Stable Homotopy Groups of Spheres [DRAFT]

Contents

1 Introduction

- Define homotopy groups
- The Eilenberg-MacLane space is $K(G,n)$, and it has the property that

$$\pi_i(K(G,n)) = \begin{cases} \mathbb{Z} & i = n, \\ 0 & i \neq n. \end{cases}$$

  They're unique up to weak homotopy equivalence (i.e. if you have another one X, there's a map between them which descends to an isomorphism on homotopy groups)

- Freudenthal's suspension theorem: if $\pi_i(X) = 0$ for $i \leq k$ (i.e. X is k-connected) then the map

$$\begin{aligned} \pi_n(X) &\to \pi_{n+1}(\Sigma X) \\ [\gamma : S^n \to X] &\mapsto [\Sigma\gamma : \Sigma S^n = S^{n+1} \to \Sigma X] \end{aligned}$$

  is an isomorphism for $n \leq 2k$ and surjective for $n = 2k+1$

- This implies $\pi_{n+k}(S^n)$ depends only on k for $n \geq k+2$
- (Obviously be careful with basepoints above)
- Suppose X is k-connected. Then, for $k \geq 0$, $0 = \pi_k(X) \cong \pi_{k+1}(\Sigma X)$, so whenever a space is k-connected its suspension is $k+1$-connected.
- As you take suspensions, then, your successive bounds are $n \leq 2k$, $n+1 \leq 2k+2 \implies n \leq 2k+1$, $n \leq 2k+2$, etc ... so the sequence $\pi_n(X) \to \pi_{n+1}(\Sigma X) \to \cdots$ will eventually stabilise.
- Thus, if you take the colimit of that direct system, it'll just equal the stable value, with the higher legs just being the inverse isomorphisms.
- [11], Cor 1.9 [not 100% convinced of how this follows, but believing it for now]: if X is a CW complex of dimension d and Y a $(k-1)$-connected space, then the suspension homomorphism $[X,Y] \to [\Sigma X, \Sigma Y]$ is bijective if $d < 2k-1$ and surjective if $d = 2k-1$.

Miscellaneous facts I might need later:

- Cohomology [possibly only of pointed CW complexes] is representable, and its representing object is the Eilenberg-MacLane space. i.e. $H^n(-;G) \cong \mathrm{Hom}(-, K(G,n))$.
- $\mathscr{A}_2$ is generated as an algebra by elements Sq^{2^k} ([6], Prop 4L.8).
- The map $\mathscr{A}_2 \to \tilde{H}^*(K(\mathbb{Z}/2\mathbb{Z}, n); \mathbb{Z}/2\mathbb{Z})$, $Sq^I \mapsto Sq^I(\iota_n)$ is an isomorphism from the degree d part of $\mathscr{A}_2$ onto $H^{n+d}(K(\mathbb{Z}/2\mathbb{Z}, n); \mathbb{Z}/2\mathbb{Z})$ for $d \geq n$. In particular, the admissible monomials Sq^I form an additive basis for $\mathscr{A}_2$. Thus, $\mathscr{A}_2$ is exactly the algebra of all $\mathbb{Z}/2\mathbb{Z}$ cohomology operations that are stable, commuting with suspension ([7], Cor 5.38).
- "Stable homotopy groups are a homology theory" (whatever that means)
- Hurewicz theorem: for any path-connected space X and $n > 0$ there exists a group homomorphism $h_* : \pi_n(X) \to H_n(X)$. For $n = 1$ this induces an isomorphism $\pi_1^{\mathrm{ab}}(X) \cong H_1(X)$. For $n \geq 2$, if X is $(n-1)$-connected then $\tilde{H}_i(X) = 0$ for all $i < n$, and the map $h_* : \pi_n(X) \to H_n(X)$ is an isomorphism.

[11], [5], [6]

2 The Steenrod algebra

The following is from [6] 4L.

- There are maps $Sq^i : H^n(-;\mathbb{Z}/2\mathbb{Z}) \to H^{n+i}(-;\mathbb{Z}/2\mathbb{Z})$ for each i, and they satisfy the following properties:
  1. $Sq^i_X(f^*(\alpha)) = f^*(Sq^i_Y(\alpha))$ for $f : X \to Y$ (i.e. Sq^i is a natural transformation).
  2. $Sq^i_X(\alpha+\beta) = Sq^i_X(\alpha) + Sq^i_X(\beta)$ (i.e. Sq^i_X respects the group operation for all X).
  3. $Sq^i(\alpha \smile \beta) = \sum\limits_{0\leq j\leq i} (Sq^j(\alpha) \smile Sq^{i-j}(\beta))$ (the Cartan formula)
  4. $Sq^i(\sigma(\alpha)) = \sigma(Sq^i(\alpha))$ where $\sigma : H^n(X;\mathbb{Z}/2\mathbb{Z}) \to H^{n+1}(\Sigma X;\mathbb{Z}/2\mathbb{Z})$ is the "suspension isomorphism given by reduced cross product with a generator of $H^1(S^1;\mathbb{Z}/2\mathbb{Z})$"
  5. $Sq^i(\alpha) = \alpha^2$ if $i = |\alpha|$ and $Sq^i(\alpha) = 0$ if $i > |\alpha|$. [Hatcher doesn't explain this notation at all, but I think he means by $|\alpha|$ the degree of α - this is what [2] says in C2]
  6. $Sq^0 = \mathrm{id}$.
  7. Sq^1 is the "$\mathbb{Z}/2\mathbb{Z}$ Bockstein homomorphism β associated with the coefficient sequence $0 \to \mathbb{Z}/2\mathbb{Z} \to \mathbb{Z}/4\mathbb{Z} \to \mathbb{Z}/2\mathbb{Z} \to 0$".
- Define $Sq := Sq^0 + Sq^1 + \cdots$. Then $Sq(\alpha \smile \beta) = Sq(\alpha) \smile Sq(\beta)$ (since $(Sq(\alpha\smile\beta))_n = \sum_i Sq^i(\alpha) \smile Sq^{n-i}(\beta) = (Sq(\alpha) \smile Sq(\beta))_n$). Thus, Sq is a ring homomorphism.
- Adem relations:

$$Sq^a Sq^b = \sum_j \binom{b-j-1}{a-2j} Sq^{a+b-j} Sq^j \quad \text{if } a < 2b,$$

  where $\binom{m}{n}$ is zero if m or n is negative, or $m < n$, and $\binom{m}{0} = 1$ for $m \geq 0$.
- The Steenrod algebra $\mathscr{A}_2$ is the algebra over $\mathbb{Z}/2\mathbb{Z}$ that is the quotient of the algebra of polynomials in the noncommuting variables $Sq^1, Sq^2, \ldots$ by the two-sided ideal generated by the Adem relations. Thus, for every space X, $H^*(X;\mathbb{Z}/2\mathbb{Z})$ is a module over $\mathscr{A}_2$, via $\alpha \cdot f = f(\alpha)$.
- $\mathscr{A}_2$ is graded, and its elements of degree k are those that map $H^n(X;\mathbb{Z}/2\mathbb{Z})$ to $H^{n_k}(X,\mathbb{Z}/2\mathbb{Z})$ for all n. [Presumably you've fixed a space X while you're doing all this?]

[1], [10], [11], [8], [6], [3]

3 Spectra may not be your friends, but I can introduce you

- [11]: There is a category $\mathcal{H}$ of finite [because the corollary wanted f.d. CW complexes] based CW complexes, with $\mathrm{Hom}(X,Y) =: [X,Y]$ the set of homotopy classes of base-point preserving maps $X \to Y$.
- There is a category $\mathbf{St}(\mathcal{H})$ of finite[?] based CW complexes, with $\mathrm{Hom}(X,Y) =: \{X,Y\}$ the set $\mathrm{colim}_i[\Sigma^i X, \Sigma^i Y]$ [it's just a colimit of sets, and **Set** is cocomplete, so we should be fine. [11] says it's a group?] [Also, how do these guys compose?]

- There is a functor $\mathcal{H} \to \mathbf{St}(\mathcal{H})$. [11] doesn't say what this is but it's presumably the one that is the identity on objects and sends $[f : X \to Y] \in [\Sigma^0 X, \Sigma^0 Y]$ to whatever it gets sent to in $\{X, Y\}$ using the universal property of the colimit. Uniqueness makes it functorial, etc.

- We have a fully faithful functor $\mathbf{St}(\mathcal{H}) \to \mathbf{St}(\mathcal{H})$ given by the suspension on objects, and the unique isomorphism $\{X, Y\} \to \{\Sigma X, \Sigma Y\}$ on maps (such an isomorphism exists, since both of those things are colimits for $[\Sigma^i X, \Sigma^i Y]$ - one of the sequences is cut off at the beginning, but it doesn't matter because both reach the stable value (see above discussion and [11] 1.9), aka the colimit).

- It's not an equivalence, because not every object is isomorphic to a suspension (e.g. anything not connected, since suspensions always connected [?])

- We can formally adjoin desuspensions $\Sigma^{-n} X$ for all n [does this mean just putting the objects there and defining $\mathrm{Hom}(Y, \Sigma^{-n} X) := \mathrm{Hom}(\Sigma^n Y, X)$ and $\mathrm{Hom}(\Sigma^{-n} X, Y) := \mathrm{Hom}(X, \Sigma^n Y)$?], but this category does not have weak colimits (i.e. colimits w/o uniqueness property). [why does it not, and why do we even want that?]

- We instead consider formal sequences of desuspensions $X_0 \to \Sigma^{-1} X_1 \to \cdots$, or sequences (X_n) and maps $\Sigma X_n \to X_{n+1}$, i.e. spectra. [and this fixes the problem?]

Below follows [7], Section 5.2.

Definition 3.0.1. A *spectrum* is a collection of pointed topological spaces $\{X_n\}_{n \in \mathbb{N}}$, together with basepoint-preserving maps $\sigma_n : \Sigma X_n \to X_{n+1}$.

Example 3.0.2. Let X be a topological space. The *suspension spectrum* of X, denoted by $\Sigma^\infty X$, has $X_n = \Sigma^n X$ and $\sigma_n = \mathrm{id} : \Sigma X_n \to X_{n+1}$.

We write $\mathbb{S} := \Sigma^\infty S^0$, and call $\mathbb{S}$ the *sphere spectrum*.

[Define EM spectrum]

Definition 3.0.3. Let $X = \{X_n\}$ be a spectrum. We define $\pi_i(X) = \mathrm{colim}_n\, \pi_{i+n}(X_n)$, where the map $\pi_{i+n}(X_n) \to \pi_{i+n+1}(X_{n+1})$ is given by the composition

$$\pi_{i+n}(X_n) \xrightarrow{\Sigma} \pi_{i+n+1}(\Sigma X_n) \xrightarrow{(\sigma_n)_*} \pi_{i+n+1}(X_{n+1}).$$

Example 3.0.4. If X is a topological space, then $\pi_i(\Sigma^\infty X) = \pi_i^S(X)$, the ith stable homotopy group of X.

Definition 3.0.5. A CW spectrum is a spectrum X consisting of CW complexes X_n with the maps $\Sigma X_n \hookrightarrow X_{n+1}$ inclusions of subcomplexes.

[Define cells and dimension of a CW spectrum]

Definition 3.0.6. A spectrum X is *connective* if its cells have dimensions which are bounded below.

[The above is phrased exactly as in hatcher - presumably he means that there is some absolute bound below which no cell has dimension, rather than a bound dependent on each cell?]

Definition 3.0.7. A CW spectrum is *finite* if it has only finitely many cells, and *of finite type* if it has only finitely many cells in each dimension.

[1], [10], [11], [8], [3], [7]

4 The Adams spectral sequence

4.1 Spectral sequences

Some notes from [11] (or maybe not)

How about some notes from [9], C2:

DEFINITION 4.1.1. A *differential bigraded module* E over a ring R is a collection of R-modules $\{E^{p,q}\}$, $p, q \in \mathbb{Z}$, together with a map $d : E^{p,q} \to E^{p+s,q-s+1}$ for each p, q and some fixed $s \in \mathbb{Z}$, satisfying $d^2 = 0$.

We can take the homology of (E, d):

$$H^{p,q}(E^{*,*}, d) = \ker(d : E^{p,q} \to E^{p+s,q-s+1}) / \operatorname{im}(d : E^{p-s,q+s-1} \to E^{p,q}).$$

DEFINITION 4.1.2. A *spectral sequence* (of *cohomological type*) is a collection of differential bigraded R-modules $\{E_r^{*,*}, d_r\}, r \in \mathbb{N}$, with the differentials d_r of bidegree $(r, 1-r)$. These satisfy the further condition that for all p, q, r, $E_{r+1}^{p,q} \cong H^{p,q}(E_r^{*,*}, d_r)$.

We will sometimes write $d_r^{p,q}$ for the differential $d_r : E^{p,q} \to E^{p+r,q-s+1}$.

Consider the term $E_2^{*,*}$. Define

$$Z_2^{p,q} := \ker d_2^{p,q} \quad \text{and} \quad B_2^{p,q} := \operatorname{im} d_2^{p-2,q+1}.$$

The condition $d^2 = 0$ implies that $B_2^{p,q} \subseteq Z_2^{p,q} \subseteq E_2^{p,q}$, and by definition we have $E_3^{p,q} \cong Z_2^{p,q}/B_2^{p,q}$.

Now, write

$$Z_3^{p,q} := \ker d_3^{p,q} \quad \text{and} \quad B_3^{p,q} := \operatorname{im} d_3^{p-3,q+2}.$$

Since $Z_3^{p,q} \subseteq E_3^{p,q}$, it can be written as $\overline{Z}_3^{p,q}/B_2^{p,q}$ for some $\overline{Z}_3^{p,q} \subseteq Z_2^{p,q}$. Similarly, $B_3^{p,q} \cong \overline{B}_3^{p,q}/B_2^{p,q}$ for some $\overline{B}_3^{p,q} \subseteq Z_2^{p,q}$. Thus,

$$E_4^{p,q} \cong Z_3^{p,q}/B_3^{p,q} \cong \frac{\overline{Z}_2^{p,q}/B_2^{p,q}}{\overline{B}_3^{p,q}/B_2^{p,q}} \cong \overline{Z}_3^{p,q}/\overline{B}_3^{p,q}.$$

Iterating this process, we present the spectral sequence as an infinite tower of submodules of $E_2^{p,q}$:

$$B_2^{p,q} \subseteq \overline{B}_3^{p,q} \subseteq \cdots \subseteq \overline{B}_n^{p,q} \subseteq \cdots \subseteq \overline{Z}_n^{p,q} \subseteq \cdots \subseteq \overline{Z}_3^{p,q} \subseteq Z_2^{p,q} \subseteq E_2^{p,q},$$

with the property that $E_{n+1}^{p,q} \cong \overline{Z}_n^{p,q}/\overline{B}_n^{p,q}$. The differential $d_{n+1}^{p,q}$ can be taken as a map $\overline{Z}_n^{p,q}/\overline{B}_n^{p,q} \to \overline{Z}_n^{p,q}/\overline{B}_n^{p,q}$ with kernel $\overline{Z}_{n+1}^{p,q}/\overline{B}_n^{p,q}$ and image $\overline{B}_{n+1}^{p,q}$. The short exact sequence induced by d_{n+1},

$$0 \to \overline{Z}_{n+1}^{p,q}/\overline{B}_n^{p,q} \to \overline{Z}_n^{p,q}/\overline{B}_n^{p,q} \xrightarrow{d_{n+1}^{p,q}} \overline{B}_{n+1}^{p,q}/\overline{B}_n^{p,q} \to 0,$$

gives rise to isomorphisms $\overline{Z}_n^{p,q}/\overline{Z}_{n+1}^{p,q} \cong \overline{B}_{n+1}^{p,q}/\overline{B}_n^{p,q}$ for all n. Conversely, a tower of submodules of E_2, together with a set of isomorphisms, gives rise to a spectral sequence.

DEFINITION 4.1.3. An element of $E_2^{p,q}$ *survives to the rth stage* if lies in $\overline{Z}_r^{p,q}$, having been in the kernel of the previous $r-2$ differentials, and is *bounded by the rth stage* if it lies in $\overline{B}_r^{p,q}$. The bigraded module $E_r^{*,*}$ is called the E_r-*term* of the spectral sequence.

We define

$$Z_\infty^{p,q} := \bigcap_n \overline{Z}_n^{p,q}, \quad B_\infty^{p,q} := \bigcup^n \overline{B}_n^{p,q}.$$

From the tower of inclusions, we see that $B_\infty^{p,q} \subseteq Z_\infty^{p,q}$, so we define $E_\infty^{p,q} := Z_\infty^{p,q}/B_\infty^{p,q}$.

Definition 4.1.4. A spectral sequence *collapses at the Nth term* if the differentials $d_r^{p,q} = 0$ for $r \geq N$.

From the short exact sequence

$$0 \to \overline{Z}_r^{p,q}/\overline{B}_{r-1}^{p,q} \to \overline{Z}_{r-1}^{p,q}/\overline{B}_{r-1}^{p,q} \xrightarrow{d_r^{p,q}} \overline{B}_r^{p,q}/\overline{B}_{r-1}^{p,q} \to 0,$$

the condition $d_r^{p,q}$ forces $\overline{Z}_r^{p,q} = \overline{Z}_{r-1}^{p,q}$ and $\overline{B}_r^{p,q} = \overline{B}_{r-1}^{p,q}$. The tower of submodules becomes

$$B_2^{p,q} \subseteq \overline{B}_3^{p,q} \subseteq \cdots \subseteq \overline{B}_{N-1}^{p,q} = B_N^{p,q} = \cdots = B_\infty^{p,q} \subseteq Z_\infty^{p,q} = \cdots = \overline{Z}_N^{p,q} = \overline{Z}_{N-1}^{p,q} \subseteq \cdots \subseteq \overline{Z}_3^{p,q} \subseteq Z_2^{p,q} \subseteq E_2^{p,q}.$$

Thus, $E_\infty^{p,q} = E_N^{p,q}$.

4.2 Exact couples

(Following [9], C2)

Definition 4.2.1. Let D, E be R-modules, and let $i : D \to D$, $j : D \to E$, $k : E \to D$ be module homomorphisms. We call $\mathcal{C} = \{D, E, i, j, k\}$ an *exact couple* if the diagram below is exact.

$$\begin{tikzcd} D \arrow[rr, "i"] & & D \arrow[dl, "j"] \\ & E \arrow[ul, "k"] & \end{tikzcd}$$

Let $d := jk$, and define the following:

$$\begin{aligned} E' &:= H(E, d) = \ker d / \operatorname{im} d \\ D' &:= i(D) = \ker j \\ i' &:= i|_{i(D)} : D' \to D' \\ j' &:= i(x) \mapsto j(x) + dE : D' \to E' \end{aligned}$$

4.3 The Adams spectral sequence

Things I need before I can set it up (according to Hatcher [7]):

Let X be a CW spectrum of finite type.

- Def: $H^*(X)$.
- Def: EM spectrum.
- Fact: $H^*(X)$ is finitely generated.
- Fact: $H^*(X)$ is an $\mathscr{A}$-module. [We know that's true for a topological space]
- Fact: We can pick generators α_i for $H^*(X)$ as an $\mathscr{A}$-module such that there are at most finitely many in each $H^n(X)$.
- Fact: There α_i determine a map $X \to K_0$, where K_0 is a wedge of EM spectra, and K_0 has finite type.
- Fact: We can replace that map with an inclusion

- Fact: A quotient of connective spectra of finite type is again a connective spectrum of finite type.
- Def: A resolution of $H^*(X)$.
- Prop: [7], 5.46.
- Def: The functor $\pi_t^Y(Z) = [\Sigma^t Y, Z]$ for a finite spectrum Y.
- Def: A cofibration.
- Def: A staircase diagram
- Fact: If Y is a finite spectrum and and Z is a connective spectrum of finite type, then $\pi_t^Y(Z)$ is finitely generated.
- Fact: I can do this. I have all the necessary skills to pull this off.
- Fact: I'm going to stop listing things I need to do and start actually doing them.

[9], [1], [10], [3], [7], [11], [4]

5 Calculating stable homotopy groups

$\mathrm{Ext}_A^s(\mathbb{F}_2, \mathbb{F}_2)_t$

[1], [10], [11]

6 Methods of resolving ambiguities

[1], [10]

READ IF YOUR CALCULATIONS AREN'T WORKING:

You are working modulo 2!!!

A Algebra

A.1 Free resolutions

Definition 1.1.1. Let M, N be modules over a ring R. A *free resolution* F of M is an exact sequence

$$\cdots \to F_2 \to F_1 \to F_0 \to M \to 0,$$

with each F_i a free R-module.

Applying $\mathrm{Hom}_R(-, N)$ gives us a chain complex

$$\cdots \leftarrow \mathrm{Hom}_R(F_2, N) \leftarrow \mathrm{Hom}_R(F_1, N) \leftarrow \mathrm{Hom}_R(F_0, N) \leftarrow \mathrm{Hom}_R(M, N) \leftarrow 0.$$

Dropping the term $\mathrm{Hom}_R(M, N)$ [why?] we get the sequence

$$\cdots \leftarrow \mathrm{Hom}_R(F_2, N) \leftarrow \mathrm{Hom}_R(F_1, N) \leftarrow \mathrm{Hom}_R(F_0, N) \leftarrow 0,$$

and we define $\mathrm{Ext}_R^n(M, N)$ to be the nth homology group of this chain complex.

[these do not depend on the choice of free resolution of M]

B Topology

B.1 Basic constructions

- Suspension and loops; the adjunction $\Sigma \dashv \Omega$, where Ω is the loop functor.
- Reduced suspension
- Smash product
- Wedge product
- 'reduced cylinder'?

B.2 Cell complexes

- The product of cell complexes is a cell complex (maybe only if one of them is finite?)
- The smash product of (pointed?) cell complexes is a cell complex (maybe only if one is them is finite?)
- Cellular maps
- Quotient if a CW complex by a subcomplex is a CW complex, where the quotient map is cellular
- The reduced suspension of a pointed cell complex is a pointed cell complex.
- CW pairs?

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