Stable Homotopy Groups of Spheres [DRAFT]

Contents

Key:

To do (likely straightforward)

To do (likely difficult)

Problems

1 Introduction

- Define homotopy groups

THEOREM 1.0.1 ([11], Thm 1.1.4, Freudenthal suspension theorem). If $\pi_i(X) = 0$ for $i \leq k$ (i.e. X is k-connected) then the map

$$\begin{aligned} \pi_n(X) &\to \pi_{n+1}(\Sigma X) \\ [\gamma : S^n \to X] &\mapsto [\Sigma\gamma : \Sigma S^n = S^{n+1} \to \Sigma X] \end{aligned}$$

is an isomorphism for $n \leq 2k$ and surjective for $n = 2k + 1$

- This implies $\pi_{n+k}(S^n)$ depends only on k for $n \geq k + 2$
- (Obviously be careful with basepoints above)
- Suppose X is k-connected. Then, for $k \geq 0$, $0 = \pi_k(X) \cong \pi_{k+1}(\Sigma X)$, so whenever a space is k-connected its suspension is $k + 1$-connected.
- As you take suspensions, then, your successive bounds are $n \leq 2k$, $n + 1 \leq 2k + 2 \implies n \leq 2k+1$, $n \leq 2k+2$, etc ... so the sequence $\pi_n(X) \to \pi_{n+1}(\Sigma X) \to \cdots$ will eventually stabilise.
- Thus, if you take the colimit of that direct system, it'll just equal the stable value, with the higher legs just being the inverse isomorphisms.

THEOREM 1.0.2 ([11], Thm 1.1.8). $\pi_{n+k}(S^n)$ is finite for $k > 0$ except when $n = 2m$, $k = 2m - 1$.

COROLLARY 1.0.3. π_i^S is finite for $i > 0$.

[13], [5]

2 The Steenrod algebra

The following is from [5] 4L.

- There are maps $Sq^i : H^n(-; \mathbb{Z}/2\mathbb{Z}) \to H^{n+i}(-; \mathbb{Z}/2\mathbb{Z})$ for each i, and they satisfy the following properties:
  1. $Sq^i_X(f^*(\alpha)) = f^*(Sq^i_Y(\alpha))$ for $f : X \to Y$ (i.e. Sq^i is a natural transformation).
  2. $Sq^i_X(\alpha + \beta) = Sq^i_X(\alpha) + Sq^i_X(\beta)$ (i.e. Sq^i_X respects the group operation for all X).
  3. $Sq^i(\alpha \smile \beta) = \sum\limits_{0 \leq j \leq i} (Sq^j(\alpha) \smile Sq^{i-j}(\beta))$ (the Cartan formula)

4. $Sq^i(\sigma(\alpha)) = \sigma(Sq^i(\alpha))$ where $\sigma : H^n(X;\mathbb{Z}/2\mathbb{Z}) \to H^{n+1}(\Sigma X;\mathbb{Z}/2\mathbb{Z})$ is the suspension isomorphism given by reduced cross product with a generator of $H^1(S^1;\mathbb{Z}/2\mathbb{Z})$. [Something is a little strange here, I think maybe it doesn't matter which suspension isomorphism you pick, though it's not immediately clear why. It's just that the wikipedia page that σ is the connecting homomorphism of the long exact sequence in cohomology. [12] just says it's the suspension isomorphism :|. Something interesting: on p219 of [5], it says "the suspension isomorphisms $\widetilde{H}^n(X;R) \cong \widetilde{H}^{n+k}(\Sigma^k X;R)$ derivable by elementary exact sequence arguments can also be obtained via cross product with a generator of $\widetilde{H}^*(S^k;R)$". Does he mean the *same* isomorphism, or just the fact that there is (at least) one?].
5. $Sq^i(\alpha) = \alpha^2$ if $i = \deg(\alpha)$ and $Sq^i(\alpha) = 0$ if $i > \deg(\alpha)$.
6. $Sq^0 = \mathrm{id}$.

- Define $Sq := Sq^0 + Sq^1 + \cdots$. Then $Sq(\alpha \smile \beta) = Sq(\alpha) \smile Sq(\beta)$ (since $(Sq(\alpha \smile \beta))_n = \sum_i Sq^i(\alpha) \smile Sq^{n-i}(\beta) = (Sq(\alpha) \smile Sq(\beta))_n$). Thus, Sq is a ring homomorphism.
- Adem relations:

$$Sq^a Sq^b = \sum_j \binom{b-j-1}{a-2j} Sq^{a+b-j} Sq^j \quad \text{if } a < 2b,$$

where $\binom{m}{n}$ is zero if m or n is negative, or $m < n$, and $\binom{m}{0} = 1$ for $m \geq 0$.

Definition 2.0.1. The *Steenrod algebra* $\mathscr{A}_2$ is the algebra over $\mathbb{Z}/2\mathbb{Z}$ that is the quotient of the algebra of polynomials in the noncommuting variables $Sq^1, Sq^2, \ldots$ by the two-sided ideal generated by the Adem relations. Thus, for every space X, $H^*(X;\mathbb{Z}/2\mathbb{Z})$ is a module over $\mathscr{A}_2$.

- $\mathscr{A}_2$ is graded, and its elements of degree k are those that map $H^n(X;\mathbb{Z}/2\mathbb{Z})$ to $H^{n+k}(X,\mathbb{Z}/2\mathbb{Z})$ for all n. [Presumably you've fixed a space X while you're doing all this?]

Definition 2.0.2. Write Sq^I for the monomial $Sq^{i_1} Sq^{i_2} \cdots Sq^{i_n}$. Then Sq^I is *admissible* if $i_j \geq 2i_{j+1}$ for all $0 \leq j < n$.

Note the admissible monomials are exactly those to which no Adem relations can be applied. Thus, $\mathscr{A}_2$ is generated as a $\mathbb{Z}/2\mathbb{Z}$ module by admissible monomials.

[5]

3 Spectra

3.1 ?Categorical nonsense

- [13]: There is a category $\mathcal{H}$ of finite based CW complexes, with $\mathrm{Hom}(X,Y) =: [X,Y]$ the set of homotopy classes of base-point preserving maps $X \to Y$.
- There is a category $\mathbf{St}(\mathcal{H})$ of finite based CW complexes, with $\mathrm{Hom}(X,Y) =: \{X,Y\}$ the set $\mathrm{colim}_i[\Sigma^i X, \Sigma^i Y]$ [it's just a colimit of sets, and **Set** is cocomplete, so we should be fine. [13] says it's a group[1]] [Also, how do these guys compose?]

[1] The colimit is equal to the stable value (which exists, by the corollary). After Σ^2, these guys are all

- There is a functor $\mathcal{H} \to \mathbf{St}(\mathcal{H})$. [13] doesn't say what this is but it's presumably the one that is the identity on objects and sends $[f : X \to Y] \in [\Sigma^0 X, \Sigma^0 Y]$ to whatever it gets sent to in $\{X, Y\}$ using the universal property of the colimit. Uniqueness makes it functorial, etc.

- We have a fully faithful functor $\mathbf{St}(\mathcal{H}) \to \mathbf{St}(\mathcal{H})$ given by the suspension on objects, and the unique isomorphism $\{X, Y\} \to \{\Sigma X, \Sigma Y\}$ on maps (such an isomorphism exists, since both of those things are colimits for $[\Sigma^i X, \Sigma^i Y]$ - one of the sequences is cut off at the beginning, but it doesn't matter because both reach the stable value (see above discussion and [13] 1.9), aka the colimit).

- It's not an equivalence, because not every object is isomorphic to a suspension (e.g. anything not connected, since suspensions always connected)

- We can formally adjoin desuspensions $\Sigma^{-n}X$ for all n [does this mean just putting the objects there and defining $\mathrm{Hom}(Y, \Sigma^{-n}X) := \mathrm{Hom}(\Sigma^n Y, X)$ and $\mathrm{Hom}(\Sigma^{-n}X, Y) := \mathrm{Hom}(X, \Sigma^n Y)$?], but this category does not have weak colimits (i.e. colimits w/o uniqueness property). [why does it not, and why do we even want that?]

- We instead consider formal sequences of desuspensions $X_0 \to \Sigma^{-1}X_1 \to \cdots$, or sequences (X_n) and maps $\Sigma X_n \to X_{n+1}$, i.e. spectra. [and this fixes the problem?]

3.2 Definitions and examples

[Some intro, below follows [6], Section 5.2. Some intuition on spectra and why we're doing this.]

Definition 3.2.1. A *spectrum* is a collection of pointed topological spaces $\{X_n\}_{n\in\mathbb{N}}$, together with basepoint-preserving maps $\sigma_n : \Sigma X_n \to X_{n+1}$.

Example 3.2.2. Let X be a topological space. The *suspension spectrum* of X, denoted by $\Sigma^\infty X$, has $X_n = \Sigma^n X$ and $\sigma_n = \mathrm{id} : \Sigma X_n \to X_{n+1}$.

We write $\mathbb{S}$ for the suspension spectrum $\Sigma^\infty S^0$, and call $\mathbb{S}$ the *sphere spectrum*. For $i > 0$, we write $\mathbb{S}^i$ for $\Sigma^\infty S^i$.

Example 3.2.3. The *Eilenberg-MacLane spectrum* $\mathbb{K}(G, m)$ has $(\mathbb{K}(G, m))_n$ a CW complex $K(G, m+n)$ and $\sigma_n : \Sigma K(G, m+n) \to K(G, m+n+1)$ is the adjoint of the CW approximation $K(G, m+n) \to \Omega K(G, m+n+1)$.

Definition 3.2.4. Let $X = \{X_n\}$ be a spectrum. We define $\pi_i(X) = \mathrm{colim}_n\, \pi_{i+n}(X_n)$, where the map $\pi_{i+n}(X_n) \to \pi_{i+n+1}(X_{n+1})$ is given by the composition

$$\pi_{i+n}(X_n) \xrightarrow{\Sigma} \pi_{i+n+1}(\Sigma X_n) \xrightarrow{(\sigma_n)_*} \pi_{i+n+1}(X_{n+1}).$$

Example 3.2.5. If X is a topological space, then $\pi_i(\Sigma^\infty X) = \pi_i^S(X)$, the ith stable homotopy group of X.

groups, so the colimit also has a group structure inherited from whatever $[\Sigma^k X, \Sigma^k Y]$ it's equal to. N.B: Remarks about cocompleteness of **Set** are misleading because that doesn't actually matter - any sequence that stabilises in any category will have a filtered colimit equal to that stable value, you don't need any extra conditions.

Definition 3.2.6. A CW spectrum is a spectrum X consisting of CW complexes X_n with the maps $\Sigma X_n \hookrightarrow X_{n+1}$ inclusions of subcomplexes.

Definition 3.2.7. Let X be a CW spectrum. Then the *k-cells* of X are the equivalence classes of non-basepoint $(k+n)$-cells in X_n, where two cells are equivalent if one is an m-fold suspension of the other, for some $m > 0$.

Definition 3.2.8. A CW spectrum X is *connective* if it has no cells below a given dimension, *finite* if it has only finitely many cells, and *of finite type* if it has only finitely many cells in each dimension.

Example 3.2.9. If X is a finite (resp. finite type) CW complex, then Σ^∞ is a finite (resp. finite type) CW spectrum. In particular, $\mathbb{S}$ is a finite CW spectrum with a unique cell in dimension 1.

Lemma 3.2.10. Let X be a connective spectrum of finite type. Then the groups $\pi_{i+n}(X_n)$ eventually stabilise; i.e. the maps $\pi_{i+n}(X_n) \xrightarrow{(\sigma_n)_* \circ \Sigma} \pi_{i+n+1}(X_{n+1})$ are isomorphisms for large enough n.

Proof. First, note that that maps $\pi_{i+n}(X_n) \xrightarrow{\Sigma} \pi_{i+n+1}(\Sigma X_n)$ are eventually isomorphisms by Theorem 1.0.1.

Recall that whenever $(X_{n+1}, \Sigma X_n)$ are such that $X_{n+1} \setminus X_n$ has no cells in dimension $\leq k$, the map $\pi_i(\Sigma X_n) \to \pi_i(X_{n+1})$ induced by the inclusion is an isomorphism ([5], Cor 4.12 and a long exact sequence). Thus, if $(\sigma_n)_* : \pi_{i+n+1}(\Sigma X_n) \to \pi_{i+n+1}$ never stabilises, there must be infinitely many natural numbers N_j such that $(X_{N_j+1}, \Sigma X_{N_j})$ is not $(i + N_j + 1)$-connected, and thus that $X_{N_j+1} \setminus \Sigma X_{N_j}$ has cells of dimension $\leq i + N_j + 2$. By connectivity, there is some fixed l such that these cells are of dimension $N_j + k + 1$ for $-l \leq k \leq i+1$. Thus, there must be some k such that infinitely many of the X_{N_j+1} have a $(k + N_j + 1)$-cell not included in ΣX_{N_j}. This then contradicts the assumption that X is of finite type, since it has infinitely many k-cells.

Thus, the maps $(\sigma_n)_* : \pi_{i+n+1}(\Sigma X_n) \to \pi_{i+n+1}$ are also eventually isomorphisms, so the groups $\pi_{i+n}(X_n)$ do stabilise. $\square$

3.3 Homology and cohomology

Recall that $C_i^{\text{cell}}(X_n; G)$ is the free abelian group on the i-cells of X_n. We have an injection

$$\begin{aligned} C_i^{\text{cell}}(X_n; G) &\to C_{i+1}^{\text{cell}}(\Sigma X_n; G) \\ e_\alpha^i &\mapsto \Sigma e_\alpha^i, \end{aligned}$$

and an injection $C_{i+1}^{\text{cell}}(\Sigma X_n; G) \to C_{i+1}^{\text{cell}}(X_{n+1}; G)$ induced by the structure map σ_n, so we get an injection $C_i^{\text{cell}}(X_n; G) \hookrightarrow C_{i+1}^{\text{cell}}(X_{n+1}; G)$.

We define

$$C_n(X; G) := \bigcup_{i \in \mathbb{Z}} C_{i+n}^{\text{cell}}(X_i; G).$$

Note that there is a G summand for every $i + n$ cell of X_i up to treating suspensions of cells as equivalent to the cells themselves, i.e. a G summand for every n-cell of X. We define $H^*(X; G)$ and $H_*(X; G)$ to be the cohomology and homology of this chain complex, respectively.

[Important example:

At the bottom of p592 of [6], it's mentioned that for a spectrum X of finite type, $H^i(X) \cong \lim\limits_{\leftarrow n} H^{i+n}(X_n)$ as $\mathscr{A}_2$-modules[2]. Thus, since $\mathbb{S}$ is of finite type (it is actually finite), we have

$$\begin{aligned} H^i(\mathbb{S}; \mathbb{Z}/2\mathbb{Z}) &= \lim_{\leftarrow n} H^{i+n}(S^n; \mathbb{Z}/2\mathbb{Z}) \\ &= \begin{cases} \mathbb{Z}/2\mathbb{Z} & i = 0, \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

(since they're all just either 0 or $\mathbb{Z}/2\mathbb{Z}$, at least eventually). So, $H^*(\mathbb{S}) = \mathbb{Z}/2\mathbb{Z}$ in degree zero and nothing else.]

Lemma 3.3.1. Let X be a connective CW spectrum of finite type. Then $H_i(X), H^i(X)$, and $\pi_i(X)$ are finitely generated for all i.

Proof. First, note that $H_i(X) = H_{i+n}(X_n)$ for sufficiently large n, since for large enough n, X_n contains all the cells of dimension $\leq i$. Similarly, $H^i = H^{i+n}(X_n)$ for sufficiently large n. Each $H_{i+n}(X_n)$ is finitely generated, since X_n has only finitely many cells in each dimension, and thus each $H^{i+n}(X_n)$ is also finitely generated ([5] Cor 3.3). Thus, $H_i(X), H^i(X)$ are finitely generated.

Now, $\pi_i(X) = \operatorname{colim}_n \pi_{i+n}(X_n)$, and the groups $\pi_{i+n}(X_n)$ stabilise by Lemma 3.2.10. The X_n must eventually be simply-connected, since X is connective. A simply-connected space has finitely generated homotopy groups if and only if it has finitely generated homology groups (see e.g. [5], Thm 5.7), and we have just seen that the $H_{i+n}(X_n)$ are finitely generated, so $\pi_i(X) = \pi_{i+n}(X_n)$ is finitely generated. □

Definition 3.3.2. Let $X = \{X_n\}$ be a CW spectrum. A *subspectrum* X' of X is a sequence of subcomplexes $\{X'_n \subseteq X_n\}$ satisfying $\Sigma X'_n \subseteq X'_{n+1}$. The subspectrum X' is *cofinal* if, for each n and each cell e^i_α of X_n, the cell $\Sigma^k e^i_\alpha$ belongs to X'_{n+k} for all sufficiently large k.

Note that if $\Sigma^k e^i_\alpha$ belongs to X'_{n+k} then $\Sigma^{k+1} e^i_\alpha$ belongs to $\Sigma X'_{n+k} \subseteq X'_{n+k+1} \subseteq X'_{k+k+2} \subseteq \cdots$. Thus, if X', X'' are cofinal spectra of X with $\Sigma^k e^i_\alpha$ a cell of X'_{n+k} and $\Sigma^l e^i_\alpha$ a cell of X''_{n+l} (with $l \geq k$) then $\Sigma^l e^i_\alpha$ is a cell of X'_{n+l} and therefore of $X'_{n+l} \cap X''_{n+l}$. In other words, the intersection of two cofinal spectra is a cofinal spectrum.

Definition 3.3.3. Let X, Y be CW spectra. A *strict map* $f : X \to Y$ is a sequence of cellular maps $f_n : X_n \to Y_n$ such that the diagram below commutes.

$$\begin{tikzcd} \Sigma X_n \arrow[r, "\sigma_n"] \arrow[d, "\sigma f_n"'] & X_{n+1} \arrow[d, "f_{n+1}"] \\ \Sigma Y_n \arrow[r, "\sigma_n"'] & Y_{n+1} \end{tikzcd}$$

[Some intuition: it's enough for maps to be 'defined eventually']

Definition 3.3.4. A *map* of CW spectra $f : X \to Y$ is an equivalence class of strict maps $f' : X' \to Y$ with X' a cofinal subspectrum of X, where two strict maps $f' : X' \to Y$ and $f'' : X'' \to Y$ are equivalent if they agree on some common cofinal subspectrum.

[2] Maybe I need to move this somewhere else. I haven't defined the $\mathscr{A}_2$-action yet...

Given two maps $f : X \to Y$, $g : Y \to Z$ represented by $f' : X' \to Y$, $g' : Y' \to Z$ respectively, we compose as follows: let X'' be the subspectrum of X', where the cells of X''_n consist of the cells of X'_n mapped to Y'_n under f'_n. Then, for any cell e^i_α of X'_n, $f_n(e^i_\alpha)$ is contained in a finite union of cells of Y_n (since the image of a compact set is compact), whose k-fold suspension lies in Y'_{n+k} for large enough k. Since f' is a strict map, $\Sigma^k f'_n(e^i_\alpha) = f'_{n+k}\Sigma^k e^i_\alpha$, so $\Sigma^k e^i_\alpha$ is a cell of X'_{n+k}. Thus, X'' is cofinal in X' and hence in X. We define $gf := [X'' \xrightarrow{f'|_{X''}} Y' \xrightarrow{g'} Z]$, which is well-defined since the intersection of cofinal subspectra is again a cofinal subspectrum.

Since any strict map $f' : X' \to Y$ can be taken to be cellular, a map $f : X \to Y$ induces a well-defined map $C_*(X) \to C_*(Y)$ (by cofinality), and thus maps on homology and cohomology.

Further, any map $\Sigma^\infty S^i \to X$ can be represented by a map $S^{i+n} \to X_n$, which has compact image and thus by Proposition B.3.7 is contained in a finite subcomplex of $\overline{X}_n$. Given any map $f : X \to Y$ represented by a strict map $f' : X' \to Y$, the kth suspension of the cells of $\overline{X}_n$ lie in X'_{n+k}, and thus f induces a map $\pi_*(X) \to \pi_*(Y)$.

Definition 3.3.5. Two spectra X, Y are *equivalent* if there are maps $f : X \to Y$ and $g : Y \to X$ such that $fg = \mathrm{id}_Y$ and $gf = \mathrm{id}_X$.

Note that a spectrum is equivalent to any of its cofinal subspectra. In particular, if $X = \{X_n\}$ is a spectrum, then $X' = \{\Sigma X_{n-1}\}$ is a cofinal subspectrum of X (where we take X_{-1} to be the basepoint of X_0). We define $\Sigma^{-1}X := \{X_{n-1}\}$, noting that $\Sigma\Sigma^{-1}X = \Sigma^{-1}\Sigma = X' \simeq X$. Thus, a spectrum is always equivalent to the suspension of some other spectrum.

Definition 3.3.6. A *homotopy* of maps between spectra is a map $X \times I \to Y$, where $X \times I$ is the spectrum with $(X \times I)_n = X_n \times_{\mathrm{red}} I$.

Note that $\Sigma(X_n \times_{\mathrm{red}} I) = \Sigma X_n \times_{\mathrm{red}} I$. The set of homotopy classes of maps $X \to Y$ is denoted by $[X, Y]$.

Remark 3.3.7. For any CW spectra Z, $[\Sigma^\infty S^t, Z] = \pi_t(Z)$.

For any CW spectra X, Y, the set $[X, Y]$ can the structure of an abelian group, since X has be written as a double suspension $\Sigma^2 X'$, and each set $[\Sigma^2 X'_n, Y_n]$ has the structure of an abelian group by Remark B.1.4.

Theorem 3.3.8. The suspension map $[X, Y] \to [\Sigma X, \Sigma Y]$ is an isomorphism of groups.

Proof. The suspension map is a homomorphism, since it is a homomorphism on maps between CW complexes. Thus, it suffices to show it is a bijection on maps between spectra.

Recall that $\Sigma^{-1}\Sigma X = \Sigma\Sigma^{-1}X \simeq X$. For any map $f : X \to Y$ given by strict maps $f_n : X'_n \to Y_n$, define $\Sigma^{-1}f : \Sigma^{-1}X \to \Sigma^{-1}Y$ by $\{f_{n-1} : X'_{n-1} \to Y_{n-1}\}$. Then $\Sigma\Sigma^{-1}f = \{\Sigma f_{n-1}\} = \{f_n|_{\Sigma X_{n-1}}\} = f$, and similarly $\Sigma^{-1}\Sigma f = f$. Thus, we have bijections $[X, Y] \cong [\Sigma\Sigma^{-1}X, \Sigma\Sigma^{-1}Y] \cong [\Sigma^{-1}\Sigma X, \Sigma^{-1}\Sigma Y]$, so Σ has a two-sided inverse. $\square$

3.4 Cofibration sequences

Definition 3.4.1. Let $X = \{X_n\}, Y = \{Y_n\}$ be spectra. Then their *wedge sum* is $X \vee Y := \{X_n \vee Y_n\}$. Note that Remark B.2.5 gives us an inclusion $\Sigma(X_n \vee Y_n) \hookrightarrow X_{n+1} \vee Y_{n+1}$.

Definition 3.4.2. Let X be a spectrum, $A \subseteq X$ a subspectrum. Then A is *closed* in X if for every cell e^n_α of X_n, if $\Sigma^k e^n_\alpha \in A_{n+k}$ then $e^n_\alpha \in A_n$.

Any subspectrum is cofinal in (and thus equivalent to) its closure. We define X/A to be the CW spectrum with $(X/A)_n = X_n/A'_n$, where $A' = \{A'_n\}$ is the closure of A. Note that a quotient of connective spectra of finite type is again a connective spectrum of finite type (since the quotient has fewer cells in each dimension than the original space).

[The map $X \cup CA \to X/A$ is a homotopy equivalence of spectra]

THEOREM 3.4.3. Let X, Y be spectra, and $A \subseteq X$ a subspectrum. Then there is an exact sequence

$$[Y, A] \to [Y, X] \to [Y, X/A] \to [Y, \Sigma A] \to [Y, \Sigma X] \to \cdots .$$

PROOF. It suffices to show that

$$[Y, A] \to [Y, X] \to [Y, X/A] \to [Y, \Sigma A]$$

is exact.

We first show exactness at $[Y, X/A]$. The composition $[Y, X] \to [Y, X/A] \to [Y, \Sigma A]$ is clearly zero. Now, if $Y \xrightarrow{f} X \cup CA \to \Sigma A$ is homotopic to the constant map, then f must be homotopic to a map contained entirely in X, and thus is in the image of $[Y, X] \to [Y, X/A]$.

Now, we show exactness at $[Y, X]$. Again, the composition $[Y, A] \to [Y, X] \to [Y, X/A]$ is clearly zero. Suppose $Y \xrightarrow{f} X \to X \cup CA$ is homotopic to the constant map. Then we have a map $h : CY \to X \cup CA$ making the solid diagram below commute.

$$\begin{tikzcd}
Y \arrow[r, "\mathrm{id}"] \arrow[d, dashed] & Y \arrow[r, hook] \arrow[d, "f"] & CY \arrow[r, hook] \arrow[d, "h"] & \Sigma Y \arrow[r, "\mathrm{id}"] \arrow[d, dashed] & \Sigma Y \arrow[d, dashed, "\Sigma f"] \\
A \arrow[r, hook, "i"'] & X \arrow[r, hook] & X \cup CA \arrow[r] & \Sigma A \arrow[r, hook, "\Sigma i"'] & \Sigma X
\end{tikzcd}$$

[[6] says that "we can then automatically fill in the next two vertical maps to make homotopy commutative squares". Vaguely, I think this is because of some sort of naturality argument, which [5] says is 'evident' on p398. If we can, then...]

By Theorem 3.3.8, we can take the map $\Sigma Y \to \Sigma A$ to be Σg for some $g : Y \to A$. Then $\Sigma f \simeq (\Sigma i)(\Sigma g) = \Sigma(ig)$, so $f \simeq ig$, as required. □

Finally, we get the lemma below, which follows from the equivalent result for CW complexes.

LEMMA 3.4.4. Let $A \xhookrightarrow{f} X \xrightarrow{i} C_f \xrightarrow{j} \Sigma A \to \cdots$ be a cofibration, where X, A are CW spectra of finite type. Then there is a long exact sequence

$$\cdots \leftarrow H^{n-1}(\Sigma A) \leftarrow H^n(X) \xleftarrow{i^*} H^n(C_f) \xleftarrow{j^*} H^n(\Sigma A) \leftarrow H^{n+1}(X) \leftarrow \cdots .$$

3.5 Eilenberg-MacLane spectra

THEOREM 3.5.1 ([6], Prop 5.45). There are natural isomorphisms $H^m(X; G) \cong [X, K(G, m)]$ for all CW spectra.

Recall that giving a map into a product is equivalent to giving a map into each of its components. We have maps $F_i : [X, \bigvee_i \mathbb{K}(G, n_i)] \to [X, \mathbb{K}(G, n_i)]$.

PROPOSITION 3.5.2 ([6], Prop 5.46). The map $F : [X, \bigvee_i \mathbb{K}(G, n_i)] \to \prod_i [X, \mathbb{K}(G, n_i)]$ is an isomorphism if X is a connective spectrum of finite type and $n_i \to \infty$ as $i \to \infty$.

3.6 p-completion of spectra

DEFINITION 3.6.1 ([8], Def 10.1.1). Let A be an abelian group. Then its *p-adic completion* is the limit

$$A_p^\wedge = \lim_{\leftarrow n}(A/p^n A).$$

If $A = \mathbb{Z}$, we instead write $\mathbb{Z}_p$ for the p-adic integers. There is a natural map $A \to A_p^\wedge$, whose component at n is reduction modulo $p^n A$.

When A is finitely generated, its p-adic completion is given by the map $A \to A \otimes \mathbb{Z}_p$; $a \mapsto a \otimes 1$.

LEMMA 3.6.2. Suppose A is finite, and write $|A| = np^r$ for $p \nmid n$. Then $A \otimes \mathbb{Z}_p \cong A/T$, where $T \subseteq A$ is the subgroup generated by all torsion coprime to p.

PROOF. Define a homomorphism $A \otimes \mathbb{Z}_p \to A/T$ sending $a \otimes z \mapsto [\hat{z}a]$, where $\hat{z} \in \mathbb{Z}$ is a lift of $q(z)$, chosen such that $0 < \hat{z} \leq p^r$, and q is the projection $\mathbb{Z}_p \twoheadrightarrow \mathbb{Z}/p^r\mathbb{Z}$. Suppose $a \otimes z \mapsto 0$. Then $\hat{z}a \in T$, so $k\hat{z}a = 0$ for some k coprime to p. Write $z = \hat{z} + p^r z'$ for some $z' \in \mathbb{Z}_p$. Then $a \otimes z = a \otimes (\hat{z} + p^r z') = \hat{z}a \otimes (1 + \frac{p^r z'}{\hat{z}}) = k\hat{z} \otimes (\frac{1}{k} + \frac{p^r z'}{k\hat{z}}) = 0$, where the second equality follows from the fact that $\hat{z}$ was chosen such that the highest power of p dividing it was less than or equal to r. Thus, $a \otimes z = 0$, so the map is injective. The map is clearly also surjective, since $a \otimes 1 \mapsto [a]$, so it is an isomorphism. $\square$

REMARK 3.6.3. If A is finite with order np^r for $p \nmid n$, then $|A_p^\wedge| = p^r$, by Cauchy's theorem.

DEFINITION 3.6.4 ([7], p129). Let X be a CW spectrum. Then a *p-completion* of X is a map $f : X \to X_p^\wedge$ such that for all i, $\pi_i f$ expresses $\pi_i(X_p^\wedge)$ as the p-completion of $\pi_i(X)$.

THEOREM 3.6.5 ([7], Thm 9.1.1). If X has finite type, then it has a p-completion unique up to equivalence.

THEOREM 3.6.6 ([7], Prop 9.2.22). Let X be a connective spectrum of finite type, and let Y be p-complete. Then the map $[X_p^\wedge, Y] \to [X, Y]$ is an isomorphism. That is, given any map $X \xrightarrow{f} Y$, there exists a unique (up to homotopy) map $X_p^\wedge \xrightarrow{\bar{f}} Y$ such that f factors as $X \to X_p^\wedge \xrightarrow{\bar{f}} Y$.

[13], [6], [8], [7]

4 The Adams spectral sequence

4.1 Spectral sequences

[Maybe add some notes from [13]]

[Some notes from [10], C2 - just here as a placeholder/reference and I'll probably completely rewrite this bit.]

DEFINITION 4.1.1. A *differential bigraded module* E over a ring R is a collection of R-modules $\{E^{p,q}\}$, $p, q \in \mathbb{Z}$, together with a map $d : E^{p,q} \to E^{p+s,q-s+1}$ for each p, q and some fixed $s \in \mathbb{Z}$, satisfying $d^2 = 0$.

We can take the homology of (E, d):

$$H^{p,q}(E^{*,*}, d) = \ker(d : E^{p,q} \to E^{p+s,q-s+1}) / \operatorname{im}(d : E^{p-s,q+s-1} \to E^{p,q}).$$

Definition 4.1.2. A *spectral sequence* (of *cohomological type*[3]) is a collection of differential bigraded R-modules $\{E_r^{*,*}, d_r\}, r \in \mathbb{N}$, with the differentials d_r of bidegree $(r, 1-r)$. These satisfy the further condition that for all p, q, r, $E_{r+1}^{p,q} \cong H^{p,q}(E_r^{*,*}, d_r)$.

We will sometimes write $d_r^{p,q}$ for the differential $d_r : E^{p,q} \to E^{p+r,q-s+1}$.

Consider the term $E_2^{*,*}$. Define

$$Z_2^{p,q} := \ker d_2^{p,q} \quad \text{and} \quad B_2^{p,q} := \operatorname{im} d_2^{p-2,q+1}.$$

The condition $d^2 = 0$ implies that $B_2^{p,q} \subseteq Z_2^{p,q} \subseteq E_2^{p,q}$, and by definition we have $E_3^{p,q} \cong Z_2^{p,q}/B_2^{p,q}$.

Now, write

$$Z_3^{p,q} := \ker d_3^{p,q} \quad \text{and} \quad B_3^{p,q} := \operatorname{im} d_3^{p-3,q+2}.$$

Since $Z_3^{p,q} \subseteq E_3^{p,q}$, it can be written as $\overline{Z}_3^{p,q}/B_2^{p,q}$ for some $\overline{Z}_3^{p,q} \subseteq Z_2^{p,q}$. Similarly, $B_3^{p,q} \cong \overline{B}_3^{p,q}/B_2^{p,q}$ for some $\overline{B}_3^{p,q} \subseteq Z_2^{p,q}$. Thus,

$$E_4^{p,q} \cong Z_3^{p,q}/B_3^{p,q} \cong \frac{\overline{Z}_2^{p,q}/B_2^{p,q}}{\overline{B}_3^{p,q}/B_2^{p,q}} \cong \overline{Z}_3^{p,q}/\overline{B}_3^{p,q}.$$

Iterating this process, we present the spectral sequence as an infinite tower of submodules of $E_2^{p,q}$:

$$B_2^{p,q} \subseteq \overline{B}_3^{p,q} \subseteq \cdots \subseteq \overline{B}_n^{p,q} \subseteq \cdots \subseteq \overline{Z}_n^{p,q} \subseteq \cdots \subseteq \overline{Z}_3^{p,q} \subseteq Z_2^{p,q} \subseteq E_2^{p,q},$$

with the property that $E_{n+1}^{p,q} \cong \overline{Z}_n^{p,q}/\overline{B}_n^{p,q}$. The differential $d_{n+1}^{p,q}$ can be taken as a map $\overline{Z}_n^{p,q}/\overline{B}_n^{p,q} \to \overline{Z}_n^{p,q}/\overline{B}_n^{p,q}$ with kernel $\overline{Z}_{n+1}^{p,q}/\overline{B}_n^{p,q}$ and image $\overline{B}_{n+1}^{p,q}$. The short exact sequence induced by d_{n+1},

$$0 \to \overline{Z}_{n+1}^{p,q}/\overline{B}_n^{p,q} \to \overline{Z}_n^{p,q}/\overline{B}_n^{p,q} \xrightarrow{d_{n+1}^{p,q}} \overline{B}_{n+1}^{p,q}/\overline{B}_n^{p,q} \to 0,$$

gives rise to isomorphisms $\overline{Z}_n^{p,q}/\overline{Z}_{n+1}^{p,q} \cong \overline{B}_{n+1}^{p,q}/\overline{B}_n^{p,q}$ for all n. Conversely, a tower of submodules of E_2, together with a set of isomorphisms, gives rise to a spectral sequence.

Definition 4.1.3. An element of $E_2^{p,q}$ *survives to the rth stage* if lies in $\overline{Z}_r^{p,q}$, having been in the kernel of the previous $r-2$ differentials, and is *bounded by the rth stage* if it lies in $\overline{B}_r^{p,q}$. The bigraded module $E_r^{*,*}$ is called the E_r*-term* of the spectral sequence.

We define

$$Z_\infty^{p,q} := \bigcap_n \overline{Z}_n^{p,q}, \quad B_\infty^{p,q} := \bigcup_n \overline{B}_n^{p,q}.$$

From the tower of inclusions, we see that $B_\infty^{p,q} \subseteq Z_\infty^{p,q}$, so we define $E_\infty^{p,q} := Z_\infty^{p,q}/B_\infty^{p,q}$.

Definition 4.1.4. A spectral sequence *collapses at the Nth term* if the differentials $d_r^{p,q} = 0$ for $r \geq N$.

From the short exact sequence

$$0 \to \overline{Z}_r^{p,q}/\overline{B}_{r-1}^{p,q} \to \overline{Z}_{r-1}^{p,q}/\overline{B}_{r-1}^{p,q} \xrightarrow{d_r^{p,q}} \overline{B}_r^{p,q}/\overline{B}_{r-1}^{p,q} \to 0,$$

the condition $d_r^{p,q}$ forces $\overline{Z}_r^{p,q} = \overline{Z}_{r-1}^{p,q}$ and $\overline{B}_r^{p,q} = \overline{B}_{r-1}^{p,q}$. The tower of submodules becomes

$$B_2^{p,q} \subseteq \overline{B}_3^{p,q} \subseteq \cdots \subseteq \overline{B}_{N-1}^{p,q} = B_N^{p,q} = \cdots = B_\infty^{p,q} \subseteq Z_\infty^{p,q} = \cdots = \overline{Z}_N^{p,q} = \overline{Z}_{N-1}^{p,q} \subseteq \cdots \subseteq \overline{Z}_3^{p,q} \subseteq Z_2^{p,q} \subseteq E_2^{p,q}.$$

Thus, $E_\infty^{p,q} = E_N^{p,q}$.

[3] I'll have to rewrite this section because the Adams spectral sequence is not a cohomological or a homological spectral sequence I don't think - the grading is $d_r : E^{s,t} \to E^{s+r,t+r-1}$.

4.2 Exact couples

(Following [10], C2)

Definition 4.2.1. Let D, E be R-modules, and let $i : D \to D$, $j : D \to E$, $k : E \to D$ be module homomorphisms. We call $\mathcal{C} = \{D, E, i, j, k\}$ an *exact couple* if the diagram below is exact.

$$\begin{tikzcd} D \arrow[rr, "i"] & & D \arrow[dl, "j"] \\ & E \arrow[ul, "k"] & \end{tikzcd}$$

Let $d := jk$, and define the following:

$$\begin{aligned} E' &:= H(E, d) = \ker d / \operatorname{im} d \\ D' &:= i(D) = \ker j \\ i' &:= i|_{i(D)} : D' \to D' \\ j' &:= i(x) \mapsto j(x) + dE : D' \to E' \\ k' &:= (e + dE) \mapsto k(e) : E' \to D' \end{aligned}$$

We call $\mathcal{C}' = \{D', E'.i', j', k'\}$ the *derived couple* of $\mathcal{C}$.

Proposition 4.2.2 ([10], Prop 2.7). If $\mathcal{C} = \{D, E, i, j, k\}$ is an exact couple, then $\mathcal{C}'$ is also an exact couple.

Theorem 4.2.3 ([10], Thm 2.8). Suppose $D^{*,*} = \{D^{p,q}\}$ and $E^{*,*} = \{E^{p,q}\}$ are bigraded modules equipped with homomorphisms i of bidegree $(-1, 1)$[4], j of bidegree $(0, 0)$, and k of bidegree $(1, 0)$, such that $\{D^{*,*}, E^{*,*}, i, j, k\}$ is an exact couple. Then these data determine a spectral sequence $\{E_r, d_r\}$ for $r \in \mathbb{Z}_+$ of cohomological type, with $E_r = (E^{*,*})^{(r-1)}$, the $(r-1)$st derived module of $E^{*,*}$ and $d_r = j^{(r-1)} \circ k^{(r-1)}$.

A bigraded exact couple may be displayed in the following diagram, known as a *staircase diagram*:

$$\begin{tikzcd} & \vdots \arrow[d, "i"] & & \vdots \arrow[d, "i"] & \\ \cdots \arrow[r, "k"] & D^{p+2,q-1} \arrow[r, "j"] \arrow[d, "i"] & E^{p+2,q-1} \arrow[r, "k"] & D^{p+3,q-1} \arrow[r, "j"] \arrow[d, "i"] & \cdots \\ \cdots \arrow[r, "k"] & D^{p+1,q} \arrow[r, "j"] \arrow[d, "i"] & E^{p+1,q} \arrow[r, "k"] & D^{p+2,q} \arrow[r, "j"] \arrow[d, "i"] & \cdots \\ \cdots \arrow[r, "k"] & D^{p,q+1} \arrow[r, "j"] \arrow[d, "i"] & E^{p,q+1} \arrow[r, "k"] & D^{p+1,q+1} \arrow[r, "j"] \arrow[d, "i"] & \cdots \\ & \vdots & & \vdots & \end{tikzcd}$$

[4]This is the wrong index. For my purposes it should be $(-1, -1)$. So, the spectral sequence will not be of cohomological type, but hopefully the proof still goes through.

[This is the same staircase diagram that Hatcher is talking about (set $D^{*,*} = \pi_*^S X_*$ and $E^{*,*} = \pi_* K_*$) - just modulo some tweak of i's bigrading because the bigrading is a bit weird in the Adams spectral sequence.]

4.3 Setting up the Adams spectral sequence

Let X be a connective CW spectrum of finite type.

[Claim: $H^*(X)$ is an $\mathscr{A}_2$-module. Either use Theorem 3.5.1 (i.e. $H^m(X;G) \cong [X, K(G,m)]$) to give the action, or the above mentioned fact that for spectra of finite type, $H^i(X;G) = H^{i+n}(X_n;G)$. To define a map $Sq^j : H^i(X;G) \to H^{i+j}(X;G)$ we can either hit $H^{i+n}(X_n;G)$ with Sq^j and then push it along the sequence of cohomology groups until it's stable, or first push it along until $H^{i+j+n}(X_n;G)$ is stable and then hit $H^{i+n}(X_n;G)$ with Sq^j. And these are the same thing, because of this handy commutative diagram:

$$\begin{tikzcd}
 & H^{i+n}(X_n;G) \arrow[dl, "\sigma"'] \arrow[dr, "Sq^j"] & \\
H^{i+n+1}(\Sigma X_n;G) \arrow[d, "i^*"'] & & H^{i+j+n}(X_n;G) \arrow[d, "\sigma"] \\
H^{i+n+1}(X_{n+1};G) \arrow[dr, "Sq^j"'] & & H^{i+j+n+1}(\Sigma X_n;G) \arrow[dl, "i^*"] \\
 & H^{i+j+n+1}(X_{n+1};G) &
\end{tikzcd}$$

which follows from the properties in Section 2. Is this an actual module action? Is the former one also a module action?]

We can pick generators α_i for $H^*(X)$ as an $\mathscr{A}_2$-module such that there are at most finitely many in each $H^n(X)$ [since each $H^n(X)$ is finitely generated (Lemma 3.3.1), and that finite generating set would certainly also generate it as an $\mathscr{A}_2$-module (we could be more efficient, though)].

[Claim: these α_i determine a map $X \to K_0$, where K_0 is a wedge of EM spectra, and K_0 has finite type. Very confused about this - each $\alpha_i \in H^{n_i}(X;G)$ gives a map $X \to K(G,n_i)$ by Theorem 3.5.1, but shouldn't they come together to give a map into a product[5] of EM spaces, not a wedge sum? Isn't a wedge sum a sort of reduced *coproduct*, not a product?? Is this something to do with the 'fibrations and cofibrations are the same' nonsense?

Vague idea: we have Proposition 3.5.2, which tells us that $[X, \bigvee_i \mathbb{K}(G,n_i)] \cong \prod_i [X, \mathbb{K}(G,n_i)]$. Now, X is connective, so $H^*(X;G)$ has some smallest degree. Let's just pretend it starts from zero for the moment, and we can reindex or whatever later. Let's say each $H^k(X;G)$ has i_k generators $\alpha_{i_0+\cdots+i_{k-1}+1}, \ldots, \alpha_{i_0+\cdots+i_k}$, and write $n_0 = \cdots = n_{i_0}$, $n_{i_0+1} = \cdots = n_{i_0+i_1}$, etc., $n_{i_0+i_1+\cdots+i_{k-1}+1} = \cdots = n_{i_0+i_1+\cdots+i_k}$. So, the $\alpha_{i_0+\cdots+i_{k-1}+j} \in H^k(X;G)$ determine maps $X \to \mathbb{K}(G,n_k)$. Putting all these maps together, one in each coordinate, gives an element of $\prod_i [X, \mathbb{K}(G,n_i)]$, which by the proposition corresponds to an element of $[X, \bigvee_i \mathbb{K}(G,n_i)]$.]

[5] And on that note, what is a product of spectra? Surely it's not a smash product? Is it just a pointed product? Are we allowed to do that?

We can replace that map with an inclusion. [I sort of believe this - Remark B.2.8 and my comment at the end of Section C.3 say we can do this for CW complexes, so I should check it for spectra. But it seems true (and I'm currently suffering from spectrum fatigue).]

[Set $X_1 = K_0/X$, and repeat the construction to get a diagram:

$$\begin{tikzcd}
X \arrow[r] & K_0 \arrow[rr] \arrow[dr] & & K_1 \arrow[rr] \arrow[dr] & & K_2 \arrow[rr] \arrow[dr] & & \cdots \\
& & K_0/X = X_1 \arrow[ur] & & K_1/X_1 = X_2 \arrow[ur] & & K_2/X_2 = X_3 \arrow[ur] &
\end{tikzcd}$$

Taking cohomology, we get a diagram

$$\begin{tikzcd}
0 & H^*(X) \arrow[l] & H^*(K_0) \arrow[l] & & H^*(K_1) \arrow[ll] \arrow[dl] & & H^*(K_2) \arrow[ll] \arrow[dl] & & \cdots \arrow[ll] \arrow[dl] \\
& & & H^*(X_1) \arrow[ul] \arrow[dl] & & H^*(X_2) \arrow[ul] \arrow[dl] & & H^*(X_3) \arrow[ul] \arrow[dl] & \\
& & 0 & & 0 \arrow[ul] & & 0 \arrow[ul] & & 0 \arrow[ul]
\end{tikzcd}$$

]

- Fact: K_0 has finite type (and thus so do all the K_s's) [I'm sure this is supposed to follow from $H^i(X)$ being finitely generated for all i, but I can't at all see how].
- Fact: the top row of the diagram above is exact. [Suffices to show that we have exact sequences:

  $$0 \leftarrow H^*(X_s) \leftarrow H^*(K_s),$$
  $$H^*(X) \leftarrow H^*(K_0) \leftarrow H^*(X_1) \leftarrow 0,$$
  $$H^*(K_s) \leftarrow H^*(K_{s+1}) \leftarrow H^*(X_{s+2}) \leftarrow 0,$$

  and that the map $H^*(K_0) \to H^*(X)$ is surjective. Actually, I think the first exact sequence will follow from $H^*(K_0) \to H^*(X)$ being surjective, since they're all constructed in the same way.

  I'm extremely confused by the supposed surjectivity - going back to the sketch of the construction for CW complexes, the corresponding statement is supposed to be that if we choose generators α_i for $H^*(X)$ as an $\mathscr{A}_2$-module giving maps $f_i : X \to K(G, \deg(\alpha_i))$, the product of these maps induces a surjection on H^*. But I can't see how it does? So, we know by [5], Thm 4.57 that there are natural bijections $T : [X, K(G,n)] \to H^n(X;G)$ with $T[f] = f^*(\beta)$ for some class $\beta \in H^n(K(G,n);G)$. So, for any given $\alpha^i \in H^n(X;G)$, it's equal to $f^*(\beta_i)$, so definitely f^* hits it. But if I have some enormous product of EM spaces, and all these maps $f_i : X \to K(G, n_i)$ come together to give a map $f : X \to \prod_i K(G, n_i)$, then Hatcher is trying to tell me I get a surjection $H^*(\prod_i K(G,n_i)) \to H^*(X)$?? I know nothing about the cohomology of an infinite product, there's no reason it would be e.g. the product of the cohomologies.]

  [Something something reduced cohomology theories for spectra and the wedge axiom?? Suppose K_0 had finite type, which I'm not convinced it does. Then for some n, $(K_0)_n = \left(\bigvee_i \mathbb{K}(G, n_i)\right)_n = \bigvee_i (\mathbb{K}(G,n_i))_n = \bigvee_i K(G, n+n_i)$, and $H^i(K_0) = H^{i+n}(\bigvee_i K(G, n + n_i)) = \prod_i H^{i+n}(K(G, n+n_i))$. Does that help?]

- Fact: each $H^*(K_s)$ is a free $\mathscr{A}_2$-module, and thus the diagram above gives a resolution of $H^*(X)$ by free $\mathscr{A}_2$-modules. [Vague idea: Hatcher claims on p582 that "the cohomology $H^*(K(\mathbb{F}_2, n))$ is free over $\mathscr{A}_2$ in dimensions less than $2n$" (this may be [13] Prop 3.22 but I'm not sure what's going on with the suspension there). Assuming this, I said earlier that if X has finite type you can act on $H^*(X)$ with Sq^i by first pushing it along the limit as far as you need to to make $H^{i+j+n}(X_{n+j})$ stable and then applying Sq^i to $H^{i+n}(X_n)$. But the point here is that I can make n as big as I like, so I can just make it big enough so that any action of $\mathscr{A}_2$ I like is free.]

[The work in Section 3.4 gives us long exact sequences

$$\cdots \to [\mathbb{S}^{t+1}, X_s] \to [\mathbb{S}^{t+1}, K_s] \to [\mathbb{S}^{t+1}, X_{s+1}] \to [\mathbb{S}^{t+1}, \Sigma X_s] \to [\mathbb{S}^{t+1}, \Sigma K_s] \to \cdots .$$

Using the isomorphism $[Y, Z] \cong [\Sigma Y, \Sigma Z]$, we get long exact sequences

$$\cdots \to [\Sigma^\infty S^{t+1}, X_s] \to [\mathbb{S}^{t+1}, K_s] \to [\mathbb{S}^{t+1}, X_{s+1}] \to [\mathbb{S}^t, X_s] \to [\mathbb{S}^t, K_s] \to \cdots ,$$

i.e.

$$\cdots \to \pi_{t+1}X_s \to \pi_{t+1}K_s \to \pi_{t+1}X_{s+1} \to \pi_t X_s \to \pi_t K_s \to \cdots .$$

These form a staircase diagram,

$$\begin{tikzcd}
 & \vdots \arrow[d, "i"] & & \vdots \arrow[d, "i"] & & \vdots \arrow[d, "i"] & \\
\cdots \arrow[r, "k"] & \pi_{t+1}X_s \arrow[r, "j"] \arrow[d, "i"] & \pi_{t+1}K_s \arrow[r, "k"] & \pi_{t+1}X_{s+1} \arrow[r, "j"] \arrow[d, "i"] & \pi_{t+1}K_{s+1} \arrow[r, "k"] & \pi_{t+1}X_{s+2} \arrow[r, "j"] \arrow[d, "i"] & \cdots \\
\cdots \arrow[r, "k"] & \pi_t X_{s-1} \arrow[r, "j"] \arrow[d, "i"] & \pi_t K_{s-1} \arrow[r, "k"] & \pi_t X_s \arrow[r, "j"] \arrow[d, "i"] & \pi_t K_s \arrow[r, "k"] & \pi_t X_{s+1} \arrow[r, "j"] \arrow[d, "i"] & \cdots \\
\cdots \arrow[r, "k"] & \pi_{t-1}X_{s-2} \arrow[r, "j"] \arrow[d, "i"] & \pi_{t-1}K_{s-2} \arrow[r, "k"] & \pi_{t-1}X_{s-1} \arrow[r, "j"] \arrow[d, "i"] & \pi_{t-1}K_{s-1} \arrow[r, "k"] & \pi_{t-1}X_s \arrow[r, "j"] \arrow[d, "i"] & \cdots \\
 & \vdots & & \vdots & & \vdots &
\end{tikzcd}$$

This gives rise to a spectral sequence, by [some version of] Theorem 4.2.3.

Now, since $K_s = \bigvee_i \mathbb{K}(G, n_{s_i})$, Proposition 3.5.2, tells us that $[\mathbb{S}, K_s] \cong \prod_i [\mathbb{S}, \mathbb{K}(G, n_{s_i})]$, which is naturally isomorphic to $\prod_i H^{n_{s_i}}(\mathbb{S}; G)$. Thus, elements of $[\mathbb{S}, K_s]$ are tuples of elements of $H^*(\mathbb{S})$.

- Def: $\mathrm{Hom}^t_{\mathscr{A}_2}$ [it's just the homs that lower the degree by t]
- Fact: there is a natural map $[\mathbb{S}, K_s] \to \mathrm{Hom}^0_{\mathscr{A}_2}(H^*(K_s), H^*(\mathbb{S}))$, and it's an isomorphism. [??]

[We thus have

$$[\Sigma^t\mathbb{S}, K_s] = \mathrm{Hom}^0_{\mathscr{A}_2}(H^*(K_s), H^*(\Sigma^t\mathbb{S})) = \mathrm{Hom}^t_{\mathscr{A}_2}(H^*(K_s), H^*(\mathbb{S})).$$

In the case of CW complexes, we have $H^*(\Sigma^t X) \cong H^{*-t}(X)$ via the map σ mentioned in Section 2. Since $\mathbb{S}$ has finite type, for i large enough we have $H^n(\Sigma^t\mathbb{S}) = H^{n+i}(\Sigma^t S^i) = H^{n+i-t}(S^i) = H^{n-t}(\mathbb{S})$.]

[$E_1^{s,t} = \pi_t K_s (= \mathrm{Hom}^t_{\mathscr{A}_2}(H^*(K_s), H^*(\mathbb{S})))$, since the staircase diagram comes from (or gives rise to, depending on your point of view) the exact couple:

$$\begin{tikzcd} \pi_* X_* \arrow[rr, "i"] & & \pi_* X_* \arrow[dl, "j"] \\ & \pi_* K_* \arrow[ul, "k"] & \end{tikzcd}$$

where $i : \pi_{t+1} X_{s+1} \to \pi_t X_s$, $j : \pi_{t+1} X_s \to \pi_{t+1} K_s$, and $k : \pi_{t+1} X_{s+1}$ all come from the diagram we set up earlier. The differential $d_1 : \pi_t(K_s) \to \pi_t K_{s+1}$ is induced by the map $K_s \to K_{s+1}$, since it's just $j \circ k$, according to the way things are set up in Section 4.2, possibly modulo some typos.]

[Then, $E_2^{s,t} = H^{s,t}(E_1^{*,*}, d_1)$, so each $E^{*,t}$ is the homology of the chain complex

$$0 \to E_1^{0,t} \to E^{1,t} \to E^{2,t} \to \cdots ,$$

or, in other words,

$$0 \to \mathrm{Hom}^t_{\mathscr{A}_2}(H^*(K_0), H^*(\mathbb{S})) \to \mathrm{Hom}^t_{\mathscr{A}_2}(H^*(K_1), H^*(\mathbb{S})) \to \cdots .$$

The homology of this is by definition $\mathrm{Ext}^{s,t}_{\mathscr{A}_2}(H^*(X), H^*(\mathbb{S}))$, so $E_2^{s,t} = \mathrm{Ext}^{s,t}_{\mathscr{A}_2}(H^*(X), H^*(\mathbb{S}))$.]

Theorem 4.3.1 ([6], Thm 5.47). There is a spectral sequence $\{E_r, d_r\}$ such that $E_2^{s,t} = \mathrm{Ext}^{s,t}_{\mathscr{A}_2}(\mathbb{F}_2, \mathbb{F}_2)$ and $\{E_r, d_r\} \implies \pi^S_{t-s}$ modulo torsion of odd order. [which is probably a lie but let's worry about that later]

[10], [6]

4.4 First computations

Lemma 4.4.1 ([6], Lem 5.49). For a minimal free resolution

$$\cdots \to F_2 \to F_1 \to F_0 \to \mathbb{F}_2 \to 0$$

of $\mathbb{F}_2 = H^*(\mathbb{S})$ as an $\mathscr{A}_2$-module, we have $\mathrm{Ext}^{s,t}_{\mathscr{A}_2}(\mathbb{F}_2, \mathbb{F}_2) = \mathrm{Hom}^t_{\mathscr{A}_2}(F_s, \mathbb{F}_2)$, where $\mathrm{Hom}^t_{\mathscr{A}_2}(F_s, \mathbb{F}_2) \subseteq \mathrm{Hom}_{\mathscr{A}_2}(F_s, \mathbb{F}_2)$ consists of the morphisms which lower the degree by t.

(proof)

[Now, since $\mathbb{F}_2$ just has stuff in degree 0, the only things that can be sent to $1 \in \mathbb{F}_2$ are the things in degree t, so for every generator for F_s in degree t, there's an $\mathbb{F}_2$'s worth of such homs.]

[13], [6], [12]

	0	1	2	3	4
0	ι	α_1^1	α_2^2	α_3^3	α_4^4
1	$Sq^1\iota$	$Sq^1\alpha_1^1$ α_2^1	$Sq^1\alpha_2^2$	$Sq^1\alpha_3^3$	$Sq^1\alpha_4^4$
2	$Sq^2\iota$	$Sq^2\alpha_1^1$ $Sq^1\alpha_2^1$	$Sq^2\alpha_2^2$ α_4^2	$Sq^2\alpha_3^3$	$Sq^2\alpha_4^4$
3	$Sq^2Sq^1\iota$ $Sq^3\iota$	$Sq^2Sq^1\alpha_1^1$ $Sq^3\alpha_1^1$ $Sq^2\alpha_2^1$ α_4^1	$Sq^2Sq^1\alpha_2^2$ $Sq^3\alpha_2^2$ $Sq^1\alpha_4^2$ α_5^2	$Sq^2Sq^1\alpha_3^3$ $Sq^3\alpha_3^3$ α_6^3	$Sq^2Sq^1\alpha_4^4$ $Sq^3\alpha_4^4$
4	$Sq^3Sq^1\iota$ $Sq^4\iota$	$Sq^3Sq^1\alpha_1^1$ $Sq^4\alpha_1^1$ $Sq^2Sq^1\alpha_2^1$ $Sq^3\alpha_2^1$ $Sq^1\alpha_4^1$	$Sq^3Sq^1\alpha_2^2$ $Sq^4\alpha_2^2$ $Sq^2\alpha_4^2$ $Sq^1\alpha_4^2$	$Sq^3Sq^1\alpha_3^3$ $Sq^4\alpha_3^3$ $Sq^1\alpha_6^3$	$Sq^3Sq^1\alpha_4^4$ $Sq^4\alpha_4^4$
5	$Sq^4Sq^1\iota$ $Sq^5\iota$	$Sq^4Sq^1\alpha_1^1$ $Sq^5\alpha_1^1$ $Sq^3Sq^1\alpha_2^1$ $Sq^4\alpha_2^1$ $Sq^2\alpha_4^1$	$Sq^4Sq^1\alpha_2^2$ $Sq^5\alpha_2^2$ $Sq^2Sq^1\alpha_4^2$ $Sq^3\alpha_4^2$ $Sq^2\alpha_5^2$	$Sq^4Sq^1\alpha_3^3$ $Sq^5\alpha_3^3$ $Sq^2\alpha_6^3$	$Sq^4Sq^1\alpha_4^4$ $Sq^5\alpha_4^4$
6	$Sq^5Sq^1\iota$ $Sq^4Sq^2\iota$ $Sq^6\iota$	$Sq^5Sq^1\alpha_1^1$ $Sq^4Sq^2\alpha_1^1$ $Sq^6\alpha_1^1$ $Sq^4Sq^1\alpha_2^1$ $Sq^5\alpha_2^1$ $Sq^2Sq^1\alpha_4^1$ $Sq^3\alpha_4^1$	$Sq^5Sq^1\alpha_2^2$ $Sq^4Sq^2\alpha_2^2$ $Sq^6\alpha_2^2$ $Sq^3Sq^1\alpha_4^2$ $Sq^4\alpha_4^2$ $Sq^2Sq^1\alpha_5^2$ $Sq^3\alpha_5^2$	$Sq^5Sq^1\alpha_3^3$ $Sq^4Sq^2\alpha_3^3$ $Sq^6\alpha_3^3$ $Sq^2Sq^1\alpha_6^3$ $Sq^3\alpha_6^3$	$Sq^5Sq^1\alpha_4^4$ $Sq^4Sq^2\alpha_4^4$ $Sq^6\alpha_4^4$
7	$Sq^4Sq^2Sq^1\iota$ $Sq^6Sq^1\iota$ $Sq^5Sq^2\iota$ $Sq^7\iota$	$Sq^4Sq^2Sq^1\alpha_1^1$ $Sq^6Sq^1\alpha_1^1$ $Sq^5Sq^2\alpha_1^1$ $Sq^7\alpha_1^1$ $Sq^5Sq^1\alpha_2^1$ $Sq^4Sq^2\alpha_2^1$ $Sq^6\alpha_2^1$ $Sq^3Sq^1\alpha_4^1$ $Sq^4\alpha_4^1$	$Sq^4Sq^2Sq^1\alpha_2^2$ $Sq^6Sq^1\alpha_2^2$ $Sq^5Sq^2\alpha_2^2$ $Sq^7\alpha_2^2$ $Sq^4Sq^1\alpha_4^2$ $Sq^5\alpha_4^2$ $Sq^3Sq^1\alpha_5^2$ $Sq^4\alpha_5^2$	$Sq^4Sq^2Sq^1\alpha_3^3$ $Sq^6Sq^1\alpha_3^3$ $Sq^5Sq^2\alpha_3^3$ $Sq^7\alpha_3^3$ $Sq^3Sq^1\alpha_3^3$ $Sq^4\alpha_3^3$	$Sq^4Sq^2Sq^1\alpha_4^4$ $Sq^6Sq^1\alpha_4^4$ $Sq^5Sq^2\alpha_4^4$ $Sq^7\alpha_4^4$

[Explain the table above]

Looking at the table above, we can see that $(\pi_3^S)_2^\wedge$ has order 8. However, we do not currently have the tools to determine whether it is isomorphic to $(\mathbb{Z}/2\mathbb{Z})^3$, $\mathbb{Z}/2\mathbb{Z}\oplus\mathbb{Z}/4\mathbb{Z}$, or $\mathbb{Z}/8\mathbb{Z}$. We will therefore spend some time describing a multiplication on the Adams spectral sequence which will allow us to resolve such ambiguities.

4.5 Multiplicative structure

4.5.1 The Yoneda product

[12]:

Definition 4.5.1. For any algebra A and A-modules L, M, N, there is a product, the *Yoneda product*

$$\circ : \mathrm{Ext}_A^{s,t}(M,N) \otimes \mathrm{Ext}_A^{u,v}(L,M) \to \mathrm{Ext}_A^{s+u}(L,N),$$

defined as follows: let

$$\cdots \xrightarrow{f_3} F_2 \xrightarrow{f_2} F_1 \xrightarrow{f_1} F_0 \xrightarrow{f_0} L \to 0,$$

$$\cdots \xrightarrow{f_3'} F_2' \xrightarrow{f_2'} F_1' \xrightarrow{f_1'} F_0' \xrightarrow{f_0'} M \to 0$$

be free resolutions for L and M. Then, given $[g] \in \mathrm{Ext}^{s,t}(M,N)$, $[h] \in \mathrm{Ext}^{u,v}(L,M)$, we inductively construct a chain map $h_\bullet : F_{u+\bullet} \to F_\bullet'[v]$, as shown in the diagram below.

$$\begin{tikzcd}
F_{u+s} \arrow[r, "f_{u+s}"] \arrow[d, dashed, "h_s"'] & F_{u+s-1} \arrow[r, "f_{u+s-1}"] \arrow[d, dashed, "h_{s-1}"'] & \cdots \arrow[r, "f_{u+1}"] & F_u \arrow[r, "f_u"] \arrow[d, dashed, "h_0"'] \arrow[dr, "h"] & F_{u-1} \arrow[r, "f_{u-1}"] & \cdots \arrow[r, "f_1"] & F_0 \arrow[r, "f_0"] & L \\
F_s'[v] \arrow[r, "f_s'"'] \arrow[d, "g"'] & F_{s-1}'[v] \arrow[r, "f_{s-1}'"'] & \cdots \arrow[r, "f_1'"'] & F_0'[v] \arrow[r, two heads, "f_0'"'] & M[v] \\
N[v+t]
\end{tikzcd}$$

The map h_0 is defined as follows: let $\alpha \in F_u$ be a generator, and consider $h(\alpha) \in M[v]$. Since f_0' is surjective, there exists some $\beta \in F_0'[v]$ such that $f_0'(\beta) = h(\alpha)$. We define $h_0(\alpha) = \beta$. Now, suppose the h_i have been constructed for $i < w$, and consider the diagram below.

$$\begin{tikzcd}
F_{u+w} \arrow[r, "f_{u+w}"] \arrow[d, dashed, "h_w"'] & F_{u+w-1} \arrow[r, "f_{u+w-1}"] \arrow[d, "h_{w-1}"] & F_{u+w-2} \arrow[d, "h_{w-2}"] \\
F_w'[v] \arrow[r, "f_w'"'] & F_{w-1}'[v] \arrow[r, "f_{w-1}'"'] & F_{w-2}'[v]
\end{tikzcd}$$

Let $\alpha \in F_{u+w}$ be a generator, and consider $f_{w-1}' h_{w-1} f_{u+w}(\alpha) \in F_{w-2}'[v]$. By induction, the right square commutes, so $f_{w-1}' h_{w-1} f_{u+w}(\alpha) = h_{w-2} f_{u+w-1} f_{u+w}(\alpha) = 0$, by exactness of the top row. Thus, $h_{w-1} f_{u+w}(\alpha) \in \ker f_{w-1}' = \operatorname{im} f_w'$. Write $h_{w-1} f_{u+w}(\alpha) = f_w'(\beta)$, and define $h_w(\alpha) = \beta$.

Now, consider the diagram below.

$$\begin{tikzcd}
F_{u+s+1} \arrow[r, "f_{u+s+1}"] \arrow[d, dashed, "h_{s+1}"'] & F_{u+s} \arrow[d, "h_s"] \\
F_{s+1}'[v] \arrow[r, "f_{s+1}'"'] & F_s'[v] \arrow[d, "g"] \\
& N[v+t]
\end{tikzcd}$$

We have $gh_s f_{u+s+1} = gf'_{s+1}h_{s+1} = 0$, since $[g] \in \mathrm{Ext}^{s,t}(F'_s, N)$, so $[gh_s] \in \mathrm{Ext}^{u+s,v+t}$. We thus define $[g] \cdot [h] = [gh_s]$.

This definition is independent of the lifts chosen, which can be seen as follows. Suppose we have two chain maps $\{h_i\}, \{h'_i\}$; we will construct a chain homotopy between them. Define $k_0 : F_{u-1} \to F'_0[v]$ to be the zero map. By construction, $f'_0 h_0 = f'_0 h'_0 = h$, so $f'_0(h_0 - h'_0) = 0$. Thus, $\mathrm{im}(h_0 - h'_0) \subseteq \ker f'_0 = \mathrm{im}\, f'_1$, so $h_0 - h'_0 = f'_1 k_1 = f'_1 k_1 + k_0 f_u$ for some map $k_1 : F_u \to F'_1[v]$. Now, suppose we have k_i, k_{i-1} such that $h_{i-1} - h'_{i-1} = f'_i k_i + k_{i-1} f_{u+i-1}$. Then $f'_i h_i = h_{i-1} f_{u+i}$ and $f'_i h'_i = h'_{i-1} f_{u+i}$, so $f'_i(h_i - h'_i) = (h_{i-1} - h'_{i-1}) f_{u+i} = (f'_i k_i + k_{i-1} f_{u+i-1}) f_{u+i} = f'_i k_i f_{u+i}$, and thus we can construct k_{i+1} such that $h_i - h'_i = f'_{i+1} k_{i+1} + k_i f_{u+i}$. Now, $g(h_s - h'_s) = g(f'_{s+1} k_{s+1} + k_s f_{u+s}) = g k_s f_{u+s}$, and therefore $[g(h_s - h'_s)] = [g k_s f_{u+s}] = [0]$.

Finally, if $h = l f_u$ for some $l : F_{u-1} \to M[v]$, with filling $\{l_i\}$, then $\{l_i f_{u+i}\}$ is a filling for h, so $[g] \cdot [h] = [g l_s f_{u+s}] = [0]$. On the other hand, if $g = m f'_s$, then $[g] \cdot [h] = [gh_s] = [m f'_s h_s] = [m h_{s-1} f_{u+s}] = [0]$. Thus, the Yoneda product is well defined.

4.5.2 The composition product

Definition 4.5.2 ([13], p47). Let X, Y, Z be spectra. The *composition pairing* $\circ : [Y,Z]_* \otimes [X,Y]_* \to [X,Z]_*$ is defined as follows:

$$\begin{aligned} \circ :\ [Y,Z]_v \ \otimes\ [X,Y]_t \ &\to\ [X,Z]_{v+t} \\ [g : \Sigma^v Y \to Z] \otimes [f : \Sigma^t X \to Y] &\mapsto [g \circ \Sigma^v f : \Sigma^{v+t} X \to Z], \end{aligned}$$

where $[X,Y]_n = [\Sigma^n X, Y]$.

In particular, if $X = Y = Z = \mathbb{S}$, we have a product $\pi^S_v \otimes \pi^S_t \to \pi^S_{v+t}$.

Lemma 4.5.3 ([5], Prop 4.56). The composition product makes π^S_* into a graded commutative ring.

(proof)

Lemma 4.5.4. There is a unique ring structure on $(\pi^S_*)^\wedge_2$ which makes the completion map $c : \pi^S_* \to (\pi^S_*)^\wedge_2$ into a ring homomorphism.

Proof. We show uniqueness first. Let $f \in (\pi^S_i)^\wedge_2$, $g \in (\pi^S_j)^\wedge_2$. If $i, j \geq 1$, then the completion map is surjective, so $f = c(\tilde{f}), g = c(\tilde{g})$ for some $\tilde{f} \in \pi^S_i, \tilde{g} \in \pi^S_j$. Then $fg = c(\tilde{f})c(\tilde{g}) = c(\tilde{f}\tilde{g})$.

If $i = 0, j \geq 1$, then let $\hat{f} \in \pi^S_0$ be a lift of $q(f) \in \pi^S_0 / 2^r \pi^S_0$, where 2^r is the highest power of 2 dividing the order of π^S_j. Then $f \equiv c(\hat{f}) \mod 2^r$, so $f = c(\hat{f}) + 2^r w$. We have $fg = f c(\tilde{g}) = (c(\hat{f}) + 2^r w) c(\tilde{g}) = c(\hat{f}) c(\tilde{g}) + 2^r (w c(\tilde{g})) = c(\hat{f}\tilde{g}) \in (\pi^S_j)^\wedge_2$.

Finally, if $i = j = 0$, we claim that any two multiplications on $\mathbb{Z}_2$ which agree on $\mathbb{Z}$ must agree on all of $\mathbb{Z}_2$, and thus the multiplication is given by the usual product on $\mathbb{Z}_2$.

Suppose not; let $\star, \cdot$ be two products on $\mathbb{Z}_2$, agreeing on $\mathbb{Z}$, with $f \star g \neq f \cdot g$. Then there is some k such that $f \star g \not\equiv f \cdot g \mod k$. Pick integers n, m such that $n \equiv f \mod k$ and $m \equiv g \mod k$. Then, modulo k, $f \cdot g \equiv n \cdot m = n \star m \equiv f \star g$, giving a contradiction.

Now, for $i, j \geq 1$, this multiplication is well defined, since if $\tilde{f}' = \tilde{f} + t$, with $nt = 0$ for odd n, then $c(\tilde{f}'\tilde{g}) = c(\tilde{f}\tilde{g} + t\tilde{g}) = c(\tilde{f}\tilde{g})$ (since $nt\tilde{g} = (nt)\tilde{g} = 0\tilde{g} = 0$). Likewise, if $\tilde{g}' = \tilde{g} + t$, then $c(\tilde{f}\tilde{g}') = c(\tilde{f}\tilde{g})$. Note that the product for $i = 0, j \geq 1$ (and vice versa) is exactly the isomorphism in the proof of Lemma 3.6.2, and the usual product on $\mathbb{Z}_2$ is of course well-defined. Finally, associativity, distributivity, and unitality are inherited from π_*^S. $\square$

Now, given spectra X, Y, Z, we can define a pairing $\circ : [Y, {}_2Z]_* \otimes [X, {}_2Y]_* \to [X, {}_2Z]$ as follows: let $f \in [Y, {}_2Z]_s, g \in [X, {}_2Y]_t$. By Theorem 3.6.6, there exists a unique (up to homotopy) map $\overline{f} : (\Sigma^s Y)_2^\wedge \to Z_2^\wedge$ such that f factors through $\overline{f}$. Now, note that $(\Sigma^s Y)_2^\wedge \simeq \Sigma^s Y_2^\wedge$, since $\pi_i(\Sigma^s Y) = \pi_{i-s}(Y)$. We can thus define the pairing of f and g to be $\overline{f} \circ \Sigma^s g$, as shown below.

$$\begin{tikzcd}
 & \Sigma^s Y \arrow[d] \arrow[dr, "f"] & \\
\Sigma^{s+t} X \arrow[r, "\Sigma^s g"'] & \Sigma^s Y_2^\wedge \arrow[r, dashed, "\overline{f}"'] & Z_2^\wedge
\end{tikzcd}$$

Lemma 4.5.5. The completion map $c_* : \pi_*^S \to \pi_*(\mathbb{S}_2^\wedge)$ is a ring homomorphism. In particular, by Lemma 4.5.4, the composition product on $\pi_*(\mathbb{S}_2^\wedge)$ coincides with the product on $(\pi_*^S)_2^\wedge$ inherited from π_*^S, so the two groups are also isomorphic as rings.

Proof. Let $f : \mathbb{S}^i \to \mathbb{S}$, $g : \mathbb{S}^j \to \mathbb{S}$ be elements of π_i^S and π_j^S respectively. Then $c_*(f)c_*(g) = (cf)(cg)$ is given by factorising $cg = \overline{cg}c$ and composing to get $\overline{cg}c\Sigma^j f$. We thus have the commutative diagram below.

$$\begin{tikzcd}
\mathbb{S}^{i+j} \arrow[r, "\Sigma^j f"] & \mathbb{S}^j \arrow[r, "g"] \arrow[d, "c"] & \mathbb{S} \arrow[d, "c"] \\
 & \Sigma^j \mathbb{S}_2^\wedge \arrow[r, "\overline{cg}"'] & \mathbb{S}_2^\wedge
\end{tikzcd}$$

The upper path is exactly $c_*(fg)$, so $c_*(f)c_*(g) = c_*(fg)$. Further, the completion map clearly preserves the identity, so it is a ring homomorphism. $\square$

4.5.3 Multiplication on the Adams spectral sequence

Definition 4.5.6 ([13], Def 5.5). Let $\{'E_r\}, \{''E_r\}, \{E_r\}$ be three spectral sequences. A *pairing* of these spectral sequences is a sequence of homomorphisms

$$\phi_r : {'E_r^{*,*}} \otimes {''E_r^{*,*}} \to E_r^{*,*},$$

such that the Leibniz rule

$$d_r \phi_r(x \otimes y) = \phi_r(d_r(x) \otimes y) + (-1)^{\deg x} \phi_r(x \otimes d_r(y))$$

holds, and

$$\phi_{r+1}([x] \otimes [y]) = [\phi_r(x \otimes y)],$$

where $[x] \in {'E_{r+1}^{*,*}}$ is the homology class of a d_r-cycle $x \in {'E_r^{*,*}}$, and similarly for y and the right hand side.

A spectral sequence pairing $\{\phi_r\}$ induces a pairing

$$\phi_\infty : {'E_\infty^{*,*}} \otimes {''E_\infty^{*,*}} \to E_\infty^{*,*}.$$

Theorem 4.5.7 ([13], Thm 5.8). Let X, Y, Z be spectra, with Y, Z connective and of finite type. There is a pairing of spectral sequences

$$E_r^{*,*}(Y, Z) \otimes E_r^{*,*}(X, Y) \to E_r^{*,*}(X, Z)$$

which agrees for $r = 2$ with the Yoneda pairing

$$\operatorname{Ext}^{*,*}_{\mathscr{A}_2}(H^*(Z), H^*(Y)) \otimes \operatorname{Ext}^{*,*}_{\mathscr{A}_2}(H^*(Y), H^*(X)) \to \operatorname{Ext}^{*,*}_{\mathscr{A}_2}(H^*(Z), H^*(X))$$

and which converges to the composition pairing

$$[Y, Z_2^\wedge]_* \otimes [X, Y_2^\wedge]_* \to [X, Z_2^\wedge]_*.$$

The pairing is associative and unital.

[Explain why if I compute a product on the E_2 page, and both terms survive to the E_3 page, that computation is still valid.]

[13], [12], [7]

5 Calculating stable homotopy groups

5.1 Resolving extensions

Proposition 5.1.1 ([13], Cor 6.5). We have the following relations:

$$\begin{aligned}
\alpha_i^i &= (\alpha_1^1)^i \\
\alpha_4^2 &= (\alpha_2^1)^2 \\
\alpha_5^2 &= \alpha_1^1\alpha_4^1 \\
\alpha_6^3 &= (\alpha_1^1)^2\alpha_4^1 = (\alpha_2^1)^3.
\end{aligned}$$

(Proof/calculation)

From now on, we will write h_i for the generator $\alpha^1_{2^i}$.

Proposition 5.1.2. Suppose $\alpha \in (\pi_i^S)_2^\wedge$ represents $a \in E_\infty$. Then 2α represents $h_0 a$. In other words, multiplication by h_0 is induced by multiplication by 2.

Proof. Recall that $\pi_0^S = \mathbb{Z}$, since $\pi_1 S^1 = \mathbb{Z}$ and $n = 1 \leq 2 = 2(1)$, so this lies in the stable region. Now, $E_r^{s,s}(\mathbb{S})$ converges to some filtration of $\mathbb{Z}_2$ whose quotients are all $\mathbb{Z}/2\mathbb{Z}$. The filtration must therefore be

$$\cdots \subseteq 4\mathbb{Z}_2 \subseteq 2\mathbb{Z}_2 \subseteq \mathbb{Z}_2,$$

since finite index subgroups of $\mathbb{Z}_p$ are of the form $p^k\mathbb{Z}_p$.

Thus, $\iota = [1] \in \mathbb{Z}_2/2\mathbb{Z}_2$, and by computing the Yoneda product we see that ι is a unit. We also have $h_0 = [2] \in 2\mathbb{Z}_2/4\mathbb{Z}_2$ so $h_0 = [2] = [2[1]] = [2\iota]$, and hence h_0 acts on ι by multiplication by 2. Now, for any $\kappa \in E_r^{s,t}(\mathbb{S})$, $h_0 \cdot \kappa = (\iota h_0) \cdot \kappa = 2\kappa \in E_r^{s+1,t+1}(\mathbb{S})$. $\square$

[The notation's a bit weird above, when I say 'multiplication by 2' I mean: take $\kappa \in E^{s,t} = F^{s,t}/F^{s+1,t+1}$. Then $2\kappa \in F^{s+1,t+1}$ since it's a bunch of $\mathbb{F}_2$'s. Take it's equivalence class to get an element of $F^{s+1,t+1}/F^{s+2,t+2}$. That's what I really mean by 2κ and it's in $E^{s+1,t+1}$.

All this is to say if I start multiplying higher things by h_0, that *is* multiplying by 2. So I can start resolving extensions this way.]

Theorem 5.1.3.

$$(\pi_i^S)_2^\wedge = \begin{cases} \mathbb{Z}/2\mathbb{Z} & i = 1, 2 \\ \mathbb{Z}/8\mathbb{Z} & i = 3 \\ 0 & i = 4, 5. \end{cases}$$

5.2 The E_2 page for $t - s \leq 15$

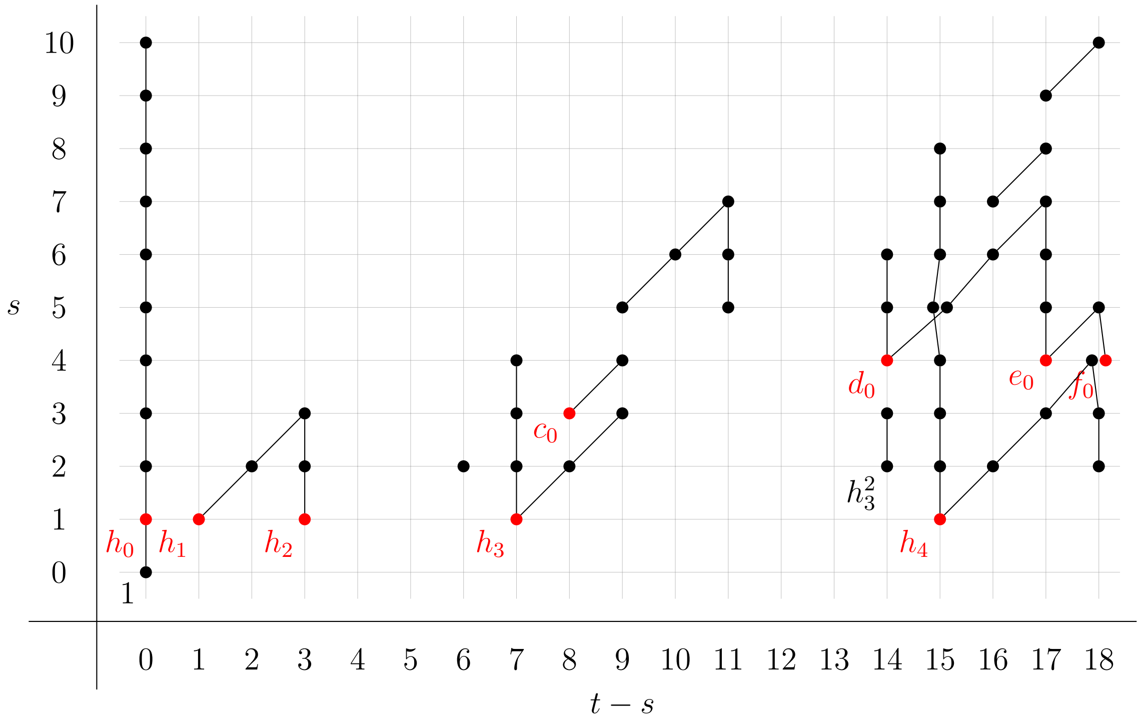

Figure 5.1: $\mathrm{Ext}^{s,t}_{\mathscr{A}_2}(\mathbb{F}_2, \mathbb{F}_2)$ for $t-s \leq 18$. The vertical and diagonal lines indicate multiplication by h_0 and h_1 respectively. Some of the algebra generators are shown in red, with their standard names.

Lemma 5.2.1. There are no nontrivial differentials for $t - s \leq 13$.

Proof. First, note that the only possible nontrivial differentials in this range are $d_r : E_r^{1,2}(\mathbb{S}) \to E_r^{1+r,1+r}(\mathbb{S})$ and $d_2 : E_2^{2,10}(\mathbb{S}) \to E_2^{4,11}(\mathbb{S})$.

Now, $0 = d_r(h_0h_1) = d_r(h_0)h_1 + h_0d_r(h_1) = h_0d_r(h_1)$, so $d_r(h_1) = 0$. Since $E_r^{1,2}(\mathbb{S})$ is generated by h_1, we must have $d_r = 0$.

On the other hand, $E_2^{2,10}(\mathbb{S})$ is generated by h_1h_3, and $d_2(h_1h_3) = d_2(h_1)h_3 + h_1d_2(h_3) = 0 + 0 = 0$ (the first factor is zero by the previous computation, and the second is an element of a trivial group). $\square$

5.3 Differentials at $14 \leq t - s \leq 15$

[The point here is that all differentials interacting with $E^{s,t}$ for $t - s < 14$ are trivial, and thus computing the stable homotopy groups is purely mechanical, because everything that appears on this part of the E_2 page has to survive to E_∞. Thus, the 'ambiguity' at $t-s = 14$ is just the fact that this is the first time you need to actually compute differentials.]

Theorem 5.3.1 ([12], Thm 11.10.2). $d_2(h_4) = h_0h_3^2$.

Proof. We have shown that h_0 detects $2 \in (\pi_*^S)^\wedge_2$ (i.e. 2 is a representative for h_0). Let $\sigma \in (\pi_7^S)^\wedge_2$ be a representative for h_3. Then $2\sigma^2$ is a representative for $h_0h_3^2$. By graded commutativity of $(\pi_*^S)^\wedge_2$, $\sigma^2 = -\sigma^2$, so $2\sigma^2 = 0$, and thus $h_0h_3^2 = 0$ in $E_\infty^{3,17}(\mathbb{S})$. Therefore, $h_0h_3^2$ is the boundary of a differential, and the only possibility is $d_2(h_4) = h_0h_3^2$. $\square$

The d_2 differentials at $E_2^{2,17}(\mathbb{S}), E_2^{3,18}(\mathbb{S}), E_2^{4,19}(\mathbb{S})$ are all trivial: $d_2(h_0^n h_4) = h_0^n d_2(h_4) = h_0^{n-1}(h_0 h_3^2) = 0$.

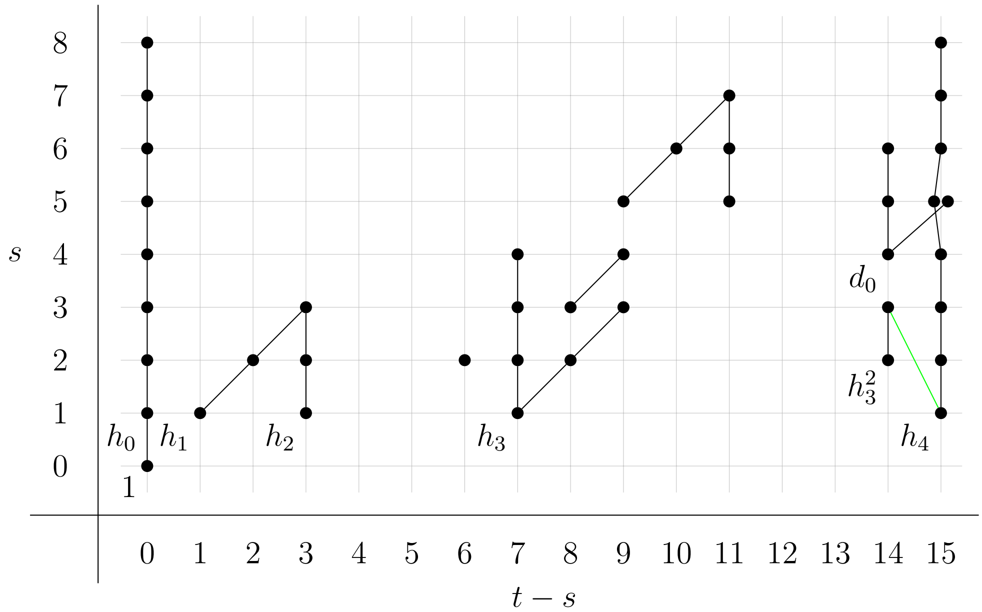

Figure 5.2: The E_2 page of the Adams spectral sequence for $\mathbb{S}$, in the range $t-s \leq 15$; the unique d_2 differential is shown in green.

[I promise we're gonna do the d_3 differential but first we need a few lemmas.]

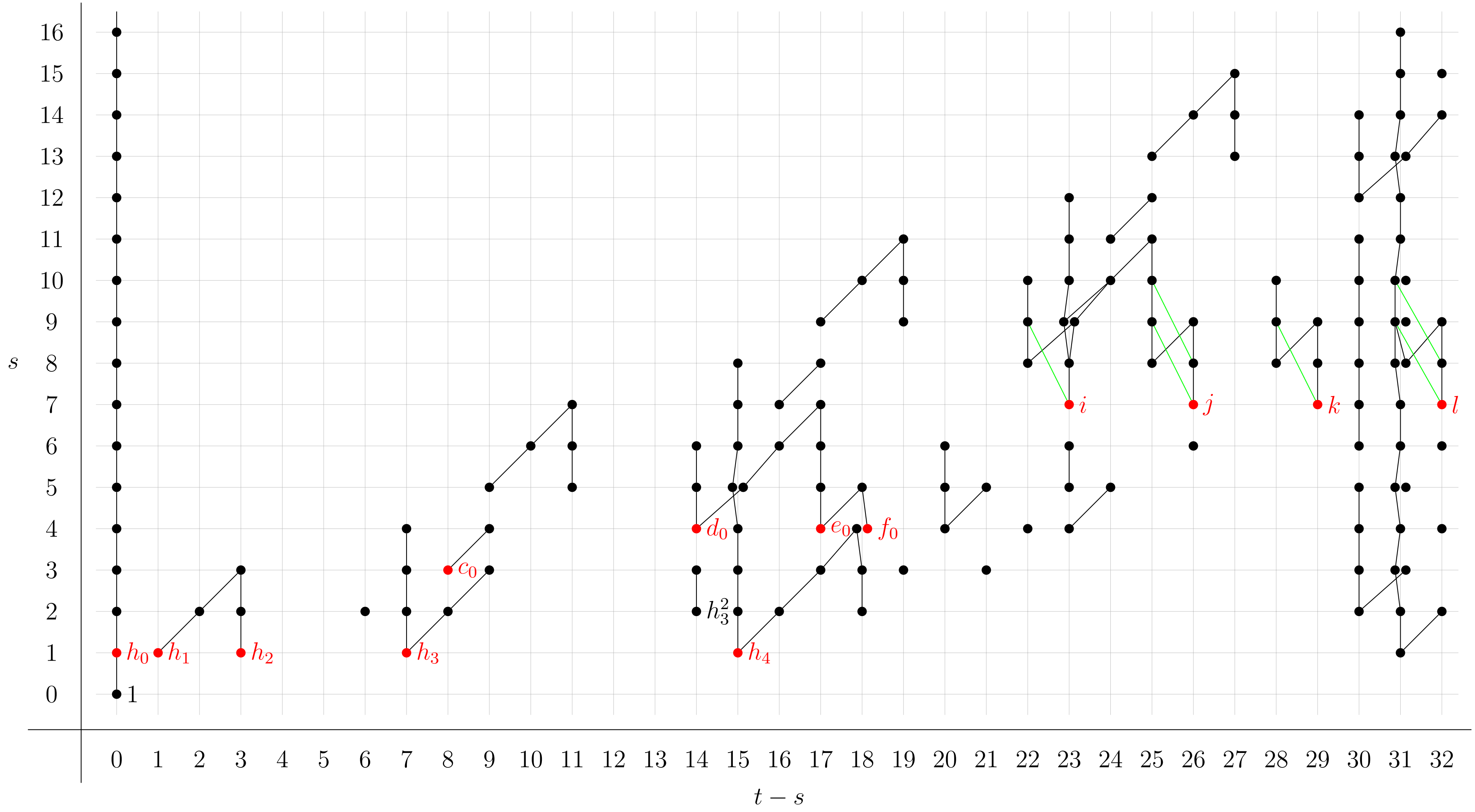

Figure 5.3: $\mathrm{Ext}^{s,t}_{\mathscr{A}_2}(\mathbb{F}_2, \mathbb{F}_2)$ for $t - s \leq 32$. The vertical and diagonal lines indicate multiplication by h_0 and h_1 respectively. Some of the algebra generators are shown in red, with their standard names. The d_2 differentials referenced in the proof of Lemma 5.3.2 are shown in green.

Lemma 5.3.2 ([12], Table 14.1 (9)). $d_2(f_0) \neq 0$.

Proof. First, note that $d_2(i)$ is nontrivial, since $h_4 i = 0$ and $h_0 h_3^2 i \neq 0$, so $0 = d_2(h_4 i) = h_0 h_3^2 i + h_4 d_2(i)$ implies that $d_2(i) \neq 0$. Now, $d_2(j) \neq 0$ since $h_0 d_2(j) = d_2(h_0 j) = d_2(h_2 i) = h_2 d_2(i) \neq 0$. An almost identical argument shows that $d_2(k), d_2(l) \neq 0$, and thus $d_2(h_0 l) = h_0 d_2(l) \neq 0$. Finally, we have $h_0 l = d_0 f_0$, so $0 \neq d_2(h_0 l) = d_2(d_0 f_0) = d_0 d_2(f_0)$. $\square$

[Probably need to move the theorem below but I'll leave it here for the moment]

Theorem 5.3.3 ([13], Cor 4.17). Let $f : Y \to Z$ be a map of connective spectra of finite type. Then there is a map

$$f_* : \{E_r(Y), d_r\}_r \to \{E_r(Z), d_r\}_r$$

of Adams spectral sequences, given at the E_2-level by the homomorphism

$$(f^*)^* : \mathrm{Ext}^{s,t}_{\mathscr{A}_2}(H^*(Y), \mathbb{F}_2) \to \mathrm{Ext}^{s,t}_{\mathscr{A}_2}(H^*(Z), \mathbb{F}_2)$$

induced by the $\mathscr{A}_2$-module homomorphism $f^* : H^*(Z) \to H^*(Y)$, with abutment the homomorphism

$$f_* : \pi_*(Y) \to \pi_*(Z).$$

Lemma 5.3.4 ([12], Table 14.9 (4)). Consider the cofibration

$$\mathbb{S}^7 \xrightarrow{\sigma} \mathbb{S} \xrightarrow{i} C_\sigma \xrightarrow{j} \mathbb{S}^8 \to \cdots .$$

Let $\overline{\overline{h_0^2 h_3}} \in E_2^{3,18}(C_\sigma)$ be the generator shown in Figure 5.3. Then $d_2(\overline{\overline{h_0^2 h_3}}) = \hat{i}(h_0 d_0)$, where $\hat{i} = (i^*)^* : \mathrm{Ext}^{s,t}(\mathbb{F}_2, \mathbb{F}_2) \to \mathrm{Ext}^{s,t}(C_\sigma, \mathbb{F}_2)$ is the map induced by $i^* : H^*(C_\sigma) \to H^*(\mathbb{S})$.

Proof.

[Some note about how we're calling $i(h_2)$ just h_2 because for anything in the range $t - s < 7$ i is injective]

We first show that $d_2(h_2 \overline{\overline{h_0^2 h_3}}) = d_2(\hat{i}(f_0))$. By Lemma 3.4.4, we have a long exact sequence

$$\cdots \leftarrow H^{n-1}(\mathbb{S}^8) \leftarrow H^n(\mathbb{S}) \xleftarrow{i^*} H^n(C_\sigma) \xleftarrow{j^*} H^n(\mathbb{S}^8) \leftarrow H^{n+1}(\mathbb{S}) \leftarrow \cdots .$$

However, any map $H^n(\mathbb{S}) \to H^{n-1}(\mathbb{S}^8)$ must be zero, so we get short exact sequences

$$0 \leftarrow H^n(\mathbb{S}) \xleftarrow{i^*} H^n(C_\sigma) \xleftarrow{j^*} H^n(\mathbb{S}^8) \leftarrow 0.$$

Taking a direct sum gives a short exact sequence

$$0 \leftarrow \mathbb{F}_2 \xleftarrow{i^*} H^*(C_\sigma) \xleftarrow{j^*} \mathbb{F}_2[8] \leftarrow 0.$$

From the short exact sequence above, we get[6] a long exact sequence

$$\begin{array}{c}\cdots \longrightarrow \mathrm{Ext}^{s,t}(\mathbb{F}_2, \mathbb{F}_2) \xrightarrow{i} \mathrm{Ext}^{s,t}(H^*(C_\sigma), \mathbb{F}_2) \xrightarrow{j} \mathrm{Ext}^{s,t-8}(\mathbb{F}_2, \mathbb{F}_2) \\ \hookrightarrow \mathrm{Ext}^{s+1,t}(\mathbb{F}_2, \mathbb{F}_2) \xrightarrow{i} \mathrm{Ext}^{s+1,t}(H^*(C_\sigma), \mathbb{F}_2) \xrightarrow{j} \mathrm{Ext}^{s+1,t-8}(\mathbb{F}_2, \mathbb{F}_2) \longrightarrow \cdots\end{array}$$

[6] This is a homological algebra fact, but maybe I should state it somewhere - trouble is I'd have to define Ext in terms of injective resolutions instead of projective ones...

Now, $f_0 \in \mathrm{Ext}^{4,22}(\mathbb{F}_2, \mathbb{F}_2)$; we consider the exact sequence

$$\mathrm{Ext}^{3,14}(\mathbb{F}_2, \mathbb{F}_2) \to \mathrm{Ext}^{4,22}(\mathbb{F}_2, \mathbb{F}_2) \xrightarrow{i} \mathrm{Ext}^{4,22}(H^*(C_\sigma), \mathbb{F}_2).$$

Figure 5.2 shows us that $\mathrm{Ext}^{3,14}(\mathbb{F}_2, \mathbb{F}_2) = 0$, so $\hat{i}$ is injective at this point, and thus $\hat{i}(f_0) \neq 0$. Similarly, $\mathrm{Ext}^{4,15}(\mathbb{F}_2, \mathbb{F}_2) = 0$ and $\mathrm{Ext}^{0,8}(\mathbb{F}_2, \mathbb{F}_2) = 0$, so $\hat{i}(h_0 f_0), \hat{i}(h_4) \neq 0$. Since $\hat{i}$ respects multiplication[7], $\hat{i}(h_0 f_0) = \hat{i}(h_0)\hat{i}(f_0) \neq 0$, so $\hat{i}(f_0)$ is equal to either $\overline{\overline{h_0^2 h_3}}$ or $\overline{\overline{h_0^2 h_3}} + h_2 \hat{i}(h_4)$. In either case, by linearity of d_2, $d_2(\hat{i}(f_0)) = d_2(\overline{\overline{h_0^2 h_3}})$ (since $d_2(h_2 \hat{i}(h_4)) = \hat{i}(d_2(h_2 h_4)) = 0$[8]).

Now, $\hat{i}(h_0^2 e_0) \neq 0$, since $\mathrm{Ext}^{5,15}(\mathbb{F}_2, \mathbb{F}_2) = 0$ (using the long exact sequence in Ext again). Thus, $h_2 d_2(\overline{\overline{h_0^2 h_3}}) = d_2(h_2 \overline{\overline{h_0^2 h_3}}) = d_2(i(f_0)) = i(d_2(f_0)) = i(h_0^2 e_0) \neq 0$, as required. □

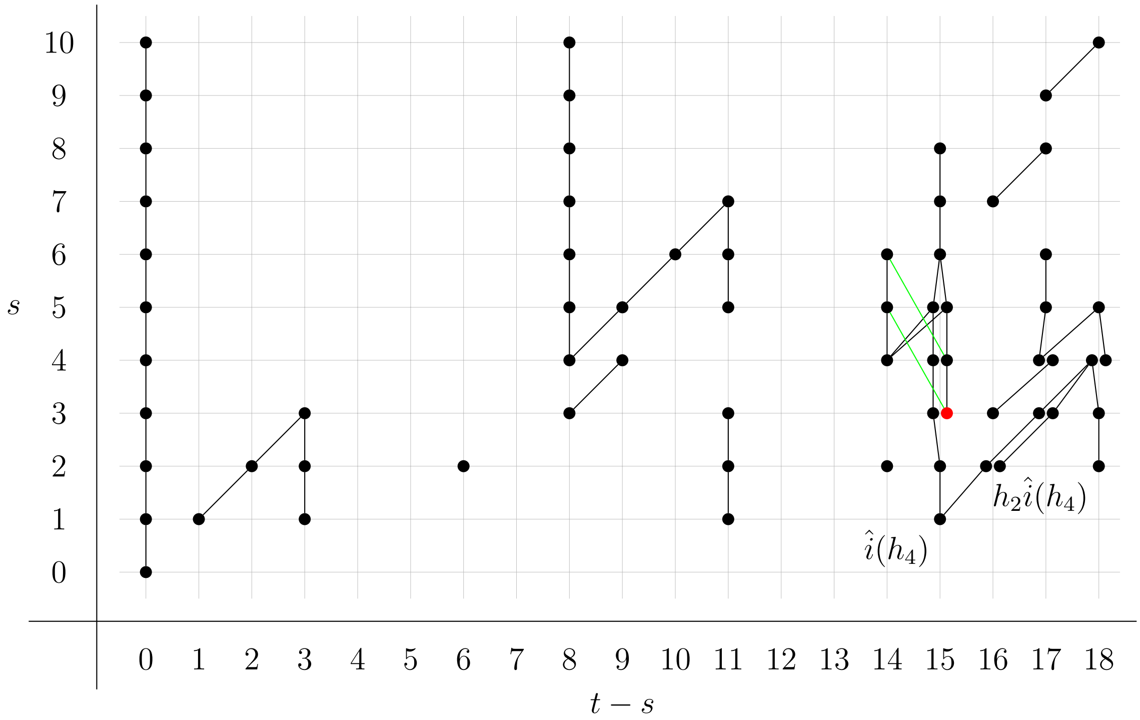

Figure 5.4: The E_2 page of the Adams spectral sequence for C_σ, in the range $t - s \leq 18$, with the generator $\overline{\overline{h_0^2 h_3}}$ shown in red, and two of the differentials shown in green.

Theorem 5.3.5 ([12], Table 14.2 (10)). *$d_3(h_0 h_4) = h_0 d_0$.*

Proof. From the cofibration

$$\mathbb{S}^7 \overset{\sigma}{\hookrightarrow} \mathbb{S} \xrightarrow{i} C_\sigma \xrightarrow{j} \mathbb{S}^8 \hookrightarrow \mathbb{S}^1 \to \cdots,$$

we get an exact sequence

$$\pi_7^S \xrightarrow{\sigma_*} \pi_{14}^S \xrightarrow{i_*} \pi_{14}(C_\sigma) \xrightarrow{j_*} \pi_6^S \to \pi_{13}^S,$$

[7] I've just drawn some diagrams, and I think that yes, $i(\alpha\beta) = i(\alpha)i(\beta)$. Essentially because the way you multiply in Ext involves writing down unique (up to chain homotopy) lifts that make certain squares commute, but both $i(\alpha\beta)$ and $i(\alpha)i(\beta)$ are lifts making the appropriate squares commute, so they must be chain homotopic and hence give rise to the same element in Ext.

[8] Explain.

by Theorem 3.4.3. Since these stable homotopy groups are all finite[9], this induces an exact sequence

$$(\pi_7^S)_2^\wedge \xrightarrow{\sigma_*} (\pi_{14}^S)_2^\wedge \xrightarrow{i_*} \pi_{14}(C_\sigma)_2^\wedge \xrightarrow{j_*} (\pi_6^S)_2^\wedge \xrightarrow{\sigma_*} (\pi_{13}^S)_2^\wedge = 0.$$

In $E_2(C_\sigma)$ we have $d_2(\overline{\overline{h_0^2h_3}}) = \hat{\imath}(h_0d_0)$ (by Lemma 5.3.4), so $\pi_{14}(C_\sigma)_2^\wedge$ has order dividing four. Let $\nu \in (\pi_3^S)_2^\wedge$ be a representative for h_2. Then $(\pi_6^S)_2^\wedge = \mathbb{Z}/2\mathbb{Z}\langle\nu^2\rangle$, and $\nu^2\sigma = 0$. By exactness, we see that j_* is surjective, so $(\pi_6^S)_2^\wedge \cong \pi_{14}(C_\sigma)_2^\wedge/\ker j_* = \pi_{14}(C_\sigma)_2^\wedge/\operatorname{im} i_*$. We know $\pi_{14}(C_\sigma)_2^\wedge$ has order dividing 4 and $(\pi_6^S)_2^\wedge$ has order 2, so $\operatorname{im} i_*$ has order dividing 2.

Now, $(\pi_7^S)_2^\wedge = \mathbb{Z}/16\mathbb{Z}\langle\sigma\rangle$, and $2\sigma^2 = 0$ by graded commutativity, so the first isomorphism theorem implies that $(\pi_{14}^S)_2^\wedge$ has order dividing four. Thus, h_0d_0 and $h_0^2d_0$ must be boundaries, and $d_3(h_0h_4) = h_0d_0$ is the only possibility. $\square$

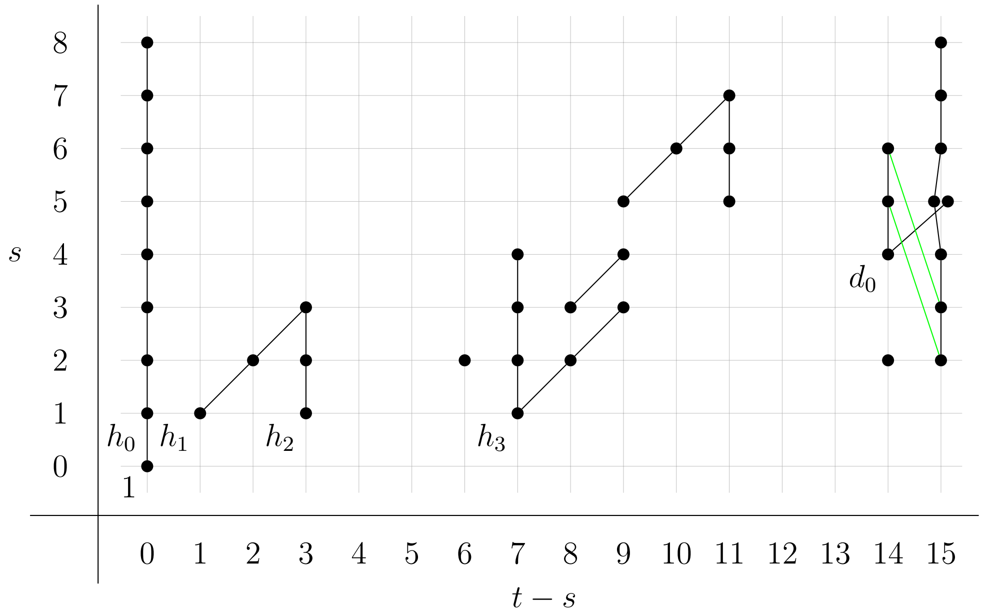

Figure 5.5: The E_3 page of the Adams spectral sequence for $\mathbb{S}$, in the range $t - s \leq 15$; the differentials are shown in green.

[9]A priori $\pi_{14}(C_\sigma)$ is only finitely generated, but from Figure 5.3 we see that its 2-completion is finite, so the group itself must be finite.

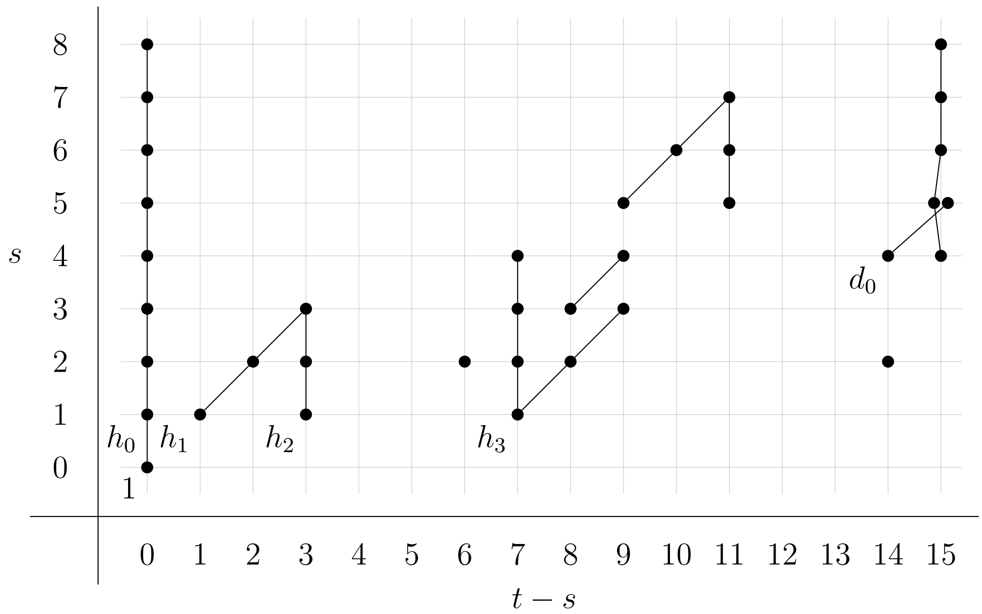

Figure 5.6: The E_4 page of the Adams spectral sequence for $\mathbb{S}$, in the range $t-s \leq 15$. There are no possible higher differentials, so this coincides with the E_∞ page for $t-s \leq 15$.

[1], [11], [13], [12].

A Algebra

A.1 Free resolutions

Definition 1.1.1. Let M, N be modules over a ring R. A *resolution* F of M is an exact sequence

$$\cdots \to F_2 \to F_1 \to F_0 \to M \to 0.$$

If in addition each F_i is a free R-module, then the resolution is called *free*.

Given a free resolution as above, applying $\mathrm{Hom}_R(-, N)$ gives us a chain complex

$$\cdots \leftarrow \mathrm{Hom}_R(F_2, N) \leftarrow \mathrm{Hom}_R(F_1, N) \leftarrow \mathrm{Hom}_R(F_0, N) \leftarrow \mathrm{Hom}_R(M, N) \leftarrow 0.$$

Dropping the term $\mathrm{Hom}_R(M, N)$ [why?] we get the sequence

$$\cdots \leftarrow \mathrm{Hom}_R(F_2, N) \leftarrow \mathrm{Hom}_R(F_1, N) \leftarrow \mathrm{Hom}_R(F_0, N) \leftarrow 0,$$

and we define $\mathrm{Ext}^n_R(M, N)$ to be the nth homology group of this chain complex.

[these do not depend on the choice of free resolution of M]

A free resolution is *minimal* if at each stage of its construction we choose the minimal number of free generators for F_i in each degree.

[The above definition is bad but I'm keeping it just for the moment.]

B Topology

All from [5] unless otherwise stated.

B.1 Suspension

Definition 2.1.1. Let X be a topological space. The *suspension* SX is the space $(X \times I)/\sim$, where $(x, 0) \sim (x', 0)$ and $(x, 1) \sim (x', 1)$ for all $x, x' \in X$.

Definition 2.1.2. Let X be a pointed topological space. The *reduced suspension* ΣX is the space $SX/\sim$, where $[x_0, t] \sim [x_0, t']$ for all $t, t' \in I$.

Given a map $f : X \to Y$, we can define $\Sigma f : \Sigma X \to \Sigma Y$ by $\Sigma f[(x, t)] = [(fx, t)]$. This makes Σ into a functor $\Sigma : \mathbf{Top} \to \mathbf{Top}$.

Remark 2.1.3. Σ is faithful, since for any maps $f, g : X \to Y$, if $\Sigma f = \Sigma g$ then in particular $[(fx, \frac{1}{2})] = [(gx, \frac{1}{2})]$, so $fx = gx$.

[below is reconstructed from [9]]

Given pointed maps $f, g : \Sigma X \to Z$, define

$$\begin{aligned} f \star g : \Sigma X &\to Z \\ [x, t] &\mapsto \begin{cases} f[x, 2t-1] & t \geq \frac{1}{2}, \\ g[x, 2t] & t \leq \frac{1}{2}. \end{cases} \end{aligned}$$

This is well defined, since both f and g are basepoint-preserving.

Remark 2.1.4. This defines a group structure on $[\Sigma X, Z]$, and thus $[\Sigma^i X, Z]$ is a group for all $i \geq 1$. For $i \geq 2$, these can be shown to be abelian, via the Eckmann-Hilton argument. The suspension map $[\Sigma X, Y] \to [\Sigma^2 X, \Sigma Y]$ is a homomorphism.[10]

Remark 2.1.5. The homotopy groups $\pi_i(Z)$ are a special case of the above construction, taking $X := S^{i-1}$.

- Loops; the adjunction $\Sigma \dashv \Omega$, where Ω is the loop functor.

[5], p395:

Remark 2.1.6. It follows that $\pi_{n+1}(X) \cong \pi_n(\Omega X)$. In particular, $\Omega K(G, n)$ is a $K(G, n-1)$.

- [5] 2.1 Ex 20 and 2.2 Ex 32: $\widetilde{H}_n(X) \cong \widetilde{H}_{n+1}(SX)$, where S is the (non-reduced) suspension. (MV?)
- Hatcher also says on p219 that $\widetilde{H}^n(X; R) \cong \widetilde{H}^{n+k}(\Sigma^k X; R)$, where Σ is reduced suspension.

B.2 Other basic constructions

Definition 2.2.1. Let $(X, x_0), (Y, y_0)$ be pointed topological spaces, and consider their product $X \times Y$. The subspaces $X \times \{y_0\} \cong X$ and $\{x_0\} \times Y \cong Y$ intersect at exactly one point, (x_0, y_0), and so can be identified with the wedge $X \vee Y$. We thus define the *smash product* $X \wedge Y := (X \times Y)/(X \vee Y)$, with the canonical basepoint (x_0, y_0).

Example 2.2.2. We have $S^n \wedge S^m \cong S^{n+m}$. [is this obvious?]

Remark 2.2.3. Note that $\Sigma X \cong X \wedge S^1$.

[10] Probably follows from the result for $\pi_*(Y)$ and induction on the cells of X, but I'll check this.

Remark 2.2.4. Observe that $X \wedge (Y \wedge Z) \cong (X \wedge Y) \wedge Z$. Combining this with the remarks above, we see that $\Sigma^k X \cong X \wedge S^k$.

Remark 2.2.5. Note that $\Sigma(X \vee Y) \cong \Sigma X \vee \Sigma Y$.

- The Eilenberg-MacLane space is $K(G, n)$, and it has the property that

$$\pi_i(K(G,n)) = \begin{cases} G & i = n, \\ 0 & i \neq n. \end{cases}$$

They're unique up to weak homotopy equivalence (i.e. if you have another one X, there's a map between them which descends to an isomorphism on homotopy groups). They can be taken to be CW complexes.

Definition 2.2.6. Let X, Y be topological spaces, where X has a basepoint x_0. Then the *reduced product* $X \times_{\mathrm{red}} Y := (X \times Y)/(x_0 \times Y)$.

Definition 2.2.7. Let $f : X \to Y$ be a map. The *mapping cylinder* M_f is defined by $((X \times I) \sqcup Y)/\sim$, where $(x, 1) \sim f(x)$ for all $x \in X$. If $(X, x_0), (Y, y_0)$ are pointed spaces, the *reduced mapping cylinder* is the quotient $M_f/\sim$, where $[x_0, t] \sim [x_0, t']$ for all $t \in I$.

Remark 2.2.8. The mapping cylinder deformation retracts onto Y via $h : M_f \times I \to M_f$; $([x, t], s) \mapsto [x, t + s(1 - t)]$.

Definition 2.2.9. Let $f : X \to Y$ be a map. The *mapping cone*[11] C_f is defined to be $Y \sqcup_f CX := (Y \sqcup CX)/(f(x) \sim [x, 1])$.

Relative Künneth Theorem:

Theorem 2.2.10 ([5]). For CW pairs $(X, A), (Y, B)$, the cross product homomorphism $H^*(X, A; R) \otimes_R H^*(Y, B; R) \to H^*(X \times Y, A \times Y \cup X \times B; R)$ is an isomorphism of rings if $H^k(Y, B)$ is a finitely generated free R-module for each k.

In particular, for pointed spaces $(X, x_0), (Y, y_0)$, we have an isomorphism

$$\bigoplus_{i+j=n} H^i(X, x_0; R) \otimes_R H^j(Y, y_0; R) \to H^n(X \times Y, X \vee Y; R).$$

Or, in other words,

$$\bigoplus_{i+j=n} \widetilde{H}^i(X; R) \otimes_R \widetilde{H}^j(Y; R) \to \widetilde{H}^n(X \wedge Y; R).$$

Setting $Y = S^1$, we get an isomorphism

$$\widetilde{H}^{n-1}(X; R) \to \widetilde{H}^n(\Sigma X; R).$$

B.3 Cell complexes

Definition 2.3.1. Let X be a cell complex, $A \subseteq X$ a subcomplex. Then the quotient X/A has a cell complex structure, with cells the cells of $X \setminus A$ along with a basepoint (the image of A in X).

[11] Why does Hatcher not insist this guy is reduced, like he does with the mapping cylinders?

DEFINITION 2.3.2. Let $f : X \to Y$ be a map between CW complexes. Then f is *cellular* if $f(X_{(n)}) \subseteq Y_{(n)}$ for all n, where $X_{(n)}$ is the n-skeleton of X.

Cellular approximation theorem:

THEOREM 2.3.3 ([5], Thm 4.8). Let $f : X \to Y$ be a map of CW complexes. Then f is homotopic to a cellular map.

LEMMA 2.3.4 ([5], Prop 0.16). Let $A \subseteq X$ be CW complexes. Then the pair (X, A) has the *homotopy extension property*; that is, for any map $f : X \to Y$ and homotopy $h : A \times I \to Y$ such that $h(a, 0) = f|_A$, there is a homotopy $\widetilde{h} : X \times I \to Y$ extending h.

- The product of cell complexes is a cell complex (maybe only if one of them is finite?)
- The smash product of (pointed?) cell complexes is a cell complex (maybe only if one is them is finite?) [[5] says "the smash product $X \wedge Y$ is a cell complex if X and Y are cell complexes with x_0 and y_0 0-cells, assuming that we give $X \times Y$ the cell-complex topology rather than the product topology in cases where these two topologies differ".]
- For a CW complex X, $SX \simeq \Sigma X$.
- The reduced suspension of a pointed cell complex (X, x_0) is another pointed cell complex ΣX with basepoint x_0 and an n-cell for each non-basepoint $n-1$ cell e_α^{n-1} of X.

DEFINITION 2.3.5. Let X is a topological space. A *CW approximation* to X is a CW complex Z equipped with a weak homotopy equivalence $f : Z \to X$.

THEOREM 2.3.6 ([5], Prop 4.13). Every space X has a CW approximation $f : Z \to X$.

- In particular, $\Omega K(G, n)$ has a CW approximation $Z \to \Omega K(G, n)$, and since $\Omega K(G, n)$ is a $K(G, n-1)$, so is Z.

Any finite CW complex is compact.

PROPOSITION 2.3.7 ([5], A.1). A compact subspace of a CW complex is contained in a finite subcomplex.

C Notes to self

C.1 Vague problems and questions....

C.1.1 ...that probably don't matter

- On p588 of [6], he says "every CW spectrum is equivalent to a suspension spectrum". Does he actually mean that, or does he mean 'equivalent to the suspension of a spectrum'? The former seems way too strong, although in fairness I still don't know what an equivalence of spectra actually *is*.
- On p586 of [6], Hatcher says "If X is of finite type then for each i there is an n such that X_n contains all the i-cells of X. It follows that $H_i(X; G) = H_i(X_n; G)$ for all sufficiently large n, and the same is true for cohomology." But from the way he set up H_* and H^* earlier, shouldn't this be $H_i(X; G) = H_{i+n}(X_n; G)$? Because $H_i(X; G) = \varinjlim H_{i+n}(X_n)$, and he talks about things stabilising in the next sentence, so shouldn't the stable point be at some H_{i+n}?

- I write $\mathscr{A}_2$ where Hatcher writes $\mathscr{A}$. We mean the same thing, right...?

C.1.2 ...that probably do matter

- I am definitely being told some lies about what the spectral sequence actually converges to. There's a strong implication/actual statement(!!) that at each i it's supposed to be a filtration of π_i^S modulo odd torsion, but I think this isn't true. I think it's actually the 2-completion of π_i^S. That coincides with the p-primary part for finite abelian groups, but for π_0^S it's supposed to be $\mathbb{Z}_2$ (i.e. the 2-adic integers), not $\mathbb{Z}$. I believe. Maybe get a source for this. Some people say it's the localisation at 2?? But I think that's also a lie.

- The Leibniz rule is $d_r(xy) = d_r(x)y \pm xd_r(y)$ (can't remember the sign). But anything I'm using that rule on is some generator of an $\mathbb{F}_2$, right? So the sign shouldn't matter. But then, shouldn't the Yoneda product be graded commutative (and thus commutative, because again, in the target signs don't matter)? So why does [13] have some comment (in Cor 6.5) about how the Yoneda product is commutative "in [some] range"??

- On p592 of [6], he says that "for spectra X of finite type [the more general] definition of an $\mathscr{A}_2$-module structure on $H^*(X)$ agrees with the definition using the usual $\mathscr{A}_2$-module structure on the cohomology of spaces and the identification of $H^*(X)$ with the inverse limit $\varprojlim H^{*+n}(X_n)$". Um? Sure, we have that each $H^{i+n}(X_n)$ stabilises eventually, but is Hatcher saying $H^{*+n}(X_n)$ stabilises? Like, as an $\mathscr{A}_2$-module? And if not, what's going on here? Because inverse limits don't commute with infinite direct sums - they're not biproducts anymore, they're coproducts and there's no reason limits should commute with them.

- There's something weird going on with products. So, things are ok in **Top**, because we have the ordinary product of two spaces, which is a categorical product. But with CW complexes, supposedly sometimes the product topology differs from the 'cell complex topology'? But, regardless, we're supposed to be working with pointed things - so in $\mathbf{Top}_*$, the pointed coproduct is the wedge sum, and the pointed product is just the normal product $X \times Y$ with the basepoint (x_0, y_0) (it's not the smash product). But what about in spectra? No one ever seems to talk about products of spectra, but for example a collection of maps $X \to \mathbb{K}(G, n_i)$ should correspond to a single map $X \to \prod_i \mathbb{K}(G, n_i)$, whatever that last object is.

  The plot thickens. From the nLab: "[some smash product] is non-canonically equivalent to a product of EM-spectra (hence a wedge sum of EM-spectra in the finite case)". ???????

- Question: so I have this multiplicative structure on the entire spectral sequence, right? And on the second page it's just the Yoneda product. But say I compute, I don't know, $\alpha_5^2 = h_0 h_2$ on the E_2 page. Since none of those guys are touched by differentials, is that computation still valid on the E_3 page? And on higher pages?

- I'm a bit suspicious of the proof of Theorem 3.3.8, because the proof is more complicated in [6]. Maybe raise this.

- Is anything lost/does anything become false if I replace every instance of the word 'free' with 'projective'?

C.2 To do

Now:

- Rewrite Sections 4.1 and 4.2 and fix the grading.
- Show the action of $\mathscr{A}_2$ on $H^*(X)$ is actually a module action.
- Figure out the whole wedge/product of EM spectra nonsense and why K_0 has finite type.
- Figure out why applying H^* to the sequence $K_0 \to K_1 \to \cdots$ makes it exact.
- Show that the $H^*(K_s)$'s are free $\mathscr{A}_2$-modules.
- Show that there is a natural map $[\mathbb{S}, K_s] \to \mathrm{Hom}^0_{\mathscr{A}_2}(H^*(K_s), H^*(\mathbb{S}))$, and that it's an isomorphism (Yoneda?).
- Find out how much of this you should be proving.
- Figure out how to state Theorem 4.3.1 without lying about 2-completion.
- Explain the Yoneda product at all coherently, and compute some examples.
- Prove that π_*^S is a graded commutative ring.
- State this whole 'replacing a map by an inclusion' business somewhere early.
- Finish the proof of Theorem 3.4.3.
- Fix Section 4.5.1.

Eventually:

- Be consistent with either cell complex or CW complex.
- Be consistent with $\mathbb{F}_2$ or $\mathbb{Z}/2\mathbb{Z}$ (don't use $\mathbb{Z}_2$, that's really bad).
- Specialise the Adams spectral sequence (i.e. set $Y = \mathbb{S}$).
- Remember that you have to hand in the tex file, so for the love of god change anything stupid that's hidden in the pdf.
- Sometimes I say π_*^S or ${}_{(2)}\pi_*^S$ (localised at 2?) instead of its completion at 2 or whatever. So make sure it's correct.
- Stick to a convention on suspension/cone/homotopy numbering. I.e. Does a homotopy start at 0 or 1? Does a suspension go from -1 to 1 with the space in the middle at 0, or 0 to 1 with the space at 1/2? Do cones go from 0 to 1, and if so, make sure when they include into suspensions they do so consistently.
- I want to at some point compute $h_0h_1h_3$ or something like that, just to show that sometimes a product can be zero even if it lands in a nonzero group (i.e. the stable groups aren't all cyclic). The trouble is by the time you get to π_8^S the free resolution is an awful mess. Maybe I can try and wrangle with the sseq program to get it to show me kernels and the like?
- Ease notation by not writing G in e.g. $H^*(X; G)$. Say we're taking coefficients in $\mathbb{F}_2$, but actually a lot of the time it doesn't matter so maybe note that.

- Have any sort of consistency in using or not using brackets (e.g. $\pi_t X_s$ v.s. $\pi_t(X_s)$).
- When I say 'spectrum' at any point after defining CW spectra I mean 'CW spectrum'. And I basically always mean 'connective CW spectrum of finite type' too.
- Make clear in the spectral sequence diagrams which towers terminate and which don't (e.g. the ones at $t-s=0$ obviously don't but unfortunately the spectral sequence package can't draw the entire infinite chain).
- Connect 1 and h_1 (if possible without messing up the labels).

C.3 Other notes

- READ IF YOUR CALCULATIONS AREN'T WORKING: You are working modulo 2!!!
- If you have a bunch of maps between graded modules/algebras, they're graded homomorphisms. So they preserve degree.
- All (co)homology is supposed to be reduced.
- Signs don't matter with the Leibniz rule either!! You are working modulo 2!!!!!!!!
- Remember, once you know that $d_2(h_4) = h_0 h_3^2$, you know h_4 *doesn't survive to the third page*. So, for example, $d_3(h_0 h_4) \neq h_0 d_3(h_4)$ because h_4 doesn't exist anymore. That's why $d_3(h_0 h_4)$ can be nonzero.
- As previously mentioned, we are working modulo 2!! What this also implies is that if anything is hit by any sort of differential, or has any nonzero differential coming out of it, it's completely killed by the next page. Because the summands are just a bunch of $\mathbb{F}_2$'s (so you don't need to worry about 'how much' of something is killed, it all is).
- Sometimes Hatcher says that you can replace any map of CW complexes by an inclusion. I think the point here is that if you have a map $f: X \to Y$, Remark B.2.8 says that M_f deformation retracts onto Y. So if you only care about X and Y up to homotopy equivalence, you can replace Y by M_f and then X definitely includes into M_f.
- Where it's ambiguous, I'm marking things I definitely need by ! and things I think I may not need by ?.
- In literature, $A_p^\wedge$ is the p-adic completion of A. Sometimes I'll write this as ${}_pA$ because of some stupid notational decisions I made earlier.
- The 'abutment' of a spectral sequence apparently means the thing it converges to (i.e. if E_∞ computes the associated graded of some H^*, the abutment of $\{E\}$ is H^* (not its associated graded)).
- [12] has some n_m notation where n_m is supposed to be the mth generator in row n. This is a bit arbitrary when there are two generators in the same row and column; I don't know how he counts them, but he's using the `ext` program, whereas I'm using sseq. Unless there's some Canonical Ordering, there's no reason why these different programs written by different people would use the same convention. In particular, even though [12] says $\overline{\overline{h_0^2 h_3}} = 3_4$, I'm pretty sure it is the one on the right (i.e. the one I would label 3_5).

Sources I've used: [11], [13], [7], [6], [5], [12], [8], [10]

Sources I probably won't use: [4], [1], [2], [3], [9] (I think the construction I need is in Hatcher)

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