Stable Homotopy Groups of Spheres [DRAFT]

Contents

Key:

To do (likely straightforward)

To do (likely difficult)

Problems

1 Introduction

- Define homotopy groups
- Freudenthal's suspension theorem ([11], Thm 1.1.4): if $\pi_i(X) = 0$ for $i \leq k$ (i.e. X is k-connected) then the map

$$\begin{aligned} \pi_n(X) &\to \pi_{n+1}(\Sigma X) \\ [\gamma : S^n \to X] &\mapsto [\Sigma\gamma : \Sigma S^n = S^{n+1} \to \Sigma X] \end{aligned}$$

  is an isomorphism for $n \leq 2k$ and surjective for $n = 2k + 1$
- This implies $\pi_{n+k}(S^n)$ depends only on k for $n \geq k + 2$
- (Obviously be careful with basepoints above)
- Suppose X is k-connected. Then, for $k \geq 0$, $0 = \pi_k(X) \cong \pi_{k+1}(\Sigma X)$, so whenever a space is k-connected its suspension is $k+1$-connected.
- As you take suspensions, then, your successive bounds are $n \leq 2k$, $n + 1 \leq 2k + 2 \implies n \leq 2k + 1$, $n \leq 2k + 2$, etc ... so the sequence $\pi_n(X) \to \pi_{n+1}(\Sigma X) \to \cdots$ will eventually stabilise.
- Thus, if you take the colimit of that direct system, it'll just equal the stable value, with the higher legs just being the inverse isomorphisms.
- ?[13], Cor 1.9 [not 100% convinced of how this follows, but believing it for now]: if X is a CW complex of dimension d and Y a $(k-1)$-connected space, then the suspension homomorphism[1] $[X, Y] \to [\Sigma X, \Sigma Y]$ is bijective if $d < 2k-1$ and surjective if $d = 2k-1$.
- [11], Thm 1.1.8: $\pi_{n+k}(S^n)$ is finite for $k > 0$ except when $n = 2m$, $k = 2m - 1$.
- Immediate corollary: π_i^S is finite for $i > 0$.

Miscellaneous facts I might need later:

- Cohomology [possibly only of pointed CW complexes] is representable[2], and its representing object is the Eilenberg-MacLane space. i.e. $H^n(-; G) \cong \mathrm{Hom}(-, K(G, n))$.
- $\mathscr{A}_2$ is generated as an algebra by elements Sq^{2^k} ([5], Prop 4L.8).
- The map $\mathscr{A}_2 \to \widetilde{H}^*(K(\mathbb{Z}/2\mathbb{Z}, n); \mathbb{Z}/2\mathbb{Z})$, $Sq^I \mapsto Sq^I(\iota_n)$ is an isomorphism from the degree d part of $\mathscr{A}_2$ onto $H^{n+d}(K(\mathbb{Z}/2\mathbb{Z}, n); \mathbb{Z}/2\mathbb{Z})$ for $d \geq n$. In particular, the admissible monomials Sq^I form an additive basis for $\mathscr{A}_2$. Thus, $\mathscr{A}_2$ is exactly the algebra of all $\mathbb{Z}/2\mathbb{Z}$ cohomology operations that are stable, commuting with suspension ([6], Cor 5.38).

[1] Hang on, the what?? X isn't the suspension of anything, why on earth would this be a group?

[2] As a set, or is this some sort of enriched thing? If it's enriched, is that over **Ab** or **Rng**? [5] says on p394 that there is a natural group structure on $\mathrm{Hom}(X, K(G, n))$ such that the natural isomorphism $\mathrm{Hom}(X, K(G, n)) \to H^n(X; G)$ is in fact an isomorphism of abelian groups. So, it's over **Ab**? N.B: There's a lot of talk about 'reduced cohomology theories', so a good next step would be to figure out what those are - if they only involve groups and not rings, maybe the cup product on H^* is not relevant here.

- ?Hurewicz theorem: for any path-connected space X and $n > 0$ there exists a group homomorphism $h_* : \pi_n(X) \to H_n(X)$. For $n = 1$ this induces an isomorphism $\pi_1^{\mathrm{ab}}(X) \cong H_1(X)$. For $n \geq 2$, if X is $(n-1)$-connected then $\widetilde{H}_i(X) = 0$ for all $i < n$, and the map $h_* : \pi_n(X) \to H_n(X)$ is an isomorphism.
- Axioms for a homology theory h_*:
  1. If $f \simeq g : X \to Y$, then $f_* = g_* : h_n(X) \to h_n(Y)$.
  2. For each $A \subseteq X$, there is a long exact sequence
  $$\cdots \xleftarrow{\partial} h_n(X/A) \xleftarrow{q_*} h_n(X) \xleftarrow{i_*} h_n(A) \xleftarrow{\partial} h_{n+1}(X/A) \xleftarrow{q_*} \cdots .$$
  3. For a wedge sum $X = \bigvee_\alpha X_\alpha$ with inclusions $i_\alpha : X_\alpha \hookrightarrow X$, the coproduct map $\bigoplus_\alpha (i_\alpha)_* : \bigoplus_\alpha h_n(X_\alpha) \to h_n(X)$ is an isomorphism for each n.

  Axioms for a (reduced) cohomology theory h^*:
  1. If $f \simeq g : X \to Y$, then $f^* = g^* : h^n(Y) \to h^n(X)$.
  2. For each $A \subseteq X$, there is a long exact sequence
  $$\cdots \xrightarrow{\delta} h^n(X/A) \xrightarrow{q^*} h^n(X) \xrightarrow{i^*} h^n(A) \xrightarrow{\delta} h^{n+1}(X/A) \xrightarrow{q^*} \cdots .$$
  3. For a wedge sum $X = \bigvee_\alpha X_\alpha$ with inclusions $i_\alpha : X_\alpha \hookrightarrow X$, the product map $(i_\alpha^*) : h^n(X) \to \prod_\alpha h^n(X_\alpha)$ is an isomorphism for each n.
- Stable homotopy groups are a homology theory.

[13], [5]

2 The Steenrod algebra

The following is from [5] 4L.

- There are maps $Sq^i : H^n(-;\mathbb{Z}/2\mathbb{Z}) \to H^{n+i}(-;\mathbb{Z}/2\mathbb{Z})$ for each i, and they satisfy the following properties:
  1. $Sq^i_X(f^*(\alpha)) = f^*(Sq^i_Y(\alpha))$ for $f : X \to Y$ (i.e. Sq^i is a natural transformation).
  2. $Sq^i_X(\alpha + \beta) = Sq^i_X(\alpha) + Sq^i_X(\beta)$ (i.e. Sq^i_X respects the group operation for all X).
  3. $Sq^i(\alpha \smile \beta) = \sum_{0 \leq j \leq i} (Sq^j(\alpha) \smile Sq^{i-j}(\beta))$ (the Cartan formula)
  4. $Sq^i(\sigma(\alpha)) = \sigma(Sq^i(\alpha))$ where $\sigma : H^n(X;\mathbb{Z}/2\mathbb{Z}) \to H^{n+1}(\Sigma X;\mathbb{Z}/2\mathbb{Z})$ is the "suspension isomorphism given by reduced cross product with a generator of $H^1(S^1;\mathbb{Z}/2\mathbb{Z})$". [see: [5], p219. N.B: I think this "relative cross product" theory is required – you can argue that there is an isomorphism via MV, but this point says that it's this specific one. Maybe they're the same[3], but Hatcher doesn't say that anywhere and there could be many isomorphisms.]
  5. $Sq^i(\alpha) = \alpha^2$ if $i = \deg(\alpha)$ and $Sq^i(\alpha) = 0$ if $i > \deg(\alpha)$.

[3] Weirdly, I think they are, though it's not immediately obvious why. It's just that the wikipedia page that σ is the connecting homomorphism of the long exact sequence in cohomology. [12] just says it's the suspension isomorphism :|

6. $Sq^0 = \mathrm{id}$.
7. Sq^1 is the "$\mathbb{Z}/2\mathbb{Z}$ Bockstein homomorphism β associated with the coefficient sequence $0 \to \mathbb{Z}/2\mathbb{Z} \to \mathbb{Z}/4\mathbb{Z} \to \mathbb{Z}/2\mathbb{Z} \to 0$".

- Define $Sq := Sq^0 + Sq^1 + \cdots$. Then $Sq(\alpha \smile \beta) = Sq(\alpha) \smile Sq(\beta)$ (since $(Sq(\alpha \smile \beta))_n = \sum_i Sq^i(\alpha) \smile Sq^{n-i}(\beta) = (Sq(\alpha) \smile Sq(\beta))_n$). Thus, Sq is a ring homomorphism.
- Adem relations:
$$Sq^a Sq^b = \sum_j \binom{b-j-1}{a-2j} Sq^{a+b-j} Sq^j \quad \text{if } a < 2b,$$
where $\binom{m}{n}$ is zero if m or n is negative, or $m < n$, and $\binom{m}{0} = 1$ for $m \geq 0$.

DEFINITION 2.0.1. The *Steenrod algebra* $\mathscr{A}_2$ is the algebra over $\mathbb{Z}/2\mathbb{Z}$ that is the quotient of the algebra of polynomials in the noncommuting variables $Sq^1, Sq^2, \ldots$ by the two-sided ideal generated by the Adem relations. Thus, for every space X, $H^*(X; \mathbb{Z}/2\mathbb{Z})$ is a module over $\mathscr{A}_2$.

- $\mathscr{A}_2$ is graded, and its elements of degree k are those that map $H^n(X; \mathbb{Z}/2\mathbb{Z})$ to $H^{n+k}(X, \mathbb{Z}/2\mathbb{Z})$ for all n. [Presumably you've fixed a space X while you're doing all this?]

DEFINITION 2.0.2. Write Sq^I for the monomial $Sq^{i_1} Sq^{i_2} \cdots Sq^{i_n}$. Then Sq^I is *admissible* if $i_j \geq 2i_{j+1}$ for all $0 \leq j < n$.

Note the admissible monomials are exactly those to which no Adem relations can be applied. Thus, $\mathscr{A}_2$ is generated as a $\mathbb{Z}/2\mathbb{Z}$ module by admissible monomials.

[5]

3 Spectra

3.1 ?Categorical nonsense

- [13]: There is a category $\mathcal{H}$ of finite based CW complexes, with $\mathrm{Hom}(X, Y) =: [X, Y]$ the set of homotopy classes of base-point preserving maps $X \to Y$.
- There is a category $\mathbf{St}(\mathcal{H})$ of finite based CW complexes, with $\mathrm{Hom}(X, Y) =: \{X, Y\}$ the set $\mathrm{colim}_i [\Sigma^i X, \Sigma^i Y]$ [it's just a colimit of sets, and **Set** is cocomplete, so we should be fine. [13] says it's a group[4]] [Also, how do these guys compose?]
- There is a functor $\mathcal{H} \to \mathbf{St}(\mathcal{H})$. [13] doesn't say what this is but it's presumably the one that is the identity on objects and sends $[f : X \to Y] \in [\Sigma^0 X, \Sigma^0 Y]$ to whatever it gets sent to in $\{X, Y\}$ using the universal property of the colimit. Uniqueness makes it functorial, etc.
- We have a fully faithful functor $\mathbf{St}(\mathcal{H}) \to \mathbf{St}(\mathcal{H})$ given by the suspension on objects, and the unique isomorphism $\{X, Y\} \to \{\Sigma X, \Sigma Y\}$ on maps (such an isomorphism exists, since both of those things are colimits for $[\Sigma^i X, \Sigma^i Y]$ - one of the sequences is cut off at the beginning, but it doesn't matter because both reach the stable value (see above discussion and [13] 1.9), aka the colimit).

[4]The colimit is equal to the stable value (which exists, by the corollary). After Σ^2, these guys are all groups, so the colimit also has a group structure inherited from whatever $[\Sigma^k X, \Sigma^k Y]$ it's equal to. N.B: Remarks about cocompleteness of **Set** are misleading because that doesn't actually matter - any sequence that stabilises in any category will have a filtered colimit equal to that stable value, you don't need any extra conditions.

- It's not an equivalence, because not every object is isomorphic to a suspension (e.g. anything not connected, since suspensions always connected)
- We can formally adjoin desuspensions $\Sigma^{-n}X$ for all n [does this mean just putting the objects there and defining $\mathrm{Hom}(Y, \Sigma^{-n}X) := \mathrm{Hom}(\Sigma^n Y, X)$ and $\mathrm{Hom}(\Sigma^{-n}X, Y) := \mathrm{Hom}(X, \Sigma^n Y)$?], but this category does not have weak colimits (i.e. colimits w/o uniqueness property). [why does it not, and why do we even want that?]
- We instead consider formal sequences of desuspensions $X_0 \to \Sigma^{-1}X_1 \to \cdots$, or sequences (X_n) and maps $\Sigma X_n \to X_{n+1}$, i.e. spectra. [and this fixes the problem?]

3.2 Definitions and examples

Below follows [6], Section 5.2.

[Maybe I could also look at [5] p454 onwards?]

Definition 3.2.1. A *spectrum* is a collection of pointed topological spaces $\{X_n\}_{n\in\mathbb{N}}$, together with basepoint-preserving maps $\sigma_n : \Sigma X_n \to X_{n+1}$.

Example 3.2.2. Let X be a topological space. The *suspension spectrum* of X, denoted by $\Sigma^\infty X$, has $X_n = \Sigma^n X$ and $\sigma_n = \mathrm{id} : \Sigma X_n \to X_{n+1}$.

We write $\mathbb{S} := \Sigma^\infty S^0$, and call $\mathbb{S}$ the *sphere spectrum*.

Example 3.2.3. The *Eilenberg-MacLane spectrum* $\mathbb{K}(G, m)$ has $(\mathbb{K}(G, m))_n$ a CW complex $K(G, m+n)$ and $\sigma_n : \Sigma K(G, m+n) \to K(G, m+n+1)$ is the adjoint of the CW approximation $K(G, m+n) \to \Omega K(G, m+n+1)$.

Definition 3.2.4. Let $X = \{X_n\}$ be a spectrum. We define $\pi_i(X) = \mathrm{colim}_n\, \pi_{i+n}(X_n)$, where the map $\pi_{i+n}(X_n) \to \pi_{i+n+1}(X_{n+1})$ is given by the composition

$$\pi_{i+n}(X_n) \xrightarrow{\Sigma} \pi_{i+n+1}(\Sigma X_n) \xrightarrow{(\sigma_n)_*} \pi_{i+n+1}(X_{n+1}).$$

Example 3.2.5. If X is a topological space, then $\pi_i(\Sigma^\infty X) = \pi_i^S(X)$, the ith stable homotopy group of X.

Definition 3.2.6. A CW spectrum is a spectrum X consisting of CW complexes X_n with the maps $\Sigma X_n \hookrightarrow X_{n+1}$ inclusions of subcomplexes.

Definition 3.2.7. Let X be a CW spectrum. Then the *k-cells* of X are the equivalence classes of non-basepoint $(k+n)$-cells in X_n, where two cells are equivalent if one is an m-fold suspension of the other.

Definition 3.2.8. A CW spectrum X is *connective* if it has no cells below a given dimension. Further, X is *finite* if it has only finitely many cells, and *of finite type* if it has only finitely many cells in each dimension.

Example 3.2.9. If X is a finite (resp. finite type) CW complex, then Σ^∞ is a finite (resp. finite type) CW spectrum. In particular, $\mathbb{S}$ is a finite CW spectrum with a unique cell in dimension 1.

3.3 Homology and cohomology

[From Hatcher: "the inclusions $\Sigma X_n \hookrightarrow X_{n+1}$ induce inclusions $C_*(X_n; G) \hookrightarrow C_*(X_{n+1}; G)$ with a dimension shift to account for the suspension". Below is my vague explanation of what I understand this to mean.

$C_i(X_n; G)$ is the free abelian group on maps $\Delta^i \to X_n$. I claim $\Sigma\Delta^i \cong \Delta^{i+1}$. If this is true, it gives a map

$$\begin{aligned} C_i(X_n; G) &\to C_{i+1}(\Sigma X_n; G) \\ f &\mapsto \Sigma f. \end{aligned}$$

This is an injection, by Remark B.1.3. We also have an injection $C_{i+1}(\Sigma X_n; G) \to C_{i+1}(X_{n+1}; G)$ induced by the structure map σ_n, so we get an injection $C_i(X_n; G) \hookrightarrow C_{i+1}(X_{n+1}; G)$, which indeed has a dimension shift.

We then have that "the union $C_*(X; G)$ of this increasing sequence of chain complexes is then a chain complex having one G summand for each cell of X". I think that

$$C_n(X; G) = \bigcup_{i \in \mathbb{Z}} C_{i+n}(X_i; G),$$

so that there's a G summand for every $i + n$ cell of X_i up to treating suspensions of cells as equivalent to the cells themselves (since they include in), i.e. a G summand for every n-cell of X, since that's how we defined them.

We define H^* and H_* to be the cohomology and homology of this chain complex, respectively.]

[Important example:

At the bottom of p592 of [6], it's mentioned that for a spectrum X of finite type, $H^i(X) \cong \lim\limits_{\leftarrow n} H^{i+n}(X_n)$ as $\mathscr{A}_2$-modules[5]. Thus, since $\mathbb{S}$ is of finite type (it is actually finite), we have

$$\begin{aligned} H^i(\mathbb{S}; \mathbb{Z}/2\mathbb{Z}) &= \lim_{\leftarrow n} H^{i+n}(S^n; \mathbb{Z}/2\mathbb{Z}) \\ &= \begin{cases} \mathbb{Z}/2\mathbb{Z} & i = 0, \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

(since they're all just either 0 or $\mathbb{Z}/2\mathbb{Z}$, at least eventually). So, $H^*(\mathbb{S}) = \mathbb{Z}/2\mathbb{Z}$ in degree zero and nothing else.]

Definition 3.3.1. Let $X = \{X_n\}$ be a CW spectrum. A *subspectrum* X' of X is a sequence of subcomplexes $\{X'_n \subseteq X_n\}$ satisfying $\Sigma X'_n \subseteq X'_{n+1}$. The subspectrum X' is *cofinal* if, for each n and each cell e^i_α of X_n, the cell $\Sigma^k e^i_\alpha$ belongs to X'_{n+k} for all sufficiently large k.

[If $\Sigma^k e^i_\alpha$ belongs to X'_{n+k} then $\Sigma^{k+1} e^i_\alpha$ belongs to $\Sigma X'_{n+k} \hookrightarrow X'_{n+k+1}$, so if it happens once it'll happen for all time after that. Thus, if X', X'' are cofinal spectra of X with $\Sigma^k e^i_\alpha$ a cell of X'_{n+k} and $\Sigma^l e^i_\alpha$ a cell of X''_{n+l} $(l \geq k)$ then $\Sigma^l e^i_\alpha$ is a cell of X'_{n+l} and therefore of $X'_{n+l} \cap X''_{n+l}$. In other words, the intersection of two cofinal spectra is a cofinal spectrum.]

Definition 3.3.2. Let X, Y be CW spectra. A *strict map* $f : X \to Y$ is a sequence of cellular maps $f_n : X_n \to Y_n$ such that the diagram below commutes.

$$\begin{tikzcd} \Sigma X_n \arrow[r, "\sigma_n"] \arrow[d, "\sigma f_n"'] & X_{n+1} \arrow[d, "f_{n+1}"] \\ \Sigma Y_n \arrow[r, "\sigma_n"'] & Y_{n+1} \end{tikzcd}$$

[this induces maps $\pi_i(X) \to \pi_i(Y)$, $H^*(Y) \to H^*(X)$, $H_*(X) \to H_*(Y)$.]

[5] Maybe I need to move this somewhere else. I haven't defined the $\mathscr{A}_2$-action yet...

Definition 3.3.3. A *map* of CW spectra $f : X \to Y$ is an equivalence class of strict maps $f' : X' \to Y$ with X' a subspectrum of X, where two strict maps $f' : X' \to Y$ and $f'' : X'' \to Y$ are equivalent if they agree on some common cofinal spectrum.

![this also induces maps $\pi_i(X) \to \pi_i(Y)$, $H^*(Y) \to H^*(X)$, $H_*(X) \to H_*(Y)$.]

![check composition is well defined]

[working definition of equivalence below:]

Definition 3.3.4. Two spectra X, Y are *equivalent* if there are maps $f : X \to Y$ and $g : Y \to X$ such that $fg = \mathrm{id}_Y$ and $gf = \mathrm{id}_X$.

[maybe something on weak equivalence?]

Note that a spectrum is equivalent to any of its cofinal subspectra.

![A spectrum is always equivalent to the suspension of some other spectrum]

Definition 3.3.5. A *homotopy* of maps between spectra is a map $X \times I \to Y$, where $X \times I$ is the spectrum with $(X \times I)_n = X_n \times_{\mathrm{red}} I$.

Note that $\Sigma(X_n \times_{\mathrm{red}} I) = \Sigma X_n \times_{\mathrm{red}} I$. The set of homotopy classes of maps $X \to Y$ is denoted by $[X, Y]$.

Remark 3.3.6. For any CW spectra Z, $[\Sigma^\infty S^t, Z] = \pi_t(Z)$. [they satisfy the same universal property]

[[1] says on p171 that "$[\Sigma X, Z]$ is obviously a group, because in ΣX we have a spare suspension coordinate out in front to manipulate. And for the same reason, $[\Sigma^2 X, Z]$ is an abelian group. But now we can give $[X, Y]$ the structure of an abelian group, because $[X, Y]$ is in 1-1 correspondence with $[\Sigma^2 X, \Sigma^2 Y]$ and we pull back the group structure on that. So now our sets of morphisms $[X, Y]$ are abelian groups, and it's easy to see that composition is bilinear".

Various claims:

- The stuff about normal CW complexes and their groups of maps (i.e. Remark B.1.5) translates to tell me the appropriate things about spectra and their groups of maps.
- After checking a lot of things, I can eventually conclude that $[X, Y]$ is an abelian group for spectra X, Y.]

![The suspension map $[X, Y] \to [\Sigma X, \Sigma Y]$ is an isomorphism of groups (important later!)]

?[Lemma B.3.4 and Theorem B.3.3 are both true for CW spectra too]

?[Whitehead's theorem: a map between CW spectra that induces isomorphisms on all homotopy groups is a homotopy equivalence.]

?[Prop: If a CW spectrum X is n-connected in the sense that $\pi_i(X) = 0$ for $i \le n$, then X is homotopy equivalent to a CW spectrum with no cells of dimension $\le n$]

3.4 Cofibration sequences

Definition 3.4.1. Let $X = \{X_n\}, Y = \{Y_n\}$ be spectra. Then their *wedge sum* is $X \vee Y := \{X_n \vee Y_n\}$. Note that Remark B.2.5 gives us an inclusion $\Sigma(X_n \vee Y_n) \hookrightarrow X_{n+1} \vee Y_{n+1}$.

Definition 3.4.2. Let X be a spectrum, $A \subseteq X$ a subspectrum. Then A is *closed* in X if for every cell e^n_α of X_n, if $\Sigma^k e^n_\alpha \in A_{n+k}$ then $e^n_\alpha \in A_n$.

Note that any subspectrum is cofinal in (and thus equivalent to) its closure. We define X/A to be the CW spectrum with $(X/A)_n = X_n/A'_n$, where $A' = \{A'_n\}$ is the closure of A.

Note that a quotient of connective spectra of finite type is again a connective spectrum of finite type. [This is obvious - the quotient has fewer cells in each dimension then the original space, it is not going to magically generate new ones.]

[The rest of the section:

- Fact: for spectra X, Y and a subspectrum $A \subseteq X$ we have exact sequences[6]

$$\cdots \to [\Sigma X, Y] \to [\Sigma A, Y] \to [X/A, Y] \to [X, Y] \to [A, Y]$$

$$[Y, A] \to [Y, X] \to [Y, X/A] \to [Y, \Sigma A] \to [Y, \Sigma X] \to \cdots$$

]

3.5 Eilenberg-MacLane spectra

THEOREM 3.5.1 ([6], Prop 5.45). There are natural isomorphisms $H^m(X; G) \cong [X, K(G, m)]$ for all CW spectra.

Recall that giving a map into a product is equivalent to giving a map into each of its components. We have maps $F_i : [X, \bigvee_i \mathbb{K}(G, n_i)] \to [X, \mathbb{K}(G, n_i)]$.

PROPOSITION 3.5.2 ([6], Prop 5.46). The map $F : [X, \bigvee_i \mathbb{K}(G, n_i)] \to \prod_i [X, \mathbb{K}(G, n_i)]$ is an isomorphism if X is a connective spectrum of finite type and $n_i \to \infty$ as $i \to \infty$.

[13], [6]

4 The Adams spectral sequence

4.1 Spectral sequences

[Maybe add some notes from [13]]

[Some notes from [10], C2 - just here as a placeholder/reference and I'll probably completely rewrite this bit.]

DEFINITION 4.1.1. A *differential bigraded module* E over a ring R is a collection of R-modules $\{E^{p,q}\}$, $p, q \in \mathbb{Z}$, together with a map $d : E^{p,q} \to E^{p+s,q-s+1}$ for each p, q and some fixed $s \in \mathbb{Z}$, satisfying $d^2 = 0$.

We can take the homology of (E, d):

$$H^{p,q}(E^{*,*}, d) = \ker(d : E^{p,q} \to E^{p+s,q-s+1}) / \operatorname{im}(d : E^{p-s,q+s-1} \to E^{p,q}).$$

DEFINITION 4.1.2. A *spectral sequence* (of *cohomological type*[7]) is a collection of differential bigraded R-modules $\{E_r^{*,*}, d_r\}$, $r \in \mathbb{N}$, with the differentials d_r of bidegree $(r, 1-r)$. These satisfy the further condition that for all p, q, r, $E_{r+1}^{p,q} \cong H^{p,q}(E_r^{*,*}, d_r)$.

We will sometimes write $d_r^{p,q}$ for the differential $d_r : E^{p,q} \to E^{p+r,q-s+1}$.

[6] Check if I actually need the first one.

[7] I'll have to rewrite this section because the Adams spectral sequence is not a cohomological or a homological spectral sequence I don't think - the grading is $d_r : E^{s,t} \to E^{s+r,t+r-1}$.

Consider the term $E_2^{*,*}$. Define

$$Z_2^{p,q} := \ker d_2^{p,q} \quad \text{and} \quad B_2^{p,q} := \operatorname{im} d_2^{p-2,q+1}.$$

The condition $d^2 = 0$ implies that $B_2^{p,q} \subseteq Z_2^{p,q} \subseteq E_2^{p,q}$, and by definition we have $E_3^{p,q} \cong Z_2^{p,q}/B_2^{p,q}$.

Now, write

$$Z_3^{p,q} := \ker d_3^{p,q} \quad \text{and} \quad B_3^{p,q} := \operatorname{im} d_3^{p-3,q+2}.$$

Since $Z_3^{p,q} \subseteq E_3^{p,q}$, it can be written as $\overline{Z}_3^{p,q}/B_2^{p,q}$ for some $\overline{Z}_3^{p,q} \subseteq Z_2^{p,q}$. Similarly, $B_3^{p,q} \cong \overline{B}_3^{p,q}/B_2^{p,q}$ for some $\overline{B}_3^{p,q} \subseteq Z_2^{p,q}$. Thus,

$$E_4^{p,q} \cong Z_3^{p,q}/B_3^{p,q} \cong \frac{\overline{Z}_2^{p,q}/B_2^{p,q}}{\overline{B}_3^{p,q}/B_2^{p,q}} \cong \overline{Z}_3^{p,q}/\overline{B}_3^{p,q}.$$

Iterating this process, we present the spectral sequence as an infinite tower of submodules of $E_2^{p,q}$:

$$B_2^{p,q} \subseteq \overline{B}_3^{p,q} \subseteq \cdots \subseteq \overline{B}_n^{p,q} \subseteq \cdots \subseteq \overline{Z}_n^{p,q} \subseteq \cdots \subseteq \overline{Z}_3^{p,q} \subseteq Z_2^{p,q} \subseteq E_2^{p,q},$$

with the property that $E_{n+1}^{p,q} \cong \overline{Z}_n^{p,q}/\overline{B}_n^{p,q}$. The differential $d_{n+1}^{p,q}$ can be taken as a map $\overline{Z}_n^{p,q}/\overline{B}_n^{p,q} \to \overline{Z}_n^{p,q}/\overline{B}_n^{p,q}$ with kernel $\overline{Z}_{n+1}^{p,q}/\overline{B}_n^{p,q}$ and image $\overline{B}_{n+1}^{p,q}$. The short exact sequence induced by d_{n+1},

$$0 \to \overline{Z}_{n+1}^{p,q}/\overline{B}_n^{p,q} \to \overline{Z}_n^{p,q}/\overline{B}_n^{p,q} \xrightarrow{d_{n+1}^{p,q}} \overline{B}_{n+1}^{p,q}/\overline{B}_n^{p,q} \to 0,$$

gives rise to isomorphisms $\overline{Z}_n^{p,q}/\overline{Z}_{n+1}^{p,q} \cong \overline{B}_{n+1}^{p,q}/\overline{B}_n^{p,q}$ for all n. Conversely, a tower of submodules of E_2, together with a set of isomorphisms, gives rise to a spectral sequence.

Definition 4.1.3. An element of $E_2^{p,q}$ *survives to the rth stage* if lies in $\overline{Z}_r^{p,q}$, having been in the kernel of the previous $r-2$ differentials, and is *bounded by the rth stage* if it lies in $\overline{B}_r^{p,q}$. The bigraded module $E_r^{*,*}$ is called the *E_r-term* of the spectral sequence.

We define

$$Z_\infty^{p,q} := \bigcap_n \overline{Z}_n^{p,q}, \quad B_\infty^{p,q} := \bigcup_n \overline{B}_n^{p,q}.$$

From the tower of inclusions, we see that $B_\infty^{p,q} \subseteq Z_\infty^{p,q}$, so we define $E_\infty^{p,q} := Z_\infty^{p,q}/B_\infty^{p,q}$.

Definition 4.1.4. A spectral sequence *collapses at the Nth term* if the differentials $d_r^{p,q} = 0$ for $r \geq N$.

From the short exact sequence

$$0 \to \overline{Z}_r^{p,q}/\overline{B}_{r-1}^{p,q} \to \overline{Z}_{r-1}^{p,q}/\overline{B}_{r-1}^{p,q} \xrightarrow{d_r^{p,q}} \overline{B}_r^{p,q}/\overline{B}_{r-1}^{p,q} \to 0,$$

the condition $d_r^{p,q}$ forces $\overline{Z}_r^{p,q} = \overline{Z}_{r-1}^{p,q}$ and $\overline{B}_r^{p,q} = \overline{B}_{r-1}^{p,q}$. The tower of submodules becomes

$$B_2^{p,q} \subseteq \overline{B}_3^{p,q} \subseteq \cdots \subseteq \overline{B}_{N-1}^{p,q} = B_N^{p,q} = \cdots = B_\infty^{p,q} \subseteq Z_\infty^{p,q} = \cdots = \overline{Z}_N^{p,q} = \overline{Z}_{N-1}^{p,q} \subseteq \cdots \subseteq \overline{Z}_3^{p,q} \subseteq Z_2^{p,q} \subseteq E_2^{p,q}.$$

Thus, $E_\infty^{p,q} = E_N^{p,q}$.

4.2 Exact couples

(Following [10], C2)

Definition 4.2.1. Let D, E be R-modules, and let $i : D \to D$, $j : D \to E$, $k : E \to D$ be module homomorphisms. We call $\mathcal{C} = \{D, E, i, j, k\}$ an *exact couple* if the diagram below is exact.

$$\begin{tikzcd} D \arrow[rr, "i"] & & D \arrow[dl, "j"] \\ & E \arrow[ul, "k"] & \end{tikzcd}$$

Let $d := jk$, and define the following:

$$\begin{aligned} E' &:= H(E, d) = \ker d / \operatorname{im} d \\ D' &:= i(D) = \ker j \\ i' &:= i|_{i(D)} : D' \to D' \\ j' &:= i(x) \mapsto j(x) + dE : D' \to E' \\ k' &:= (e + dE) \mapsto k(e) : E' \to D' \end{aligned}$$

We call $\mathcal{C}' = \{D', E'.i', j', k'\}$ the *derived couple* of $\mathcal{C}$.

Proposition 4.2.2 ([10], Prop 2.7). If $\mathcal{C} = \{D, E, i, j, k\}$ is an exact couple, then $\mathcal{C}'$ is also an exact couple.

Theorem 4.2.3 ([10], Thm 2.8). Suppose $D^{*,*} = \{D^{p,q}\}$ and $E^{*,*} = \{E^{p,q}\}$ are bigraded modules equipped with homomorphisms i of bidegree $(-1, 1)$[8], j of bidegree $(0, 0)$, and k of bidegree $(1, 0)$, such that $\{D^{*,*}, E^{*,*}, i, j, k\}$ is an exact couple. Then these data determine a spectral sequence $\{E_r, d_r\}$ for $r \in \mathbb{Z}_+$ of cohomological type, with $E_r = (E^{*,*})^{(r-1)}$, the $(r-1)$st derived module of $E^{*,*}$ and $d_r = j^{(r)} \circ k^{(r)}$. [9]

A bigraded exact couple may be displayed in the following diagram, known as a *staircase diagram*:

$$\begin{tikzcd} & \vdots \arrow[d, "i"] & & \vdots \arrow[d, "i"] & \\ \cdots \arrow[r, "k"] & D^{p+2,q-1} \arrow[r, "j"] \arrow[d, "i"] & E^{p+2,q-1} \arrow[r, "k"] & D^{p+3,q-1} \arrow[r, "j"] \arrow[d, "i"] & \cdots \\ \cdots \arrow[r, "k"] & D^{p+1,q} \arrow[r, "j"] \arrow[d, "i"] & E^{p+1,q} \arrow[r, "k"] & D^{p+2,q} \arrow[r, "j"] \arrow[d, "i"] & \cdots \\ \cdots \arrow[r, "k"] & D^{p,q+1} \arrow[r, "j"] \arrow[d, "i"] & E^{p,q+1} \arrow[r, "k"] & D^{p+1,q+1} \arrow[r, "j"] \arrow[d, "i"] & \cdots \\ & \vdots & & \vdots & \end{tikzcd}$$

[8]This is the wrong index. For my purposes it should be $(-1, -1)$. So, the spectral sequence will not be of cohomological type, but hopefully the proof still goes through.

[9]Is this supposed to be $d_r = j^{(r-1)} \circ k^{(r-1)}$? Because d_1 is supposed to be a map on $E_1 = E$, but j' and k' are between $E' = E_2$ and D'.

[This is the same staircase diagram that Hatcher is talking about (set $D^{*,*} = \pi_*^S X_*$ and $E^{*,*} = \pi_* K_*$) - just modulo some tweak of i's bigrading because the bigrading is a bit weird in the Adams spectral sequence.]

4.3 The Adams spectral sequence

Let X be a connective CW spectrum of finite type.

[Each $H^i(X)$ is finitely generated: Hatcher says on p586 of [6] that $H^i(X) = H^i(X_n)$ for sufficiently large n as long as X is of finite type (which I think should actually read $H^{i+n}(X_n)$, but it doesn't matter for this particular point), and $H^i(X_n)$ (or $H^{i+n}(X_n)$) is supposed to be finitely generated because all the $H_j(X_n)$ are finitely generated[10], since X_n should have finitely many cells in each dimension (if it didn't, X certainly wouldn't).]

[Claim: $H^*(X)$ is an $\mathscr{A}_2$-module. Either use Theorem 3.5.1 (i.e. $H^m(X;G) \cong [X, K(G,m)]$) to give the action, or the above mentioned fact that for spectra of finite type, $H^i(X;G) = H^{i+n}(X_n;G)$. To define a map $Sq^j : H^i(X;G) \to H^{i+j}(X;G)$ we can either hit $H^{i+n}(X_n;G)$ with Sq^j and then push it along the sequence of cohomology groups until it's stable, or first push it along until $H^{i+j+n}(X_n;G)$ is stable and then hit $H^{i+n}(X_n;G)$ with Sq^j. And these are the same thing, because of this handy commutative diagram:

$$\begin{tikzcd}
& H^{i+n}(X_n;G) \arrow[dl, "\sigma"'] \arrow[dr, "Sq^j"] & \\
H^{i+n+1}(\Sigma X_n;G) \arrow[d, "i^*"'] & & H^{i+j+n}(X_n;G) \arrow[d, "\sigma"] \\
H^{i+n+1}(X_{n+1};G) \arrow[dr, "Sq^j"'] & & H^{i+j+n+1}(\Sigma X_n;G) \arrow[dl, "i^*"] \\
& H^{i+j+n+1}(X_{n+1};G) &
\end{tikzcd}$$

which follows from the properties in Section 2. Is this an actual module action? Is the former one also a module action?]

We can pick generators α_i for $H^*(X)$ as an $\mathscr{A}_2$-module such that there are at most finitely many in each $H^n(X)$ [since each $H^n(X)$ is finitely generated, and that finite generating set would certainly also generate it as an $\mathscr{A}_2$-module (we could be more efficient, though)].

[Claim: these α_i determine a map $X \to K_0$, where K_0 is a wedge of EM spectra, and K_0 has finite type. Very confused about this - each $\alpha_i \in H^{n_i}(X;G)$ gives a map $X \to K(G, n_i)$ by Theorem 3.5.1, but shouldn't they come together to give a map into a product[11] of EM spaces, not a wedge sum? Isn't a wedge sum a sort of reduced *coproduct*, not a product?? Is this something to do with the 'fibrations and cofibrations are the same' nonsense?

Vague idea: we have Proposition 3.5.2, which tells us that $[X, \bigvee_i \mathbb{K}(G,n_i)] \cong \prod_i [X, \mathbb{K}(G,n_i)]$. Now, X is connective, so $H^*(X;G)$ has some smallest degree. Let's just pretend it starts from zero for the moment, and we can reindex or whatever later. Let's say each $H^k(X;G)$ has i_k generators $\alpha_{i_0+\cdots+i_{k-1}+1}, \ldots, \alpha_{i_0+\cdots+i_k}$, and write $n_0 = \cdots = n_{i_0}$, $n_{i_0+1} = \cdots = n_{i_0+i_1}$, etc., $n_{i_0+i_1+\cdots+i_{k-1}+1} = \cdots = n_{i_0+i_1+\cdots+i_k}$. So, the $\alpha_{i_0+\cdots+i_{k-1}+j} \in H^k(X;G)$ determine maps

[10] The cohomology groups are finitely generated if the homology ones are, see [5] Cor 3.3.

[11] And on that note, what is a product of spectra? Surely it's not a smash product? Is it just a pointed product? Are we allowed to do that?

$X \to \mathbb{K}(G, n_k)$. Putting all these maps together, one in each coordinate, gives an element of $\prod_i [X, \mathbb{K}(G, n_i)]$, which by the proposition corresponds to an element of $[X, \bigvee_i \mathbb{K}(G, n_i)]$.]

We can replace that map with an inclusion. [I sort of believe this - Remark B.2.8 and my comment at the end of Section C.3 say we can do this for CW complexes, so I should check it for spectra. But it seems true (and I'm currently suffering from spectrum fatigue).]

[Set $X_1 = K_0/X$, and repeat the construction to get a diagram:

$$\begin{tikzcd}
X \arrow[r] & K_0 \arrow[rr] \arrow[dr] & & K_1 \arrow[rr] \arrow[dr] & & K_2 \arrow[rr] \arrow[dr] & & \cdots \\
& & K_0/X = X_1 \arrow[ur] & & K_1/X_1 = X_2 \arrow[ur] & & K_2/X_2 = X_3 \arrow[ur] &
\end{tikzcd}$$

Taking cohomology, we get a diagram

$$\begin{tikzcd}
0 & H^*(X) \arrow[l] & H^*(K_0) \arrow[l] \arrow[dr] & & H^*(K_1) \arrow[ll] \arrow[dl] \arrow[dr] & & H^*(K_2) \arrow[ll] \arrow[dl] \arrow[dr] & & \cdots \arrow[ll] \arrow[dl] \\
& & & H^*(X_1) \arrow[ul] \arrow[dl] & & H^*(X_2) \arrow[ul] \arrow[dl] & & H^*(X_3) \arrow[ul] \arrow[dl] & \\
& & 0 & & 0 \arrow[ul] & & 0 \arrow[ul] & & 0 \arrow[ul]
\end{tikzcd}$$

]

- Fact: K_0 has finite type (and thus so do all the K_s's) [I'm sure this is supposed to follow from $H^i(X)$ being finitely generated for all i, but I can't at all see how].
- Fact: the top row of the diagram above is exact. [Suffices to show that we have exact sequences:

$$0 \leftarrow H^*(X_s) \leftarrow H^*(K_s),$$
$$H^*(X) \leftarrow H^*(K_0) \leftarrow H^*(X_1) \leftarrow 0,$$
$$H^*(K_s) \leftarrow H^*(K_{s+1}) \leftarrow H^*(X_{s+2}) \leftarrow 0,$$

  and that the map $H^*(K_0) \to H^*(X)$ is surjective. Actually, I think the first exact sequence will follow from $H^*(K_0) \to H^*(X)$ being surjective, since they're all constructed in the same way.

  I'm extremely confused by the supposed surjectivity - going back to the sketch of the construction for CW complexes, the corresponding statement is supposed to be that if we choose generators α_i for $H^*(X)$ as an $\mathscr{A}_2$-module giving maps $f_i : X \to K(G, \deg(\alpha_i))$, the product of these maps induces a surjection on H^*. But I can't see how it does? So, we know by [5], Thm 4.57 that there are natural bijections $T : [X, K(G, n)] \to H^n(X; G)$ with $T[f] = f^*(\beta)$ for some class $\beta \in H^n(K(G, n); G)$. So, for any given $\alpha^i \in H^n(X; G)$, it's equal to $f^*(\beta_i)$, so definitely f^* hits it. But if I have some enormous product of EM spaces, and all these maps $f_i : X \to K(G, n_i)$ come together to give a map $f : X \to \prod_i K(G, n_i)$, then Hatcher is trying to tell me I get a surjection $H^*(\prod_i K(G, n_i)) \to H^*(X)$?? I know nothing about the cohomology of an infinite product, there's no reason it would be e.g. the product of the cohomologies.]

  [Something something reduced cohomology theories for spectra and the wedge axiom?? Suppose K_0 had finite type, which I'm not convinced it does. Then for some n, $(K_0)_n = (\bigvee_i \mathbb{K}(G, n_i))_n = \bigvee_i (\mathbb{K}(G, n_i))_n = \bigvee_i K(G, n + n_i)$, and $H^i(K_0) = H^{i+n}(\bigvee_i K(G, n + n_i)) = \prod_i H^{i+n}(K(G, n + n_i))$. Does that help?]

- Fact: each $H^*(K_s)$ is a free $\mathscr{A}_2$-module, and thus the diagram above gives a resolution of $H^*(X)$ by free $\mathscr{A}_2$-modules. [Vague idea: Hatcher claims on p582 that "the cohomology $H^*(K(\mathbb{F}_2,n))$ is free over $\mathscr{A}_2$ in dimensions less than $2n$" (this may be [13] Prop 3.22 but I'm not sure what's going on with the suspension there). Assuming this, I said earlier that if X has finite type you can act on $H^*(X)$ with Sq^i by first pushing it along the limit as far as you need to to make $H^{i+j+n}(X_{n+j})$ stable and then applying Sq^i to $H^{i+n}(X_n)$. But the point here is that I can make n as big as I like, so I can just make it big enough so that any action of $\mathscr{A}_2$ I like is free.]

[The work in Section 3.4 gives us long exact sequences

$$\cdots \to [\Sigma^\infty S^{t+1}, X_s] \to [\Sigma^\infty S^{t+1}, K_s] \to [\Sigma^\infty S^{t+1}, X_{s+1}] \to [\Sigma^\infty S^{t+1}, \Sigma X_s] \to [\Sigma^\infty S^{t+1}, \Sigma K_s] \to \cdots .$$

Using the isomorphism $[Y,Z] \cong [\Sigma Y, \Sigma Z]$, we get long exact sequences

$$\cdots \to [\Sigma^\infty S^{t+1}, X_s] \to [\Sigma^\infty S^{t+1}, K_s] \to [\Sigma^\infty S^{t+1}, X_{s+1}] \to [\Sigma^\infty S^{t}, X_s] \to [\Sigma^\infty S^{t}, K_s] \to \cdots ,$$

i.e.

$$\cdots \to \pi_{t+1}X_s \to \pi_{t+1}K_s \to \pi_{t+1}X_{s+1} \to \pi_t X_s \to \pi_t K_s \to \cdots .$$

These form a staircase diagram,

$$\begin{tikzcd}
 & \vdots \arrow[d, "i"] & & \vdots \arrow[d, "i"] & & \vdots \arrow[d, "i"] & \\
\cdots \arrow[r, "k"] & \pi_{t+1}X_s \arrow[r, "j"] \arrow[d, "i"] & \pi_{t+1}K_s \arrow[r, "k"] & \pi_{t+1}X_{s+1} \arrow[r, "j"] \arrow[d, "i"] & \pi_{t+1}K_{s+1} \arrow[r, "k"] & \pi_{t+1}X_{s+2} \arrow[r, "j"] \arrow[d, "i"] & \cdots \\
\cdots \arrow[r, "k"] & \pi_t X_{s-1} \arrow[r, "j"] \arrow[d, "i"] & \pi_t K_{s-1} \arrow[r, "k"] & \pi_t X_s \arrow[r, "j"] \arrow[d, "i"] & \pi_t K_s \arrow[r, "k"] & \pi_t X_{s+1} \arrow[r, "j"] \arrow[d, "i"] & \cdots \\
\cdots \arrow[r, "k"] & \pi_{t-1}X_{s-2} \arrow[r, "j"] \arrow[d, "i"] & \pi_{t-1}K_{s-2} \arrow[r, "k"] & \pi_{t-1}X_{s-1} \arrow[r, "j"] \arrow[d, "i"] & \pi_{t-1}K_{s-1} \arrow[r, "k"] & \pi_{t-1}X_s \arrow[r, "j"] \arrow[d, "i"] & \cdots \\
 & \vdots & & \vdots & & \vdots &
\end{tikzcd}$$

This gives rise to a spectral sequence, by [some version of] Theorem 4.2.3.

- ?Fact: If Z is a connective CW spectrum of finite type, then $\pi_t(Z)$ is finitely generated. [Halfway down p586 of [6], Hatcher says that "for a connective CW spectrum X of finite type the groups $\pi_{i+n}(X_n)$ do eventually stabilise by the Freudenthal theorem". Since the homotopy groups are just the colimits of these guys, they're equal to the stable values. [5], Thm 5.7 says in particular that a simply-connected space has finitely generated homotopy groups if and only if it has finitely generated homology groups. Now, the X_n must eventually by simply-connected because otherwise X would not be connective[12], and we've already seen the H_i are finitely generated, so yes, the π_i are also finitely generated.]

Now, since $K_s = \bigvee_i \mathbb{K}(G, n_{s_i})$, Proposition 3.5.2, tells us that $[\mathbb{S}, K_s] \cong \prod_i [\mathbb{S}, \mathbb{K}(G, n_{s_i})]$, which is naturally isomorphic to $\prod_i H^{n_{s_i}}(\mathbb{S}; G)$. Thus, elements of $[\mathbb{S}, K_s]$ are tuples of elements of $H^*(\mathbb{S})$.

[12]Claim: if a CW complex Y is not simply-connected, it has at least one 1-cell. Proof: suppose Y has no 1-cells, and let $f : S^1 \to Y$. By Theorem B.3.3, $f \simeq g : S^1 \to Y$, where g is cellular. We have $g(S^1) = g(S^1_{(1)}) \subseteq Y_{(1)} = Y_{(0)}$, so g is constant. Thus, $\pi_1(Y)$ is trivial. Now, suppose each X_n has at least one 1-cell. Then for every $n < 0$, X has an n-cell: the 1-cell of X_{1-n}, so X is not connective.

- Def: $\mathrm{Hom}^t_{\mathscr{A}_2}$ [it's just the homs that lower the degree by t]
- Fact: there is a natural map $[\mathbb{S}, K_s] \to \mathrm{Hom}^0_{\mathscr{A}_2}(H^*(K_s), H^*(\mathbb{S}))$, and it's an isomorphism. [??]

[We thus have

$$[\Sigma^t\mathbb{S}, K_s] = \mathrm{Hom}^0_{\mathscr{A}_2}(H^*(K_s), H^*(\Sigma^t\mathbb{S})) = \mathrm{Hom}^t_{\mathscr{A}_2}(H^*(K_s), H^*(\mathbb{S})).$$

In the case of CW complexes, we have $H^*(\Sigma^t X) \cong H^{*-t}(X)$ via the map σ mentioned in Section 2. Since $\mathbb{S}$ has finite type, for i large enough we have $H^n(\Sigma^t\mathbb{S}) = H^{n+i}(\Sigma^t S^i) = H^{n+i-t}(S^i) = H^{n-t}(\mathbb{S})$.]

[$E_1^{s,t} = \pi_t K_s (= \mathrm{Hom}^t_{\mathscr{A}_2}(H^*(K_s), H^*(\mathbb{S})))$, since the staircase diagram comes from (or gives rise to, depending on your point of view) the exact couple:

$$\begin{tikzcd} \pi_* X_* \arrow[rr, "i"] & & \pi_* X_* \arrow[dl, "j"] \\ & \pi_* K_* \arrow[ul, "k"] & \end{tikzcd}$$

where $i : \pi_{t+1}X_{s+1} \to \pi_t X_s$, $j : \pi_{t+1}X_s \to \pi_{t+1}K_s$, and $k : \pi_{t+1}X_{s+1}$ all come from the diagram we set up earlier. The differential $d_1 : \pi_t(K_s) \to \pi_t K_{s+1}$ is induced by the map $K_s \to K_{s+1}$, since it's just $j \circ k$, according to the way things are set up in Section 4.2, possibly modulo some typos.]

[Then, $E_2^{s,t} = H^{s,t}(E_1^{*,*}, d_1)$, so each $E^{*,t}$ is the homology of the chain complex

$$0 \to E_1^{0,t} \to E^{1,t} \to E^{2,t} \to \cdots ,$$

or, in other words,

$$0 \to \mathrm{Hom}^t_{\mathscr{A}_2}(H^*(K_0), H^*(\mathbb{S})) \to \mathrm{Hom}^t_{\mathscr{A}_2}(H^*(K_1), H^*(\mathbb{S})) \to \cdots .$$

The homology of this is by definition $\mathrm{Ext}^{s,t}_{\mathscr{A}_2}(H^*(X), H^*(\mathbb{S}))$, so $E_2^{s,t} = \mathrm{Ext}^{s,t}_{\mathscr{A}_2}(H^*(X), H^*(\mathbb{S}))$.]

Theorem 4.3.1 ([6], Thm 5.47). There is a spectral sequence $\{E_r, d_r\}$ such that $E_2^{s,t} = \mathrm{Ext}^{s,t}_{\mathscr{A}_2}(\mathbb{F}_2, \mathbb{F}_2)$ and $\{E_r, d_r\} \implies \pi^S_{t-s}$ modulo torsion of odd order. [which is probably a lie but let's worry about that later]

[10], [6]

4.4 Multiplicative structure

4.4.1 The Yoneda product

- There is a multiplication, the *Yoneda product*, on $E_2 = \mathrm{Ext}(\text{whatever})$
- More generally, [12]: For any algebra A and A-modules L, M, N, there is a natural Yoneda composition product

  $$\circ : \mathrm{Ext}^s_A(M, N) \otimes \mathrm{Ext}^u_A(L, M) \to \mathrm{Ext}^{s+u}_A(L, N)$$

  [doing what it should].

[The stuff below is old and needs to be rewritten but I'm putting it here for the moment]

Using Lemma 5.1.1 (which comes later but whatever, I'll reorder things), we get $E_2^{s,t} = \mathrm{Hom}^t_{\mathscr{A}_2}(F_s, \mathbb{F}_2)$. So, if we have a minimal free resolution

$$\cdots \to F_2 \to F_1 \to F_0 \to \mathbb{F}_2$$

of $\mathbb{F}_2$ as an $\mathscr{A}_2$-module, and we have $f \in \mathrm{Hom}^t_{\mathscr{A}_2}(F_s, \mathbb{F}_2)$, $g \in \mathrm{Hom}^{t'}_{\mathscr{A}_2}(F_{s'}, \mathbb{F}_2)$, we multiply them as follows:

First, we put these resolutions side by side.

$$\begin{tikzcd}
F_{s+s'} \arrow[r, dashed] \arrow[d] & F_{s'}[t] \arrow[r, "g"] \arrow[d] & \mathbb{F}_2[t+t'] \\
F_{s+s'-1} \arrow[r, dashed] \arrow[d] & F_{s'-1}[t] \arrow[d] & \\
\vdots \arrow[d] & \vdots \arrow[d] & \\
F_s \arrow[r, dashed] \arrow[d] \arrow[dr, "f"] & F_0[t] \arrow[d, two heads] & \\
F_{s-1} \arrow[d] & \mathbb{F}_2[t] & \\
\vdots \arrow[d] & & \\
F_0 \arrow[d] & & \\
\mathbb{F}_2 & &
\end{tikzcd}$$

(The $[t]$ means shifted by t i.e. in degree t because the maps are.)

Then we inductively fill in the dotted arrows from the bottom up. So you can fill the first one because the modules are all free so you just need to say where the generator goes (a priori maybe there are infinitely many generators but it's ok because these are degree t maps and there are only finitely many in each degree), and the right map is surjective, so just do it the way you would do it. Likewise for the higher ones.

This thing is hopefully well-defined.

4.4.2 The composition product

From [13], 5.1

DEFINITION 4.4.1. Let X, Y, Z be spectra. The *composition pairing* $\circ : [Y, Z]_* \otimes [X, Y]_* \to [X, Z]_*$ is defined as follows:

$$\begin{aligned} \circ :\ [Y,Z]_v \ \otimes \ [X,Y]_t \ &\to \ [X,Z]_{v+t} \\ [g : \Sigma^v Y \to Z] \otimes [f : \Sigma^t X \to Y] &\mapsto [g \circ \Sigma^v f : \Sigma^{v+t} X \to Z], \end{aligned}$$

where $[X,Y]_n = [S^n \wedge X, Y]$ [which is the same as $[\Sigma^n X, Y]$ by the version of Remark B.2.4 for spectra[13]]

In particular, if $X = Y = Z = \mathbb{S}$, we have a product $\pi_v\mathbb{S} \otimes \pi_t\mathbb{S} \to \pi_{v+t}\mathbb{S}$, or, in other words, a product $\pi_v^S \otimes \pi_t^S \to \pi_{v+t}^S$.

- Lemma: this makes π_*^S into a graded commutative ring.
- There is a multiplication, also called the composition product, on ${}_2\pi_*^S$ (i.e. $\bigoplus_i {}_2\pi_*^S$). It's inherited from the one on π_*^S: we have a product

$$\pi_i(S)_2^\wedge \otimes \pi_j(S)_2^\wedge \to \pi_{i+j}(S)_2^\wedge.$$

1. If $i, j \geq 1$, then $\pi_i(S)_2^\wedge \cong \pi_i(S)/T_i$, where T_i is the subgroup of odd torsion. Then we have a map

$$\begin{aligned} \pi_i(S)/T_i \otimes \pi_j(S)/T_j &\to \pi_{i+j}(S)/T_{i+j} \\ [a] \otimes [b] &\to [ab], \end{aligned}$$

   which is well defined, since if $a' = a+t$ with $nt = 0$ for odd n, then $[a'b] = [ab+tb] = [ab]$ (since $ntb = (nt)b = 0b = 0$). Likewise, if $b' = b + t$ with $nt = 0$ for odd n, then $[ab'] = [ab + at] = [ab]$.

2. If exactly one of i, j is zero, then without loss of generality $i = 0$ and we have a map

$$\mathbb{Z}_2 \otimes \pi_j(S)/T_j \to \pi_j(S)/T_j$$

   defined via the isomorphism $\mathbb{Z}_2 \otimes \pi_j(S)/T_j \cong \pi_j(S)/T_j$ of Lemma A.2.2.

3. If $i = j = 0$, we have the usual product on $\mathbb{Z}_2$:

$$\mathbb{Z}_2 \otimes \mathbb{Z}_2 \to \mathbb{Z}_2.$$

Lemma 4.4.2. Lemma: the multiplication described above is the unique multiplication on ${}_2\pi_*^S$ which makes the completion map $c : \pi_*^S \to {}_2\pi_*^S$ into a ring homomorphism.

Proof. Let $f \in {}_2\pi_i^S$, $g \in {}_2\pi_j^S$. If $i, j \geq 1$, then $f = c(\tilde{f}), g = c(\tilde{g})$ for some $\tilde{f} \in \pi_i^S, \tilde{g} \in \pi_j^S$. Then $fg = c(\tilde{f})c(\tilde{g}) = c(\tilde{f}\tilde{g})$.

If $i = 0, j \geq 1$, then let $\hat{f} \in \pi_0^S$ be a lift of $q(f) \in \pi_0^S/2^r\pi_0^S$, where 2^r is the highest power of 2 dividing the order of π_j^S. Then $f \equiv c(\hat{f}) \mod 2^r$, so $f = c(\hat{f}) + 2^r w$. We have $fg = fc(\tilde{g}) = (c(\hat{f}) + 2^r w)c(\tilde{g}) = c(\hat{f})c(\tilde{g}) + 2^r(wc(\tilde{g})) = c(\hat{f}\tilde{g}) \in {}_2\pi_j^S$.

Finally, if $i = j = 0$, we claim that any two multiplications on $\mathbb{Z}_2$ which agree on $\mathbb{Z}$ must agree on all of $\mathbb{Z}_2$.

Suppose not; let $\star, \cdot$ be two products on $\mathbb{Z}_2$, with $f \star g \neq f \cdot g$. Then there is some k such that $f \star g \not\equiv f \cdot g \mod k$. Pick integers n, m such that $n \equiv f \mod k$ and $m \equiv g \mod k$. Then, modulo k, $f \cdot g \equiv n \cdot m = n \star m \equiv f \star g$, giving a contradiction. □

The following definition is from [7], Chapter 9.

Definition 4.4.3. Let X be a CW spectrum. Then a *p-completion* of X is a map $f : X \to {}_pX$ such that for all i, $\pi_i f$ expresses $\pi_i({}_2X)$ as the p-completion of $\pi_i(X)$.

Theorem 4.4.4 ([7], Thm 9.1.1). If X has finite type, then it has a p-completion unique up to equivalence.

[13]Note, by the way, that this notation is defined in [12] Def 11.1.1, and yes, he really means S^n, as in the normal n-dimensional sphere, so we're taking the smash product of a spectrum and a space, which is fine.

- p-completion has a universal property: whenever $Y \xrightarrow{f} T$ is a map where Z is p-complete, there exists a unique (up to homotopy) factorisation $Y \to {}_pY \xrightarrow{\overline{f}} T$ of f.
- Given spectra X, Y, Z, we have a pairing $\circ : [Y, {}_2Z]_* \otimes [X, {}_2Y]_* \to [X, {}_2Z]$: given $f \in [Y, {}_2Z], g \in [X, {}_2Y]$, we can 'compose' to get $\overline{f}g \in [X, {}_2Z]$.
- Note that suspension commutes with p-completion; the ith suspension of the canonical map $\Sigma^i Y \to \Sigma^i Y_p^\wedge$ descends to a map $\pi_j(\Sigma^i Y) \to \pi_j(\Sigma^i Y_p^\wedge)$, but $\pi_j(\Sigma^i Y)$ is just $\pi_{i+j}(Y)$ and $\pi_j(\Sigma^i Y_p^\wedge)$ is $\pi_{i+j}(Y_p^\wedge)$, so the map on homotopy groups is the one witnessing p-completion.
- Lemma: the completion map $\pi_*(S) \to \pi_*(S_2^\wedge)$ is a ring homomorphism. In particular, the composition product on $\pi_*(S_2^\wedge)$ coincides with the product on $\pi_*(S)_2^\wedge$ described above, so the two groups are also isomorphic as rings.

4.4.3 Multiplication on the spectral sequence

- There is a multiplication on E_r which coincides with the Yoneda product on E_2 and converges (whatever that means) to the composition product on ${}_2\pi_*^S$. It's associative and unital, and plays nicely with the differentials.

Questions/problems (briefly):

- If I compute a product on the E_2 page, and both terms survive to the E_3 page, is that computation still valid?

Definition 4.4.5 ([13], Def 5.5). Let $\{'E_r\}, \{''E_r\}, \{E_r\}$ be three spectral sequences. A *pairing* of these spectral sequences is a sequence of homomorphisms

$$\phi_r : {}'E_r^{*,*} \otimes {}''E_r^{*,*} \to E_r^{*,*}$$

such that the Leibniz rule

$$d_r\phi_r(x \otimes y) = \phi_r(d_r(x) \otimes y) + (-1)^{\deg x}\phi_r(x \otimes d_r(y))$$

holds, and

$$\phi_{r+1}([x] \otimes [y]) = [\phi_r(x \otimes y)],$$

where $[x] \in {}'E_{r+1}^{*,*}$ is the homology class of a d_r-cycle $x \in {}'E_r^{*,*}$, and similarly for y and the right hand side.

A spectral sequence pairing $\{\phi_r\}$ induces a pairing

$$\phi_\infty : {}'E_\infty^{*,*} \otimes {}''E_\infty^{*,*} \to E_\infty^{*,*}.$$

What [12] says:

- 11.8.5: For spectra X, Y, Z, consider the Adams spectral sequences

$$'E_2 = \mathrm{Ext}_A(H_*(Y), H_*(Z)) \implies [Y, Z]_*$$

$$''E_2 = \mathrm{Ext}_A(H_*(X), H_*(Y)) \implies [X, Y]_*$$

$$E_2 = \mathrm{Ext}_A(H_*(X), H_*(Z)) \implies [X, Z]_*$$

1. There is a natural pairing

$$\circ_r : ('E_r, '' E_r) \to E_r$$

of Adams spectral sequences, with abutment the filtration-preserving pairing

$$\circ : [Y, Z]_* \otimes [X, Y]_* \to [X, Z]_*$$

mapping $g \otimes f$ to $g \circ \Sigma^{|g|} f$

2. The pairing of E_2-terms

$$\circ_2 : \mathrm{Ext}_{A_*}(H_*(Y), H_*(Z)) \otimes \mathrm{Ext}_{A_*}(H_*(X), H_*(Y)) \to \mathrm{Ext}_{A_*}(H_*(X), H_*(Z))$$

is the composition product.

3. If Y/p and Z/p are bounded below of finite type, then the E_2-pairing

$$\circ_2 : \mathrm{Ext}_{A_*}(H^*(Z), H^*(Y)) \otimes \mathrm{Ext}_{A_*}(H^*(Y), H^*(X)) \to \mathrm{Ext}_{A_*}(H^*(Z), H^*(X))$$

is the twisted composition product, mapping $y \otimes x$ to $(-1)^{|x||y|} x \circ y$, where $|x| = v - u$ and $|y| = t - s$ for $x \in {''E}_2^{u,v}$ and $y \in {'E}_2^{s,t}$.

What [13] says:

Theorem 4.4.6 ([13], Thm 5.8). Let X, Y, Z be spectra, with Y, Z connective and of finite type. There is a pairing of spectral sequences

$$E_r^{*,*}(Y, Z) \otimes E_r^{*,*}(X, Y) \to E_r^{*,*}(X, Z)$$

which agrees for $r = 2$ with the Yoneda pairing

$$\mathrm{Ext}_{\mathscr{A}_2}^{*,*}(H^*(Z), H^*(Y)) \otimes \mathrm{Ext}_{\mathscr{A}_2}^{*,*}(H^*(Y), H^*(X)) \to \mathrm{Ext}_{\mathscr{A}_2}^{*,*}(H^*(Z), H^*(X))$$

and which convergesto the composition pairing

$$[Y, {}_2Z]_* \otimes [X, {}_2Y]_* \to [X, {}_2Z]_*.$$

The pairing is associative and unital.

[Leibniz rule?]

[13], [12], [7]

5 Calculating stable homotopy groups

5.1 The E_2 page

Lemma 5.1.1 ([6], Lem 5.49). For a minimal free resolution

$$\cdots \to F_2 \to F_1 \to F_0 \to \mathbb{F}_2 \to 0$$

of $\mathbb{F}_2 = H^*(\mathbb{S})$ as an $\mathscr{A}_2$-module, we have $\mathrm{Ext}_{\mathscr{A}_2}^{s,t}(\mathbb{F}_2, \mathbb{F}_2) = \mathrm{Hom}_{\mathscr{A}_2}^{t}(F_s, \mathbb{F}_2)$, where $\mathrm{Hom}_{\mathscr{A}_2}^{t}(F_s, \mathbb{F}_2) \subseteq \mathrm{Hom}_{\mathscr{A}_2}(F_s, \mathbb{F}_2)$ consists of the morphisms which lower the degree by t.

[Now, since $\mathbb{F}_2$ just has stuff in degree 0, the only things that can be sent to $1 \in \mathbb{F}_2$ are the things in degree t, so for every generator for F_s in degree t, there's an $\mathbb{F}_2$'s worth of such homs.]

[13], [6], [12]

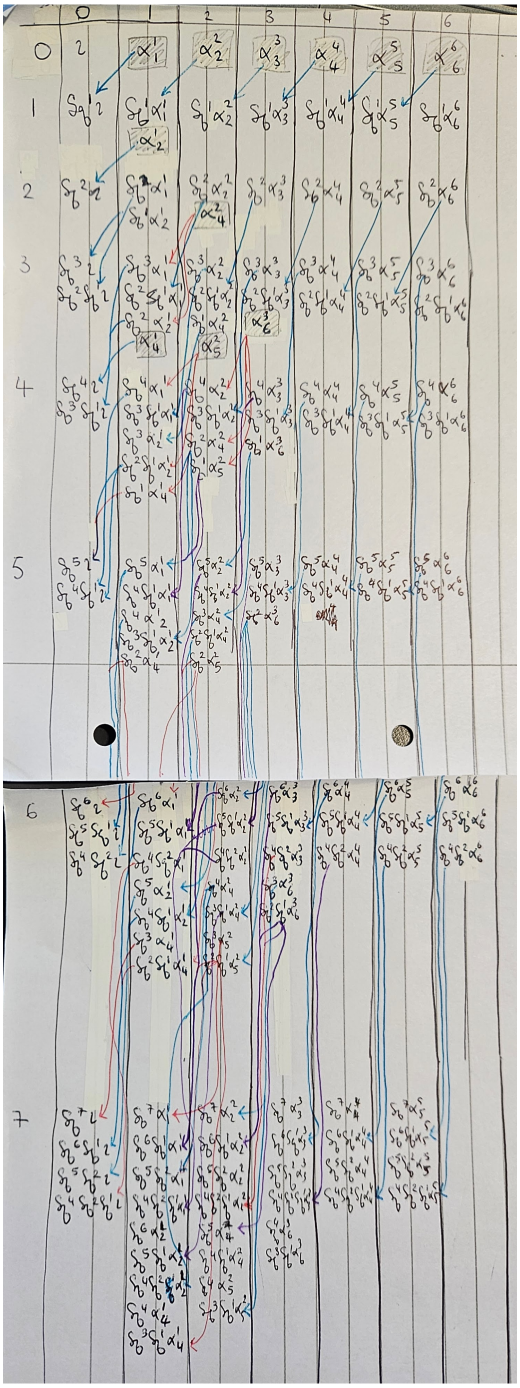

Figure 5.1: Calculating the first 5 rows

Lemma 5.1.2. There are no nontrivial differentials for $t - s \leq 13$.

Proof. First, note that the only possible nontrivial differentials in this range are $d_r : E_r^{1,2} \to E_r^{1+r,1+r}$ and $d_2 : E_2^{2,10} \to E_2^{4,11}$.

Now, $0 = d_r(h_0 h_1) = d_r(h_0)h_1 + h_0 d_r(h_1) = h_0 d_r(h_1)$, so $d_r(h_1) = 0$. Since $E_r^{1,2}$ is generated by h_1, we must have $d_r = 0$.

On the other hand, $E^{2,10}$ is generated by $h_1 h_3$, and $d_2(h_1 h_3) = d_2(h_1)h_3 + h_1 d_2(h_3) = 0 + 0 = 0$ (the first factor is zero by the previous computation, and the second is an element of a trivial group). $\square$

[Obviously justify the h notation and the claims above that the generators are what I say they are]

5.2 Resolving extensions

PROPOSITION 5.2.1 ([13], Cor 6.5). We have the following relations:

$$\begin{aligned} \alpha_i^i &= (\alpha_1^1)^i \\ \alpha_4^2 &= (\alpha_2^1)^2 \\ \alpha_5^2 &= \alpha_1^1\alpha_4^1 \\ \alpha_6^3 &= (\alpha_1^1)^2\alpha_4^1 = (\alpha_2^1)^3. \end{aligned}$$

A few other things:

1. We know that $\pi_0^S = \mathbb{Z}$ since $\pi_1 S^1 = \mathbb{Z}$ and $n = 1 \leq 2 = 2(1)$, so this lies in the stable region. Now, $E^{s,t}$ is actually supposed to converge to some filtration of $\mathbb{Z}_2$ (i.e. the 2-adics) (i.e. ${}_2\mathbb{Z}$) I think? But either way, we know that for our filtration all the quotients are $\mathbb{Z}/2\mathbb{Z}$. I claim[14] then that the filtration must be $\cdots 4\mathbb{Z}_2 \subseteq 2\mathbb{Z}_2 \subseteq \mathbb{Z}_2$.
2. If that's true then $\iota = [1] \in \mathbb{Z}_2/2\mathbb{Z}_2$, and using the multiplication above we see ι is a unit. We also have $h_0 = [2] \in 2\mathbb{Z}_2/4\mathbb{Z}_2$ so $h_0 = [2] = [2[1]] = [2\iota]$, so h_0 acts on ι by multiplication by 2.
3. Now, for any $\kappa \in E_2^{s,t}$, $h_0 \cdot \kappa = (\iota h_0) \cdot \kappa = 2\kappa$.

The notation's a bit weird above, when I say 'multiplication by 2' I mean: take $\kappa \in E^{s,t} = F^{s,t}/F^{s+1,t+1}$. Then $2\kappa \in F^{s+1,t+1}$ since it's a bunch of $\mathbb{F}_2$'s. Take it's equivalence class to get an element of $F^{s+1,t+1}/F^{s+2,t+2}$. That's what I really mean by 2κ and it's in $E^{s+1,t+1}$.

All this is to say if I start multiplying higher things by h_0, that *is* multiplying by 2. So I can start resolving extensions this way.]

[Justify the below with sseq, not quite sure how to do this in a satisfying way though. See note in Section C.2]

THEOREM 5.2.2.

$$ {}_2\pi_i^S = \begin{cases} \mathbb{Z}/2\mathbb{Z} & i = 1,2,6,10 \\ \mathbb{Z}/8\mathbb{Z} & i = 3,11 \\ 0 & i = 4,5,12,13 \\ \mathbb{Z}/16\mathbb{Z} & i = 7 \\ (\mathbb{Z}/2\mathbb{Z})^2 & i = 8 \\ (\mathbb{Z}/2\mathbb{Z})^3 & i = 9. \end{cases}$$

5.3 Nontrivial differentials

[The point here is that all differentials interacting with $E^{s,t}$ for $t - s = 14$ are trivial, and thus computing the stable homotopy groups is purely mechanical, because everything that appears

[14]Here is an absolutely ridiculous argument for that: $(\mathbb{Z}_2, +)$ is a topological group. For any subgroup H of finite index $n = 2^k m$, H is open (since by Lagrange's theorem $n\mathbb{Z}_2 \subseteq H$ and thus $n\mathbb{Z}_2 = 2^k\mathbb{Z}_2$, so $x + 2^k\mathbb{Z}_2$ is an open ball around any $x \in H$). Now, every open subgroup of $\mathbb{Z}_2$ is also closed (since it's complement is a union of open cosets), and it's apparently the case [elaborate] that the closed subgroups of $\mathbb{Z}_2$ are the ideals $2^k\mathbb{Z}_2$. These facts then imply that the filtration I give is the only possible one. Short answer: finite index subgroups of $\mathbb{Z}_p$ are of the form $p^k\mathbb{Z}_p$.

on this part of the E_2 page has to survive to E_∞. Thus, the 'ambiguity' at $t-s=14$ is just the fact that this is the first time you need to actually compute differentials.

1. The d_2 differential at $E_2^{1,16}$ is nontrivial: [12], Thm 11.10.2:
   - h_0 detects[15] $2 \in {}_2\pi_0^S$.
   - h_3 detects some $\sigma \in {}_2\pi_7^S$.
   - By multiplicativity [?] $h_0h_3^2$ detects $2\sigma^2$.
   - ${}_2\pi_*^S$ is graded commutative.
   - Thus, $\sigma^2 = -\sigma^2$, so $2\sigma^2 = 0$.
   - Thus, $h_0h_3^2 = 0$ in E_∞, so it is the boundary of a differential. The only possibility is $d_2(h_4) = h_0h_3^2$.
2. The d_3 differential at $E_3^{2,17}$ is nontrivial: [12], Thm 11.10.7 (which he doesn't prove, so really it's [12] Table 14.2 (10), which uses a differential d_2 of $E_2(C\sigma)$ which is on table 14.9 (4)(!)).

[1], [11], [13], [12].

A Algebra

A.1 Free resolutions

Definition 1.1.1. Let M, N be modules over a ring R. A *resolution* F of M is an exact sequence

$$\cdots \to F_2 \to F_1 \to F_0 \to M \to 0.$$

If in addition each F_i is a free R-module, then the resolution is called *free*.

Given a free resolution as above, applying $\mathrm{Hom}_R(-,N)$ gives us a chain complex

$$\cdots \leftarrow \mathrm{Hom}_R(F_2,N) \leftarrow \mathrm{Hom}_R(F_1,N) \leftarrow \mathrm{Hom}_R(F_0,N) \leftarrow \mathrm{Hom}_R(M,N) \leftarrow 0.$$

Dropping the term $\mathrm{Hom}_R(M,N)$ [why?] we get the sequence

$$\cdots \leftarrow \mathrm{Hom}_R(F_2,N) \leftarrow \mathrm{Hom}_R(F_1,N) \leftarrow \mathrm{Hom}_R(F_0,N) \leftarrow 0,$$

and we define $\mathrm{Ext}^n_R(M,N)$ to be the nth homology group of this chain complex.

[these do not depend on the choice of free resolution of M]

A free resolution is *minimal* if at each stage of its construction we choose the minimal number of free generators for F_i in each degree.

[The above definition is bad but I'm keeping it just for the moment.]

A.2 Completion of abelian groups

Definition 1.2.1 ([8], Def 10.1.1). Let A be an abelian group. Then its *p-adic completion* is the limit

$${}_pA = \lim_{\leftarrow n}(A/p^nA).$$

[15] Make precise.

If $A = \mathbb{Z}$, we instead write $\mathbb{Z}_p$ for the p-adic integers.

There is a natural map $A \to {}_pA$, whose component at n is reduction modulo p^nA.

[[8] p154]:

?When A is finitely generated, its p-adic completion is given by the map $A \to A \otimes \mathbb{Z}_p$; $a \mapsto a \otimes 1$.

Lemma 1.2.2. Suppose A is finite, and write $|A| = np^r$ for $p \not| n$. Then $A \otimes \mathbb{Z}_p \cong A/T$, where $T \subseteq A$ is the subgroup generated by all torsion coprime to p.

Proof. Define a homomorphism $A \otimes \mathbb{Z}_p \to A/T$ sending $a \otimes z \mapsto [q(z)a]$, where q is the projection $\mathbb{Z}_p \twoheadrightarrow \mathbb{Z}/p^r\mathbb{Z}$. Suppose $a \otimes z \mapsto 0$. Then $q(z)a \in T$, so $kq(z)a = 0$ for some k coprime to p, and thus $q(z)a \otimes z = kq(z)a \otimes \frac{1}{k}z = 0$. On the other hand, $z = q(z)z'$ for some $z' \in \mathbb{Z}_p$, so $a \otimes z = q(z) \otimes z'$. Thus, $a \otimes z = 0$, so the map is injective. The map is clearly also surjective, since $a \otimes 1 \mapsto [a]$, so it is an isomorphism. $\square$

Remark 1.2.3. If A is finite with order np^r for $p \not| n$, then $|{}_pA| = p^r$, by Cauchy's theorem.

B Topology

All from [5] unless otherwise stated.

B.1 Suspension

Definition 2.1.1. Let X be a topological space. The *suspension* SX is the space $(X \times I)/\sim$, where $(x, 0) \sim (x', 0)$ and $(x, 1) \sim (x', 1)$ for all $x, x' \in X$.

Definition 2.1.2. Let X be a pointed topological space. The *reduced suspension* ΣX is the space $SX/\sim$, where $[x_0, t] \sim [x_0, t']$ for all $t, t' \in I$.

Given a map $f : X \to Y$, we can define $\Sigma f : \Sigma X \to \Sigma Y$ by $\Sigma f[(x, t)] = [(fx, t)]$. This makes Σ into a functor $\Sigma : \mathbf{Top} \to \mathbf{Top}$.

Remark 2.1.3. Σ is faithful, since for any maps $f, g : X \to Y$, if $\Sigma f = \Sigma g$ then in particular $[(fx, \frac{1}{2})] = [(gx, \frac{1}{2})]$, so $fx = gx$.

[below is reconstructed from [9]]

Given pointed maps $f, g : \Sigma X \to Z$, define

$$\begin{aligned} f \star g : \Sigma X &\to Z \\ [x, t] &\mapsto \begin{cases} f[x, 2t-1] & t \geq \frac{1}{2}, \\ g[x, 2t] & t \leq \frac{1}{2}. \end{cases} \end{aligned}$$

This is well defined, since both f and g are basepoint-preserving.

Remark 2.1.4. This defines a group structure on $[\Sigma X, Z]$, and thus $[\Sigma^i X, Z]$ is a group for all $i \geq 1$. For $i \geq 2$, these can be shown to be abelian, via the Eckmann-Hilton argument.

Remark 2.1.5. The homotopy groups $\pi_i(Z)$ are a special case of the above construction, taking $X := S^{i-1}$.

- Loops; the adjunction $\Sigma \dashv \Omega$, where Ω is the loop functor.

[5], p395:

Remark 2.1.6. It follows that $\pi_{n+1}(X) \cong \pi_n(\Omega X)$. In particular, $\Omega K(G,n)$ is a $K(G,n-1)$.

- [5] 2.1 Ex 20 and 2.2 Ex 32: $\widetilde{H}_n(X) \cong \widetilde{H}_{n+1}(SX)$, where S is the (non-reduced) suspension. (MV?)
- Hatcher also says on p219 that $\widetilde{H}^n(X;R) \cong \widetilde{H}^{n+k}(\Sigma^k X;R)$, where Σ is reduced suspension.

B.2 Other basic constructions

Definition 2.2.1. Let $(X,x_0),(Y,y_0)$ be pointed topological spaces, and consider their product $X \times Y$. The subspaces $X \times \{y_0\} \cong X$ and $\{x_0\} \times Y \cong Y$ intersect at exactly one point, (x_0,y_0), and so can be identified with the wedge $X \vee Y$. We thus define the *smash product* $X \wedge Y := (X \times Y)/(X \vee Y)$, with the canonical basepoint (x_0,y_0).

Example 2.2.2. We have $S^n \wedge S^m \cong S^{n+m}$. [is this obvious?]

Remark 2.2.3. Note that $\Sigma X \cong X \wedge S^1$.

Remark 2.2.4. Observe that $X \wedge (Y \wedge Z) \cong (X \wedge Y) \wedge Z$. Combining this with the remarks above, we see that $\Sigma^k X \cong X \wedge S^k$.

Remark 2.2.5. Note that $\Sigma(X \vee Y) \cong \Sigma X \vee \Sigma Y$.

- The Eilenberg-MacLane space is $K(G,n)$, and it has the property that

$$\pi_i(K(G,n)) = \begin{cases} G & i = n, \\ 0 & i \neq n. \end{cases}$$

  They're unique up to weak homotopy equivalence (i.e. if you have another one X, there's a map between them which descends to an isomorphism on homotopy groups). They can be taken to be CW complexes.

Definition 2.2.6. Let X, Y be topological spaces, where X has a basepoint x_0. Then the *reduced product* $X \times_{\mathrm{red}} Y := (X \times Y)/(x_0 \times Y)$.

Definition 2.2.7. Let $f : X \to Y$ be a map. The *mapping cylinder* M_f is defined by $((X \times I) \sqcup Y)/\sim$, where $(x,1) \sim f(x)$ for all $x \in X$. If $(X,x_0),(Y,y_0)$ are pointed spaces, the *reduced mapping cylinder* is the quotient $M_f/\sim$, where $[x_0,t] \sim [x_0,t']$ for all $t \in I$.

Remark 2.2.8. The mapping cylinder deformation retracts onto Y via $h : M_f \times I \to M_f$; $([x,t],s) \mapsto [x,t+s(1-t)]$.

Definition 2.2.9. Let $f : X \to Y$ be a map. The *mapping cone*[16] C_f is defined to be $Y \sqcup_f CX := (Y \sqcup CX)/(f(x) \sim [x,1])$.

B.3 Cell complexes

Definition 2.3.1. Let X be a cell complex, $A \subseteq X$ a subcomplex. Then the quotient X/A has a cell complex structure, with cells the cells of $X \setminus A$ along with a basepoint (the image of A in X).

Definition 2.3.2. Let $f : X \to Y$ be a map between CW complexes. Then f is *cellular* if $f(X_{(n)}) \subseteq Y_{(n)}$ for all n, where $X_{(n)}$ is the n-skeleton of X.

[16] Why does Hatcher not insist this guy is reduced, like he does with the mapping cylinders?

Cellular approximation theorem:

THEOREM 2.3.3 ([5], Thm 4.8). Let $f : X \to Y$ be a map of CW complexes. Then f is homotopic to a cellular map.

LEMMA 2.3.4 ([5], Prop 0.16). Let $A \subseteq X$ be CW complexes. Then the pair (X, A) has the *homotopy extension property*; that is, for any map $f : X \to Y$ and homotopy $h : A \times I \to Y$ such that $h(a, 0) = f|_A$, there is a homotopy $\widetilde{h} : X \times I \to Y$ extending h.

- The product of cell complexes is a cell complex (maybe only if one of them is finite?)
- The smash product of (pointed?) cell complexes is a cell complex (maybe only if one is them is finite?) [[5] says "the smash product $X \wedge Y$ is a cell complex if X and Y are cell complexes with x_0 and y_0 0-cells, assuming that we give $X \times Y$ the cell-complex topology rather than the product topology in cases where these two topologies differ".]
- For a CW complex X, $SX \simeq \Sigma X$.
- The reduced suspension of a pointed cell complex (X, x_0) is another pointed cell complex ΣX with basepoint x_0 and an n-cell for each non-basepoint $n-1$ cell e_α^{n-1} of X.

DEFINITION 2.3.5. Let X is a topological space. A *CW approximation* to X is a CW complex Z equipped with a weak homotopy equivalence $f : Z \to X$.

THEOREM 2.3.6 ([5], Prop 4.13). Every space X has a CW approximation $f : Z \to X$.

- In particular, $\Omega K(G, n)$ has a CW approximation $Z \to \Omega K(G, n)$, and since $\Omega K(G, n)$ is a $K(G, n-1)$, so is Z.
- Something along the lines of 'compact $\leftrightarrow$ finite number of cells'.

C Notes to self

C.1 Vague problems and questions....

C.1.1 ...that probably don't matter

- On p588 of [6], he says "every CW spectrum is equivalent to a suspension spectrum". Does he actually mean that, or does he mean 'equivalent to the suspension of a spectrum'? The former seems way too strong, although in fairness I still don't know what an equivalence of spectra actually *is*.
- On p586 of [6], Hatcher says "If X is of finite type then for each i there is an n such that X_n contains all the i-cells of X. It follows that $H_i(X; G) = H_i(X_n; G)$ for all sufficiently large n, and the same is true for cohomology." But from the way he set up H_* and H^* earlier, shouldn't this be $H_i(X; G) = H_{i+n}(X_n; G)$? Because $H_i(X; G) = \varinjlim H_{i+n}(X_n)$, and he talks about things stabilising in the next sentence, so shouldn't the stable point be at some H_{i+n}?
- I write $\mathscr{A}_2$ where Hatcher writes $\mathscr{A}$. We mean the same thing, right...?

C.1.2 ...that probably do matter

- I am definitely being told some lies about what the spectral sequence actually converges to. There's a strong implication/actual statement(!!) that at each i it's supposed to be a filtration of π_i^S modulo odd torsion, but I think this isn't true. I think it's actually the

2-completion of π_i^S. That coincides with the p-primary part for finite abelian groups, but for π_0^S it's supposed to be $\mathbb{Z}_2$ (i.e. the 2-adic integers), not $\mathbb{Z}$. I believe. Maybe get a source for this. Some people say it's the localisation at 2?? But I think that's also a lie.

- The Leibniz rule is $d_r(xy) = d_r(x)y \pm xd_r(y)$ (can't remember the sign). But anything I'm using that rule on is some generator of an $\mathbb{F}_2$, right? So the sign shouldn't matter. But then, shouldn't the Yoneda product be graded commutative (and thus commutative, because again, in the target signs don't matter)? So why does [13] have some comment (in Cor 6.5) about how the Yoneda product is commutative "in [some] range"??
- On p592 of [6], he says that "for spectra X of finite type [the more general] definition of an $\mathscr{A}_2$-module structure on $H^*(X)$ agrees with the definition using the usual $\mathscr{A}_2$-module structure on the cohomology of spaces and the identification of $H^*(X)$ with the inverse limit $\varprojlim H^{*+n}(X_n)$". Um? Sure, we have that each $H^{i+n}(X_n)$ stabilises eventually, but is Hatcher saying $H^{*+n}(X_n)$ stabilises? Like, as an $\mathscr{A}_2$-module? And if not, what's going on here? Because inverse limits don't commute with infinite direct sums - they're not biproducts anymore, they're coproducts and there's no reason limits should commute with them.
- There's something weird going on with products. So, things are ok in **Top**, because we have the ordinary product of two spaces, which is a categorical product. But with CW complexes, supposedly sometimes the product topology differs from the 'cell complex topology'? But, regardless, we're supposed to be working with pointed things - so in $\mathbf{Top}_*$, the pointed coproduct is the wedge sum, and the pointed product is just the normal product $X \times Y$ with the basepoint (x_0, y_0) (it's not the smash product). But what about in spectra? No one ever seems to talk about products of spectra, but for example a collection of maps $X \to \mathbb{K}(G, n_i)$ should correspond to a single map $X \to \prod_i \mathbb{K}(G, n_i)$, whatever that last object is.

  The plot thickens. From the nLab: "[some smash product] is non-canonically equivalent to a product of EM-spectra (hence a wedge sum of EM-spectra in the finite case)". ???????
- Question: so I have this multiplicative structure on the entire spectral sequence, right? And on the second page it's just the Yoneda product. But say I compute, I don't know, $\alpha_5^2 = h_0 h_2$ on the E_2 page. Since none of those guys are touched by differentials, is that computation still valid on the E_3 page? And on higher pages?

C.2 To do

Now:

- Find out what the "suspension isomorphism given by reduced cross product with a generator of $H^1(S^1; \mathbb{F}_2)$" is.
- Find out what the $\mathbb{F}_2$ Bockstein homomorphism is.
- Write down the induced maps on π_* and H^*/H_* from maps of spectra.
- Check composition of maps of spectra is well-defined.
- Prove that a spectrum is equivalent to the suspension of some other spectrum.
- Prove that $[X, Y]$ is an abelian group for X, Y spectra.
- Prove that the suspension map $[X, Y] \to [\Sigma X, \Sigma Y]$ is a group isomorphism.

- Derive the exact sequence

$$[Y,A] \to [Y,X] \to [Y,X/A] \to [Y,\Sigma A] \to [Y,\Sigma X].$$

- Rewrite Sections 4.1 and 4.2 and fix the grading.
- Show the action of $\mathscr{A}_2$ on $H^*(X)$ is actually a module action.
- Figure out the whole wedge/product of EM spectra nonsense and why K_0 has finite type.
- Figure out when and why maps can be replaced by inclusions.
- Figure out why applying H^* to the sequence $k_0 \to K_1 \to \cdots$ makes it exact.
- Show that the $H*(K_s)$'s are free $\mathscr{A}_2$-modules.
- Show that there is a natural map $[\mathbb{S}, K_s] \to \mathrm{Hom}^0_{\mathscr{A}_2}(H^*(K_s), H^*(\mathbb{S}))$, and that it's an isomorphism (Yoneda?).
- Find out how much of this you should be proving.
- Explain the Yoneda product at all coherently, and compute some examples.
- Prove that π_*^S is a graded commutative ring.
- Find out where the universal property of p-completion comes from.
- Find a way to translate the calculations in 5.1 into readable and coherent LaTeX'd calculations, and make some figures of pages of the spectral sequence (up to 14 I guess..)
- Make the α^i_j notation not appalling - link it to the usual h notation.
- Make the language of 'detection' precise.
- Prove that the d_3 differential at $E^{2,17}$ is nontrivial.

Eventually:

- Be consistent with either cell complex or CW complex.
- Be consistent with $\mathbb{F}_2$ or $\mathbb{Z}/2\mathbb{Z}$ (don't use $\mathbb{Z}_2$, that's really bad).
- Specialise the Adams spectral sequence (i.e. set $Y = \mathbb{S}$).
- Remember that you have to hand in the tex file, so for the love of god change anything stupid that's hidden in the pdf.
- Sometimes I say π_*^S or ${}_{(2)}\pi_*^S$ (localised at 2?) instead of its completion at 2 or whatever. So make sure it's correct.
- Stick to a convention on suspension/cone/homotopy numbering. I.e. Does a homotopy start at 0 or 1? Does a suspension go from -1 to 1 with the space in the middle at 0, or 0 to 1 with the space at 1/2? Do cones go from 0 to 1, and if so, make sure when they include into suspensions they do so consistently.
- I want to at some point compute $h_0h_1h_3$ or something like that, just to show that sometimes a product can be zero even if it lands in a nonzero group (i.e. the stable groups aren't all cyclic). The trouble is by the time you get to π_8^S the free resolution is an awful mess. Maybe I can try and wrangle with the sseq program to get it to show me kernels and the like?

- Ease notation by not writing G in e.g. $H^*(X; G)$. Say we're taking coefficients in $\mathbb{F}_2$, but actually a lot of the time it doesn't matter so maybe note that.
- Have any sort of consistency in using or not using brackets (e.g. $\pi_t X_s$ v.s. $\pi_t(X_s)$).
- When I say 'spectrum' at any point after defining CW spectra I mean 'CW spectrum'. And I basically always mean 'connective CW spectrum of finite type' too.

C.3 Other notes

- READ IF YOUR CALCULATIONS AREN'T WORKING: You are working modulo 2!!!
- If you have a bunch of maps between graded modules/algebras, they're graded homomorphisms. So they preserve degree.
- All (co)homology is supposed to be reduced.
- Signs don't matter with the Leibniz rule either!! You are working modulo 2!!!!!!!!
- Remember, once you know that $d_2(h_4) = h_0 h_3^2$, you know h_4 *doesn't survive to the third page*. So, for example, $d_3(h_0 h_4) \neq h_0 d_3(h_4)$ because h_4 doesn't exist anymore. That's why $d_3(h_0 h_4)$ can be nonzero.
- As previously mentioned, we are working modulo 2!! What this also implies is that if anything is hit by any sort of differential, or has any nonzero differential coming out of it, it's completely killed by the next page. Because the summands are just a bunch of $\mathbb{F}_2$'s (so you don't need to worry about 'how much' of something is killed, it all is).
- Sometimes Hatcher says that you can replace any map of CW complexes by an inclusion. I think the point here is that if you have a map $f : X \to Y$, Remark B.2.8 says that M_f deformation retracts onto Y. So if you only care about X and Y up to homotopy equivalence, you can replace Y by M_f and then X definitely includes into M_f.
- Where it's ambiguous, I'm marking things I definitely need by ! and things I think I may not need by ?.
- By the way, the argument at footnote 12 looks a little unsettling, because it seems like you could use it to show that *all* homotopy groups are trivial eventually. And that's true! But when you take the colimit $\pi_i(X) = \mathrm{colim}_n\, \pi_{i+n} X_n$, notice that it's over π_{i+n}, not π_i. So yes, the connectivity of X_n increases with n, but the index of π also does. Basically, it's all fine. I'm putting this note here because I will invariably forget this and have a heart attack a day before this is due thinking everything is broken.
- In literature, $A_p^\wedge$ is the p-adic completion of A. Sometimes I'll write this as ${}_pA$ because of some stupid notational decisions I made earlier.
- The 'abutment' of a spectral sequence apparently means the thing it converges to (i.e. if E_∞ computes the associated graded of some H^*, the abutment of $\{E\}$ is H^* (not its associated graded)).

Sources I've used: [11], [13], [7], [6], [5], [12], [8], [10]

Sources I probably won't use: [4], [1], [2], [3], [9] (I think the construction I need is in Hatcher)

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