the cell  $\Sigma^k e_\alpha^i$  belongs to  $X'_{n+k}$  for all sufficiently large  $k$ .

Note that if  $\Sigma^k e_\alpha^i$  belongs to  $X'_{n+k}$  then  $\Sigma^{k+1} e_\alpha^i$  belongs to  $\Sigma X'_{n+k} \subseteq X'_{n+k+1} \subseteq X'_{k+k+2} \subseteq \dots$ . Thus, if  $X', X''$  are cofinal spectra of  $X$  with  $\Sigma^k e_\alpha^i$  a cell of  $X'_{n+k}$  and  $\Sigma^l e_\alpha^i$  a cell of  $X''_{n+l}$  (with  $l \geq k$ ) then  $\Sigma^l e_\alpha^i$  is a cell of  $X'_{n+l}$  and therefore of  $X'_{n+l} \cap X''_{n+l}$ . In other words, the intersection of two cofinal spectra is a cofinal spectrum.

**DEFINITION 3.3.3.** Let  $X, Y$  be CW spectra. A *strict map*  $f : X \rightarrow Y$  is a sequence of cellular maps  $f_n : X_n \rightarrow Y_n$  such that the diagram below commutes.

$$\begin{array}{ccc} \Sigma X_n & \xrightarrow{\sigma_n} & X_{n+1} \\ \sigma f_n \downarrow & & \downarrow f_{n+1} \\ \Sigma Y_n & \xrightarrow{\sigma_n} & Y_{n+1} \end{array}$$

[Some intuition: it's enough for maps to be 'defined eventually']

**DEFINITION 3.3.4.** A *map* of CW spectra  $f : X \rightarrow Y$  is an equivalence class of strict maps  $f' : X' \rightarrow Y$  with  $X'$  a cofinal subspectrum of  $X$ , where two strict maps  $f' : X' \rightarrow Y$  and  $f'' : X'' \rightarrow Y$  are equivalent if they agree on some common cofinal subspectrum.

Given two maps  $f : X \rightarrow Y, g : Y \rightarrow Z$  represented by  $f' : X' \rightarrow Y, g' : Y' \rightarrow Z$  respectively, we compose as follows: let  $X''$  be the subspectrum of  $X'$ , where the cells of  $X''_n$  consist of the cells of  $X'_n$  mapped to  $Y'_n$  under  $f'_n$ . Then, for any cell  $e_\alpha^i$  of  $X'_n$ ,  $f_n(e_\alpha^i)$  is contained in a finite union of cells<sup>5</sup> of  $Y_n$ , whose  $k$ -fold suspension lies in  $Y'_{n+k}$  for large enough  $k$ . Since  $f'$  is a strict map,  $\Sigma^k f'_n(e_\alpha^i) = f'_{n+k} \Sigma^k e_\alpha^i$ , so  $\Sigma^k e_\alpha^i$  is a cell of  $X'_{n+k}$ . Thus,  $X''$  is cofinal in  $X'$  and hence in  $X$ . We define  $gf := [X'' \xrightarrow{f'|_{X''}} Y' \xrightarrow{g'} Z]$ , which is well-defined since the intersection of cofinal subspectra is again a cofinal subspectrum.

Since any strict map  $f' : X' \rightarrow Y$  can be taken to be cellular, a map  $f : X \rightarrow Y$  induces a well-defined map  $C_*(X) \rightarrow C_*(Y)$  (by cofinality), and thus maps on homology and cohomology.

Further, any map  $\Sigma^\infty S^i \rightarrow X$  can be represented by a map  $S^{i+n} \rightarrow X_n$ , which has compact image and thus by [Proposition B.3.7](#) is contained in a finite subcomplex of  $\overline{X}_n$ . Given any map  $f : X \rightarrow Y$  represented by a strict map  $f' : X' \rightarrow Y$ , the  $k$ th suspension of the cells of  $\overline{X}_n$  lie in  $X'_{n+k}$ , and thus  $f$  induces a map  $\pi_*(X) \rightarrow \pi_*(Y)$ .

**DEFINITION 3.3.5.** Two spectra  $X, Y$  are *equivalent* if there are maps  $f : X \rightarrow Y$  and  $g : Y \rightarrow X$  such that  $fg = \operatorname{id}_Y$  and  $gf = \operatorname{id}_X$ .

Note that a spectrum is equivalent to any of its cofinal subspectra. In particular, if  $X = \{X_n\}$  is a spectrum, then  $X' = \{\Sigma X_{n-1}\}$  is a cofinal subspectrum of  $X$  (where we take  $X_{-1}$  to be the basepoint of  $X_0$ ). We define  $\Sigma^{-1}X := \{X_{n-1}\}$ , noting that  $\Sigma \Sigma^{-1}X = \Sigma^{-1} \Sigma = X' \simeq X$ . Thus, a spectrum is always equivalent to the suspension of some other spectrum.

<sup>5</sup>Since the image of a compact set is compact.



DEFINITION 3.3.6. A *homotopy* of maps between spectra is a map  $X \times I \rightarrow Y$ , where  $X \times I$  is the spectrum with  $(X \times I)_n = X_n \times_{\text{red}} I$ .

Note that  $\Sigma(X_n \times_{\text{red}} I) = \Sigma X_n \times_{\text{red}} I$ . The set of homotopy classes of maps  $X \rightarrow Y$  is denoted by  $[X, Y]$ .

REMARK 3.3.7. For any CW spectra  $Z$ ,  $[\Sigma^\infty S^t, Z] = \pi_t(Z)$ .

For any CW spectra  $X, Y$ , the set  $[X, Y]$  can be the structure of an abelian group, since  $X$  can be written as a double suspension  $\Sigma^2 X'$ , and each set  $[\Sigma^2 X'_n, Y_n]$  has the structure of an abelian group by Remark B.1.4.

THEOREM 3.3.8. The suspension map  $[X, Y] \rightarrow [\Sigma X, \Sigma Y]$  is an isomorphism of groups.

PROOF. The suspension map is a homomorphism, since it is a homomorphism on maps between CW complexes. Thus, it suffices to show it is a bijection on maps between spectra.

Recall that  $\Sigma^{-1}\Sigma X = \Sigma\Sigma^{-1}X \simeq X$ . For any map  $f : X \rightarrow Y$  given by strict maps  $f_n : X'_n \rightarrow Y_n$ , define  $\Sigma^{-1}f : \Sigma^{-1}X \rightarrow \Sigma^{-1}Y$  by  $\{f_{n-1} : X'_{n-1} \rightarrow Y_{n-1}\}$ . Then  $\Sigma\Sigma^{-1}f = \{\Sigma f_{n-1}\} = \{f_n|_{\Sigma X_{n-1}}\} = f$ , and similarly  $\Sigma^{-1}\Sigma f = f$ . Thus, we have bijections  $[X, Y] \cong [\Sigma\Sigma^{-1}X, \Sigma\Sigma^{-1}Y] \cong [\Sigma^{-1}\Sigma X, \Sigma^{-1}\Sigma Y]$ , so  $\Sigma$  has a two-sided inverse.  $\square$

### 3.4 Cofibration sequences

DEFINITION 3.4.1. Let  $X = \{X_n\}, Y = \{Y_n\}$  be spectra. Then their *wedge sum* is  $X \vee Y := \{X_n \vee Y_n\}$ . Note that Remark B.2.5 gives us an inclusion  $\Sigma(X_n \vee Y_n) \hookrightarrow X_{n+1} \vee Y_{n+1}$ .

DEFINITION 3.4.2. Let  $X$  be a spectrum,  $A \subseteq X$  a subspectrum. Then  $A$  is *closed* in  $X$  if for every cell  $e_\alpha^n$  of  $X_n$ , if  $\Sigma^k e_\alpha^n \in A_{n+k}$  then  $e_\alpha^n \in A_n$ .

Any subspectrum is cofinal in (and thus equivalent to) its closure. We define  $X/A$  to be the CW spectrum with  $(X/A)_n = X_n/A'_n$ , where  $A' = \{A'_n\}$  is the closure of  $A$ . Note that a quotient of connective spectra of finite type is again a connective spectrum of finite type (since the quotient has fewer cells in each dimension than the original space).

[The map  $X \cup CA \rightarrow X/A$  is a homotopy equivalence of spectra]

THEOREM 3.4.3. Let  $X, Y$  be spectra, and  $A \subseteq X$  a subspectrum. Then there is an exact sequence

$$[Y, A] \rightarrow [Y, X] \rightarrow [Y, X/A] \rightarrow [Y, \Sigma A] \rightarrow [Y, \Sigma X] \rightarrow \cdots$$

PROOF. It suffices to show that

$$[Y, A] \rightarrow [Y, X] \rightarrow [Y, X/A] \rightarrow [Y, \Sigma A]$$

is exact.

We first show exactness at  $[Y, X/A]$ . The composition  $[Y, X] \rightarrow [Y, X/A] \rightarrow [Y, \Sigma A]$  is clearly zero. Now, if  $Y \xrightarrow{f} X \cup CA \rightarrow \Sigma A$  is homotopic to the constant map, then  $f$  must be homotopic to a map contained entirely in  $X$ , and thus is in the image of  $[Y, X] \rightarrow [Y, X/A]$ .

Now, we show exactness at  $[Y, X]$ . Again, the composition  $[Y, A] \rightarrow [Y, X] \rightarrow [Y, X/A]$  is clearly zero. Suppose  $Y \xrightarrow{f} X \rightarrow X \cup CA$  is homotopic to the constant map. Then we have a map



$h : CY \rightarrow X \cup CA$  making the solid diagram below commute.

$$\begin{array}{ccccccccc}
Y & \xrightarrow{\text{id}} & Y & \hookrightarrow & CY & \hookrightarrow & \Sigma Y & \xrightarrow{\text{id}} & \Sigma Y \\
\downarrow & & \downarrow f & & \downarrow h & & \downarrow & & \downarrow \Sigma f \\
A & \xrightarrow{i} & X & \hookrightarrow & X \cup CA & \longrightarrow & \Sigma A & \xrightarrow{\Sigma i} & \Sigma X
\end{array}$$

[6] says that “we can then automatically fill in the next two vertical maps to make homotopy commutative squares”. Vaguely, I think this is because of some sort of naturality argument, which [5] says is ‘evident’ on p398. If we can, then...

By Theorem 3.3.8, we can take the map  $\Sigma Y \rightarrow \Sigma A$  to be  $\Sigma g$  for some  $g : Y \rightarrow A$ . Then  $\Sigma f \simeq (\Sigma i)(\Sigma g) = \Sigma(ig)$ , so  $f \simeq ig$ , as required.  $\square$

Finally, we get the lemma below, which follows from the equivalent result for CW complexes.

LEMMA 3.4.4. Let  $A \xrightarrow{f} X \xrightarrow{i} C_f \xrightarrow{j} \Sigma A \rightarrow \dots$  be a cofibration, where  $X, A$  are CW spectra of finite type. Then there is a long exact sequence

$$\dots \leftarrow H^{n-1}(\Sigma A) \leftarrow H^n(X) \xleftarrow{i^*} H^n(C_f) \xleftarrow{j^*} H^n(\Sigma A) \leftarrow H^{n+1}(X) \leftarrow \dots$$

### 3.5 Eilenberg-MacLane spectra

THEOREM 3.5.1 ([6], Prop 5.45). There are natural isomorphisms  $H^m(X; G) \cong [X, K(G, m)]$  for all CW spectra.

Recall that giving a map into a product is equivalent to giving a map into each of its components. We have maps  $F_i : [X, \bigvee_i \mathbb{K}(G, n_i)] \rightarrow [X, \mathbb{K}(G, n_i)]$ .

PROPOSITION 3.5.2 ([6], Prop 5.46). The map  $F : [X, \bigvee_i \mathbb{K}(G, n_i)] \rightarrow \prod_i [X, \mathbb{K}(G, n_i)]$  is an isomorphism if  $X$  is a connective spectrum of finite type and  $n_i \rightarrow \infty$  as  $i \rightarrow \infty$ .

### 3.6 $p$ -completion of spectra

DEFINITION 3.6.1 ([8], Def 10.1.1). Let  $A$  be an abelian group. Then its  $p$ -adic completion is the limit

$${}_pA = \varprojlim_n (A/p^n A).$$

If  $A = \mathbb{Z}$ , we instead write  $\mathbb{Z}_p$  for the  $p$ -adic integers. There is a natural map  $A \rightarrow {}_pA$ , whose component at  $n$  is reduction modulo  $p^n A$ .

When  $A$  is finitely generated, its  $p$ -adic completion is given by the map  $A \rightarrow A \otimes \mathbb{Z}_p$ ;  $a \mapsto a \otimes 1$ .

LEMMA 3.6.2. Suppose  $A$  is finite, and write  $|A| = np^r$  for  $p \nmid n$ . Then  $A \otimes \mathbb{Z}_p \cong A/T$ , where  $T \subseteq A$  is the subgroup generated by all torsion coprime to  $p$ .

PROOF. Define a homomorphism  $A \otimes \mathbb{Z}_p \rightarrow A/T$  sending  $a \otimes z \mapsto [\hat{z}a]$ , where  $\hat{z} \in \mathbb{Z}$  is a lift of  $q(z)$ , chosen such that  $0 < \hat{z} \leq p^r$ , and  $q$  is the projection  $\mathbb{Z}_p \twoheadrightarrow \mathbb{Z}/p^r\mathbb{Z}$ . Suppose  $a \otimes z \mapsto 0$ . Then  $\hat{z}a \in T$ , so  $k\hat{z}a = 0$  for some  $k$  coprime to  $p$ . Write  $z = \hat{z} + p^r z'$  for some  $z' \in \mathbb{Z}_p$ . Then  $a \otimes z = a \otimes (\hat{z} + p^r z') = \hat{z}a \otimes (1 + \frac{p^r z'}{\hat{z}}) = k\hat{z} \otimes (\frac{1}{k} + \frac{p^r z'}{k\hat{z}}) = 0$ , where the second equality follows from the fact that  $\hat{z}$  was chosen such that the highest power of  $p$  dividing it was less than or equal to  $r$ . Thus,  $a \otimes z = 0$ , so the map is injective. The map is clearly also surjective, since  $a \otimes 1 \mapsto [a]$ , so it is an isomorphism.  $\square$

REMARK 3.6.3. If  $A$  is finite with order  $np^r$  for  $p \nmid n$ , then  $|_p A| = p^r$ , by Cauchy's theorem.

DEFINITION 3.6.4 ([7], p129). Let  $X$  be a CW spectrum. Then a  $p$ -completion of  $X$  is a map  $f : X \rightarrow {}_pX$  such that for all  $i$ ,  $\pi_i f$  expresses  $\pi_i({}_pX)$  as the  $p$ -completion of  $\pi_i(X)$ .

THEOREM 3.6.5 ([7], Thm 9.1.1). If  $X$  has finite type, then it has a  $p$ -completion unique up to equivalence.

THEOREM 3.6.6 ([7], Prop 9.2.22). Let  $X$  be a connective spectrum of finite type, and let  $Y$  be  $p$ -complete. Then the map  $[X_p^\wedge, Y] \rightarrow [X, Y]$  is an isomorphism. That is, given any map  $X \xrightarrow{f} Y$ , there exists a unique (up to homotopy) map  $X_p^\wedge \xrightarrow{\bar{f}} Y$  such that  $f$  factors as  $X \rightarrow X_p^\wedge \xrightarrow{\bar{f}} Y$ .

[13], [6], [8], [7]

## 4 The Adams spectral sequence

### 4.1 Spectral sequences

[Maybe add some notes from [13]]

[Some notes from [10], C2 - just here as a placeholder/reference and I'll probably completely rewrite this bit.]

DEFINITION 4.1.1. A *differential bigraded module*  $E$  over a ring  $R$  is a collection of  $R$ -modules  $\{E^{p,q}\}$ ,  $p, q \in \mathbb{Z}$ , together with a map  $d : E^{p,q} \rightarrow E^{p+s, q-s+1}$  for each  $p, q$  and some fixed  $s \in \mathbb{Z}$ , satisfying  $d^2 = 0$ .

We can take the homology of  $(E, d)$ :

$$H^{p,q}(E^{*,*}, d) = \ker(d : E^{p,q} \rightarrow E^{p+s, q-s+1}) / \operatorname{im}(d : E^{p-s, q+s-1} \rightarrow E^{p,q}).$$

DEFINITION 4.1.2. A *spectral sequence* (of *cohomological type*<sup>6</sup>) is a collection of differential bigraded  $R$ -modules  $\{E_r^{*,*}, d_r\}$ ,  $r \in \mathbb{N}$ , with the differentials  $d_r$  of bidegree  $(r, 1-r)$ . These satisfy the further condition that for all  $p, q, r$ ,  $E_{r+1}^{p,q} \cong H^{p,q}(E_r^{*,*}, d_r)$ .

We will sometimes write  $d_r^{p,q}$  for the differential  $d_r : E^{p,q} \rightarrow E^{p+r, q-s+1}$ .

Consider the term  $E_2^{*,*}$ . Define

$$Z_2^{p,q} := \ker d_2^{p,q} \quad \text{and} \quad B_2^{p,q} := \operatorname{im} d_2^{p-2, q+1}.$$

The condition  $d^2 = 0$  implies that  $B_2^{p,q} \subseteq Z_2^{p,q} \subseteq E_2^{p,q}$ , and by definition we have  $E_3^{p,q} \cong Z_2^{p,q}/B_2^{p,q}$ .

Now, write

$$Z_3^{p,q} := \ker d_3^{p,q} \quad \text{and} \quad B_3^{p,q} := \operatorname{im} d_3^{p-3, q+2}.$$

Since  $Z_3^{p,q} \subseteq E_3^{p,q}$ , it can be written as  $\bar{Z}_3^{p,q}/B_2^{p,q}$  for some  $\bar{Z}_3^{p,q} \subseteq Z_2^{p,q}$ . Similarly,  $B_3^{p,q} \cong \bar{B}_3^{p,q}/B_2^{p,q}$  for some  $\bar{B}_3^{p,q} \subseteq Z_2^{p,q}$ . Thus,

$$E_4^{p,q} \cong Z_3^{p,q}/B_3^{p,q} \cong \frac{\bar{Z}_3^{p,q}/B_2^{p,q}}{\bar{B}_3^{p,q}/B_2^{p,q}} \cong \bar{Z}_3^{p,q}/\bar{B}_3^{p,q}.$$

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<sup>6</sup>I'll have to rewrite this section because the Adams spectral sequence is not a cohomological or a homological spectral sequence I don't think - the grading is  $d_r : E^{s,t} \rightarrow E^{s+r, t+r-1}$ .

Iterating this process, we present the spectral sequence as an infinite tower of submodules of  $E_2^{p,q}$ :

$$B_2^{p,q} \subseteq \overline{B}_3^{p,q} \subseteq \cdots \subseteq \overline{B}_n^{p,q} \subseteq \cdots \subseteq \overline{Z}_n^{p,q} \subseteq \cdots \subseteq \overline{Z}_3^{p,q} \subseteq Z_2^{p,q} \subseteq E_2^{p,q},$$

with the property that  $E_{n+1}^{p,q} \cong \overline{Z}_n^{p,q} / \overline{B}_n^{p,q}$ . The differential  $d_{n+1}^{p,q}$  can be taken as a map  $\overline{Z}_n^{p,q} / \overline{B}_n^{p,q} \rightarrow \overline{Z}_n^{p,q} / \overline{B}_n^{p,q}$  with kernel  $\overline{Z}_{n+1}^{p,q} / \overline{B}_n^{p,q}$  and image  $\overline{B}_{n+1}^{p,q}$ . The short exact sequence induced by  $d_{n+1}$ ,

$$0 \rightarrow \overline{Z}_{n+1}^{p,q} / \overline{B}_n^{p,q} \rightarrow \overline{Z}_n^{p,q} / \overline{B}_n^{p,q} \xrightarrow{d_{n+1}^{p,q}} \overline{B}_{n+1}^{p,q} / \overline{B}_n^{p,q} \rightarrow 0,$$

gives rise to isomorphisms  $\overline{Z}_n^{p,q} / \overline{Z}_{n+1}^{p,q} \cong \overline{B}_{n+1}^{p,q} / \overline{B}_n^{p,q}$  for all  $n$ . Conversely, a tower of submodules of  $E_2$ , together with a set of isomorphisms, gives rise to a spectral sequence.

DEFINITION 4.1.3. An element of  $E_2^{p,q}$  *survives to the  $r$ th stage* if lies in  $\overline{Z}_r^{p,q}$ , having been in the kernel of the previous  $r - 2$  differentials, and is *bounded by the  $r$ th stage* if it lies in  $\overline{B}_r^{p,q}$ . The bigraded module  $E_r^{*,*}$  is called the  $E_r$ -term of the spectral sequence.

We define

$$Z_\infty^{p,q} := \bigcap_n \overline{Z}_n^{p,q}, \quad B_\infty^{p,q} := \bigcup_n \overline{B}_n^{p,q}.$$

From the tower of inclusions, we see that  $B_\infty^{p,q} \subseteq Z_\infty^{p,q}$ , so we define  $E_\infty^{p,q} := Z_\infty^{p,q} / B_\infty^{p,q}$ .

DEFINITION 4.1.4. A spectral sequence *collapses at the  $N$ th term* if the differentials  $d_r^{p,q} = 0$  for  $r \geq N$ .

From the short exact sequence

$$0 \rightarrow \overline{Z}_r^{p,q} / \overline{B}_{r-1}^{p,q} \rightarrow \overline{Z}_{r-1}^{p,q} / \overline{B}_{r-1}^{p,q} \xrightarrow{d_r^{p,q}} \overline{B}_r^{p,q} / \overline{B}_{r-1}^{p,q} \rightarrow 0,$$

the condition  $d_r^{p,q}$  forces  $\overline{Z}_r^{p,q} = \overline{Z}_{r-1}^{p,q}$  and  $\overline{B}_r^{p,q} = \overline{B}_{r-1}^{p,q}$ . The tower of submodules becomes

$$B_2^{p,q} \subseteq \overline{B}_3^{p,q} \subseteq \cdots \subseteq \overline{B}_{N-1}^{p,q} = B_N^{p,q} = \cdots = B_\infty^{p,q} \subseteq Z_\infty^{p,q} = \cdots = \overline{Z}_N^{p,q} = \overline{Z}_{N-1}^{p,q} \subseteq \cdots \subseteq \overline{Z}_3^{p,q} \subseteq Z_2^{p,q} \subseteq E_2^{p,q}.$$

Thus,  $E_\infty^{p,q} = E_N^{p,q}$ .

## 4.2 Exact couples

(Following [10], C2)

DEFINITION 4.2.1. Let  $D, E$  be  $R$ -modules, and let  $i : D \rightarrow D$ ,  $j : D \rightarrow E$ ,  $k : E \rightarrow D$  be module homomorphisms. We call  $\mathcal{C} = \{D, E, i, j, k\}$  an *exact couple* if the diagram below is exact.

$$\begin{array}{ccc} D & \xrightarrow{i} & D \\ & \swarrow k & \searrow j \\ & E & \end{array}$$

Let  $d := jk$ , and define the following:

$$\begin{aligned} E' &:= H(E, d) = \ker d / \operatorname{im} d \\ D' &:= i(D) = \ker j \\ i' &:= i|_{D'} : D' \rightarrow D' \\ j' &:= i(x) \mapsto j(x) + dE : D' \rightarrow E' \\ k' &:= (e + dE) \mapsto k(e) : E' \rightarrow D' \end{aligned}$$

We call  $\mathcal{C}' = \{D', E', i', j', k'\}$  the *derived couple* of  $\mathcal{C}$ .

PROPOSITION 4.2.2 ([10], Prop 2.7). If  $\mathcal{C} = \{D, E, i, j, k\}$  is an exact couple, then  $\mathcal{C}'$  is also an exact couple.

THEOREM 4.2.3 ([10], Thm 2.8). Suppose  $D^{*,*} = \{D^{p,q}\}$  and  $E^{*,*} = \{E^{p,q}\}$  are bigraded modules equipped with homomorphisms  $i$  of bidegree  $(-1, 1)$ <sup>7</sup>,  $j$  of bidegree  $(0, 0)$ , and  $k$  of bidegree  $(1, 0)$ , such that  $\{D^{*,*}, E^{*,*}, i, j, k\}$  is an exact couple. Then these data determine a spectral sequence  $\{E_r, d_r\}$  for  $r \in \mathbb{Z}_+$  of cohomological type, with  $E_r = (E^{*,*})^{(r-1)}$ , the  $(r-1)$ st derived module of  $E^{*,*}$  and  $d_r = j^{(r-1)} \circ k^{(r-1)}$ .

A bigraded exact couple may be displayed in the following diagram, known as a *staircase diagram*:

$$\begin{array}{ccccccc}
 & & \vdots & & \vdots & & \\
 & & \downarrow i & & \downarrow i & & \\
 \dots & \xrightarrow{k} & D^{p+2,q-1} & \xrightarrow{j} & E^{p+2,q-1} & \xrightarrow{k} & D^{p+3,q-1} \xrightarrow{j} \dots \\
 & & \downarrow i & & \downarrow i & & \\
 \dots & \xrightarrow{k} & D^{p+1,q} & \xrightarrow{j} & E^{p+1,q} & \xrightarrow{k} & D^{p+2,q} \xrightarrow{j} \dots \\
 & & \downarrow i & & \downarrow i & & \\
 \dots & \xrightarrow{k} & D^{p,q+1} & \xrightarrow{j} & E^{p,q+1} & \xrightarrow{k} & D^{p+1,q+1} \xrightarrow{j} \dots \\
 & & \downarrow i & & \downarrow i & & \\
 & & \vdots & & \vdots & & 
 \end{array}$$

[This is the same staircase diagram that Hatcher is talking about (set  $D^{*,*} = \pi_*^S X_*$  and  $E^{*,*} = \pi_* K_*$ ) - just modulo some tweak of  $i$ 's bigrading because the bigrading is a bit weird in the Adams spectral sequence.]

### 4.3 Setting up the Adams spectral sequence

Let  $X$  be a connective CW spectrum of finite type.

[Claim:  $H^*(X)$  is an  $\mathcal{A}_2$ -module. Either use Theorem 3.5.1 (i.e.  $H^m(X; G) \cong [X, K(G, m)]$ ) to give the action, or the above mentioned fact that for spectra of finite type,  $H^i(X; G) = H^{i+n}(X_n; G)$ . To define a map  $Sq^j : H^i(X; G) \rightarrow H^{i+j}(X; G)$  we can either hit  $H^{i+n}(X_n; G)$  with  $Sq^j$  and then push it along the sequence of cohomology groups until it's stable, or first push it along until  $H^{i+j+n}(X_n; G)$  is stable and then hit  $H^{i+n}(X_n; G)$  with  $Sq^j$ . And these are

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<sup>7</sup>This is the wrong index. For my purposes it should be  $(-1, -1)$ . So, the spectral sequence will not be of cohomological type, but hopefully the proof still goes through.

the same thing, because of this handy commutative diagram:

$$\begin{array}{ccccc}
& & H^{i+n}(X_n; G) & & \\
& \swarrow \sigma & & \searrow Sq^j & \\
H^{i+n+1}(\Sigma X_n; G) & & & & H^{i+j+n}(X_n; G) \\
& \downarrow i^* & & & \downarrow \sigma \\
H^{i+n+1}(X_{n+1}; G) & & & & H^{i+j+n+1}(\Sigma X_n; G) \\
& \searrow Sq^j & & \swarrow i^* & \\
& & H^{i+j+n+1}(X_{n+1}; G) & & 
\end{array}$$

which follows from the properties in Section 2. Is this an actual module action? Is the former one also a module action?

We can pick generators  $\alpha_i$  for  $H^*(X)$  as an  $\mathcal{A}_2$ -module such that there are at most finitely many in each  $H^n(X)$  [since each  $H^n(X)$  is finitely generated (Lemma 3.3.1), and that finite generating set would certainly also generate it as an  $\mathcal{A}_2$ -module (we could be more efficient, though)].

[Claim: these  $\alpha_i$  determine a map  $X \rightarrow K_0$ , where  $K_0$  is a wedge of EM spectra, and  $K_0$  has finite type. Very confused about this - each  $\alpha_i \in H^{n_i}(X; G)$  gives a map  $X \rightarrow K(G, n_i)$  by Theorem 3.5.1, but shouldn't they come together to give a map into a product<sup>8</sup> of EM spaces, not a wedge sum? Isn't a wedge sum a sort of reduced *coproduct*, not a product?? Is this something to do with the 'fibrations and cofibrations are the same' nonsense?

Vague idea: we have Proposition 3.5.2, which tells us that  $[X, \bigvee_i \mathbb{K}(G, n_i)] \cong \prod_i [X, \mathbb{K}(G, n_i)]$ . Now,  $X$  is connective, so  $H^*(X; G)$  has some smallest degree. Let's just pretend it starts from zero for the moment, and we can reindex or whatever later. Let's say each  $H^k(X; G)$  has  $i_k$  generators  $\alpha_{i_0+\dots+i_{k-1}+1}, \dots, \alpha_{i_0+\dots+i_k}$ , and write  $n_0 = \dots = n_{i_0}$ ,  $n_{i_0+1} = \dots = n_{i_0+i_1}$ , etc.,  $n_{i_0+i_1+\dots+i_{k-1}+1} = \dots = n_{i_0+i_1+\dots+i_k}$ . So, the  $\alpha_{i_0+\dots+i_{k-1}+j} \in H^k(X; G)$  determine maps  $X \rightarrow \mathbb{K}(G, n_k)$ . Putting all these maps together, one in each coordinate, gives an element of  $\prod_i [X, \mathbb{K}(G, n_i)]$ , which by the proposition corresponds to an element of  $[X, \bigvee_i \mathbb{K}(G, n_i)]$ .

We can replace that map with an inclusion. [I sort of believe this - Remark B.2.8 and my comment at the end of Section C.3 say we can do this for CW complexes, so I should check it for spectra. But it seems true (and I'm currently suffering from spectrum fatigue).]

[Set  $X_1 = K_0/X$ , and repeat the construction to get a diagram:

$$\begin{array}{ccccccc}
X & \longrightarrow & K_0 & \longrightarrow & K_1 & \longrightarrow & K_2 & \longrightarrow & \dots \\
& & \searrow & & \swarrow & & \searrow & & \swarrow \\
& & K_0/X = X_1 & & K_1/X_1 = X_2 & & K_2/X_2 = X_3 & & 
\end{array}$$

Taking cohomology, we get a diagram

$$\begin{array}{ccccccc}
0 & \longleftarrow & H^*(X) & \longleftarrow & H^*(K_0) & \longleftarrow & H^*(K_1) & \longleftarrow & H^*(K_2) & \longleftarrow & \dots \\
& & & & \swarrow & & \swarrow & & \swarrow & & \swarrow \\
& & & & H^*(X_1) & & H^*(X_2) & & H^*(X_3) & & \\
& \swarrow & & \swarrow & \swarrow & & \swarrow & & \swarrow & & \swarrow \\
& 0 & & 0 & 0 & & 0 & & 0 & & 0
\end{array}$$

<sup>8</sup>And on that note, what is a product of spectra? Surely it's not a smash product? Is it just a pointed product? Are we allowed to do that?

]

- **Fact:  $K_0$  has finite type** (and thus so do all the  $K_s$ 's) [I'm sure this is supposed to follow from  $H^i(X)$  being finitely generated for all  $i$ , but I can't at all see how].
- **Fact: the top row of the diagram above is exact.** [Suffices to show that we have exact sequences:

$$\begin{aligned} 0 \leftarrow H^*(X_s) \leftarrow H^*(K_s), \\ H^*(X) \leftarrow H^*(K_0) \leftarrow H^*(X_1) \leftarrow 0, \\ H^*(K_s) \leftarrow H^*(K_{s+1}) \leftarrow H^*(X_{s+2}) \leftarrow 0, \end{aligned}$$

and that the map  $H^*(K_0) \rightarrow H^*(X)$  is surjective. Actually, I think the first exact sequence will follow from  $H^*(K_0) \rightarrow H^*(X)$  being surjective, since they're all constructed in the same way.

I'm extremely confused by the supposed surjectivity - going back to the sketch of the construction for CW complexes, the corresponding statement is supposed to be that if we choose generators  $\alpha_i$  for  $H^*(X)$  as an  $\mathcal{A}_2$ -module giving maps  $f_i : X \rightarrow K(G, \deg(\alpha_i))$ , the product of these maps induces a surjection on  $H^*$ . But I can't see how it does? So, we know by [5], Thm 4.57 that there are natural bijections  $T : [X, K(G, n)] \rightarrow H^n(X; G)$  with  $T[f] = f^*(\beta)$  for some class  $\beta \in H^n(K(G, n); G)$ . So, for any given  $\alpha^i \in H^n(X; G)$ , it's equal to  $f^*(\beta_i)$ , so definitely  $f^*$  hits it. But if I have some enormous product of EM spaces, and all these maps  $f_i : X \rightarrow K(G, n_i)$  come together to give a map  $f : X \rightarrow \prod_i K(G, n_i)$ , then Hatcher is trying to tell me I get a surjection  $H^*(\prod_i K(G, n_i)) \rightarrow H^*(X)$ ?? I know nothing about the cohomology of an infinite product, there's no reason it would be e.g. the product of the cohomologies.]

[Something something reduced cohomology theories for spectra and the wedge axiom?? Suppose  $K_0$  had finite type, which I'm not convinced it does. Then for some  $n$ ,  $(K_0)_n = (\bigvee_i \mathbb{K}(G, n_i))_n = \bigvee_i (\mathbb{K}(G, n_i))_n = \bigvee_i K(G, n + n_i)$ , and  $H^i(K_0) = H^{i+n}(\bigvee_i K(G, n + n_i)) = \prod_i H^{i+n}(K(G, n + n_i))$ . Does that help?]

- **Fact: each  $H^*(K_s)$  is a free  $\mathcal{A}_2$ -module**, and thus the diagram above gives a resolution of  $H^*(X)$  by free  $\mathcal{A}_2$ -modules. [Vague idea: Hatcher claims on p582 that "the cohomology  $H^*(K(\mathbb{F}_2, n))$  is free over  $\mathcal{A}_2$  in dimensions less than  $2n$ " (this may be [13] Prop 3.22 but I'm not sure what's going on with the suspension there). Assuming this, I said earlier that if  $X$  has finite type you can act on  $H^*(X)$  with  $Sq^i$  by first pushing it along the limit as far as you need to to make  $H^{i+j+n}(X_{n+j})$  stable and then applying  $Sq^i$  to  $H^{i+n}(X_n)$ . But the point here is that I can make  $n$  as big as I like, so I can just make it big enough so that any action of  $\mathcal{A}_2$  I like is free.]

[The work in Section 3.4 gives us long exact sequences

$$\cdots \rightarrow [\mathbb{S}^{t+1}, X_s] \rightarrow [\mathbb{S}^{t+1}, K_s] \rightarrow [\mathbb{S}^{t+1}, X_{s+1}] \rightarrow [\mathbb{S}^{t+1}, \Sigma X_s] \rightarrow [\mathbb{S}^{t+1}, \Sigma K_s] \rightarrow \cdots$$

Using the isomorphism  $[Y, Z] \cong [\Sigma Y, \Sigma Z]$ , we get long exact sequences

$$\cdots \rightarrow [\Sigma^\infty \mathbb{S}^{t+1}, X_s] \rightarrow [\mathbb{S}^{t+1}, K_s] \rightarrow [\mathbb{S}^{t+1}, X_{s+1}] \rightarrow [\mathbb{S}^t, X_s] \rightarrow [\mathbb{S}^t, K_s] \rightarrow \cdots,$$

i.e.

$$\cdots \rightarrow \pi_{t+1} X_s \rightarrow \pi_{t+1} K_s \rightarrow \pi_{t+1} X_{s+1} \rightarrow \pi_t X_s \rightarrow \pi_t K_s \rightarrow \cdots$$

These form a staircase diagram,

$$\begin{array}{ccccccccccc}
& \vdots & & \vdots & & \vdots & & \vdots & & \vdots & \\
& \downarrow i & & \downarrow i & & \downarrow i & & \downarrow i & & \downarrow i & \\
\cdots & \xrightarrow{k} & \pi_{t+1}X_s & \xrightarrow{j} & \pi_{t+1}K_s & \xrightarrow{k} & \pi_{t+1}X_{s+1} & \xrightarrow{j} & \pi_{t+1}K_{s+1} & \xrightarrow{k} & \pi_{t+1}X_{s+2} & \xrightarrow{j} & \cdots \\
& \downarrow i & & \downarrow i & & \downarrow i & & \downarrow i & & \downarrow i & \\
\cdots & \xrightarrow{k} & \pi_tX_{s-1} & \xrightarrow{j} & \pi_tK_{s-1} & \xrightarrow{k} & \pi_tX_s & \xrightarrow{j} & \pi_tK_s & \xrightarrow{k} & \pi_tX_{s+1} & \xrightarrow{j} & \cdots \\
& \downarrow i & & \downarrow i & & \downarrow i & & \downarrow i & & \downarrow i & \\
\cdots & \xrightarrow{k} & \pi_{t-1}X_{s-2} & \xrightarrow{j} & \pi_{t-1}K_{s-2} & \xrightarrow{k} & \pi_{t-1}X_{s-1} & \xrightarrow{j} & \pi_{t-1}K_{s-1} & \xrightarrow{k} & \pi_{t-1}X_s & \xrightarrow{j} & \cdots \\
& \downarrow i & & \downarrow i & & \downarrow i & & \downarrow i & & \downarrow i & \\
& \vdots & & \vdots & & \vdots & & \vdots & & \vdots & 
\end{array}$$

This gives rise to a spectral sequence, by [some version of] [Theorem 4.2.3](#).

Now, since  $K_s = \bigvee_i \mathbb{K}(G, n_{s_i})$ , [Proposition 3.5.2](#), tells us that  $[\mathbb{S}, K_s] \cong \prod_i [\mathbb{S}, \mathbb{K}(G, n_{s_i})]$ , which is naturally isomorphic to  $\prod_i H^{n_{s_i}}(\mathbb{S}; G)$ . Thus, elements of  $[\mathbb{S}, K_s]$  are tuples of elements of  $H^*(\mathbb{S})$ .

- Def:  $\text{Hom}_{\mathcal{A}_2}^t$  [it's just the homs that lower the degree by  $t$ ]
- **Fact:** there is a natural map  $[\mathbb{S}, K_s] \rightarrow \text{Hom}_{\mathcal{A}_2}^0(H^*(K_s), H^*(\mathbb{S}))$ , and it's an isomorphism. [??]

[We thus have

$$[\Sigma^t \mathbb{S}, K_s] = \text{Hom}_{\mathcal{A}_2}^0(H^*(K_s), H^*(\Sigma^t \mathbb{S})) = \text{Hom}_{\mathcal{A}_2}^t(H^*(K_s), H^*(\mathbb{S})).$$

In the case of CW complexes, we have  $H^*(\Sigma^t X) \cong H^{*-t}(X)$  via the map  $\sigma$  mentioned in [Section 2](#). Since  $\mathbb{S}$  has finite type, for  $i$  large enough we have  $H^n(\Sigma^t \mathbb{S}) = H^{n+i}(\Sigma^t S^i) = H^{n+i-t}(S^i) = H^{n-t}(\mathbb{S})$ .]

$[E_1^{s,t} = \pi_t K_s (= \text{Hom}_{\mathcal{A}_2}^t(H^*(K_s), H^*(\mathbb{S})))$ , since the staircase diagram comes from (or gives rise to, depending on your point of view) the exact couple:

$$\begin{array}{ccc}
\pi_* X_* & \xrightarrow{i} & \pi_* X_* \\
\swarrow k & & \searrow j \\
& \pi_* K_* & 
\end{array}$$

where  $i : \pi_{t+1}X_{s+1} \rightarrow \pi_t X_s$ ,  $j : \pi_{t+1}X_s \rightarrow \pi_{t+1}K_s$ , and  $k : \pi_{t+1}X_{s+1} \rightarrow \pi_{t+1}K_s$  all come from the diagram we set up earlier. The differential  $d_1 : \pi_t(K_s) \rightarrow \pi_t K_{s+1}$  is induced by the map  $K_s \rightarrow K_{s+1}$ , since it's just  $j \circ k$ , according to the way things are set up in [Section 4.2](#), possibly modulo some typos.]

[Then,  $E_2^{s,t} = H^{s,t}(E_1^{*,*}, d_1)$ , so each  $E^{*,t}$  is the homology of the chain complex

$$0 \rightarrow E_1^{0,t} \rightarrow E_1^{1,t} \rightarrow E_1^{2,t} \rightarrow \cdots,$$

or, in other words,

$$0 \rightarrow \text{Hom}_{\mathcal{A}_2}^t(H^*(K_0), H^*(\mathbb{S})) \rightarrow \text{Hom}_{\mathcal{A}_2}^t(H^*(K_1), H^*(\mathbb{S})) \rightarrow \cdots.$$



The homology of this is by definition  $\text{Ext}_{\mathcal{A}_2}^{s,t}(H^*(X), H^*(\mathbb{S}))$ , so  $E_2^{s,t} = \text{Ext}_{\mathcal{A}_2}^{s,t}(H^*(X), H^*(\mathbb{S}))$ .]

THEOREM 4.3.1 ([6], Thm 5.47). There is a spectral sequence  $\{E_r, d_r\}$  such that  $E_2^{s,t} = \text{Ext}_{\mathcal{A}_2}^{s,t}(\mathbb{F}_2, \mathbb{F}_2)$  and  $\{E_r, d_r\} \implies \pi_{t-s}^S$  modulo torsion of odd order. [which is probably a lie but let's worry about that later]

[10], [6]

## 4.4 Multiplicative structure

### 4.4.1 The Yoneda product

- There is a multiplication, the *Yoneda product*, on  $E_2 = \text{Ext}(\text{whatever})$
- More generally, [12]: For any algebra  $A$  and  $A$ -modules  $L, M, N$ , there is a natural Yoneda composition product

$$\circ : \text{Ext}_A^s(M, N) \otimes \text{Ext}_A^u(L, M) \rightarrow \text{Ext}_A^{s+u}(L, N)$$

[doing what it should].

[The stuff below is old and needs to be rewritten but I'm putting it here for the moment. Essentially I've specialised to the case where the resolution is free so  $\text{Ext} = \text{Hom}$ .]

Using Lemma 5.1.1 (which comes later but whatever, I'll reorder things), we get  $E_2^{s,t} = \text{Hom}_{\mathcal{A}_2}^t(F_s, \mathbb{F}_2)$ . So, if we have a minimal free resolution

$$\cdots \rightarrow F_2 \rightarrow F_1 \rightarrow F_0 \rightarrow \mathbb{F}_2$$

of  $\mathbb{F}_2$  as an  $\mathcal{A}_2$ -module, and we have  $f \in \text{Hom}_{\mathcal{A}_2}^t(F_s, \mathbb{F}_2), g \in \text{Hom}_{\mathcal{A}_2}^{t'}(F_{s'}, \mathbb{F}_2)$ , we multiply them as follows:

First, we put these resolutions side by side.

$$\begin{array}{ccccccccccccccc} F_{s+s'} & \longrightarrow & F_{s+s'-1} & \longrightarrow & \cdots & \longrightarrow & F_s & \longrightarrow & F_{s-1} & \longrightarrow & \cdots & \longrightarrow & F_0 & \longrightarrow & \mathbb{F}_2 \\ \downarrow & & \downarrow & & & & \downarrow & \searrow f & & & & & & & \\ F_{s'}[t] & \longrightarrow & F_{s'-1}[t] & \longrightarrow & \cdots & \longrightarrow & F_0[t] & \twoheadrightarrow & \mathbb{F}_2[t] & & & & & & \\ \downarrow g & & & & & & & & & & & & & & \\ \mathbb{F}_2[t+t'] & & & & & & & & & & & & & & \end{array}$$

(The  $[t]$  means shifted by  $t$  i.e. in degree  $t$  because the maps are.)

Then we inductively fill in the dotted arrows from the bottom up. So you can fill the first one because the modules are all free so you just need to say where the generator goes (a priori maybe there are infinitely many generators but it's ok because these are degree  $t$  maps and there are only finitely many in each degree), and the right map is surjective, so just do it the way you would do it. Likewise for the higher ones.

This thing is hopefully well-defined.

#### 4.4.2 The composition product

From [13], 5.1

DEFINITION 4.4.1. Let  $X, Y, Z$  be spectra. The *composition pairing*  $\circ : [Y, Z]_* \otimes [X, Y]_* \rightarrow [X, Z]_*$  is defined as follows:

$$\begin{aligned} \circ : [Y, Z]_v \otimes [X, Y]_t &\rightarrow [X, Z]_{v+t} \\ [g : \Sigma^v Y \rightarrow Z] \otimes [f : \Sigma^t X \rightarrow Y] &\mapsto [g \circ \Sigma^v f : \Sigma^{v+t} X \rightarrow Z], \end{aligned}$$

where  $[X, Y]_n = [S^n \wedge X, Y]$  [which is the same as  $[\Sigma^n X, Y]$  by the version of Remark B.2.4 for spectra<sup>9</sup>]

In particular, if  $X = Y = Z = \mathbb{S}$ , we have a product  $\pi_v \mathbb{S} \otimes \pi_t \mathbb{S} \rightarrow \pi_{v+t} \mathbb{S}$ , or, in other words, a product  $\pi_v^S \otimes \pi_t^S \rightarrow \pi_{v+t}^S$ .

LEMMA 4.4.2. The composition product makes  $\pi_*^S$  into a graded commutative ring.

(proof)

LEMMA 4.4.3. There is a unique ring structure on  $2\pi_*^S$  which makes the completion map  $c : \pi_*^S \rightarrow 2\pi_*^S$  into a ring homomorphism.

PROOF. We show uniqueness first. Let  $f \in 2\pi_i^S, g \in 2\pi_j^S$ . If  $i, j \geq 1$ , then the completion map is surjective, so  $f = c(\tilde{f}), g = c(\tilde{g})$  for some  $\tilde{f} \in \pi_i^S, \tilde{g} \in \pi_j^S$ . Then  $fg = c(\tilde{f})c(\tilde{g}) = c(\tilde{f}\tilde{g})$ .

If  $i = 0, j \geq 1$ , then let  $\hat{f} \in \pi_0^S$  be a lift of  $q(f) \in \pi_0^S/2^r\pi_0^S$ , where  $2^r$  is the highest power of 2 dividing the order of  $\pi_j^S$ . Then  $f \equiv c(\hat{f}) \pmod{2^r}$ , so  $f = c(\hat{f}) + 2^r w$ . We have  $fg = f c(\tilde{g}) = (c(\hat{f}) + 2^r w) c(\tilde{g}) = c(\hat{f}) c(\tilde{g}) + 2^r (w c(\tilde{g})) = c(\hat{f}\tilde{g}) \in 2\pi_j^S$ .

Finally, if  $i = j = 0$ , we claim that any two multiplications on  $\mathbb{Z}_2$  which agree on  $\mathbb{Z}$  must agree on all of  $\mathbb{Z}_2$ , and thus the multiplication is given by the usual product on  $\mathbb{Z}_2$ .

Suppose not; let  $\star, \cdot$  be two products on  $\mathbb{Z}_2$ , agreeing on  $\mathbb{Z}$ , with  $f \star g \neq f \cdot g$ . Then there is some  $k$  such that  $f \star g \not\equiv f \cdot g \pmod{k}$ . Pick integers  $n, m$  such that  $n \equiv f \pmod{k}$  and  $m \equiv g \pmod{k}$ . Then, modulo  $k$ ,  $f \cdot g \equiv n \cdot m = n \star m \equiv f \star g$ , giving a contradiction.

Now, for  $i, j \geq 1$ , this multiplication is well defined, since if  $\tilde{f}' = \tilde{f} + t$ , with  $nt = 0$  for odd  $n$ , then  $c(\tilde{f}'\tilde{g}) = c(\tilde{f}\tilde{g} + t\tilde{g}) = c(\tilde{f}\tilde{g})$  (since  $nt\tilde{g} = (nt)\tilde{g} = 0\tilde{g} = 0$ ). Likewise, if  $\tilde{g}' = \tilde{g} + t$ , then  $c(\tilde{f}\tilde{g}') = c(\tilde{f}\tilde{g})$ . Note that the product for  $i = 0, j \geq 1$  (and vice versa) is exactly the isomorphism in the proof of Lemma 3.6.2, and the usual product on  $\mathbb{Z}_2$  is of course well-defined. Finally, associativity, distributivity, and unitality are inherited from  $\pi_*^S$ .  $\square$

- Given spectra  $X, Y, Z$ , we have a pairing  $\circ : [Y, {}_2Z]_* \otimes [X, {}_2Y]_* \rightarrow [X, {}_2Z]$ : given  $f \in [Y, {}_2Z], g \in [X, {}_2Y]$ , we can ‘compose’ to get  $\bar{f}g \in [X, {}_2Z]$ .
- Note that suspension commutes with  $p$ -completion; the  $i$ th suspension of the canonical map  $\Sigma^i Y \rightarrow \Sigma^i Y_p^\wedge$  descends to a map  $\pi_j(\Sigma^i Y) \rightarrow \pi_j(\Sigma^i Y_p^\wedge)$ , but  $\pi_j(\Sigma^i Y)$  is just  $\pi_{i+j}(Y)$  and  $\pi_j(\Sigma^i Y_p^\wedge)$  is  $\pi_{i+j}(Y_p^\wedge)$ , so the map on homotopy groups is the one witnessing  $p$ -completion.

LEMMA 4.4.4. The completion map  $\pi_*(S) \rightarrow \pi_*(S_2^\wedge)$  is a ring homomorphism. In particular, by Lemma 4.4.3, the composition product on  $\pi_*(S_2^\wedge)$  coincides with the product on  $\pi_*(S)_2^\wedge$  inherited from  $\pi_*^S$ , so the two groups are also isomorphic as rings.

<sup>9</sup>Note, by the way, that this notation is defined in [12] Def 11.1.1, and yes, he really means  $S^n$ , as in the normal  $n$ -dimensional sphere, so we’re taking the smash product of a spectrum and a space, which is fine.

PROOF. Let  $f : S^i \rightarrow S$ ,  $g : S^j \rightarrow S$  be elements of  $\pi_i(S)$  and  $\pi_j(S)$  respectively. Then  $c(f)c(g) = (cf)(cg)$ <sup>10</sup> is given by factorising  $cg = \overline{cg}c$  and composing to get  $\overline{cg}c\Sigma^j f$ . We thus have the commutative diagram below.

$$\begin{array}{ccccc} S^{i+j} & \xrightarrow{\Sigma^j f} & S^j & \xrightarrow{g} & S \\ & & \downarrow c & & \downarrow c \\ & & \Sigma^j S_2^\wedge & \xrightarrow{\overline{cg}} & S_2^\wedge \end{array}$$

The upper path is exactly  $c(fg)$ , so  $c(f)c(g) = c(fg)$ . Further, the completion map clearly preserves the identity, so it is a ring homomorphism.  $\square$

[Throughout this section I use  $c$  to mean, at various points, literally any completion map I like on spectra (including two different maps in the same proof), and also any completion map on groups. It's not a huge deal because it's obvious what I mean, but I should either decorate them or make a remark about this.]

#### 4.4.3 Multiplication on the spectral sequence

- There is a multiplication on  $E_r$  which coincides with the Yoneda product on  $E_2$  and converges (whatever that means) to the composition product on  ${}_2\pi_*^S$ . It's associative and unital, and plays nicely with the differentials.

Questions/problems (briefly):

- If I compute a product on the  $E_2$  page, and both terms survive to the  $E_3$  page, is that computation still valid?

DEFINITION 4.4.5 ([13], Def 5.5). Let  $\{^{\prime}E_r\}, \{^{\prime\prime}E_r\}, \{E_r\}$  be three spectral sequences. A *pairing* of these spectral sequences is a sequence of homomorphisms

$$\phi_r : ^{\prime}E_r^{*,*} \otimes ^{\prime\prime}E_r^{*,*} \rightarrow E_r^{*,*}$$

such that the Leibniz rule

$$d_r \phi_r(x \otimes y) = \phi_r(d_r(x) \otimes y) + (-1)^{\deg x} \phi_r(x \otimes d_r(y))$$

holds, and

$$\phi_{r+1}([x] \otimes [y]) = [\phi_r(x \otimes y)],$$

where  $[x] \in ^{\prime}E_{r+1}^{*,*}$  is the homology class of a  $d_r$ -cycle  $x \in ^{\prime}E_r^{*,*}$ , and similarly for  $y$  and the right hand side.

A spectral sequence pairing  $\{\phi_r\}$  induces a pairing

$$\phi_\infty : ^{\prime}E_\infty^{*,*} \otimes ^{\prime\prime}E_\infty^{*,*} \rightarrow E_\infty^{*,*}.$$

What [12] says:

- 11.8.5: For spectra  $X, Y, Z$ , consider the Adams spectral sequences

$$^{\prime}E_2 = \text{Ext}_A(H_*(Y), H_*(Z)) \implies [Y, Z]_*$$

$$^{\prime\prime}E_2 = \text{Ext}_A(H_*(X), H_*(Y)) \implies [X, Y]_*$$

$$E_2 = \text{Ext}_A(H_*(X), H_*(Z)) \implies [X, Z]_*$$

---

<sup>10</sup>Oh god I'm so sorry I didn't realise that notation was so appalling, I'll fix it later!!

1. There is a natural pairing

$$\circ_r : ({}^{\prime}E_r, {}^{\prime\prime}E_r) \rightarrow E_r$$

of Adams spectral sequences, with abutment the filtration-preserving pairing

$$\circ : [Y, Z]_* \otimes [X, Y]_* \rightarrow [X, Z]_*$$

mapping  $g \otimes f$  to  $g \circ \Sigma^{|g|} f$

2. The pairing of  $E_2$ -terms

$$\circ_2 : \text{Ext}_{A_*}(H_*(Y), H_*(Z)) \otimes \text{Ext}_{A_*}(H_*(X), H_*(Y)) \rightarrow \text{Ext}_{A_*}(H_*(X), H_*(Z))$$

is the composition product.

3. If  $Y/p$  and  $Z/p$  are bounded below of finite type, then the  $E_2$ -pairing

$$\circ_2 : \text{Ext}_{A_*}(H^*(Z), H^*(Y)) \otimes \text{Ext}_{A_*}(H^*(Y), H^*(X)) \rightarrow \text{Ext}_{A_*}(H^*(Z), H^*(X))$$

is the twisted composition product, mapping  $y \otimes x$  to  $(-1)^{|x||y|} x \circ y$ , where  $|x| = v - u$  and  $|y| = t - s$  for  $x \in {}^{\prime\prime}E_2^{u,v}$  and  $y \in {}^{\prime}E_2^{s,t}$ .

What [13] says:

THEOREM 4.4.6 ([13], Thm 5.8). Let  $X, Y, Z$  be spectra, with  $Y, Z$  connective and of finite type. There is a pairing of spectral sequences

$$E_r^{*,*}(Y, Z) \otimes E_r^{*,*}(X, Y) \rightarrow E_r^{*,*}(X, Z)$$

which agrees for  $r = 2$  with the Yoneda pairing

$$\text{Ext}_{\mathcal{A}_2}^{*,*}(H^*(Z), H^*(Y)) \otimes \text{Ext}_{\mathcal{A}_2}^{*,*}(H^*(Y), H^*(X)) \rightarrow \text{Ext}_{\mathcal{A}_2}^{*,*}(H^*(Z), H^*(X))$$

and which converges to the composition pairing

$$[Y, {}_2Z]_* \otimes [X, {}_2Y]_* \rightarrow [X, {}_2Z]_*.$$

The pairing is associative and unital.

[Leibniz rule?]

[13], [12], [7]

## 5 Calculating stable homotopy groups

### 5.1 The $E_2$ page

LEMMA 5.1.1 ([6], Lem 5.49). For a minimal free resolution

$$\cdots \rightarrow F_2 \rightarrow F_1 \rightarrow F_0 \rightarrow \mathbb{F}_2 \rightarrow 0$$

of  $\mathbb{F}_2 = H^*(\mathbb{S})$  as an  $\mathcal{A}_2$ -module, we have  $\text{Ext}_{\mathcal{A}_2}^{s,t}(\mathbb{F}_2, \mathbb{F}_2) = \text{Hom}_{\mathcal{A}_2}^t(F_s, \mathbb{F}_2)$ , where  $\text{Hom}_{\mathcal{A}_2}^t(F_s, \mathbb{F}_2) \subseteq \text{Hom}_{\mathcal{A}_2}(F_s, \mathbb{F}_2)$  consists of the morphisms which lower the degree by  $t$ .

[Now, since  $\mathbb{F}_2$  just has stuff in degree 0, the only things that can be sent to  $1 \in \mathbb{F}_2$  are the things in degree  $t$ , so for every generator for  $F_s$  in degree  $t$ , there's an  $\mathbb{F}_2$ 's worth of such homs.]

[13], [6], [12]

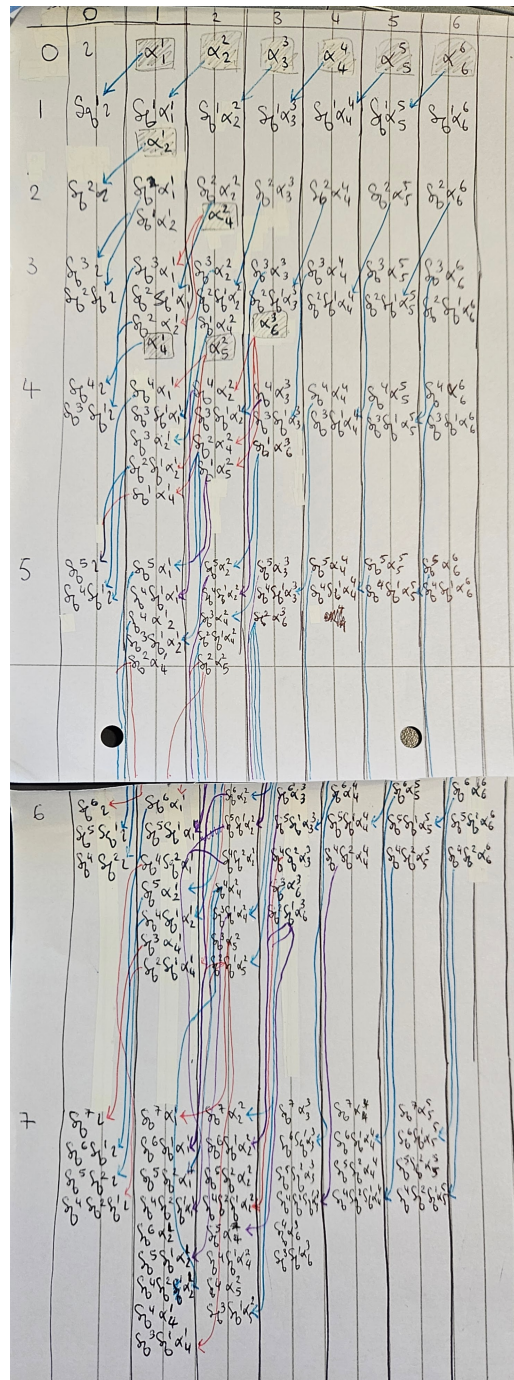


Figure 5.1: Calculating the first 5 rows

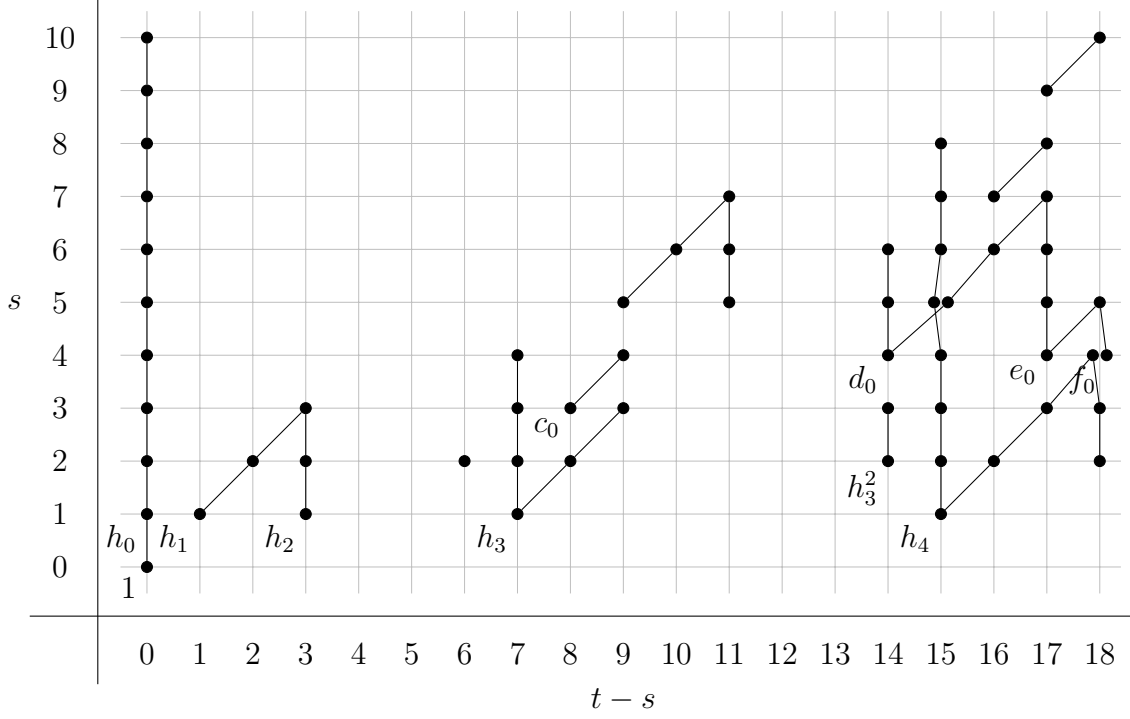


Figure 5.2:  $\text{Ext}_{\mathcal{A}_2}^{s,t}(\mathbb{F}_2, \mathbb{F}_2)$  for  $t - s \leq 18$ .<sup>11</sup>

LEMMA 5.1.2. There are no nontrivial differentials for  $t - s \leq 13$ .

PROOF. First, note that the only possible nontrivial differentials in this range are  $d_r : E_r^{1,2} \rightarrow E_r^{1+r,1+r}$  and  $d_2 : E_2^{2,10} \rightarrow E_2^{4,11}$ .

Now,  $0 = d_r(h_0 h_1) = d_r(h_0)h_1 + h_0 d_r(h_1) = h_0 d_r(h_1)$ , so  $d_r(h_1) = 0$ . Since  $E_r^{1,2}$  is generated by  $h_1$ , we must have  $d_r = 0$ .

On the other hand,  $E^{2,10}$  is generated by  $h_1 h_3$ , and  $d_2(h_1 h_3) = d_2(h_1)h_3 + h_1 d_2(h_3) = 0 + 0 = 0$  (the first factor is zero by the previous computation, and the second is an element of a trivial group).  $\square$

[Obviously justify the  $h$  notation and the claims above that the generators are what I say they are]

## 5.2 Resolving extensions

PROPOSITION 5.2.1 ([13], Cor 6.5). We have the following relations:

$$\begin{aligned} \alpha_i^i &= (\alpha_1^1)^i \\ \alpha_4^2 &= (\alpha_2^1)^2 \\ \alpha_5^2 &= \alpha_1^1 \alpha_4^1 \\ \alpha_6^3 &= (\alpha_1^1)^2 \alpha_4^1 = (\alpha_2^1)^3. \end{aligned}$$

A few other things:

1. We know that  $\pi_0^S = \mathbb{Z}$  since  $\pi_1 S^1 = \mathbb{Z}$  and  $n = 1 \leq 2 = 2(1)$ , so this lies in the stable region. Now,  $E^{s,t}$  is actually supposed to converge to some filtration of  $\mathbb{Z}_2$  (i.e. the 2-

<sup>11</sup>By the way,  $f_0$  is the node on the right, i.e. the node at (18,4).

adics) (i.e.  ${}_2\mathbb{Z}$ ) I think? But either way, we know that for our filtration all the quotients are  $\mathbb{Z}/2\mathbb{Z}$ . I claim<sup>12</sup> then that the filtration must be  $\cdots 4\mathbb{Z}_2 \subseteq 2\mathbb{Z}_2 \subseteq \mathbb{Z}_2$ .

2. If that's true then  $\iota = [1] \in \mathbb{Z}_2/2\mathbb{Z}_2$ , and using the multiplication above we see  $\iota$  is a unit. We also have  $h_0 = [2] \in 2\mathbb{Z}_2/4\mathbb{Z}_2$  so  $h_0 = [2] = [2[1]] = [2\iota]$ , so  $h_0$  acts on  $\iota$  by multiplication by 2.

3. Now, for any  $\kappa \in E_2^{s,t}$ ,  $h_0 \cdot \kappa = (\iota h_0) \cdot \kappa = 2\kappa$ .

The notation's a bit weird above, when I say 'multiplication by 2' I mean: take  $\kappa \in E^{s,t} = F^{s,t}/F^{s+1,t+1}$ . Then  $2\kappa \in F^{s+1,t+1}$  since it's a bunch of  $\mathbb{F}_2$ 's. Take it's equivalence class to get an element of  $F^{s+1,t+1}/F^{s+2,t+2}$ . That's what I really mean by  $2\kappa$  and it's in  $E^{s+1,t+1}$ .

All this is to say if I start multiplying higher things by  $h_0$ , that *is* multiplying by 2. So I can start resolving extensions this way.]

[Justify the below with sseq, not quite sure how to do this in a satisfying way though. See note in Section C.2]

THEOREM 5.2.2.

$${}_2\pi_i^S = \begin{cases} \mathbb{Z}/2\mathbb{Z} & i = 1, 2, 6, 10 \\ \mathbb{Z}/8\mathbb{Z} & i = 3, 11 \\ 0 & i = 4, 5, 12, 13 \\ \mathbb{Z}/16\mathbb{Z} & i = 7 \\ (\mathbb{Z}/2\mathbb{Z})^2 & i = 8 \\ (\mathbb{Z}/2\mathbb{Z})^3 & i = 9. \end{cases}$$

### 5.3 Differentials at $14 \leq t - s \leq 15$

[The point here is that all differentials interacting with  $E^{s,t}$  for  $t - s < 14$  are trivial, and thus computing the stable homotopy groups is purely mechanical, because everything that appears on this part of the  $E_2$  page has to survive to  $E_\infty$ . Thus, the 'ambiguity' at  $t - s = 14$  is just the fact that this is the first time you need to actually compute differentials.]

THEOREM 5.3.1 ([12], Thm 11.10.2).  $d_2(h_4) = h_0 h_3^2$ .

PROOF. We have shown that  $h_0$  detects  $2 \in {}_2\pi_*^S$  (i.e. 2 is a representative for  $h_0$ ). Let  $\sigma \in {}_2\pi_7^S$  be a representative for  $h_3$ . Then  $2\sigma^2$  is a representative for  $h_0 h_3^2$ . By graded commutativity of  ${}_2\pi_*^S$ ,  $\sigma^2 = -\sigma^2$ , so  $2\sigma^2 = 0$ , and thus  $h_0 h_3^2 = 0$  in  $E_\infty$ . Therefore,  $h_0 h_3^2$  is the boundary of a differential, and the only possibility is  $d_2(h_4) = h_0 h_3^2$ .  $\square$

The  $d_2$  differentials at  $E_2^{2,17}, E_2^{3,18}, E_2^{4,19}$  are all trivial:  $d_2(h_0^n h_4) = h_0^n d_2(h_4) = h_0^{n-1} (h_0 h_3^2) = 0$ .

<sup>12</sup>Here is an absolutely ridiculous argument for that:  $(\mathbb{Z}_2, +)$  is a topological group. For any subgroup  $H$  of finite index  $n = 2^k m$ ,  $H$  is open (since by Lagrange's theorem  $n\mathbb{Z}_2 \subseteq H$  and thus  $n\mathbb{Z}_2 = 2^k \mathbb{Z}_2$ , so  $x + 2^k \mathbb{Z}_2$  is an open ball around any  $x \in H$ ). Now, every open subgroup of  $\mathbb{Z}_2$  is also closed (since it's complement is a union of open cosets), and it's apparently the case [elaborate] that the closed subgroups of  $\mathbb{Z}_2$  are the ideals  $2^k \mathbb{Z}_2$ . These facts then imply that the filtration I give is the only possible one. Short answer: finite index subgroups of  $\mathbb{Z}_p$  are of the form  $p^k \mathbb{Z}_p$ .



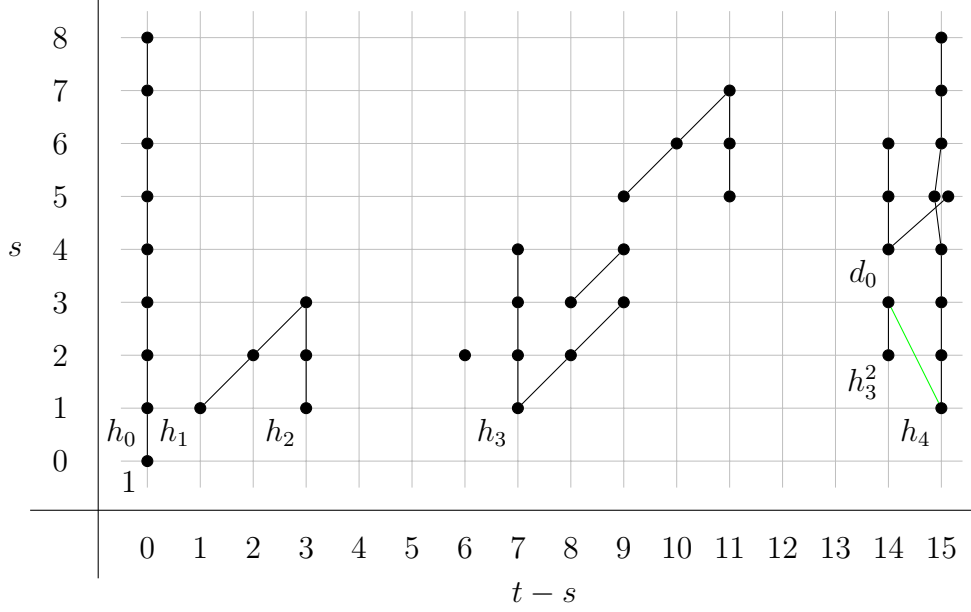


Figure 5.3: The  $E_2$  page of the Adams spectral sequence for  $\mathbb{S}$ , in the range  $t - s \leq 15$ ; the unique  $d_2$  differential is shown in green.

[I promise we're gonna do the  $d_3$  differential but first we need a few lemmas.]

LEMMA 5.3.2 ([12], Table 14.1 (9)).  $d_2(f_0) \neq 0$ .

PROOF. First, note that  $d_2(i)$  is nontrivial, since  $h_4i = 0$  and  $h_0h_3^2i \neq 0$ , so  $0 = d_2(h_4i) = h_0h_3^2i + h_4d_2(i)$  implies that  $d_2(i) \neq 0$ . Now,  $d_2(j) \neq 0$  since  $h_0d_2(j) = d_2(h_0j) = d_2(h_2i) = h_2d_2(i) \neq 0$ . An almost identical argument shows that  $d_2(k), d_2(l) \neq 0$ , and thus  $d_2(h_0l) = h_0d_2(l) \neq 0$ . Finally, we have  $h_0l = d_0f_0$ , so  $0 \neq d_2(h_0l) = d_2(d_0f_0) = d_0d_2(f_0)$ .  $\square$

[Probably need to move the theorem below but I'll leave it here for the moment]

THEOREM 5.3.3 ([13], Cor 4.17). Let  $f : Y \rightarrow Z$  be a map of connective spectra of finite type. Then there is a map

$$f_* : \{E_r(Y), d_r\}_r \rightarrow \{E_r(Z), d_r\}_r$$

of Adams spectral sequences, given at the  $E_2$ -level by the homomorphism

$$(f^*)^* : \text{Ext}_{\mathcal{A}_2}^{s,t}(H^*(Y), \mathbb{F}_2) \rightarrow \text{Ext}_{\mathcal{A}_2}^{s,t}(H^*(Z), \mathbb{F}_2)$$

induced by the  $\mathcal{A}_2$ -module homomorphism  $f^* : H^*(Z) \rightarrow H^*(Y)$ , with abutment the homomorphism

$$f_* : \pi_*(Y) \rightarrow \pi_*(Z).$$

LEMMA 5.3.4 ([12], Table 14.9 (4)). Consider the cofibration

$$\mathbb{S}^7 \xrightarrow{\sigma} \mathbb{S} \xrightarrow{i} C_\sigma \xrightarrow{j} \mathbb{S}^8 \rightarrow \dots$$

Let  $\overline{\overline{h_0^2 h_3}} \in E_2^{3,18}(C_\sigma)$  be the generator shown in Figure 5.3. Then  $d_2(\overline{\overline{h_0^2 h_3}}) = \hat{i}(h_0 d_0)$ , where  $\hat{i} : \text{Ext}^{s,t}(\mathbb{S}) \rightarrow \text{Ext}^{s,t}(C_\sigma)$  is the map induced by filling in the diagram below with the unique

dotted lifts.<sup>13</sup>

$$\begin{array}{ccccccc}
\cdots & \longrightarrow & F_2 & \longrightarrow & F_1 & \longrightarrow & F_0 \longrightarrow H^*(C_\sigma) \\
& & \downarrow & & \downarrow & & \downarrow \\
\cdots & \longrightarrow & P_2 & \longrightarrow & P_1 & \longrightarrow & P_0 \longrightarrow H^*(\mathbb{S})
\end{array}$$

$\downarrow i^*$

PROOF.

[Some note about how we're calling  $i(h_2)$  just  $h_2$  because for anything in the range  $t - s < 7$   $i$  is injective]

We first show that  $d_2(h_2 \overline{h_0^2 h_3}) = d_2(\hat{i}(f_0))$ . By Lemma 3.4.4, we have a long exact sequence

$$\cdots \leftarrow H^{n-1}(\mathbb{S}^8) \leftarrow H^n(\mathbb{S}) \xleftarrow{i^*} H^n(C_\sigma) \xleftarrow{j^*} H^n(\mathbb{S}^8) \leftarrow H^{n+1}(\mathbb{S}) \leftarrow \cdots$$

However, any map  $H^n(\mathbb{S}) \rightarrow H^{n-1}(\mathbb{S}^8)$  must be zero, so we get short exact sequences

$$0 \leftarrow H^n(\mathbb{S}) \xleftarrow{i^*} H^n(C_\sigma) \xleftarrow{j^*} H^n(\mathbb{S}^8) \leftarrow 0.$$

Taking a direct sum gives a short exact sequence

$$0 \leftarrow \mathbb{F}_2 \xleftarrow{i^*} H^*(C_\sigma) \xleftarrow{j^*} \mathbb{F}_2[8] \leftarrow 0.$$

From the short exact sequence above, we get<sup>14</sup> a long exact sequence

$$\begin{array}{ccccccc}
\cdots & \longrightarrow & \text{Ext}^{s,t}(\mathbb{F}_2, \mathbb{F}_2) & \xrightarrow{i} & \text{Ext}^{s,t}(H^*(C_\sigma), \mathbb{F}_2) & \xrightarrow{j} & \text{Ext}^{s,t-8}(\mathbb{F}_2, \mathbb{F}_2) \\
& & & & & & \downarrow \\
& & & & & & \text{Ext}^{s+1,t}(\mathbb{F}_2, \mathbb{F}_2) \xrightarrow{i} \text{Ext}^{s+1,t}(H^*(C_\sigma), \mathbb{F}_2) \xrightarrow{j} \text{Ext}^{s+1,t-8}(\mathbb{F}_2, \mathbb{F}_2) \longrightarrow \cdots
\end{array}$$

Now,  $f_0 \in \text{Ext}^{4,22}(\mathbb{F}_2, \mathbb{F}_2)$ ; we consider the exact sequence

$$\text{Ext}^{3,14}(\mathbb{F}_2, \mathbb{F}_2) \rightarrow \text{Ext}^{4,22}(\mathbb{F}_2, \mathbb{F}_2) \xrightarrow{i} \text{Ext}^{4,22}(H^*(C_\sigma), \mathbb{F}_2).$$

Figure 5.1 shows us that  $\text{Ext}^{3,14} = 0$ , so  $\hat{i}$  is injective at this point, and thus  $\hat{i}(f_0) \neq 0$ . Similarly,  $\text{Ext}^{4,15} = 0$  and  $\text{Ext}^{0,8} = 0$ , so  $\hat{i}(h_0 f_0), \hat{i}(h_4) \neq 0$ . Since  $\hat{i}$  respects multiplication<sup>15</sup>,  $\hat{i}(h_0 f_0) = \hat{i}(h_0) \hat{i}(f_0) \neq 0$ , so  $\hat{i}(f_0)$  is equal to either  $\overline{h_0^2 h_3}$  or  $\overline{h_0^2 h_3} + h_2 \hat{i}(h_4)$ . In either case, by linearity of  $d_2$ ,  $d_2(\hat{i}(f_0)) = d_2(\overline{h_0^2 h_3})$  (since  $d_2(h_2 \hat{i}(h_4)) = \hat{i}(d_2(h_2 h_4)) = 0$ <sup>16</sup>).

Now,  $\hat{i}(h_0^2 e_0) \neq 0$ , since  $\text{Ext}^{5,15}(\mathbb{F}_2, \mathbb{F}_2) = 0$  (using the long exact sequence in Ext again). Thus,  $h_2 d_2(\overline{h_0^2 h_3}) = d_2(h_2 \overline{h_0^2 h_3}) = d_2(\hat{i}(f_0)) = i(d_2(f_0)) = i(h_0^2 e_0) \neq 0$ , as required.  $\square$

<sup>13</sup>Make this explanation make any sense.

<sup>14</sup>This is a homological algebra fact, but maybe I should state it somewhere.

<sup>15</sup>I've just drawn some diagrams, and I think that yes,  $i(\alpha\beta) = i(\alpha)i(\beta)$ . Essentially because the way you multiply in Ext involves writing down unique lifts that make certain squares commute, but both  $i(\alpha\beta)$  and  $i(\alpha)i(\beta)$  are lifts making the appropriate squares commute, so by uniqueness they must be the same.

<sup>16</sup>Explain.

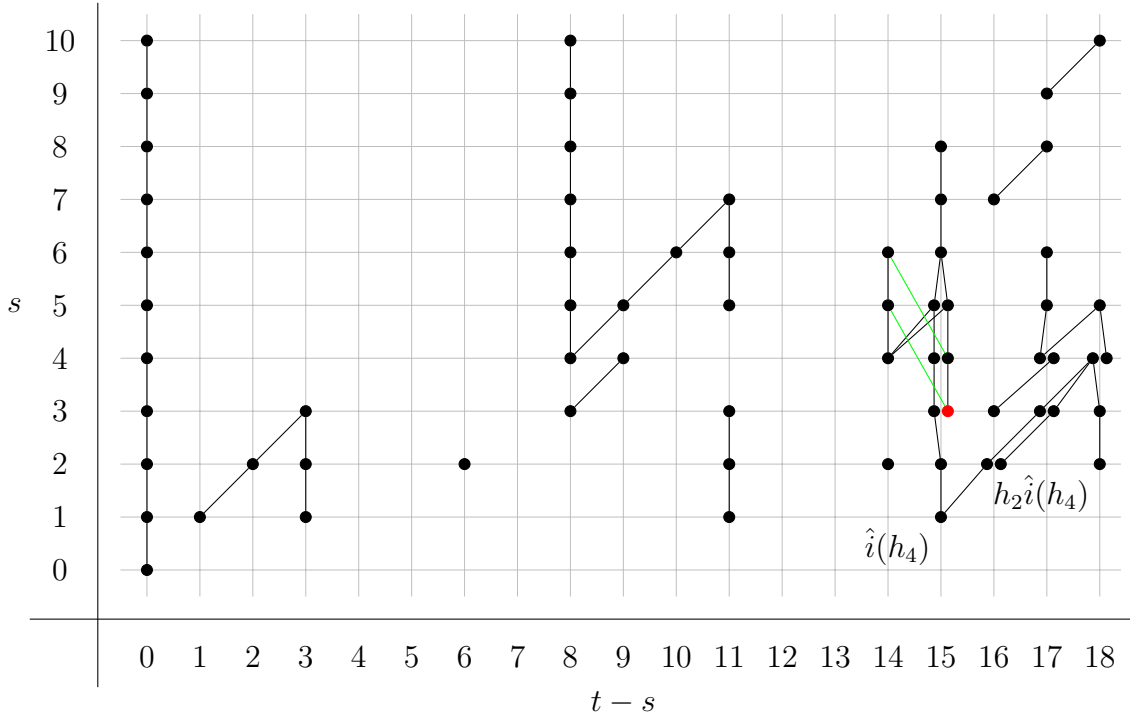


Figure 5.4: The  $E_2$  page of the Adams spectral sequence for  $C_\sigma$ , in the range  $t - s \leq 18$ , with the generator  $\overline{h_0^2 h_3}$  shown in red.

THEOREM 5.3.5 ([12], Table 14.2 (10)).  $d_3(h_0 h_4) = h_0 d_0$ .

PROOF. From the cofibration

$$\mathbb{S}^7 \xrightarrow{\sigma} \mathbb{S} \xrightarrow{i} C_\sigma \xrightarrow{j} \mathbb{S}^8 \hookrightarrow \mathbb{S}^1 \rightarrow \dots,$$

we get an exact sequence

$$\pi_7(S) \xrightarrow{\sigma} \pi_{14}(S) \xrightarrow{i} \pi_{14}(C_\sigma) \xrightarrow{j} \pi_6(S) \rightarrow \pi_{13}(S),$$

by [the work in Section 3.4]. Since these stable homotopy groups are all finite<sup>17</sup>, this induces<sup>18</sup> a exact sequence

$$\pi_7(S)_2^\wedge \xrightarrow{\sigma} \pi_{14}(S)_2^\wedge \xrightarrow{i} \pi_{14}(C_\sigma)_2^\wedge \xrightarrow{j} \pi_6(S)_2^\wedge \xrightarrow{\sigma} \pi_{13}(S)_2^\wedge = 0.$$

In  $E_2(C_\sigma)$  we have  $d_2(\overline{h_0^2 h_3}) = i(h_0 d_0)$  (by Lemma 5.3.4), so  $\pi_{14}(C_\sigma)_2^\wedge$  has order dividing four. Let  $\nu \in \pi_3(S)_2^\wedge$  be a representative for  $h_2$ . Then  $\pi_6(S)_2^\wedge = \mathbb{Z}/2\mathbb{Z}\langle \nu^2 \rangle$ , and  $\nu^2 \sigma = 0$ . By exactness, we see that  $j$  is surjective, so  $\pi_6(S)_2^\wedge \cong \pi_{14}(C_\sigma)_2^\wedge / \ker j = \pi_{14}(C_\sigma)_2^\wedge / \text{im } i$ . We know  $\pi_{14}(C_\sigma)_2^\wedge$  has order dividing 4 and  $\pi_6(S)_2^\wedge$  has order 2, so  $\text{im } i$  has order dividing 2.

Now,  $\pi_7(S)_2^\wedge = \mathbb{Z}/16\mathbb{Z}\langle \sigma \rangle$ , and  $2\sigma^2 = 0$  by graded commutativity, so the first isomorphism theorem implies that  $\pi_{14}(S)_2^\wedge$  has order dividing four. Thus,  $h_0 d_0$  and  $h_0^2 d_0$  must be boundaries, and  $d_3(h_0 h_4) = h_0 d_0$  is the only possibility.  $\square$

<sup>17</sup>A priori  $\pi_{14}(C_\sigma)$  is only finitely generated, but from Figure 5.3 we see that its 2-completion is finite, so the group itself must be finite.

<sup>18</sup>Explain or have as a lemma somewhere else (I've commented it out for the moment).

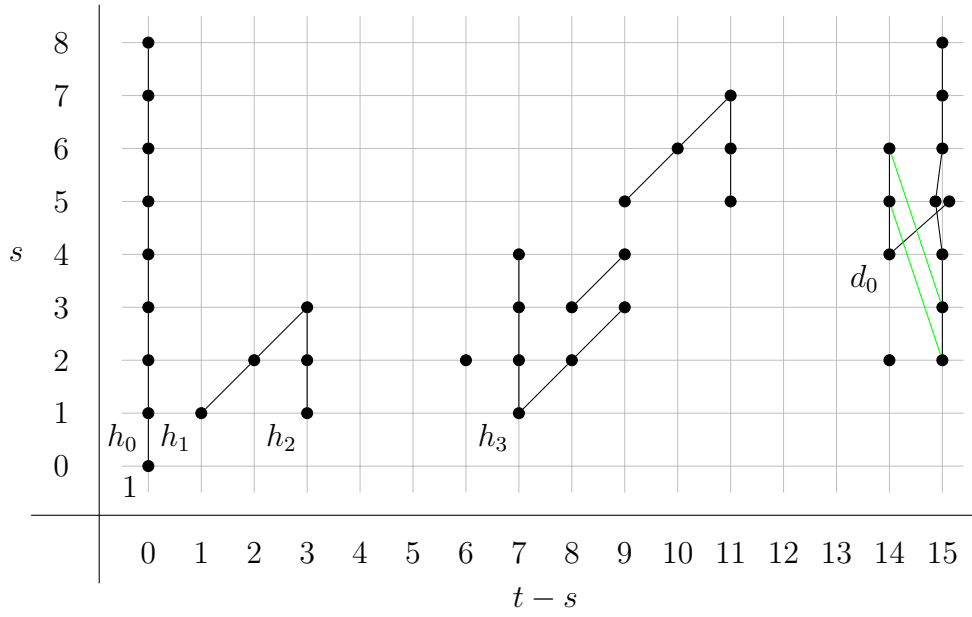


Figure 5.5: The  $E_3$  page of the Adams spectral sequence for  $\mathbb{S}$ , in the range  $t - s \leq 15$ ; the differentials are shown in green.

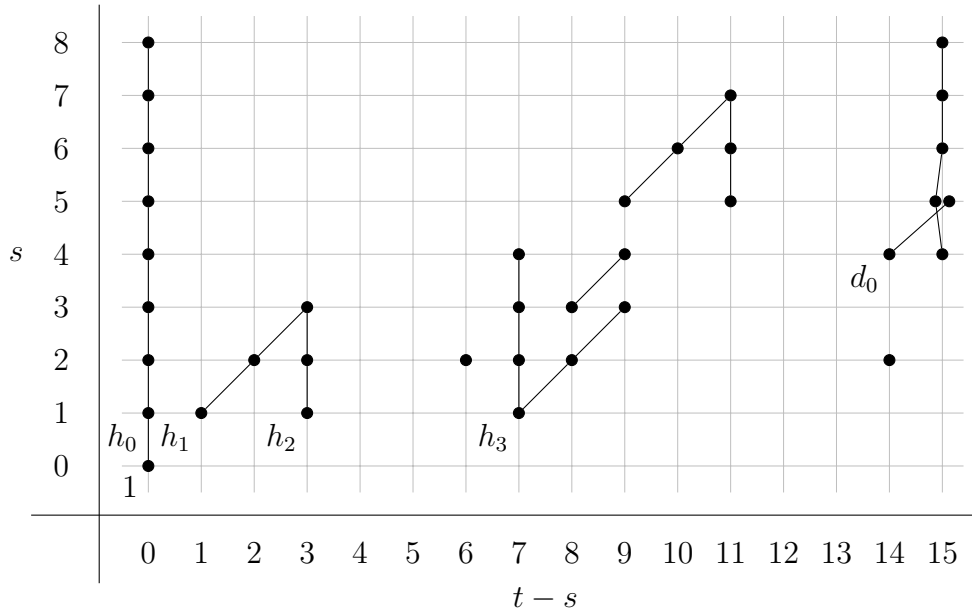


Figure 5.6: The  $E_4$  page of the Adams spectral sequence for  $\mathbb{S}$ , in the range  $t - s \leq 15$ . There are no possible higher differentials, so this coincides with the  $E_\infty$  page for  $t - s \leq 15$ .

[1], [11], [13], [12].

## A Algebra

### A.1 Free resolutions

DEFINITION 1.1.1. Let  $M, N$  be modules over a ring  $R$ . A *resolution*  $F$  of  $M$  is an exact sequence

$$\cdots \rightarrow F_2 \rightarrow F_1 \rightarrow F_0 \rightarrow M \rightarrow 0.$$

If in addition each  $F_i$  is a free  $R$ -module, then the resolution is called *free*.

Given a free resolution as above, applying  $\text{Hom}_R(-, N)$  gives us a chain complex

$$\cdots \leftarrow \text{Hom}_R(F_2, N) \leftarrow \text{Hom}_R(F_1, N) \leftarrow \text{Hom}_R(F_0, N) \leftarrow \text{Hom}_R(M, N) \leftarrow 0.$$

Dropping the term  $\text{Hom}_R(M, N)$  [why?] we get the sequence

$$\cdots \leftarrow \text{Hom}_R(F_2, N) \leftarrow \text{Hom}_R(F_1, N) \leftarrow \text{Hom}_R(F_0, N) \leftarrow 0,$$

and we define  $\text{Ext}_R^n(M, N)$  to be the  $n$ th homology group of this chain complex.

[these do not depend on the choice of free resolution of  $M$ ]

A free resolution is *minimal* if at each stage of its construction we choose the minimal number of free generators for  $F_i$  in each degree.

[The above definition is bad but I'm keeping it just for the moment.]

## B Topology

All from [5] unless otherwise stated.

### B.1 Suspension

DEFINITION 2.1.1. Let  $X$  be a topological space. The *suspension*  $SX$  is the space  $(X \times I) / \sim$ , where  $(x, 0) \sim (x', 0)$  and  $(x, 1) \sim (x', 1)$  for all  $x, x' \in X$ .

DEFINITION 2.1.2. Let  $X$  be a pointed topological space. The *reduced suspension*  $\Sigma X$  is the space  $SX / \sim$ , where  $[x_0, t] \sim [x_0, t']$  for all  $t, t' \in I$ .

Given a map  $f : X \rightarrow Y$ , we can define  $\Sigma f : \Sigma X \rightarrow \Sigma Y$  by  $\Sigma f[(x, t)] = [(fx, t)]$ . This makes  $\Sigma$  into a functor  $\Sigma : \mathbf{Top} \rightarrow \mathbf{Top}$ .

REMARK 2.1.3.  $\Sigma$  is faithful, since for any maps  $f, g : X \rightarrow Y$ , if  $\Sigma f = \Sigma g$  then in particular  $[(fx, \frac{1}{2})] = [(gx, \frac{1}{2})]$ , so  $fx = gx$ .

[below is reconstructed from [9]]

Given pointed maps  $f, g : \Sigma X \rightarrow Z$ , define

$$f \star g : \Sigma X \rightarrow Z$$

$$[x, t] \mapsto \begin{cases} f[x, 2t - 1] & t \geq \frac{1}{2}, \\ g[x, 2t] & t \leq \frac{1}{2}. \end{cases}$$

This is well defined, since both  $f$  and  $g$  are basepoint-preserving.

REMARK 2.1.4. This defines a group structure on  $[\Sigma X, Z]$ , and thus  $[\Sigma^i X, Z]$  is a group for all  $i \geq 1$ . For  $i \geq 2$ , these can be shown to be abelian, via the Eckmann-Hilton argument. The suspension map  $[\Sigma X, Y] \rightarrow [\Sigma^2 X, \Sigma Y]$  is a homomorphism.<sup>19</sup>

REMARK 2.1.5. The homotopy groups  $\pi_i(Z)$  are a special case of the above construction, taking  $X := S^{i-1}$ .

- Loops; the adjunction  $\Sigma \dashv \Omega$ , where  $\Omega$  is the loop functor.

---

<sup>19</sup>Probably follows from the result for  $\pi_*(Y)$  and induction on the cells of  $X$ , but I'll check this.

[5], p395:

REMARK 2.1.6. It follows that  $\pi_{n+1}(X) \cong \pi_n(\Omega X)$ . In particular,  $\Omega K(G, n)$  is a  $K(G, n-1)$ .

- [5] 2.1 Ex 20 and 2.2 Ex 32:  $\tilde{H}_n(X) \cong \tilde{H}_{n+1}(SX)$ , where  $S$  is the (non-reduced) suspension. (MV?)
- Hatcher also says on p219 that  $\tilde{H}^n(X; R) \cong \tilde{H}^{n+k}(\Sigma^k X; R)$ , where  $\Sigma$  is reduced suspension.

## B.2 Other basic constructions

DEFINITION 2.2.1. Let  $(X, x_0), (Y, y_0)$  be pointed topological spaces, and consider their product  $X \times Y$ . The subspaces  $X \times \{y_0\} \cong X$  and  $\{x_0\} \times Y \cong Y$  intersect at exactly one point,  $(x_0, y_0)$ , and so can be identified with the wedge  $X \vee Y$ . We thus define the *smash product*  $X \wedge Y := (X \times Y)/(X \vee Y)$ , with the canonical basepoint  $(x_0, y_0)$ .

EXAMPLE 2.2.2. We have  $S^n \wedge S^m \cong S^{n+m}$ . [is this obvious?]

REMARK 2.2.3. Note that  $\Sigma X \cong X \wedge S^1$ .

REMARK 2.2.4. Observe that  $X \wedge (Y \wedge Z) \cong (X \wedge Y) \wedge Z$ . Combining this with the remarks above, we see that  $\Sigma^k X \cong X \wedge S^k$ .

REMARK 2.2.5. Note that  $\Sigma(X \vee Y) \cong \Sigma X \vee \Sigma Y$ .

- The Eilenberg-MacLane space is  $K(G, n)$ , and it has the property that

$$\pi_i(K(G, n)) = \begin{cases} G & i = n, \\ 0 & i \neq n. \end{cases}$$

They're unique up to weak homotopy equivalence (i.e. if you have another one  $X$ , there's a map between them which descends to an isomorphism on homotopy groups). They can be taken to be CW complexes.

DEFINITION 2.2.6. Let  $X, Y$  be topological spaces, where  $X$  has a basepoint  $x_0$ . Then the *reduced product*  $X \times_{\text{red}} Y := (X \times Y)/(x_0 \times Y)$ .

DEFINITION 2.2.7. Let  $f : X \rightarrow Y$  be a map. The *mapping cylinder*  $M_f$  is defined by  $((X \times I) \sqcup Y) / \sim$ , where  $(x, 1) \sim f(x)$  for all  $x \in X$ . If  $(X, x_0), (Y, y_0)$  are pointed spaces, the *reduced mapping cylinder* is the quotient  $M_f / \sim$ , where  $[x_0, t] \sim [x_0, t']$  for all  $t \in I$ .

REMARK 2.2.8. The mapping cylinder deformation retracts onto  $Y$  via  $h : M_f \times I \rightarrow M_f$ ;  $([x, t], s) \mapsto [x, t + s(1 - t)]$ .

DEFINITION 2.2.9. Let  $f : X \rightarrow Y$  be a map. The *mapping cone*<sup>20</sup>  $C_f$  is defined to be  $Y \sqcup_f CX := (Y \sqcup CX)/(f(x) \sim [x, 1])$ .

Relative Künneth Theorem:

THEOREM 2.2.10 ([5]). For CW pairs  $(X, A), (Y, B)$ , the cross product homomorphism  $H^*(X, A; R) \otimes_R H^*(Y, B; R) \rightarrow H^*(X \times Y, A \times Y \cup X \times B; R)$  is an isomorphism of rings if  $H^k(Y, B)$  is a finitely generated free  $R$ -module for each  $k$ .

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<sup>20</sup>Why does Hatcher not insist this guy is reduced, like he does with the mapping cylinders?

In particular, for pointed spaces  $(X, x_0), (Y, y_0)$ , we have an isomorphism

$$\bigoplus_{i+j=n} H^i(X, x_0; R) \otimes_R H^j(Y, y_0; R) \rightarrow H^n(X \times Y, X \vee Y; R).$$

Or, in other words,

$$\bigoplus_{i+j=n} \tilde{H}^i(X; R) \otimes_R \tilde{H}^j(Y; R) \rightarrow \tilde{H}^n(X \wedge Y; R).$$

Setting  $Y = S^1$ , we get an isomorphism

$$\tilde{H}^{n-1}(X; R) \rightarrow \tilde{H}^n(\Sigma X; R).$$

### B.3 Cell complexes

**DEFINITION 2.3.1.** Let  $X$  be a cell complex,  $A \subseteq X$  a subcomplex. Then the quotient  $X/A$  has a cell complex structure, with cells the cells of  $X \setminus A$  along with a basepoint (the image of  $A$  in  $X$ ).

**DEFINITION 2.3.2.** Let  $f : X \rightarrow Y$  be a map between CW complexes. Then  $f$  is *cellular* if  $f(X_{(n)}) \subseteq Y_{(n)}$  for all  $n$ , where  $X_{(n)}$  is the  $n$ -skeleton of  $X$ .

Cellular approximation theorem:

**THEOREM 2.3.3** ([5], Thm 4.8). Let  $f : X \rightarrow Y$  be a map of CW complexes. Then  $f$  is homotopic to a cellular map.

**LEMMA 2.3.4** ([5], Prop 0.16). Let  $A \subseteq X$  be CW complexes. Then the pair  $(X, A)$  has the *homotopy extension property*; that is, for any map  $f : X \rightarrow Y$  and homotopy  $h : A \times I \rightarrow Y$  such that  $h(a, 0) = f|_A$ , there is a homotopy  $\tilde{h} : X \times I \rightarrow Y$  extending  $h$ .

- The product of cell complexes is a cell complex (maybe only if one of them is finite?)
- The smash product of (pointed?) cell complexes is a cell complex (maybe only if one is them is finite?) [[5] says “the smash product  $X \wedge Y$  is a cell complex if  $X$  and  $Y$  are cell complexes with  $x_0$  and  $y_0$  0-cells, assuming that we give  $X \times Y$  the cell-complex topology rather than the product topology in cases where these two topologies differ”.]
- For a CW complex  $X$ ,  $SX \simeq \Sigma X$ .
- The reduced suspension of a pointed cell complex  $(X, x_0)$  is another pointed cell complex  $\Sigma X$  with basepoint  $x_0$  and an  $n$ -cell for each non-basepoint  $n - 1$  cell  $e_\alpha^{n-1}$  of  $X$ .

**DEFINITION 2.3.5.** Let  $X$  is a topological space. A *CW approximation* to  $X$  is a CW complex  $Z$  equipped with a weak homotopy equivalence  $f : Z \rightarrow X$ .

**THEOREM 2.3.6** ([5], Prop 4.13). Every space  $X$  has a CW approximation  $f : Z \rightarrow X$ .

- In particular,  $\Omega K(G, n)$  has a CW approximation  $Z \rightarrow \Omega K(G, n)$ , and since  $\Omega K(G, n)$  is a  $K(G, n - 1)$ , so is  $Z$ .

Any finite CW complex is compact.

**PROPOSITION 2.3.7** ([5], A.1). A compact subspace of a CW complex is contained in a finite subcomplex.



## C Notes to self

### C.1 Vague problems and questions....

#### C.1.1 ...that probably don't matter

- On p588 of [6], he says “every CW spectrum is equivalent to a suspension spectrum”. Does he actually mean that, or does he mean ‘equivalent to the suspension of a spectrum’? The former seems way too strong, although in fairness I still don’t know what an equivalence of spectra actually *is*.
- On p586 of [6], Hatcher says “If  $X$  is of finite type then for each  $i$  there is an  $n$  such that  $X_n$  contains all the  $i$ -cells of  $X$ . It follows that  $H_i(X; G) = H_i(X_n; G)$  for all sufficiently large  $n$ , and the same is true for cohomology.” But from the way he set up  $H_*$  and  $H^*$  earlier, shouldn’t this be  $H_i(X; G) = H_{i+n}(X_n; G)$ ? Because  $H_i(X; G) = \varinjlim H_{i+n}(X_n)$ , and he talks about things stabilising in the next sentence, so shouldn’t the stable point be at some  $H_{i+n}$ ?
- I write  $\mathcal{A}_2$  where Hatcher writes  $\mathcal{A}$ . We mean the same thing, right...?

#### C.1.2 ...that probably do matter

- I am definitely being told some lies about what the spectral sequence actually converges to. There’s a strong implication/actual statement(!) that at each  $i$  it’s supposed to be a filtration of  $\pi_i^S$  modulo odd torsion, but I think this isn’t true. I think it’s actually the 2-completion of  $\pi_i^S$ . That coincides with the  $p$ -primary part for finite abelian groups, but for  $\pi_0^S$  it’s supposed to be  $\mathbb{Z}_2$  (i.e. the 2-adic integers), not  $\mathbb{Z}$ . I believe. Maybe get a source for this. Some people say it’s the localisation at 2?? But I think that’s also a lie.
- The Leibniz rule is  $d_r(xy) = d_r(x)y \pm xd_r(y)$  (can’t remember the sign). But anything I’m using that rule on is some generator of an  $\mathbb{F}_2$ , right? So the sign shouldn’t matter. But then, shouldn’t the Yoneda product be graded commutative (and thus commutative, because again, in the target signs don’t matter)? So why does [13] have some comment (in Cor 6.5) about how the Yoneda product is commutative “in [some] range”??
- On p592 of [6], he says that “for spectra  $X$  of finite type [the more general] definition of an  $\mathcal{A}_2$ -module structure on  $H^*(X)$  agrees with the definition using the usual  $\mathcal{A}_2$ -module structure on the cohomology of spaces and the identification of  $H^*(X)$  with the inverse limit  $\varprojlim H^{i+n}(X_n)$ ”. Um? Sure, we have that each  $H^{i+n}(X_n)$  stabilises eventually, but is Hatcher saying  $H^{i+n}(X_n)$  stabilises? Like, as an  $\mathcal{A}_2$ -module? And if not, what’s going on here? Because inverse limits don’t commute with infinite direct sums - they’re not biproducts anymore, they’re coproducts and there’s no reason limits should commute with them.
- There’s something weird going on with products. So, things are ok in **Top**, because we have the ordinary product of two spaces, which is a categorical product. But with CW complexes, supposedly sometimes the product topology differs from the ‘cell complex topology’? But, regardless, we’re supposed to be working with pointed things - so in **Top**<sub>\*</sub>, the pointed coproduct is the wedge sum, and the pointed product is just the normal product  $X \times Y$  with the basepoint  $(x_0, y_0)$  (it’s not the smash product). But what about in spectra? No one ever seems to talk about products of spectra, but for example a collection of maps  $X \rightarrow \mathbb{K}(G, n_i)$  should correspond to a single map  $X \rightarrow \prod_i \mathbb{K}(G, n_i)$ , whatever that last object is.

The plot thickens. From [the nLab](#): “[some smash product] is non-canonically equivalent to a product of EM-spectra (hence a wedge sum of EM-spectra in the finite case)”. ???????

- Question: so I have this multiplicative structure on the entire spectral sequence, right? And on the second page it’s just the Yoneda product. But say I compute, I don’t know,  $\alpha_5^2 = h_0 h_2$  on the  $E_2$  page. Since none of those guys are touched by differentials, is that computation still valid on the  $E_3$  page? And on higher pages?
- I’m a bit suspicious of the proof of [Theorem 3.3.8](#), because the proof is more complicated in [6]. Maybe raise this.

## C.2 To do

Now:

- Rewrite Sections [4.1](#) and [4.2](#) and fix the grading.
- Show the action of  $\mathcal{A}_2$  on  $H^*(X)$  is actually a module action.
- Figure out the whole wedge/product of EM spectra nonsense and why  $K_0$  has finite type.
- Figure out why applying  $H^*$  to the sequence  $K_0 \rightarrow K_1 \rightarrow \dots$  makes it exact.
- Show that the  $H^*(K_s)$ ’s are free  $\mathcal{A}_2$ -modules.
- Show that there is a natural map  $[\mathbb{S}, K_s] \rightarrow \text{Hom}_{\mathcal{A}_2}^0(H^*(K_s), H^*(\mathbb{S}))$ , and that it’s an isomorphism (Yoneda?).
- Find out how much of this you should be proving.
- Figure out how to state [Theorem 4.3.1](#) without lying about 2-completion.
- Explain the Yoneda product at all coherently, and compute some examples.
- Prove that  $\pi_*^S$  is a graded commutative ring.
- Find out where the universal property of  $p$ -completion comes from.
- Find a way to translate the calculations in [Figure 5.1](#) into readable and coherent LaTeX’d calculations.
- Make the  $\alpha_j^i$  notation not appalling - link it to the usual  $h$  notation.
- State this whole ‘replacing a map by an inclusion’ business somewhere early.
- Finish the proof of [Theorem 3.4.3](#).

Eventually:

- Be consistent with either cell complex or CW complex.
- Be consistent with  $\mathbb{F}_2$  or  $\mathbb{Z}/2\mathbb{Z}$  (don’t use  $\mathbb{Z}_2$ , that’s really bad).
- Specialise the Adams spectral sequence (i.e. set  $Y = \mathbb{S}$ ).
- Remember that you have to hand in the tex file, so for the love of god change anything stupid that’s hidden in the pdf.
- Sometimes I say  $\pi_*^S$  or  ${}_{(2)}\pi_*^S$  (localised at 2?) instead of its completion at 2 or whatever. So make sure it’s correct.

- Stick to a convention on suspension/cone/homotopy numbering. I.e. Does a homotopy start at 0 or 1? Does a suspension go from -1 to 1 with the space in the middle at 0, or 0 to 1 with the space at 1/2? Do cones go from 0 to 1, and if so, make sure when they include into suspensions they do so consistently.
- I want to at some point compute  $h_0h_1h_3$  or something like that, just to show that sometimes a product can be zero even if it lands in a nonzero group (i.e. the stable groups aren't all cyclic). The trouble is by the time you get to  $\pi_8^S$  the free resolution is an awful mess. Maybe I can try and wrangle with the sseq program to get it to show me kernels and the like?
- Ease notation by not writing  $G$  in e.g.  $H^*(X; G)$ . Say we're taking coefficients in  $\mathbb{F}_2$ , but actually a lot of the time it doesn't matter so maybe note that.
- Have any sort of consistency in using or not using brackets (e.g.  $\pi_t X_s$  v.s.  $\pi_t(X_s)$ ).
- When I say 'spectrum' at any point after defining CW spectra I mean 'CW spectrum'. And I basically always mean 'connective CW spectrum of finite type' too.
- Make clear in the spectral sequence diagrams which towers terminate and which don't (e.g. the ones at  $t-s=0$  obviously don't but unfortunately the spectral sequence package can't draw the entire infinite chain).
- Connect 1 and  $h_1$  (if possible without messing up the labels).

### C.3 Other notes

- READ IF YOUR CALCULATIONS AREN'T WORKING: You are working modulo 2!!!
- If you have a bunch of maps between graded modules/algebras, they're graded homomorphisms. So they preserve degree.
- All (co)homology is supposed to be reduced.
- Signs don't matter with the Leibniz rule either!! You are working modulo 2!!!!!!!
- Remember, once you know that  $d_2(h_4) = h_0h_3^2$ , you know  $h_4$  *doesn't survive to the third page*. So, for example,  $d_3(h_0h_4) \neq h_0d_3(h_4)$  because  $h_4$  doesn't exist anymore. That's why  $d_3(h_0h_4)$  can be nonzero.
- As previously mentioned, we are working modulo 2!! What this also implies is that if anything is hit by any sort of differential, or has any nonzero differential coming out of it, it's completely killed by the next page. Because the summands are just a bunch of  $\mathbb{F}_2$ 's (so you don't need to worry about 'how much' of something is killed, it all is).
- Sometimes Hatcher says that you can replace any map of CW complexes by an inclusion. I think the point here is that if you have a map  $f : X \rightarrow Y$ , [Remark B.2.8](#) says that  $M_f$  deformation retracts onto  $Y$ . So if you only care about  $X$  and  $Y$  up to homotopy equivalence, you can replace  $Y$  by  $M_f$  and then  $X$  definitely includes into  $M_f$ .
- Where it's ambiguous, I'm marking things I definitely need by ! and things I think I may not need by ?.
- In literature,  $A_p^\wedge$  is the  $p$ -adic completion of  $A$ . Sometimes I'll write this as  ${}_pA$  because of some stupid notational decisions I made earlier.

- The ‘abutment’ of a spectral sequence apparently means the thing it converges to (i.e. if  $E_\infty$  computes the associated graded of some  $H^*$ , the abutment of  $\{E\}$  is  $H^*$  (not its associated graded)).
- [12] has some  $n_m$  notation where  $n_m$  is supposed to be the  $m$ th generator in row  $n$ . This is a bit arbitrary when there are two generators in the same row and column; I don’t know how he counts them, but he’s using the `ext` program, whereas I’m using `sseq`. Unless there’s some Canonical Ordering, there’s no reason why these different programs written by different people would use the same convention. In particular, even though [12] says  $\overline{h_0^2 h_3} = 3_4$ , I’m pretty sure it is the one on the right (i.e. the one I would label  $3_5$ ).

Sources I’ve used: [11], [13], [7], [6], [5], [12], [8], [10]

Sources I probably won’t use: [4], [1], [2], [3], [9] (I think the construction I need is in Hatcher)

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