

Stable Homotopy Groups of Spheres [DRAFT]

Contents

1	Introduction	2
2	The Steenrod algebra	3
3	Spectra may not be your friends, but I can introduce you	4
3.1	Categorical nonsense	4
3.2	Definitions and examples	4
3.3	Homology and cohomology	5
4	The Adams spectral sequence	7
4.1	Spectral sequences	7
4.2	Exact couples	8
4.3	The Adams spectral sequence	9
5	Calculating stable homotopy groups	10
5.1	The E_2 term	10
5.2	Nontrivial differentials	11
A	Algebra	11
A.1	Free resolutions	11
B	Topology	11
B.1	Suspension	11
B.2	Other basic constructions	12
B.3	Cell complexes	13
C	Notes to self	13
C.1	Vague problems and Questions	13
C.2	To do	14
C.3	Other notes	14

1 Introduction

- Define homotopy groups
- Freudenthal's suspension theorem: if $\pi_i(X) = 0$ for $i \leq k$ (i.e. X is k -connected) then the map

$$\begin{aligned} \pi_n(X) &\rightarrow \pi_{n+1}(\Sigma X) \\ [\gamma : S^n \rightarrow X] &\mapsto [\Sigma \gamma : \Sigma S^n = S^{n+1} \rightarrow \Sigma X] \end{aligned}$$

is an isomorphism for $n \leq 2k$ and surjective for $n = 2k + 1$

- This implies $\pi_{n+k}(S^n)$ depends only on k for $n \geq k + 2$
- (Obviously be careful with basepoints above)
- Suppose X is k -connected. Then, for $k \geq 0$, $0 = \pi_k(X) \cong \pi_{k+1}(\Sigma X)$, so whenever a space is k -connected its suspension is $k + 1$ -connected.
- As you take suspensions, then, your successive bounds are $n \leq 2k$, $n + 1 \leq 2k + 2 \implies n \leq 2k + 1$, $n \leq 2k + 2$, etc ... so the sequence $\pi_n(X) \rightarrow \pi_{n+1}(\Sigma X) \rightarrow \dots$ will eventually stabilise.
- Thus, if you take the colimit of that direct system, it'll just equal the stable value, with the higher legs just being the inverse isomorphisms.
- [11], Cor 1.9 [not 100% convinced of how this follows, but believing it for now]: if X is a CW complex of dimension d and Y a $(k - 1)$ -connected space, then the suspension homomorphism¹ $[X, Y] \rightarrow [\Sigma X, \Sigma Y]$ is bijective if $d < 2k - 1$ and surjective if $d = 2k - 1$.

Miscellaneous facts I might need later:

- Cohomology [possibly only of pointed² CW complexes] is representable³, and its representing object is the Eilenberg-MacLane space. i.e. $H^n(-; G) \cong \text{Hom}(-, K(G, n))$.
- \mathcal{A}_2 is generated as an algebra by elements Sq^{2^k} ([5], Prop 4L.8).
- The map $\mathcal{A}_2 \rightarrow \tilde{H}^*(K(\mathbb{Z}/2\mathbb{Z}, n); \mathbb{Z}/2\mathbb{Z})$, $Sq^I \mapsto Sq^I(\iota_n)$ is an isomorphism from the degree d part of \mathcal{A}_2 onto $H^{n+d}(K(\mathbb{Z}/2\mathbb{Z}, n); \mathbb{Z}/2\mathbb{Z})$ for $d \geq n$. In particular, the admissible monomials Sq^I form an additive basis for \mathcal{A}_2 . Thus, \mathcal{A}_2 is exactly the algebra of all $\mathbb{Z}/2\mathbb{Z}$ cohomology operations that are stable, commuting with suspension ([6], Cor 5.38).
- “Stable homotopy groups are a homology theory” (whatever that means)
- Hurewicz theorem: for any path-connected space X and $n > 0$ there exists a group homomorphism $h_* : \pi_n(X) \rightarrow H_n(X)$. For $n = 1$ this induces an isomorphism $\pi_1^{\text{ab}}(X) \cong H_1(X)$. For $n \geq 2$, if X is $(n - 1)$ -connected then $\tilde{H}_i(X) = 0$ for all $i < n$, and the map $h_* : \pi_n(X) \rightarrow H_n(X)$ is an isomorphism.

[11], [4], [5]

¹Hang on, the what?? X isn't the suspension of anything, why on earth would this be a group?

²What's the relevance of the ‘pointedness’ when you're only taking cohomology?? See C.

³As a set, or is this some sort of enriched thing? If it's enriched, is that over **Ab** or **Rng**? [5] says on p394 that there is a natural group structure on $\text{Hom}(X, K(G, n))$ such that the natural isomorphism $\text{Hom}(X, K(G, n)) \rightarrow H^n(X; G)$ is in fact an isomorphism of abelian groups. So, it's over **Ab**? N.B: There's a lot of talk about ‘reduced cohomology theories’, so a good next step would be to figure out what those are - if they only involve groups and not rings, maybe the cup product on H^* is not relevant here.

2 The Steenrod algebra

The following is from [5] 4L.

- There are maps $Sq^i : H^n(-; \mathbb{Z}/2\mathbb{Z}) \rightarrow H^{n+i}(-; \mathbb{Z}/2\mathbb{Z})$ for each i , and they satisfy the following properties:
 1. $Sq_X^i(f^*(\alpha)) = f^*(Sq_Y^i(\alpha))$ for $f : X \rightarrow Y$ (i.e. Sq^i is a natural transformation).
 2. $Sq_X^i(\alpha + \beta) = Sq_X^i(\alpha) + Sq_X^i(\beta)$ (i.e. Sq_X^i respects the group operation for all X).
 3. $Sq^i(\alpha \smile \beta) = \sum_{0 \leq j \leq i} (Sq^j(\alpha) \smile Sq^{i-j}(\beta))$ (the Cartan formula)
 4. $Sq^i(\sigma(\alpha)) = \sigma(Sq^i(\alpha))$ where $\sigma : H^n(X; \mathbb{Z}/2\mathbb{Z}) \rightarrow H^{n+1}(\Sigma X; \mathbb{Z}/2\mathbb{Z})$ is the “suspension isomorphism given by reduced cross product with a generator of $H^1(S^1; \mathbb{Z}/2\mathbb{Z})$ ”. [see: [5], p219. N.B: I think this “relative cross product” theory is required – you can argue that there is an isomorphism via MV, but this point says that it’s this specific one. Maybe they’re the same, but Hatcher doesn’t say that anywhere and there could be many isomorphisms.]
 5. $Sq^i(\alpha) = \alpha^2$ if $i = \deg(\alpha)$ and $Sq^i(\alpha) = 0$ if $i > \deg(\alpha)$.
 6. $Sq^0 = \text{id}$.
 7. Sq^1 is the “ $\mathbb{Z}/2\mathbb{Z}$ Bockstein homomorphism β associated with the coefficient sequence $0 \rightarrow \mathbb{Z}/2\mathbb{Z} \rightarrow \mathbb{Z}/4\mathbb{Z} \rightarrow \mathbb{Z}/2\mathbb{Z} \rightarrow 0$ ”.
- Define $Sq := Sq^0 + Sq^1 + \dots$. Then $Sq(\alpha \smile \beta) = Sq(\alpha) \smile Sq(\beta)$ (since $(Sq(\alpha \smile \beta))_n = \sum_i Sq^i(\alpha) \smile Sq^{n-i}(\beta) = (Sq(\alpha) \smile Sq(\beta))_n$). Thus, Sq is a ring homomorphism.
- Adem relations:

$$Sq^a Sq^b = \sum_j \binom{b-j-1}{a-2j} Sq^{a+b-j} Sq^j \quad \text{if } a < 2b,$$

where $\binom{m}{n}$ is zero if m or n is negative, or $m < n$, and $\binom{m}{0} = 1$ for $m \geq 0$.

DEFINITION 2.0.1. The *Steenrod algebra* \mathcal{A}_2 is the algebra over $\mathbb{Z}/2\mathbb{Z}$ that is the quotient of the algebra of polynomials in the noncommuting variables Sq^1, Sq^2, \dots by the two-sided ideal generated by the Adem relations. Thus, for every space X , $H^*(X; \mathbb{Z}/2\mathbb{Z})$ is a module over \mathcal{A}_2 .

- \mathcal{A}_2 is graded, and its elements of degree k are those that map $H^n(X; \mathbb{Z}/2\mathbb{Z})$ to $H^{n+k}(X; \mathbb{Z}/2\mathbb{Z})$ for all n . [Presumably you’ve fixed a space X while you’re doing all this?]

DEFINITION 2.0.2. Write Sq^I for the monomial $Sq^{i_1} Sq^{i_2} \dots Sq^{i_n}$. Then Sq^I is *admissible* if $i_j \geq 2i_{j+1}$ for all $0 \leq j < n$. Note the admissible monomials are exactly those to which no Adem relations can be applied.

Thus, \mathcal{A}_2 is generated as a $\mathbb{Z}/2\mathbb{Z}$ module by admissible monomials.

[1], [10], [11], [7], [5], [2]

3 Spectra may not be your friends, but I can introduce you

3.1 Categorical nonsense

- [11]: There is a category \mathcal{H} of finite based CW complexes, with $\text{Hom}(X, Y) =: [X, Y]$ the set of homotopy classes of base-point preserving maps $X \rightarrow Y$.
- There is a category $\mathbf{St}(\mathcal{H})$ of finite based CW complexes, with $\text{Hom}(X, Y) =: \{X, Y\}$ the set $\text{colim}_i [\Sigma^i X, \Sigma^i Y]$ [it's just a colimit of sets, and **Set** is cocomplete, so we should be fine. [11] says it's a group⁴] [Also, how do these guys compose?]
- There is a functor $\mathcal{H} \rightarrow \mathbf{St}(\mathcal{H})$. [11] doesn't say what this is but it's presumably the one that is the identity on objects and sends $[f : X \rightarrow Y] \in [\Sigma^0 X, \Sigma^0 Y]$ to whatever it gets sent to in $\{X, Y\}$ using the universal property of the colimit. Uniqueness makes it functorial, etc.
- We have a fully faithful functor $\mathbf{St}(\mathcal{H}) \rightarrow \mathbf{St}(\mathcal{H})$ given by the suspension on objects, and the unique isomorphism $\{X, Y\} \rightarrow \{\Sigma X, \Sigma Y\}$ on maps (such an isomorphism exists, since both of those things are colimits for $[\Sigma^i X, \Sigma^i Y]$ - one of the sequences is cut off at the beginning, but it doesn't matter because both reach the stable value (see above discussion and [11] 1.9), aka the colimit).
- It's not an equivalence, because not every object is isomorphic to a suspension (e.g. anything not connected, since suspensions always connected [?])
- We can formally adjoin desuspensions $\Sigma^{-n} X$ for all n [does this mean just putting the objects there and defining $\text{Hom}(Y, \Sigma^{-n} X) := \text{Hom}(\Sigma^n Y, X)$ and $\text{Hom}(\Sigma^{-n} X, Y) := \text{Hom}(X, \Sigma^n Y)$?], but this category does not have weak colimits (i.e. colimits w/o uniqueness property). [why does it not, and why do we even want that?]
- We instead consider formal sequences of desuspensions $X_0 \rightarrow \Sigma^{-1} X_1 \rightarrow \dots$, or sequences (X_n) and maps $\Sigma X_n \rightarrow X_{n+1}$, i.e. spectra. [and this fixes the problem?]

3.2 Definitions and examples

Below follows [6], Section 5.2.

[Maybe I could also look at [5] p454 onwards?]

DEFINITION 3.2.1. A *spectrum* is a collection of pointed topological spaces $\{X_n\}_{n \in \mathbb{N}}$, together with basepoint-preserving maps $\sigma_n : \Sigma X_n \rightarrow X_{n+1}$.

EXAMPLE 3.2.2. Let X be a topological space. The *suspension spectrum* of X , denoted by $\Sigma^\infty X$, has $X_n = \Sigma^n X$ and $\sigma_n = \text{id} : \Sigma X_n \rightarrow X_{n+1}$.

We write $\mathbb{S} := \Sigma^\infty S^0$, and call \mathbb{S} the *sphere spectrum*.

EXAMPLE 3.2.3. The *Eilenberg-MacLane spectrum* has X_n a CW complex $K(G, n)$ and $\sigma_n : \Sigma K(G, n) \rightarrow K(G, n+1)$ is the adjoint of the CW approximation $K(G, n) \rightarrow \Omega K(G, n+1)$.

⁴The colimit is equal to the stable value (which exists, by the corollary). After Σ^2 , these guys are all groups, so the colimit also has a group structure inherited from whatever $[\Sigma^k X, \Sigma^k Y]$ it's equal to. N.B: Remarks about cocompleteness of **Set** are misleading because that doesn't actually matter - any sequence that stabilises in any category will have a filtered colimit equal to that stable value, you don't need any extra conditions.

[N.B. the important point of the above is not that it's 'an' EM space rather than 'the' (they're all homotopy equivalent). It's that a) we can definitely construct one that's a CW complex, and b) even though $\Omega(\text{a CW complex})$ is not necessarily a CW complex, we can make it one via CW approximation.]

DEFINITION 3.2.4. Let $X = \{X_n\}$ be a spectrum. We define $\pi_i(X) = \text{colim}_n \pi_{i+n}(X_n)$, where the map $\pi_{i+n}(X_n) \rightarrow \pi_{i+n+1}(X_{n+1})$ is given by the composition

$$\pi_{i+n}(X_n) \xrightarrow{\Sigma} \pi_{i+n+1}(\Sigma X_n) \xrightarrow{(\sigma_n)_*} \pi_{i+n+1}(X_{n+1}).$$

EXAMPLE 3.2.5. If X is a topological space, then $\pi_i(\Sigma^\infty X) = \pi_i^S(X)$, the i th stable homotopy group of X .

DEFINITION 3.2.6. A CW spectrum is a spectrum X consisting of CW complexes X_n with the maps $\Sigma X_n \hookrightarrow X_{n+1}$ inclusions of subcomplexes.

DEFINITION 3.2.7. Let X be a CW spectrum. Then the k -cells of X are the equivalence classes of non-basepoint $k + n$ -cells in X_n , where two cells are equivalent if one is an m -fold suspension of the other.

DEFINITION 3.2.8. A CW spectrum X is *connective* if it has no cells below a given dimension. Further, X is *finite* if it has only finitely many cells, and *of finite type* if it has only finitely many cells in each dimension.

EXAMPLE 3.2.9. If X is a finite (resp. finite type) CW complex, then Σ^∞ is a finite (resp. finite type) CW spectrum. In particular, \mathbb{S}^n is a finite CW spectrum with a unique cell in dimension 1.

3.3 Homology and cohomology

[From Hatcher: "the inclusions $\Sigma X_n \hookrightarrow X_{n+1}$ induce inclusions $C_*(X_n; G) \hookrightarrow C_*(X_{n+1}; G)$ with a dimension shift to account for the suspension". Below is my vague explanation of what I understand this to mean.

$C_i(X_n; G)$ is the free abelian group on maps $\Delta^i \rightarrow X_n$. I claim $\Sigma \Delta^i \cong \Delta^{i+1}$. If this is true, it gives a map

$$\begin{aligned} C_i(X_n; G) &\rightarrow C_{i+1}(\Sigma X_n; G) \\ f &\mapsto \Sigma f. \end{aligned}$$

This is an injection, by Remark B.1.3. We also have an injection $C_{i+1}(\Sigma X_n; G) \rightarrow C_{i+1}(X_{n+1}; G)$ induced by the structure map σ_n , so we get an injection $C_i(X_n; G) \hookrightarrow C_{i+1}(X_{n+1}; G)$, which indeed has a dimension shift.

Some issues:

- The way it's phrased, it seems to be that this is a morphism of chain complexes - i.e. these maps commute with the ∂ s. Why would they?]

[Define H^* and H_* of a spectrum]

DEFINITION 3.3.1. Let $X = \{X_n\}$ be a CW spectrum. A *subspectrum* X' of X is a sequence of subcomplexes $\{X'_n \subseteq X_n\}$ satisfying $\Sigma X'_n \subseteq X'_{n+1}$. The subspectrum X' is *cofinal* if, for each n and each cell e_α^i of X_n , the cell $\Sigma^k e_\alpha^i$ belongs to X'_{n+k} for all sufficiently large k .

⁵Assuming it is actually a cell, see note in B.3.

[N.B. If $\Sigma^k e_\alpha^i$ belongs to X'_{n+k} then $\Sigma^{k+1} e_\alpha^i$ belongs to $\Sigma X'_{n+k} \hookrightarrow X'_{n+k+1}$ (I think!!), so if it happens once it'll happen for all time after that. Thus, if X', X'' are cofinal spectra of X with $\Sigma^k e_\alpha^i$ a cell of X'_{n+k} and $\Sigma^l e_\alpha^i$ a cell of X''_{n+l} ($l \geq k$) then $\Sigma^l e_\alpha^i$ is a cell of X'_{n+l} and therefore of $X'_{n+l} \cap X''_{n+l}$. In other words, the intersection of two cofinal spectra is a cofinal spectrum.]

DEFINITION 3.3.2. Let X, Y be CW spectra. A *strict map* $f : X \rightarrow Y$ is a sequence of cellular maps $f_n : X_n \rightarrow Y_n$ such that the diagram below commutes.

$$\begin{array}{ccc} \Sigma X_n & \xrightarrow{\sigma_n} & X_{n+1} \\ \sigma f_n \downarrow & & \downarrow f_{n+1} \\ \Sigma Y_n & \xrightarrow{\sigma_n} & Y_{n+1} \end{array}$$

[this induces maps $\pi_i(X) \rightarrow \pi_i(Y)$, $H^*(Y) \rightarrow H^*(X)$, $H_*(X) \rightarrow H_*(Y)$.]

DEFINITION 3.3.3. A *map* of CW spectra $f : X \rightarrow Y$ is an equivalence class of strict maps $f' : X' \rightarrow Y$ with X' a subspectrum of X , where two strict maps $f' : X' \rightarrow Y$ and $f'' : X'' \rightarrow Y$ are equivalent if they agree on some common cofinal spectrum.

[this also induces maps $\pi_i(X) \rightarrow \pi_i(Y)$, $H^*(Y) \rightarrow H^*(X)$, $H_*(X) \rightarrow H_*(Y)$.]

[check composition is well defined]

[working definition of equivalence below:]

DEFINITION 3.3.4. Two spectra X, Y are *equivalent* if there are maps $f : X \rightarrow Y$ and $g : Y \rightarrow X$ such that $fg = \text{id}_Y$ and $gf = \text{id}_X$.

[A spectrum is always equivalent to the suspension of some other spectrum]

DEFINITION 3.3.5. A *homotopy* of maps between spectra is a map $X \times I \rightarrow Y$, where $X \times I$ is the spectrum with $(X \times I)_n = X_n \times_{\text{red}} I$.

Note that $\Sigma(X_n \times_{\text{red}} I) = \Sigma X_n \times_{\text{red}} I$. The set of homotopy classes of maps $X \rightarrow Y$ is denoted by $[X, Y]$.

[Claim: (probably very obvious): $[\Sigma^\infty S^t, Z] = \pi_t(Z)$.]

[1] says on p171 that “[$\Sigma X, Z$] is obviously a group, because in ΣX we have a spare suspension coordinate out in front to manipulate. And for the same reason, $[\Sigma^2 X, Z]$ is an abelian group. But now we can give $[X, Y]$ the structure of an abelian group, because $[X, Y]$ is in 1-1 correspondence with $[\Sigma^2 X, \Sigma^2 Y]$ and we pull back the group structure on that. So now our sets of morphisms $[X, Y]$ are abelian groups, and it's easy to see that composition is bilinear”.

Various claims:

- The stuff about normal CW complexes and their groups of maps (i.e. [Remark B.1.5](#)) translates to tell me the appropriate things about spectra and their groups of maps.
- After checking a lot of things, I can eventually conclude that $[X, Y]$ is an abelian group for spectra X, Y .]

[The suspension map $[X, Y] \rightarrow [\Sigma X, \Sigma Y]$ is an isomorphism of groups]

[[Lemma B.3.3](#) and [Theorem B.3.2](#) are both true for CW spectra too]

[Whitehead's theorem: a map between CW spectra that induces isomorphisms on all homotopy groups is a homotopy equivalence.]

[Prop: If a CW spectrum X is n -connected in the sense that $\pi_i(X) = 0$ for $i \leq n$, then X is homotopy equivalent to a CW spectrum with no cells of dimension $\leq n$]

DEFINITION 3.3.6. Let $X = \{X_n\}, Y = \{Y_n\}$ be spectra. Then their *wedge sum*⁶ is $X \vee Y := \{X_n \vee Y_n\}$. Note that Remark B.2.5 gives us an inclusion $\Sigma(X_n \vee Y_n) \hookrightarrow X_{n+1} \vee Y_{n+1}$.

[The rest of the section:

- Def: closed subspectrum
- Def: quotient of spectra
- Fact: for spectra X, Y and a subspectrum $A \subseteq X$ we have exact sequences

$$\begin{aligned} \cdots \rightarrow [\Sigma X, Y] &\rightarrow [\Sigma A, Y] \rightarrow [X/A, Y] \rightarrow [X, Y] \rightarrow [A, Y] \\ [Y, A] &\rightarrow [Y, X] \rightarrow [Y, X/A] \rightarrow [Y, \Sigma A] \rightarrow [Y, \Sigma X] \rightarrow \cdots \end{aligned}$$

]

[1], [10], [11], [7], [2], [6]

4 The Adams spectral sequence

4.1 Spectral sequences

Some notes from [11] (or maybe not)

How about some notes from [9], C2:

DEFINITION 4.1.1. A *differential bigraded module* E over a ring R is a collection of R -modules $\{E^{p,q}\}$, $p, q \in \mathbb{Z}$, together with a map $d : E^{p,q} \rightarrow E^{p+s, q-s+1}$ for each p, q and some fixed $s \in \mathbb{Z}$, satisfying $d^2 = 0$.

We can take the homology of (E, d) :

$$H^{p,q}(E^{*,*}, d) = \ker(d : E^{p,q} \rightarrow E^{p+s, q-s+1}) / \operatorname{im}(d : E^{p-s, q+s-1} \rightarrow E^{p,q}).$$

DEFINITION 4.1.2. A *spectral sequence* (of *cohomological type*)⁷ is a collection of differential bigraded R -modules $\{E_r^{*,*}, d_r\}$, $r \in \mathbb{N}$, with the differentials d_r of bidegree $(r, 1-r)$. These satisfy the further condition that for all p, q, r , $E_{r+1}^{p,q} \cong H^{p,q}(E_r^{*,*}, d_r)$.

We will sometimes write $d_r^{p,q}$ for the differential $d_r : E^{p,q} \rightarrow E^{p+r, q-s+1}$.

Consider the term $E_2^{*,*}$. Define

$$Z_2^{p,q} := \ker d_2^{p,q} \quad \text{and} \quad B_2^{p,q} := \operatorname{im} d_2^{p-2, q+1}.$$

The condition $d^2 = 0$ implies that $B_2^{p,q} \subseteq Z_2^{p,q} \subseteq E_2^{p,q}$, and by definition we have $E_3^{p,q} \cong Z_2^{p,q} / B_2^{p,q}$.

Now, write

$$Z_3^{p,q} := \ker d_3^{p,q} \quad \text{and} \quad B_3^{p,q} := \operatorname{im} d_3^{p-3, q+2}.$$

⁶This definition comes from [12], but I'm a little suspicious because the nLab page says a wedge sum should always identify the basepoints, but the basepoint of a spectra surely isn't given by just a basepoint from each X_n ? That's the only thing being identified here.

⁷I think(?) I actually want one of homological type. I'll check this.

Since $Z_3^{p,q} \subseteq E_3^{p,q}$, it can be written as $\overline{Z}_3^{p,q}/B_2^{p,q}$ for some $\overline{Z}_3^{p,q} \subseteq Z_2^{p,q}$. Similarly, $B_3^{p,q} \cong \overline{B}_3^{p,q}/B_2^{p,q}$ for some $\overline{B}_3^{p,q} \subseteq Z_2^{p,q}$. Thus,

$$E_4^{p,q} \cong Z_3^{p,q}/B_3^{p,q} \cong \frac{\overline{Z}_2^{p,q}/B_2^{p,q}}{\overline{B}_3^{p,q}/B_2^{p,q}} \cong \overline{Z}_3^{p,q}/\overline{B}_3^{p,q}.$$

Iterating this process, we present the spectral sequence as an infinite tower of submodules of $E_2^{p,q}$:

$$B_2^{p,q} \subseteq \overline{B}_3^{p,q} \subseteq \cdots \subseteq \overline{B}_n^{p,q} \subseteq \cdots \subseteq \overline{Z}_n^{p,q} \subseteq \cdots \subseteq \overline{Z}_3^{p,q} \subseteq Z_2^{p,q} \subseteq E_2^{p,q},$$

with the property that $E_{n+1}^{p,q} \cong \overline{Z}_n^{p,q}/\overline{B}_n^{p,q}$. The differential $d_{n+1}^{p,q}$ can be taken as a map $\overline{Z}_n^{p,q}/\overline{B}_n^{p,q} \rightarrow \overline{Z}_n^{p,q}/\overline{B}_n^{p,q}$ with kernel $\overline{Z}_{n+1}^{p,q}/\overline{B}_n^{p,q}$ and image $\overline{B}_{n+1}^{p,q}$. The short exact sequence induced by d_{n+1} ,

$$0 \rightarrow \overline{Z}_{n+1}^{p,q}/\overline{B}_n^{p,q} \rightarrow \overline{Z}_n^{p,q}/\overline{B}_n^{p,q} \xrightarrow{d_{n+1}^{p,q}} \overline{B}_{n+1}^{p,q}/\overline{B}_n^{p,q} \rightarrow 0,$$

gives rise to isomorphisms $\overline{Z}_n^{p,q}/\overline{Z}_{n+1}^{p,q} \cong \overline{B}_{n+1}^{p,q}/\overline{B}_n^{p,q}$ for all n . Conversely, a tower of submodules of E_2 , together with a set of isomorphisms, gives rise to a spectral sequence.

DEFINITION 4.1.3. An element of $E_2^{p,q}$ *survives to the r th stage* if lies in $\overline{Z}_r^{p,q}$, having been in the kernel of the previous $r-2$ differentials, and is *bounded by the r th stage* if it lies in $\overline{B}_r^{p,q}$. The bigraded module $E_r^{*,*}$ is called the E_r -term of the spectral sequence.

We define

$$Z_\infty^{p,q} := \bigcap_n \overline{Z}_n^{p,q}, \quad B_\infty^{p,q} := \bigcup_n \overline{B}_n^{p,q}.$$

From the tower of inclusions, we see that $B_\infty^{p,q} \subseteq Z_\infty^{p,q}$, so we define $E_\infty^{p,q} := Z_\infty^{p,q}/B_\infty^{p,q}$.

DEFINITION 4.1.4. A spectral sequence *collapses at the N th term* if the differentials $d_r^{p,q} = 0$ for $r \geq N$.

From the short exact sequence

$$0 \rightarrow \overline{Z}_r^{p,q}/\overline{B}_{r-1}^{p,q} \rightarrow \overline{Z}_{r-1}^{p,q}/\overline{B}_{r-1}^{p,q} \xrightarrow{d_r^{p,q}} \overline{B}_r^{p,q}/\overline{B}_{r-1}^{p,q} \rightarrow 0,$$

the condition $d_r^{p,q}$ forces $\overline{Z}_r^{p,q} = \overline{Z}_{r-1}^{p,q}$ and $\overline{B}_r^{p,q} = \overline{B}_{r-1}^{p,q}$. The tower of submodules becomes

$$B_2^{p,q} \subseteq \overline{B}_3^{p,q} \subseteq \cdots \subseteq \overline{B}_{N-1}^{p,q} = B_N^{p,q} = \cdots = B_\infty^{p,q} \subseteq Z_\infty^{p,q} = \cdots = \overline{Z}_N^{p,q} = \overline{Z}_{N-1}^{p,q} \subseteq \cdots \subseteq \overline{Z}_3^{p,q} \subseteq Z_2^{p,q} \subseteq E_2^{p,q}.$$

Thus, $E_\infty^{p,q} = E_N^{p,q}$.

4.2 Exact couples

(Following [9], C2)

DEFINITION 4.2.1. Let D, E be R -modules, and let $i : D \rightarrow D$, $j : D \rightarrow E$, $k : E \rightarrow D$ be module homomorphisms. We call $\mathcal{C} = \{D, E, i, j, k\}$ an *exact couple* if the diagram below is exact.

$$\begin{array}{ccc} D & \xrightarrow{i} & D \\ & \nwarrow k & \nearrow j \\ & E & \end{array}$$

Let $d := jk$, and define the following:

$$\begin{aligned} E' &:= H(E, d) = \ker d / \operatorname{im} d \\ D' &:= i(D) = \ker j \\ i' &:= i|_{i(D)} : D' \rightarrow D' \\ j' &:= i(x) \mapsto j(x) + dE : D' \rightarrow E' \\ k' &:= (e + dE) \mapsto k(e) : E' \rightarrow D' \end{aligned}$$

We call $\mathcal{C}' = \{D', E', i', j', k'\}$ the *derived couple* of \mathcal{C} .

PROPOSITION 4.2.2 ([9], Prop 2.7). If $\mathcal{C} = \{D, E, i, j, k\}$ is an exact couple, then \mathcal{C}' is also an exact couple.

THEOREM 4.2.3 ([9], Thm 2.8). Suppose $D^{*,*} = \{D^{p,q}\}$ and $E^{*,*} = \{E^{p,q}\}$ are bigraded modules equipped with homomorphisms i of bidegree $(-1, 1)$, j of bidegree $(0, 0)$, and k of bidegree $(1, 0)$, such that $\{D^{*,*}, E^{*,*}, i, j, k\}$ is an exact couple. Then these data determine a spectral sequence $\{E_r, d_r\}$ for $r \in \mathbb{Z}_+$ of cohomological type, with $E_r = (E^{*,*})^{(r-1)}$, the $(r-1)$ st derived module of $E^{*,*}$ and $d_r = j^{(r)} \circ k^{(r)}$.

A bigraded exact couple may be displayed in the following diagram, known as a *staircase diagram*:

$$\begin{array}{ccccccc} & & \vdots & & \vdots & & \\ & & \downarrow i & & \downarrow i & & \\ \cdots & \xrightarrow{k} & D^{p+2,q-1} & \xrightarrow{j} & E^{p+2,q-1} & \xrightarrow{k} & D^{p+3,q-1} \xrightarrow{j} \cdots \\ & & \downarrow i & & \downarrow i & & \\ \cdots & \xrightarrow{k} & D^{p+1,q} & \xrightarrow{j} & E^{p+1,q} & \xrightarrow{k} & D^{p+2,q} \xrightarrow{j} \cdots \\ & & \downarrow i & & \downarrow i & & \\ \cdots & \xrightarrow{k} & D^{p,q+1} & \xrightarrow{j} & E^{p,q+1} & \xrightarrow{k} & D^{p+1,q+1} \xrightarrow{j} \cdots \\ & & \downarrow i & & \downarrow i & & \\ & & \vdots & & \vdots & & \end{array}$$

[Check this is actually the same staircase diagram that Hatcher is talking about]

4.3 The Adams spectral sequence

Things I need before I can set it up (according to Hatcher [6]):

Let X be a CW spectrum of finite type.

- Def: $H^*(X)$.
- Fact: $H^*(X)$ is finitely generated.
- Fact: $H^*(X)$ is an \mathcal{A}_2 -module. [We know that's true for a topological space]
- Fact: We can pick generators α_i for $H^*(X)$ as an \mathcal{A}_2 -module such that there are at most finitely many in each $H^n(X)$.

- Fact: There α_i determine a map $X \rightarrow K_0$, where K_0 is a wedge of EM spectra, and K_0 has finite type.
- Fact: We can replace that map with an inclusion.
- Fact: A quotient of connective spectra of finite type is again a connective spectrum of finite type.
- Prop: [6], 5.46.
- Def: The functor $\pi_t^Y(Z) = [\Sigma^t Y, Z]$ for a finite spectrum Y .
- Def: A cofibration. [see : [5], p398.]
- Fact: If Y is a finite spectrum and Z is a connective spectrum of finite type, then $\pi_t^Y(Z)$ is finitely generated.
- Fact: I can do this. I have all the necessary skills to pull this off.
- Fact: I'm going to stop listing things I need to do and start actually doing them.

[9], [1], [10], [2], [6], [11], [3]

5 Calculating stable homotopy groups

5.1 The E_2 term

[Some miscellaneous and vague rambling:

[6] Prop 5.47 says that there is a spectral sequence $\{E_r, d_r\}$ such that $E_2^{s,t} = \text{Ext}_{\mathcal{A}_2}(H^*(\mathbb{S}), H^*(\mathbb{S}))$, and $\{E_r, d_r\} \implies \pi_*^S$ modulo any odd torsion.

At the bottom of p592 of [6], it's mentioned that for a spectrum X of finite type, $H^i(X) \cong \lim_{\leftarrow n} H^{i+n}(X_n)$ as \mathcal{A}_2 -modules. Thus, since \mathbb{S} is of finite type (it is actually finite), we have

$$\begin{aligned} H^i(\mathbb{S}; \mathbb{Z}/2\mathbb{Z}) &= \lim_{\leftarrow n} H^{i+n}(S^n; \mathbb{Z}/2\mathbb{Z}) \\ &= \begin{cases} \mathbb{Z}/2\mathbb{Z} & i = 0, \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

(since they're all just either 0 or $\mathbb{Z}/2\mathbb{Z}$, at least eventually). So, $H^*(\mathbb{S}) = \mathbb{Z}/2\mathbb{Z}$ in degree zero and nothing else.

Lemma 5.49 says that if I have a minimal free resolution

$$\cdots \rightarrow F_2 \rightarrow F_1 \rightarrow F_0 \rightarrow \mathbb{F}_2 \rightarrow 0,$$

where the F_i are \mathcal{A}_2 -modules, then $\text{Ext}_{\mathcal{A}_2}^{s,t}(\mathbb{F}_2, \mathbb{F}_2) = \text{Hom}_{\mathcal{A}_2}^t(F_s, \mathbb{F}_2)$, where the left hand side is the group of \mathcal{A}_2 -module homomorphisms $F_s \rightarrow \mathbb{F}_2$ which lower the degree by t . Now, since \mathbb{F}_2 just has stuff in degree 0, the only things that can be sent to $1 \in \mathbb{F}_2$ are the things in degree t , so for every generator for F_s in degree t , there's an \mathbb{F}_2 's worth of such homs.]

[Note: there is a multiplicative structure on the E_2 page of the spectral sequence, which Hatcher says is there and then proceeds to never ever mention again. So I'll have to find another source for this bit. It should correspond to the composition product on π_*^S , i.e. given $[f] \in \pi_i^S$, $[g] \in \pi_j^S$, we have $[f][g] = [g \circ \Sigma^j f] \in \pi_{i+j}^S$.]

[1], [10], [11], [6].

5.2 Nontrivial differentials

[The point here is that (I claim) all differentials interacting with $E^{s,t}$ for $t - s = 14$ are trivial, and thus computing the stable homotopy groups is purely mechanical, because everything that appears on this part of the E_2 page has to survive to E_∞ . Thus, the ‘ambiguity’ at $t - s = 14$ is just the fact that this is the first time you need to actually compute differentials.]

For $t - s < 14$ there are only actually the ones emanating from $E^{1,2}$ and the d_2 differential starting at $E^{2,10}$ to worry about, because all the others either enter or leave 0.]

[1], [10], [11].

A Algebra

A.1 Free resolutions

DEFINITION 1.1.1. Let M, N be modules over a ring R . A *resolution* F of M is an exact sequence

$$\cdots \rightarrow F_2 \rightarrow F_1 \rightarrow F_0 \rightarrow M \rightarrow 0.$$

If in addition each F_i is a free R -module, then the resolution is called *free*.

Given a free resolution as above, applying $\text{Hom}_R(-, N)$ gives us a chain complex

$$\cdots \leftarrow \text{Hom}_R(F_2, N) \leftarrow \text{Hom}_R(F_1, N) \leftarrow \text{Hom}_R(F_0, N) \leftarrow \text{Hom}_R(M, N) \leftarrow 0.$$

Dropping the term $\text{Hom}_R(M, N)$ [why?] we get the sequence

$$\cdots \leftarrow \text{Hom}_R(F_2, N) \leftarrow \text{Hom}_R(F_1, N) \leftarrow \text{Hom}_R(F_0, N) \leftarrow 0,$$

and we define $\text{Ext}_R^n(M, N)$ to be the n th homology group of this chain complex.

[these do not depend on the choice of free resolution of M]

A free resolution is *minimal* if at each stage of its construction we choose the minimal number of free generators for F_i in each degree.

[The above definition is stupid but I’m keeping it just for the moment.]

B Topology

All from [5] unless otherwise stated.

B.1 Suspension

DEFINITION 2.1.1. Let X be a topological space. The *suspension* SX is the space $(X \times I) / \sim$, where $(x, 0) \sim (x', 0)$ and $(x, 1) \sim (x', 1)$ for all $x, x' \in X$.

DEFINITION 2.1.2. Let X be a pointed topological space. The *reduced suspension* ΣX is the space SX / \sim , where $[x_0, t] \sim [x_0, t']$ for all $t, t' \in I$.

Given a map $f : X \rightarrow Y$, we can define $\Sigma f : \Sigma X \rightarrow \Sigma Y$ by $\Sigma f[(x, t)] = [(fx, t)]$. This makes Σ into a functor $\Sigma : \mathbf{Top} \rightarrow \mathbf{Top}$.

REMARK 2.1.3. Σ is faithful, since for any maps $f, g : X \rightarrow Y$, if $\Sigma f = \Sigma g$ then in particular $[(fx, \frac{1}{2})] = [(gx, \frac{1}{2})]$, so $fx = gx$.

[below is reconstructed from [8]]

Given pointed maps $f, g : \Sigma X \rightarrow Z$, define

$$f \star g : \Sigma X \rightarrow Z$$

$$[x, t] \mapsto \begin{cases} f[x, 2t - 1] & t \geq \frac{1}{2}, \\ g[x, 2t] & t \leq \frac{1}{2}. \end{cases}$$

This is well defined, since both f and g are basepoint-preserving.

REMARK 2.1.4. This defines a group structure on $[\Sigma X, Z]$, and thus $[\Sigma^i X, Z]$ is a group for all $i \geq 1$. For $i \geq 2$, these can be shown to be abelian, via the Eckmann-Hilton argument.

REMARK 2.1.5. The homotopy groups $\pi_i(Z)$ are a special case of the above construction, taking $X := S^{i-1}$.

- Loops; the adjunction $\Sigma \dashv \Omega$, where Ω is the loop functor.

[5], p395:

REMARK 2.1.6. It follows that $\pi_{n+1}(X) \cong \pi_n(\Omega X)$. In particular, $\Omega K(G, n)$ is a $K(G, n-1)$.

- [5] 2.1 Ex 20 and 2.2 Ex 32: $\tilde{H}_n(X) \cong \tilde{H}_{n+1}(SX)$, where S is the (non-reduced) suspension. (MV?)
- Hatcher also says on p219 that $\tilde{H}^n(X; R) \cong \tilde{H}^{n+k}(\Sigma^k X; R)$, where Σ is reduced suspension.

B.2 Other basic constructions

DEFINITION 2.2.1. Let $(X, x_0), (Y, y_0)$ be pointed topological spaces, and consider their product $X \times Y$. The subspaces $X \times \{y_0\} \cong X$ and $\{x_0\} \times Y \cong Y$ intersect at exactly one point, (x_0, y_0) , and so can be identified with the wedge $X \vee Y$. We thus define the *smash product* $X \wedge Y := (X \times Y)/(X \vee Y)$, with the canonical basepoint (x_0, y_0) .

EXAMPLE 2.2.2. We have $S^n \wedge S^m \cong S^{n+m}$. [is this obvious?]

REMARK 2.2.3. Note that $\Sigma X \cong X \wedge S^1$.

REMARK 2.2.4. Observe that $X \wedge (Y \wedge Z) \cong (X \wedge Y) \wedge Z$. Combining this with the remarks above, we see that $\Sigma^k X = X \wedge S^k$.

REMARK 2.2.5. Note that $\Sigma(X \vee Y) \cong \Sigma X \vee \Sigma Y$.

- ‘Reduced mapping cylinder’?
- Mapping cones?
- The Eilenberg-MacLane space is $K(G, n)$, and it has the property that

$$\pi_i(K(G, n)) = \begin{cases} G & i = n, \\ 0 & i \neq n. \end{cases}$$

They’re unique up to weak homotopy equivalence (i.e. if you have another one X , there’s a map between them which descends to an isomorphism on homotopy groups). They can be taken to be CW complexes.

DEFINITION 2.2.6. Let X, Y be topological spaces, where X has a basepoint x_0 . Then the *reduced product* $X \times_{\text{red}} Y := (X \times Y)/(x_0 \times Y)$.

B.3 Cell complexes

DEFINITION 2.3.1. Let $f : X \rightarrow Y$ be a map between CW complexes. Then f is *cellular* if $f(X_{(n)}) \subseteq Y_{(n)}$ for all n , where $X_{(n)}$ is the n -skeleton of X .

Cellular approximation theorem:

THEOREM 2.3.2 ([5], Thm 4.8). Let $f : X \rightarrow Y$ be a map of CW complexes. Then f is homotopic to a cellular map.

LEMMA 2.3.3 ([5], Prop 0.16). Let $A \subseteq X$ be CW complexes. Then the pair (X, A) has the *homotopy extension property*; that is, for any map $f : X \rightarrow Y$ and homotopy $h : A \times I \rightarrow Y$ such that $h(a, 0) = f|_A$, there is a homotopy $\tilde{h} : X \times I \rightarrow Y$ extending h .

- The product of cell complexes is a cell complex (maybe only if one of them is finite?)
- The smash product of (pointed?) cell complexes is a cell complex (maybe only if one is finite?) [5] says “the smash product $X \wedge Y$ is a cell complex if X and Y are cell complexes with x_0 and y_0 0-cells, assuming that we give $X \times Y$ the cell-complex topology rather than the product topology in cases where these two topologies differ”.]
- Quotient of a CW complex by a subcomplex is a CW complex, where the quotient map is cellular
- For a CW complex X , $SX \simeq \Sigma X$.
- The suspension of a pointed cell complex (X, x_0) is another pointed cell complex ΣX with basepoint x_0 and an n -cell for each non-basepoint $n - 1$ cell e_α^{n-1} of X .

DEFINITION 2.3.4. Let X is a topological space. A *CW approximation* to X is a CW complex Z equipped with a weak homotopy equivalence $f : Z \rightarrow X$.

THEOREM 2.3.5 ([5], Prop 4.13). Every space X has a CW approximation $f : Z \rightarrow X$.

- In particular, $\Omega K(G, n)$ has a CW approximation $Z \rightarrow \Omega K(G, n)$, and since $\Omega K(G, n)$ is a $K(G, n - 1)$, so is Z .

C Notes to self

C.1 Vague problems and Questions

- Is ‘pointed’ (co)homology just reduced (co)homology? I’ve noticed ‘pointed things’ ($\Sigma, \Omega, \wedge, \dots$) seem to happen to/in reduced (co)homology, and ‘unpointed things’ (S, \times, \dots) happen to/in normal (co)homology. I want to do pointed things.
- What does Hatcher mean when he says two spectra are ‘equivalent’?
- On p588 of [6], he says “every CW spectrum is equivalent to a suspension spectrum”. Does he actually mean that, or does he mean ‘equivalent to the suspension of a spectrum’? The former seems way too strong, although in fairness I still don’t know what an equivalence of spectra actually *is*.
- On p592 of [6], he says that “for spectra X of finite type [the more general] definition of an \mathcal{A}_2 -module structure on $H^*(X)$ agrees with the definition using the usual \mathcal{A}_2 -module structure on the cohomology of spaces and the identification of $H^*(X)$ with the inverse limit $\lim_{\leftarrow} H^{*+n}(X_n)$ ”. Absolutely everything relevant to spheres in the construction of the Adams spectral sequence seems to only use spectra of finite type, so why does Hatcher

say on p585 that we can't just take the inverse limit because it doesn't work for the 'more general spectra' used when constructing the Adams spectral sequence? Is that just because I could take the role of \mathbb{S} and give it to some non-finite-type spectrum (actually, it cannot be this!! Hatcher specifically says the spectrum Y has to be finite), or is there something I'm missing? Can I just take limits and be done with it?

- Suppose I have the E_2 page for the appropriate spectral sequence. That in principle only gives me the 'dots' and possibly the differentials if I've been clever. But if I just want to calculate, say, π_3^S , I get \mathbb{Z}_2 summands at $E_2^{1,4}$, $E_2^{2,5}$, and $E_2^{3,6}$. So, that says I have a filtration

$$0 \subseteq \mathbb{Z}_2 \subseteq F^{2,5} \subseteq F^{1,4} = \pi_3^S,$$

and $F^{2,5}/\mathbb{Z}_2 \cong \mathbb{Z}_2$, $F^{1,4}/F^{2,5} \cong \mathbb{Z}_2$. So π_3^S has order 8 and a filtration by \mathbb{Z}_2 's, but that still doesn't tell me whether it's $\mathbb{Z}_8, \mathbb{Z}_4 \times \mathbb{Z}_2$ or \mathbb{Z}_2^3 ? Thm 5.47 of [6] tells me that $F^{s,t}$ is the image of the map $\pi_t(\mathbb{S}_s) \rightarrow \pi_3^S$, so do I have to go back to the constructing and figure out what that map actually is, or is there something algebraic I can do?⁸

C.2 To do

Now:

- The definition of Ext in A.1 has only one superscript, but the Ext in the Adams spectral sequence has a bigrading - figure out where the extra index came from and write it down so I don't forget it.
- Read about fibration and cofibration sequences of CW complexes.
- Find out what a (co)homology theory is.
- Find out what the deal with reduced cohomology and being stable under suspension is.

Eventually:

- Be consistent with either cell complex or CW complex (eventually)
- Be consistent with \mathbb{F}_2 , \mathbb{Z}_2 , or $\mathbb{Z}/2\mathbb{Z}$ (I'm trying to use the latter, but old habits die hard).

C.3 Other notes

- READ IF YOUR CALCULATIONS AREN'T WORKING: You are working modulo 2!!!
- There are two ways I can play the chapter on spectra. The first is to constantly draw analogues with CW complexes, carry over all the theorems and prove them in a similar way, really emphasising that these are the 'stable' versions of CW complexes and we should definitely think about them that way. The other is really stupid I just realised so I'll just do the first thing I said.
- If you have a bunch of maps between graded modules/algebras, they're graded homomorphisms. So they preserve degree.

References

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⁸I think this related to something [11] calls the Yoneda product on p47-50 and 61-62 - this is a multiplicative structure on E_2 and should tell me something about the additive structure of π_*^S .

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