Stable Homotopy Groups of Spheres [DRAFT]

Contents

1 Introduction

- Define homotopy groups
- Freudenthal's suspension theorem: if $\pi_i(X) = 0$ for $i \le k$ (i.e. X is k-connected) then the map

$$\begin{array}{rcl} \pi_n(X) & \to & \pi_{n+1}(\Sigma X) \\ [\gamma : S^n \to X] & \mapsto & [\Sigma\gamma : \Sigma S^n = S^{n+1} \to \Sigma X] \end{array}$$

  is an isomorphism for $n \le 2k$ and surjective for $n = 2k+1$
- This implies $\pi_{n+k}(S^n)$ depends only on k for $n \ge k+2$
- (Obviously be careful with basepoints above)
- Suppose X is k-connected. Then, for $k \ge 0$, $0 = \pi_k(X) \cong \pi_{k+1}(\Sigma X)$, so whenever a space is k-connected its suspension is $k+1$-connected.
- As you take suspensions, then, your successive bounds are $n \le 2k$, $n+1 \le 2k+2 \implies n \le 2k+1$, $n \le 2k+2$, etc ... so the sequence $\pi_n(X) \to \pi_{n+1}(\Sigma X) \to \cdots$ will eventually stabilise.
- Thus, if you take the colimit of that direct system, it'll just equal the stable value, with the higher legs just being the inverse isomorphisms.
- [11], Cor 1.9 [not 100% convinced of how this follows, but believing it for now]: if X is a CW complex of dimension d and Y a $(k-1)$-connected space, then the suspension homomorphism $[X, Y] \to [\Sigma X, \Sigma Y]$ is bijective if $d < 2k-1$ and surjective if $d = 2k-1$.

Miscellaneous facts I might need later:

- Cohomology [possibly only of pointed[1] CW complexes] is representable[2], and its representing object is the Eilenberg-MacLane space. i.e. $H^n(-;G) \cong \mathrm{Hom}(-, K(G,n))$.
- $\mathscr{A}_2$ is generated as an algebra by elements Sq^{2^k} ([5], Prop 4L.8).
- The map $\mathscr{A}_2 \to \tilde{H}^*(K(\mathbb{Z}/2\mathbb{Z}, n); \mathbb{Z}/2\mathbb{Z})$, $Sq^I \mapsto Sq^I(\iota_n)$ is an isomorphism from the degree d part of $\mathscr{A}_2$ onto $H^{n+d}(K(\mathbb{Z}/2\mathbb{Z}, n); \mathbb{Z}/2\mathbb{Z})$ for $d \ge n$. In particular, the admissible monomials Sq^I form an additive basis for $\mathscr{A}_2$. Thus, $\mathscr{A}_2$ is exactly the algebra of all $\mathbb{Z}/2\mathbb{Z}$ cohomology operations that are stable, commuting with suspension ([6], Cor 5.38).
- "Stable homotopy groups are a homology theory" (whatever that means)
- Hurewicz theorem: for any path-connected space X and $n > 0$ there exists a group homomorphism $h_* : \pi_n(X) \to H_n(X)$. For $n = 1$ this induces an isomorphism $\pi_1^{\mathrm{ab}}(X) \cong H_1(X)$. For $n \ge 2$, if X is $(n-1)$-connected then $\tilde{H}_i(X) = 0$ for all $i < n$, and the map $h_* : \pi_n(X) \to H_n(X)$ is an isomorphism.

[11], [4], [5]

[1] What's the relevance of the 'pointedness' when you're only taking cohomology?? See C.

[2] As a set, or is this some sort of enriched thing? If it's enriched, is that over **Ab** or **Rng**? [5] says on p394 that there is a natural group structure on $\mathrm{Hom}(X, K(G,n))$ such that the natural isomorphism $\mathrm{Hom}(X, K(G,n)) \to H^n(X;G)$ is in fact an isomorphism of abelian groups. So, it's over **Ab**? N.B: There's a lot of talk about 'reduced cohomology theories', so a good next step would be to figure out what those are - if they only involve groups and not rings, maybe the cup product on H^* is not relevant here.

2 The Steenrod algebra

The following is from [5] 4L.

- There are maps $Sq^i : H^n(-;\mathbb{Z}/2\mathbb{Z}) \to H^{n+i}(-;\mathbb{Z}/2\mathbb{Z})$ for each i, and they satisfy the following properties:
  1. $Sq^i_X(f^*(\alpha)) = f^*(Sq^i_Y(\alpha))$ for $f : X \to Y$ (i.e. Sq^i is a natural transformation).
  2. $Sq^i_X(\alpha+\beta) = Sq^i_X(\alpha) + Sq^i_X(\beta)$ (i.e. Sq^i_X respects the group operation for all X).
  3. $Sq^i(\alpha \smile \beta) = \sum_{0\le j\le i} (Sq^j(\alpha) \smile Sq^{i-j}(\beta))$ (the Cartan formula)
  4. $Sq^i(\sigma(\alpha)) = \sigma(Sq^i(\alpha))$ where $\sigma : H^n(X;\mathbb{Z}/2\mathbb{Z}) \to H^{n+1}(\Sigma X;\mathbb{Z}/2\mathbb{Z})$ is the "suspension isomorphism given by reduced cross product with a generator of $H^1(S^1;\mathbb{Z}/2\mathbb{Z})$". [see: [5], p219. N.B: I think this "relative cross product" theory is required – you can argue that there is an isomorphism via MV, but this point says that it's this specific one. Maybe they're the same, but Hatcher doesn't say that anywhere and there could be many isomorphisms.]
  5. $Sq^i(\alpha) = \alpha^2$ if $i = \deg(\alpha)$ and $Sq^i(\alpha) = 0$ if $i > \deg(\alpha)$.
  6. $Sq^0 = \mathrm{id}$.
  7. Sq^1 is the "$\mathbb{Z}/2\mathbb{Z}$ Bockstein homomorphism β associated with the coefficient sequence $0 \to \mathbb{Z}/2\mathbb{Z} \to \mathbb{Z}/4\mathbb{Z} \to \mathbb{Z}/2\mathbb{Z} \to 0$".
- Define $Sq := Sq^0 + Sq^1 + \cdots$. Then $Sq(\alpha \smile \beta) = Sq(\alpha) \smile Sq(\beta)$ (since $(Sq(\alpha\smile\beta))_n = \sum_i Sq^i(\alpha) \smile Sq^{n-i}(\beta) = (Sq(\alpha) \smile Sq(\beta))_n$). Thus, Sq is a ring homomorphism.
- Adem relations:

$$Sq^a Sq^b = \sum_j \binom{b-j-1}{a-2j} Sq^{a+b-j} Sq^j \quad \text{if } a < 2b,$$

  where $\binom{m}{n}$ is zero if m or n is negative, or $m < n$, and $\binom{m}{0} = 1$ for $m \ge 0$.
- The Steenrod algebra $\mathscr{A}_2$ is the algebra over $\mathbb{Z}/2\mathbb{Z}$ that is the quotient of the algebra of polynomials in the noncommuting variables $Sq^1, Sq^2, \ldots$ by the two-sided ideal generated by the Adem relations. Thus, for every space X, $H^*(X;\mathbb{Z}/2\mathbb{Z})$ is a module over $\mathscr{A}_2$.
- $\mathscr{A}_2$ is graded, and its elements of degree k are those that map $H^n(X;\mathbb{Z}/2\mathbb{Z})$ to $H^{n+k}(X,\mathbb{Z}/2\mathbb{Z})$ for all n. [Presumably you've fixed a space X while you're doing all this?]

[1], [10], [11], [7], [5], [2]

3 Spectra may not be your friends, but I can introduce you

3.1 Categorical nonsense

- [11]: There is a category $\mathcal{H}$ of finite based CW complexes, with $\mathrm{Hom}(X,Y) =: [X,Y]$ the set of homotopy classes of base-point preserving maps $X \to Y$.

- There is a category $\mathbf{St}(\mathcal{H})$ of finite based CW complexes, with $\mathrm{Hom}(X, Y) =: \{X, Y\}$ the set $\mathrm{colim}_i[\Sigma^i X, \Sigma^i Y]$ [it's just a colimit of sets, and **Set** is cocomplete, so we should be fine. [11] says it's a group[3]] [Also, how do these guys compose?]
- There is a functor $\mathcal{H} \to \mathbf{St}(\mathcal{H})$. [11] doesn't say what this is but it's presumably the one that is the identity on objects and sends $[f : X \to Y] \in [\Sigma^0 X, \Sigma^0 Y]$ to whatever it gets sent to in $\{X, Y\}$ using the universal property of the colimit. Uniqueness makes it functorial, etc.
- We have a fully faithful functor $\mathbf{St}(\mathcal{H}) \to \mathbf{St}(\mathcal{H})$ given by the suspension on objects, and the unique isomorphism $\{X, Y\} \to \{\Sigma X, \Sigma Y\}$ on maps (such an isomorphism exists, since both of those things are colimits for $[\Sigma^i X, \Sigma^i Y]$ - one of the sequences is cut off at the beginning, but it doesn't matter because both reach the stable value (see above discussion and [11] 1.9), aka the colimit).
- It's not an equivalence, because not every object is isomorphic to a suspension (e.g. anything not connected, since suspensions always connected [?])
- We can formally adjoin desuspensions $\Sigma^{-n}X$ for all n [does this mean just putting the objects there and defining $\mathrm{Hom}(Y, \Sigma^{-n}X) := \mathrm{Hom}(\Sigma^n Y, X)$ and $\mathrm{Hom}(\Sigma^{-n}X, Y) := \mathrm{Hom}(X, \Sigma^n Y)$?], but this category does not have weak colimits (i.e. colimits w/o uniqueness property). [why does it not, and why do we even want that?]
- We instead consider formal sequences of desuspensions $X_0 \to \Sigma^{-1}X_1 \to \cdots$, or sequences (X_n) and maps $\Sigma X_n \to X_{n+1}$, i.e. spectra. [and this fixes the problem?]

3.2 Definitions and examples

Below follows [6], Section 5.2.

[Maybe I could also look at [5] p454 onwards?]

Definition 3.2.1. A *spectrum* is a collection of pointed topological spaces $\{X_n\}_{n\in\mathbb{N}}$, together with basepoint-preserving maps $\sigma_n : \Sigma X_n \to X_{n+1}$.

Example 3.2.2. Let X be a topological space. The *suspension spectrum* of X, denoted by $\Sigma^\infty X$, has $X_n = \Sigma^n X$ and $\sigma_n = \mathrm{id} : \Sigma X_n \to X_{n+1}$.

We write $\mathbb{S} := \Sigma^\infty S^0$, and call $\mathbb{S}$ the *sphere spectrum.*

Example 3.2.3. The *Eilenberg-MacLane spectrum* has X_n a CW complex $K(G, n)$ and $\sigma_n : \Sigma K(G, n) \to K(G, n+1)$ is the adjoint of the CW approximation $K(G, n) \to \Omega K(G, n+1)$.

[N.B. the important point of the above is not that it's 'an' EM space rather than 'the' (they're all homotopy equivalent). It's that a) we can definitely construct one that's a CW complex, and b) even though Ω(a CW complex) is not necessarily a CW complex, we can make it one via CW approximation.]

Definition 3.2.4. Let $X = \{X_n\}$ be a spectrum. We define $\pi_i(X) = \mathrm{colim}_n\, \pi_{i+n}(X_n)$, where the map $\pi_{i+n}(X_n) \to \pi_{i+n+1}(X_{n+1})$ is given by the composition

$$\pi_{i+n}(X_n) \xrightarrow{\Sigma} \pi_{i+n+1}(\Sigma X_n) \xrightarrow{(\sigma_n)_*} \pi_{i+n+1}(X_{n+1}).$$

[3] The colimit is equal to the stable value (which exists, by the corollary). After Σ^2, these guys are all groups, so the colimit also has a group structure inherited from whatever $[\Sigma^k X, \Sigma^k Y]$ it's equal to. N.B: Remarks about cocompleteness of **Set** are misleading because that doesn't actually matter - any sequence that stabilises in any category will have a filtered colimit equal to that stable value, you don't need any extra conditions.

EXAMPLE 3.2.5. If X is a topological space, then $\pi_i(\Sigma^\infty X) = \pi_i^S(X)$, the ith stable homotopy group of X.

DEFINITION 3.2.6. A CW spectrum is a spectrum X consisting of CW complexes X_n with the maps $\Sigma X_n \hookrightarrow X_{n+1}$ inclusions of subcomplexes.

[Define cells and dimension of a CW spectrum]

[Insert brief but enlightening observation of the effect of suspension on cells of CW complexes here]

DEFINITION 3.2.7. A spectrum X is *connective* if its cells have dimensions which are bounded below.

[The above is phrased exactly as in hatcher - presumably he means that there is some absolute bound below which no cell has dimension, rather than a bound dependent on each cell?]

DEFINITION 3.2.8. A CW spectrum is *finite* if it has only finitely many cells, and *of finite type* if it has only finitely many cells in each dimension.

3.3 Homology and cohomology

[From Hatcher: "the inclusions $\Sigma X_n \hookrightarrow X_{n+1}$ induce inclusions $C_*(X_n; G) \hookrightarrow C_*(X_{n+1}; G)$ with a dimension shift to account for the suspension". Below is my vague explanation of what I understand this to mean.

$C_i(X_n; G)$ is the free abelian group on maps $\Delta^i \to X_n$. I claim $\Sigma\Delta^i \cong \Delta^{i+1}$. If this is true, it gives a map

$$\begin{aligned} C_i(X_n; G) &\to C_{i+1}(\Sigma X_n; G) \\ f &\mapsto \Sigma f. \end{aligned}$$

This is an injection, by Remark B.1.3. We also have an injection $C_{i+1}(\Sigma X_n; G) \to C_{i+1}(X_{n+1}; G)$ induced by the structure map σ_n, so we get an injection $C_i(X_n; G) \hookrightarrow C_{i+1}(X_{n+1}; G)$, which indeed has a dimension shift.

Some issues:

- The way it's phrased, it seems to be that this is a morphism of chain complexes - i.e. these maps commute with the ∂s. Why would they?]

[Define H^* and H_* of a spectrum]

DEFINITION 3.3.1. Let $X = \{X_n\}$ be a CW spectrum. A *subspectrum* X' of X is a sequence of subcomplexes $\{X'_n \subseteq X_n\}$ satisfying $\Sigma X'_n \subseteq X'_{n+1}$. The subspectrum X' is *cofinal* if, for each n and each cell e^i_α of X_n, the cell[4] $\Sigma^k e^i_\alpha$ belongs to X'_{n+k} for all sufficiently large k.

[N.B. If $\Sigma^k e^i_\alpha$ belongs to X'_{n+k} then $\Sigma^{k+1} e^i_\alpha$ belongs to $\Sigma X'_{n+k} \hookrightarrow X'_{n+k+1}$ (I think!!), so if it happens once it'll happen for all time after that. Thus, if X', X'' are cofinal spectra of X with $\Sigma^k e^i_\alpha$ a cell of X'_{n+k} and $\Sigma^l e^i_\alpha$ a cell of X''_{n+l} ($l \geq k$) then $\Sigma^l e^i_\alpha$ is a cell of X'_{n+l} and therefore of $X'_{n+l} \cap X''_{n+l}$. In other words, the intersection of two cofinal spectra is a cofinal spectrum.]

DEFINITION 3.3.2. Let X, Y be CW spectra. A *strict map* $f : X \to Y$ is a sequence of

[4] Assuming it is actually a cell, see note in B.3.

cellular maps $f_n : X_n \to Y_n$ such that the diagram below commutes.

$$\begin{tikzcd} \Sigma X_n \arrow[r, "\sigma_n"] \arrow[d, "\sigma f_n"'] & X_{n+1} \arrow[d, "f_{n+1}"] \\ \Sigma Y_n \arrow[r, "\sigma_n"'] & Y_{n+1} \end{tikzcd}$$

[this induces maps $\pi_i(X) \to \pi_i(Y)$, $H^*(Y) \to H^*(X)$, $H_*(X) \to H_*(Y)$.]

DEFINITION 3.3.3. A *map* of CW spectra $f : X \to Y$ is an equivalence class of strict maps $f' : X' \to Y$ with X' a subspectrum of X, where two strict maps $f' : X' \to Y$ and $f'' : X'' \to Y$ are equivalent if they agree on some common cofinal spectrum.

[this also induces maps $\pi_i(X) \to \pi_i(Y)$, $H^*(Y) \to H^*(X)$, $H_*(X) \to H_*(Y)$.]

[check composition is well defined]

[working definition of equivalence below:]

DEFINITION 3.3.4. Two spectra X, Y are *equivalent* if there are maps $f : X \to Y$ and $g : Y \to X$ such that $fg = \mathrm{id}_Y$ and $gf = \mathrm{id}_X$.

[A spectrum is always equivalent to the suspension of some other spectrum]

DEFINITION 3.3.5. A *homotopy* of maps between spectra is a map $X \times I \to Y$, where $X \times I$ is the spectrum with $(X \times I)_n = X_n \times_{\mathrm{red}} I$.

Note that $\Sigma(X_n \times_{\mathrm{red}} I) = \Sigma X_n \times_{\mathrm{red}} I$. The set of homotopy classes of maps $X \to Y$ is denoted by $[X, Y]$.

[[1] says on p171 that "$[\Sigma X, Z]$ is obviously a group, because in ΣX we have a spare suspension coordinate out in front to manipulate. And for the same reason, $[\Sigma^2 X, Z]$ is an abelian group. But now we can give $[X, Y]$ the structure of an abelian group, because $[X, Y]$ is in 1-1 correspondence with $[\Sigma^2 X, \Sigma^2 Y]$ and we pull back the group structure on that. So now our sets of morphisms $[X, Y]$ are abelian groups, and it's easy to see that composition is bilinear".

Various claims:

- The stuff about normal CW complexes and their groups of maps (i.e. Remark B.1.5) translates to tell me the appropriate things about spectra and their groups of maps.
- After checking a lot of things, I can eventually conclude that $[X, Y]$ is an abelian group for spectra X, Y.]

[1], [10], [11], [7], [2], [6]

4 The Adams spectral sequence

4.1 Spectral sequences

Some notes from [11] (or maybe not)

How about some notes from [9], C2:

DEFINITION 4.1.1. A *differential bigraded module* E over a ring R is a collection of R-modules $\{E^{p,q}\}$, $p, q \in \mathbb{Z}$, together with a map $d : E^{p,q} \to E^{p+s,q-s+1}$ for each p, q and some fixed $s \in \mathbb{Z}$, satisfying $d^2 = 0$.

We can take the homology of (E, d):

$$H^{p,q}(E^{*,*}, d) = \ker(d : E^{p,q} \to E^{p+s,q-s+1}) / \operatorname{im}(d : E^{p-s,q+s-1} \to E^{p,q}).$$

DEFINITION 4.1.2. A *spectral sequence* (of *cohomological type*) is a collection of differential bigraded R-modules $\{E_r^{*,*}, d_r\}, r \in \mathbb{N}$, with the differentials d_r of bidegree $(r, 1-r)$. These satisfy the further condition that for all p, q, r, $E_{r+1}^{p,q} \cong H^{p,q}(E_r^{*,*}, d_r)$.

We will sometimes write $d_r^{p,q}$ for the differential $d_r : E^{p,q} \to E^{p+r,q-s+1}$.

Consider the term $E_2^{*,*}$. Define

$$Z_2^{p,q} := \ker d_2^{p,q} \quad \text{and} \quad B_2^{p,q} := \operatorname{im} d_2^{p-2,q+1}.$$

The condition $d^2 = 0$ implies that $B_2^{p,q} \subseteq Z_2^{p,q} \subseteq E_2^{p,q}$, and by definition we have $E_3^{p,q} \cong Z_2^{p,q}/B_2^{p,q}$.

Now, write

$$Z_3^{p,q} := \ker d_3^{p,q} \quad \text{and} \quad B_3^{p,q} := \operatorname{im} d_3^{p-3,q+2}.$$

Since $Z_3^{p,q} \subseteq E_3^{p,q}$, it can be written as $\overline{Z}_3^{p,q}/B_2^{p,q}$ for some $\overline{Z}_3^{p,q} \subseteq Z_2^{p,q}$. Similarly, $B_3^{p,q} \cong \overline{B}_3^{p,q}/B_2^{p,q}$ for some $\overline{B}_3^{p,q} \subseteq Z_2^{p,q}$. Thus,

$$E_4^{p,q} \cong Z_3^{p,q}/B_3^{p,q} \cong \frac{\overline{Z}_2^{p,q}/B_2^{p,q}}{\overline{B}_3^{p,q}/B_2^{p,q}} \cong \overline{Z}_3^{p,q}/\overline{B}_3^{p,q}.$$

Iterating this process, we present the spectral sequence as an infinite tower of submodules of $E_2^{p,q}$:

$$B_2^{p,q} \subseteq \overline{B}_3^{p,q} \subseteq \cdots \subseteq \overline{B}_n^{p,q} \subseteq \cdots \subseteq \overline{Z}_n^{p,q} \subseteq \cdots \subseteq \overline{Z}_3^{p,q} \subseteq Z_2^{p,q} \subseteq E_2^{p,q},$$

with the property that $E_{n+1}^{p,q} \cong \overline{Z}_n^{p,q}/\overline{B}_n^{p,q}$. The differential $d_{n+1}^{p,q}$ can be taken as a map $\overline{Z}_n^{p,q}/\overline{B}_n^{p,q} \to \overline{Z}_n^{p,q}/\overline{B}_n^{p,q}$ with kernel $\overline{Z}_{n+1}^{p,q}/\overline{B}_n^{p,q}$ and image $\overline{B}_{n+1}^{p,q}$. The short exact sequence induced by d_{n+1},

$$0 \to \overline{Z}_{n+1}^{p,q}/\overline{B}_n^{p,q} \to \overline{Z}_n^{p,q}/\overline{B}_n^{p,q} \xrightarrow{d_{n+1}^{p,q}} \overline{B}_{n+1}^{p,q}/\overline{B}_n^{p,q} \to 0,$$

gives rise to isomorphisms $\overline{Z}_n^{p,q}/\overline{Z}_{n+1}^{p,q} \cong \overline{B}_{n+1}^{p,q}/\overline{B}_n^{p,q}$ for all n. Conversely, a tower of submodules of E_2, together with a set of isomorphisms, gives rise to a spectral sequence.

DEFINITION 4.1.3. An element of $E_2^{p,q}$ *survives to the rth stage* if lies in $\overline{Z}_r^{p,q}$, having been in the kernel of the previous $r-2$ differentials, and is *bounded by the rth stage* if it lies in $\overline{B}_r^{p,q}$. The bigraded module $E_r^{*,*}$ is called the *E_r-term* of the spectral sequence.

We define

$$Z_\infty^{p,q} := \bigcap_n \overline{Z}_n^{p,q}, \quad B_\infty^{p,q} := \bigcup^n \overline{B}_n^{p,q}.$$

From the tower of inclusions, we see that $B_\infty^{p,q} \subseteq Z_\infty^{p,q}$, so we define $E_\infty^{p,q} := Z_\infty^{p,q}/B_\infty^{p,q}$.

DEFINITION 4.1.4. A spectral sequence *collapses at the Nth term* if the differentials $d_r^{p,q} = 0$ for $r \geq N$.

From the short exact sequence

$$0 \to \overline{Z}_r^{p,q}/\overline{B}_{r-1}^{p,q} \to \overline{Z}_{r-1}^{p,q}/\overline{B}_{r-1}^{p,q} \xrightarrow{d_r^{p,q}} \overline{B}_r^{p,q}/\overline{B}_{r-1}^{p,q} \to 0,$$

the condition $d_r^{p,q}$ forces $\overline{Z}_r^{p,q} = \overline{Z}_{r-1}^{p,q}$ and $\overline{B}_r^{p,q} = \overline{B}_{r-1}^{p,q}$. The tower of submodules becomes

$$B_2^{p,q} \subseteq \overline{B}_3^{p,q} \subseteq \cdots \subseteq \overline{B}_{N-1}^{p,q} = B_N^{p,q} = \cdots = B_\infty^{p,q} \subseteq Z_\infty^{p,q} = \cdots = \overline{Z}_N^{p,q} = \overline{Z}_{N-1}^{p,q} \subseteq \cdots \subseteq \overline{Z}_3^{p,q} \subseteq Z_2^{p,q} \subseteq E_2^{p,q}.$$

Thus, $E_\infty^{p,q} = E_N^{p,q}$.

4.2 Exact couples

(Following [9], C2)

Definition 4.2.1. Let D, E be R-modules, and let $i : D \to D$, $j : D \to E$, $k : E \to D$ be module homomorphisms. We call $\mathcal{C} = \{D, E, i, j, k\}$ an *exact couple* if the diagram below is exact.

$$\begin{tikzcd} D \arrow[rr, "i"] & & D \arrow[dl, "j"] \\ & E \arrow[ul, "k"] & \end{tikzcd}$$

Let $d := jk$, and define the following:

$$\begin{aligned} E' &:= H(E, d) = \ker d / \operatorname{im} d \\ D' &:= i(D) = \ker j \\ i' &:= i|_{i(D)} : D' \to D' \\ j' &:= i(x) \mapsto j(x) + dE : D' \to E' \\ k' &:= (e + dE) \mapsto k(e) : E' \to D' \end{aligned}$$

We call $\mathcal{C}' = \{D', E'.i', j', k'\}$ the *derived couple* of $\mathcal{C}$.

Proposition 4.2.2 ([9], Prop 2.7). If $\mathcal{C} = \{D, E, i, j, k\}$ is an exact couple, then $\mathcal{C}'$ is also an exact couple.

Theorem 4.2.3 ([9], Thm 2.8). Suppose $D^{*,*} = \{D^{p,q}\}$ and $E^{*,*} = \{E^{p,q}\}$ are bigraded modules equipped with homomorphisms i of bidegree $(-1, 1)$, j of bidegree $(0, 0)$, and k of bidegree $(1, 0)$, such that $\{D^{*,*}, E^{*,*}, i, j, k\}$ is an exact couple. Then these data determine a spectral sequence $\{E_r, d_r\}$ for $r \in \mathbb{Z}_+$ of cohomological type, with $E_r = (E^{*,*})^{(r-1)}$, the $(r-1)$st derived module of $E^{*,*}$ and $d_r = j^{(r)} \circ k^{(r)}$.

A bigraded exact couple may be displayed in the following diagram, known as a *staircase diagram*:

$$\begin{tikzcd} & \vdots \arrow[d, "i"] & & \vdots \arrow[d, "i"] & \\ \cdots \arrow[r, "k"'] & D^{p+2,q-1} \arrow[r, "j"] \arrow[d, "i"] & E^{p+2,q-1} \arrow[r, "k"] & D^{p+3,q-1} \arrow[r, "j"] \arrow[d, "i"] & \cdots \\ \cdots \arrow[r, "k"'] & D^{p+1,q} \arrow[r, "j"] \arrow[d, "i"] & E^{p+1,q} \arrow[r, "k"] & D^{p+2,q} \arrow[r, "j"] \arrow[d, "i"] & \cdots \\ \cdots \arrow[r, "k"'] & D^{p,q+1} \arrow[r, "j"] \arrow[d, "i"] & E^{p,q+1} \arrow[r, "k"] & D^{p+1,q+1} \arrow[r, "j"] \arrow[d, "i"] & \cdots \\ & \vdots & & \vdots & \end{tikzcd}$$

4.3 The Adams spectral sequence

Things I need before I can set it up (according to Hatcher [6]):

Let X be a CW spectrum of finite type.

- Def: $H^*(X)$.
- Def: A wedge of spectra.
- Fact: $H^*(X)$ is finitely generated.
- Fact: $H^*(X)$ is an $\mathscr{A}$-module. [We know that's true for a topological space]
- Fact: We can pick generators α_i for $H^*(X)$ as an $\mathscr{A}$-module such that there are at most finitely many in each $H^n(X)$.
- Fact: There α_i determine a map $X \to K_0$, where K_0 is a wedge of EM spectra, and K_0 has finite type.
- Fact: We can replace that map with an inclusion.
- Def: A quotient of (connective) spectra.
- Fact: A quotient of connective spectra of finite type is again a connective spectrum of finite type.
- Prop: [6], 5.46.
- Def: The functor $\pi_t^Y(Z) = [\Sigma^t Y, Z]$ for a finite spectrum Y.
- Def: A cofibration. [see : [5], p398.]
- Fact: If Y is a finite spectrum and and Z is a connective spectrum of finite type, then $\pi_t^Y(Z)$ is finitely generated.
- Fact: I can do this. I have all the necessary skills to pull this off.
- Fact: I'm going to stop listing things I need to do and start actually doing them.

[9], [1], [10], [2], [6], [11], [3]

5 Calculating stable homotopy groups

$\mathrm{Ext}^s_A(\mathbb{F}_2, \mathbb{F}_2)_t$

[1], [10], [11]

6 Methods of resolving ambiguities

[1], [10]

A Algebra

A.1 Free resolutions

Definition 1.1.1. Let M, N be modules over a ring R. A *resolution* F of M is an exact sequence

$$\cdots \to F_2 \to F_1 \to F_0 \to M \to 0.$$

If in addition each F_i is a free R-module, then the resolution is called *free*.

Given a free resolution as above, applying $\mathrm{Hom}_R(-, N)$ gives us a chain complex

$$\cdots \leftarrow \mathrm{Hom}_R(F_2, N) \leftarrow \mathrm{Hom}_R(F_1, N) \leftarrow \mathrm{Hom}_R(F_0, N) \leftarrow \mathrm{Hom}_R(M, N) \leftarrow 0.$$

Dropping the term $\mathrm{Hom}_R(M, N)$ [why?] we get the sequence

$$\cdots \leftarrow \mathrm{Hom}_R(F_2, N) \leftarrow \mathrm{Hom}_R(F_1, N) \leftarrow \mathrm{Hom}_R(F_0, N) \leftarrow 0,$$

and we define $\mathrm{Ext}^n_R(M, N)$ to be the nth homology group of this chain complex.

[these do not depend on the choice of free resolution of M]

B Topology

All from [5] unless otherwise stated.

B.1 Suspension

Definition 2.1.1. Let X be a topological space. The *suspension* SX is the space $(X \times I)/\sim$, where $(x, 0) \sim (x', 0)$ and $(x, 1) \sim (x', 1)$ for all $x, x' \in X$.

Definition 2.1.2. Let X be a pointed topological space. The *reduced suspension* ΣX is the space $SX/\sim$, where $[x_0, t] \sim [x_0, t']$ for all $t, t' \in I$.

Given a map $f : X \to Y$, we can define $\Sigma f : \Sigma X \to \Sigma Y$ by $\Sigma f[(x, t)] = [(fx, t)]$. This makes Σ into a functor $\Sigma : \mathbf{Top} \to \mathbf{Top}$.

Remark 2.1.3. Σ is faithful, since for any maps $f, g : X \to Y$, if $\Sigma f = \Sigma g$ then in particular $[(fx, \frac{1}{2})] = [(gx, \frac{1}{2})]$, so $fx = gx$.

[below is reconstructed from [8]]

Given pointed maps $f, g : \Sigma X \to Z$, define

$$f \star g : \Sigma X \to Z$$
$$[x, t] \mapsto \begin{cases} f[x, 2t-1] & t \geq \frac{1}{2}, \\ g[x, 2t] & t \leq \frac{1}{2}. \end{cases}$$

This is well defined, since both f and g are basepoint-preserving.

Remark 2.1.4. This defines a group structure on $[\Sigma X, Z]$, and thus $[\Sigma^i X, Z]$ is a group for all $i \geq 1$. For $i \geq 2$, these can be shown to be abelian, via the Eckmann-Hilton argument.

Remark 2.1.5. The homotopy groups $\pi_i(Z)$ are a special case of the above construction, taking $X := S^{i-1}$.

- Loops; the adjunction $\Sigma \dashv \Omega$, where Ω is the loop functor.

[5], p395:

Remark 2.1.6. It follows that $\pi_{n+1}(X) \cong \pi_n(\Omega X)$. In particular, $\Omega K(G, n)$ is a $K(G, n-1)$.

- [5] 2.1 Ex 20 and 2.2 Ex 32: $\tilde{H}_n(X) \cong \tilde{H}_{n+1}(SX)$, where S is the (non-reduced) suspension. (MV?)
- Hatcher also says on p219 that $\tilde{H}^n(X; R) \cong \tilde{H}^{n+k}(\Sigma^k X; R)$, where Σ is reduced suspension.

B.2 Other basic constructions

Definition 2.2.1. Let $(X, x_0), (Y, y_0)$ be pointed topological spaces, and consider their product $X \times Y$. The subspaces $X \times \{y_0\} \cong X$ and $\{x_0\} \times Y \cong Y$ intersect at exactly one point, (x_0, y_0), and so can be identified with the wedge $X \vee Y$. We thus define the *smash product* $X \wedge Y := (X \times Y)/(X \vee Y)$, with the canonical basepoint (x_0, y_0).

Example 2.2.2. We have $S^n \wedge S^m \cong S^{n+m}$. [is this obvious?]

Remark 2.2.3. Note that $\Sigma X \cong X \wedge S^1$.

Remark 2.2.4. Observe that $X \wedge (Y \wedge Z) \cong (X \wedge Y) \wedge Z$. Combining this with the remarks above, we see that $\Sigma^k X = X \wedge S^k$.

- 'Reduced mapping cylinder'?
- Mapping cones?
- The Eilenberg-MacLane space is $K(G, n)$, and it has the property that

$$\pi_i(K(G,n)) = \begin{cases} G & i = n, \\ 0 & i \neq n. \end{cases}$$

  They're unique up to weak homotopy equivalence (i.e. if you have another one X, there's a map between them which descends to an isomorphism on homotopy groups). They can be taken to be CW complexes.

Definition 2.2.5. Let X, Y be topological spaces, where Xhas a basepoint x_0. Then the *reduced product* $X \times_{\text{red}} Y := (X \times Y)/(x_0 \times Y)$.

B.3 Cell complexes

Definition 2.3.1. Let $f : X \to Y$ be a map between CW complexes. Then f is *cellular* if $f(X_{(n)}) \subseteq Y_{(n)}$ for all n, where $X_{(n)}$ is the n-skeleton of X.

Cellular approximation theorem:

Theorem 2.3.2 ([5], Thm 4.8). Let $f : X \to Y$ be a map of CW complexes. Then f is homotopic to a cellular map.

- The product of cell complexes is a cell complex (maybe only if one of them is finite?)
- The smash product of (pointed?) cell complexes is a cell complex (maybe only if one is them is finite?) [[5] says "the smash product $X \wedge Y$ is a cell complex if X and Y are cell complexes with x_0 and y_0 0-cells, assuming that we give $X \times Y$ the cell-complex topology rather than the product topology in cases where these two topologies differ".]
- Quotient of a CW complex by a subcomplex is a CW complex, where the quotient map is cellular
- The reduced suspension of a pointed cell complex is a pointed cell complex.
- CW pairs?
- For a CW complex X, $SX \simeq \Sigma X$.
- Something along the lines of "the suspension of a cell is a cell of the suspension" should be true.

Definition 2.3.3. Let X is a topological space. A *CW approximation* to X is a CW complex Z equipped with a weak homotopy equivalence $f : Z \to X$.

Theorem 2.3.4 ([5], Prop 4.13). Every space X has a CW approximation $f : Z \to X$.

- In particular, $\Omega K(G, n)$ has a CW approximation $Z \to \Omega K(G, n)$, and since $\Omega K(G, n)$ is a $K(G, n-1)$, so is Z.

C Notes to self

C.1 Vague problems and Questions

- Is 'pointed' (co)homology just reduced (co)homology? I've noticed 'pointed things' ($\Sigma, \Omega, \wedge$, ...) seem to happen to/in reduced (co)homology, and 'unpointed things' ($S, \times, ...$) happen to/in normal (co)homology. I want to do pointed things.
- What does Hatcher mean when he says two spectra are 'equivalent'?
- On p588 of [6], he says "every CW spectrum is equivalent to a suspension spectrum". Does he actually mean that, or does he mean 'equivalent to the suspension of a spectrum'? The former seems way too strong, although in fairness I still don't know what an equivalence of spectra actually *is*.
- On p592 of [6], he says that "for spectra X of finite type [the more general] definition of an $\mathscr{A}_2$-module structure on $H^*(X)$ agrees with the definition using the usual $\mathscr{A}_2$-module structure on the cohomology of spaces and the identification of $H^*(X)$ with the inverse limit $\varprojlim H^{*+n}(X_n)$". Absolutely everything relevant to spheres in the construction of the Adams spectral sequence seems to only use spectra of finite type, so why does Hatcher say on p585 that we can't just take the inverse limit because it doesn't work for the 'more general spectra' used when constructing the Adams spectral sequence? Is that just because I could take the role of $\mathbb{S}$ and give it to some non-finite-type spectrum, or is there something I'm missing? Can I just take limits and be done with it?

C.2 To do/other notes

- READ IF YOUR CALCULATIONS AREN'T WORKING: You are working modulo 2!!!
- Be consistent with either cell complex or CW complex (eventually)

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