En esta práctica se implementa un programa que resuelve mediante el método de elementos finitos la ecuación diferencial siguiente:

$$-u'' + c(x)u' + d(x)u = f(x), \qquad x \in [a,b],$$

con condiciones de contorno de Dirichlet ($u(a) = \alpha$, $u(b) = \beta$) o de Neumann ($u'(a) = \alpha$, $u'(b) = \beta$). Multiplicamos la ecuación por una función test suficientemente regular e integramos, quedando

$$\int_a^b (-u''v + c(x)u'v + d(x)uv)\,dx = \int_a^b f(x)v\,dx.$$

Integrando por partes,

$$\int_a^b u''v\,dx = [u'v]_{x=a}^{x=b} - \int_a^b u'v'\,dx.$$

Para que se tenga $[u'v]_{x=a}^{x=b} = 0$, le pediremos a la función test que verifique $v(a) = v(b) = 0$. Por tanto, sustituyendo en la ecuación,

$$\int_a^b (u'v' + c(x)u'v + d(x)uv)\,dx = \int_a^b f(x)v\,dx,$$

es decir,

$$\int_a^b u'(v' + c(x)v)\,dx + \int_a^b d(x)uv\,dx = \int_a^b f(x)v\,dx.$$

Consideramos una partición equiespaciada $x_0 = a < x_1 < x_2 < \cdots < x_n = b$ del intervalo $[a,b]$. Sea $h = x_i - x_{i-1}$ y consideramos el espacio vectorial

$$W_h = \{w_h \in \mathcal{C}^0([0,1]) \colon w_h|_{[x_{i-1},x_i]} \in \mathcal{P}_1,\ i = 1,2,\ldots,n\},$$

que es de dimensión $n+1$. Sea $\{\varphi_0, \varphi_1, \ldots, \varphi_n\}$ la base canónica de este espacio vectorial, y consideremos el subespacio vectorial

$$V_h = \{v_h \in W_h \colon v(a) = v(b) = 0\},$$

que es de dimensión $n-1$. Una base de este subespacio es $\{\varphi_1, \varphi_2, \ldots, \varphi_{n-1}\}$. Por tanto, la solución aproximada del problema debe ser de la forma

$$u_h = \alpha\varphi_0 + \sum_{i=1}^{n-1} v_h(x_i)\varphi_i + \beta\varphi_n.$$

Esta función debe verificar

$$\int_a^b u_h'(v' + c(x)v)\,dx + \int_a^b d(x)u_h v\,dx = \int_a^b f(x)v\,dx.$$

para toda $v \in V_h$. Como $\{\varphi_1, \varphi_2, \ldots, \varphi_{n-1}\}$ es base de V_h, debe tenerse

$$\int_a^b u_h'(\varphi_i' + c(x)\varphi_i)\,dx + \int_a^b d(x)u_h\varphi_i\,dx = \int_a^b f(x)\varphi_i\,dx. \tag{$*$}$$

para todo $i = 1,2,\ldots,n-1$. Llamemos $v_h(x_j) = v_j$ para cada $j = 1,2,\ldots,n-1$. Se tiene que

$$\begin{aligned}\int_a^b u_h'(\varphi_i' + c(x)\varphi_i)\,dx &= \int_a^b \Big(\alpha\varphi_0' + \sum_{j=1}^{n-1} v_j\varphi_j' + \beta\varphi_n'\Big)(\varphi_i' + c(x)\varphi_i)\,dx \\ &= \sum_{j=1}^{n-1} v_j \int_a^b \varphi_j'(\varphi_i' + c(x)\varphi_i)\,dx + \int_a^b (\alpha\varphi_0' + \beta\varphi_n')(\varphi_i' + c(x)\varphi_i)\,dx.\end{aligned}$$

Por otro lado,

$$
\begin{aligned}
\int_a^b d(x)u_h\varphi_i\,dx &= \int_a^b d(x)\Big(\alpha\varphi_0 + \sum_{j=1}^{n-1} v_j\varphi_j + \beta\varphi_n\Big)\varphi_i\,dx \\
&= \sum_{j=1}^{n} v_j \int_a^b d(x)\varphi_j\varphi_i\,dx + \int_a^b d(x)(\alpha\varphi_0+\beta\varphi_n)\varphi_i\,dx.
\end{aligned}
$$

Sustituyendo todo esto en $(*)$ y reagrupando,

$$
\begin{aligned}
&\sum_{j=1}^{n} v_j\Big(\int_a^b (\varphi_j'\varphi_i' + c(x)\varphi_j'\varphi_i + d(x)\varphi_j\varphi_i)\,dx\Big) = \\
&= \int_a^b \Big(f(x)\varphi_i - \alpha\varphi_0'\varphi_i' - \alpha c(x)\varphi_0'\varphi_i - \beta\varphi_n'\varphi_i' - \beta c(x)\varphi_n'\varphi_i - \alpha d(x)\varphi_0\varphi_i - \beta d(x)\varphi_n\varphi_i\,dx\Big).
\end{aligned}
$$

Obtenemos un sistema lineal de la forma $AV=B$. Hallemos la matriz A y el término independiente B. En primer lugar, $\varphi_j'\varphi_i' \neq 0$ si y solo si $i-1\leq j\leq i+1$, y lo mismo para $\varphi_j'\varphi_i$ y $\varphi_j\varphi_i$, ya que

$$
\begin{aligned}
&\varphi_i(x) = \frac{x-x_{i-1}}{h}, && \varphi_i'(x) = \frac{1}{h}, && \text{si } x_{i-1}<x<x_i, \\
&\varphi_i(x) = \frac{x_{i+1}-x}{h}, && \varphi_i'(x) = -\frac{1}{h}, && \text{si } x_i<x<x_{i+1}, \\
&\varphi_i(x) = 0, && \varphi_i'(x) = 0, && \text{si } x\leq x_{i-1} \text{ o } x\geq x_{i+1}.
\end{aligned}
$$

Por tanto, $a_{i,j}=0$ para $j<i-1$ o $j>i+1$. Para cada $i=1,2,\dots,n-1$, aproximaremos $c(x)$, $d(x)$ y $f(x)$ en el intervalo $[x_{i-1},x_{i+1}]$ mediante $c(x)\approx c(x_i)=c_i$, $d(x)\approx d(x_i)=d_i$ y $f(x)\approx f(x_i)=f_i$, respectivamente. Se tiene que

$$
\begin{aligned}
a_{i,i} &= \int_a^b (\varphi_i'\varphi_i' + c(x)\varphi_i'\varphi_i + d(x)\varphi_i\varphi_i)\,dx \\
&\approx \int_{x_{i-1}}^{x_i}\Big(\frac{1}{h^2} + c_i\frac{x-x_{i-1}}{h^2} + d_i\frac{(x-x_{i-1})^2}{h^2}\Big)\,dx + \int_{x_i}^{x_{i+1}}\Big(\frac{1}{h^2} - c_i\frac{x_{i+1}-x}{h^2} + d_i\frac{(x_{i+1}-x)^2}{h^2}\Big)\,dx \\
&= \frac{2h}{h^2} + \frac{c_i}{h^2}\int_{x_{i-1}}^{x_i}(x-x_{i-1})\,dx + \frac{d_i}{h^2}\int_{x_{i-1}}^{x_i}(x-x_{i-1})^2\,dx \\
&\quad - \frac{c_i}{h^2}\int_{x_i}^{x_{i+1}}(x_{i+1}-x)\,dx + \frac{d_i}{h^2}\int_{x_i}^{x_{i+1}}(x_{i+1}-x)^2\,dx \\
&= \frac{2}{h} + \frac{c_i}{h^2}\Big[\frac{(x-x_{i-1})^2}{2}\Big]_{x=x_{i-1}}^{x=x_i} + \frac{d_i}{h^2}\Big[\frac{(x-x_{i-1})^3}{3}\Big]_{x=x_{i-1}}^{x=x_i} \\
&\quad - \frac{c_i}{h^2}\Big[-\frac{(x_{i+1}-x)^2}{2}\Big]_{x=x_i}^{x=x_{i+1}} + \frac{d_i}{h^2}\Big[-\frac{(x_{i+1}-x)^3}{3}\Big]_{x=x_i}^{x=x_{i+1}} \\
&= \frac{2}{h} + \frac{c_i}{\cancel{h^2}}\frac{\cancel{h^2}}{2} + \frac{d_i}{h^2}\frac{h^3}{3} - \frac{c_i}{\cancel{h^2}}\frac{\cancel{h^2}}{2} + \frac{d_i}{h^2}\frac{h^3}{3} = \frac{2}{h} + \frac{2d_ih}{3}. \\
a_{i,i+1} &= \int_a^b (\varphi_{i+1}'\varphi_i' + c(x)\varphi_{i+1}'\varphi_i + d(x)\varphi_{i+1}\varphi_i)\,dx \\
&= \int_{x_i}^{x_{i+1}} (\varphi_{i+1}'\varphi_i' + c(x)\varphi_{i+1}'\varphi_i + d(x)\varphi_{i+1}\varphi_i)\,dx \\
&\approx \int_{x_i}^{x_{i+1}}\Big(-\frac{1}{h^2} - c_i\frac{x_{i+1}-x}{h^2} + d_i\frac{(x-x_i)(x_{i+1}-x)}{h^2}\Big)\,dx \\
&= -\frac{h}{h^2} - \frac{c_i}{h^2}\int_{x_i}^{x_{i+1}}(x_{i+1}-x)\,dx + \frac{d_i}{h^2}\Big(x_{i+1}\int_{x_i}^{x_{i+1}}x\,dx - x_{i+1}x_ih - \int_{x_i}^{x_{i+1}}x^2\,dx + x_i\int_{x_i}^{x_{i+1}}x\,dx\Big) \\
&= -\frac{1}{h} - \frac{c_i}{h^2}\Big[-\frac{(x_{i+1}-x)^2}{2}\Big]_{x=x_i}^{x=x_{i+1}} + \frac{d_i}{h^2}\Big(x_{i+1}\Big[\frac{x^2}{2}\Big]_{x=x_i}^{x=x_{i+1}} - x_{i+1}x_ih - \Big[\frac{x^3}{3}\Big]_{x=x_i}^{x=x_{i+1}} + x_i\Big[\frac{x^2}{2}\Big]_{x=x_i}^{x=x_{i+1}}\Big) \\
&= -\frac{1}{h} - \frac{c_i}{h^2}\frac{h^2}{2} + \frac{d_i}{h^2}\Big(x_{i+1}\frac{x_{i+1}^2-x_i^2}{2} - x_{i+1}x_ih - \frac{x_{i+1}^3-x_i^3}{3} + x_i\frac{x_{i+1}^2-x_i^2}{2}\Big)
\end{aligned}
$$

$$
\begin{aligned}
&= -\frac{1}{h} - \frac{c_i}{2} + \frac{d_i}{h^2}\Big(x_{i+1}\frac{x_{i+1}^2 - x_i^2}{2} - x_{i+1}x_i h - \frac{x_{i+1}^3 - x_i^3}{3} + x_i\frac{x_{i+1}^2 - x_i^2}{2}\Big).\\
a_{i,i-1} &= \int_a^b (\varphi_{i-1}'\varphi_i' + c(x)\varphi_{i-1}'\varphi_i + d(x)\varphi_{i-1}\varphi_i)\,dx\\
&= \int_{x_{i-1}}^{x_i} (\varphi_{i-1}'\varphi_i' + c(x)\varphi_{i-1}'\varphi_i + d(x)\varphi_{i-1}\varphi_i)\,dx\\
&\approx \int_{x_{i-1}}^{x_i} \Big(-\frac{1}{h^2} + c_i\frac{x - x_{i-1}}{h^2} + d_i\frac{(x_i - x)(x - x_{i-1})}{h^2}\Big)\,dx\\
&= -\frac{h}{h^2} + \frac{c_i}{h^2}\int_{x_{i-1}}^{x_i}(x - x_{i-1})\,dx + \frac{d_i}{h^2}\Big(x_i\int_{x_{i-1}}^{x_i} x\,dx - x_{i-1}x_i h - \int_{x_{i-1}}^{x_i} x^2\,dx + x_{i-1}\int_{x_{i-1}}^{x_i} x\,dx\Big)\\
&= -\frac{1}{h} + \frac{c_i}{h^2}\Big[\frac{(x - x_{i-1})^2}{2}\Big]_{x=x_{i-1}}^{x=x_i} + \frac{d_i}{h^2}\Big(x_i\Big[\frac{x^2}{2}\Big]_{x=x_{i-1}}^{x=x_i} - x_{i-1}x_i h - \Big[\frac{x^3}{3}\Big]_{x=x_{i-1}}^{x=x_i} + x_{i-1}\Big[\frac{x^2}{2}\Big]_{x=x_{i-1}}^{x=x_i}\Big)\\
&= -\frac{1}{h} + \frac{c_i}{h^2}\frac{h^2}{2} + \frac{d_i}{h^2}\Big(x_i\frac{x_i^2 - x_{i-1}^2}{2} - x_{i-1}x_i h - \frac{x_i^3 - x_{i-1}^3}{3} + x_{i-1}\frac{x_i^2 - x_{i-1}^2}{2}\Big)\\
&= -\frac{1}{h} + \frac{c_i}{2} + \frac{d_i}{h^2}\Big(x_i\frac{x_i^2 - x_{i-1}^2}{2} - x_{i-1}x_i h - \frac{x_i^3 - x_{i-1}^3}{3} + x_{i-1}\frac{x_i^2 - x_{i-1}^2}{2}\Big).
\end{aligned}
$$

Con esto finaliza el cálculo de la matriz A. Respecto al segundo miembro, si $1 < i$, entonces $\varphi_0'\varphi_i' = 0$, $\varphi_0'\varphi_i = 0$ y $\varphi_0\varphi_i = 0$, mientras que si $i < n-1$, entonces $\varphi_n'\varphi_i' = 0$, $\varphi_n'\varphi_i = 0$ y $\varphi_n\varphi_i = 0$. Teniendo esto en cuenta, si $1 < i < n-1$,

$$
\begin{aligned}
b_i &= \int_a^b f(x)\varphi_i\,dx = f_i\Big(\int_{x_{i-1}}^{x_i}\frac{x - x_{i-1}}{h}\,dx + \int_{x_i}^{x_{i+1}}\frac{x_{i+1} - x}{h}\,dx\Big)\\
&\approx \frac{f_i}{h}\Big(\Big[\frac{(x - x_{i-1})^2}{2}\Big]_{x=x_{i-1}}^{x=x_i} + \Big[-\frac{(x_{i+1} - x)^2}{2}\Big]_{x=x_i}^{x=x_{i+1}}\Big) = \frac{f_i}{h}\Big(\frac{h^2}{2} + \frac{h^2}{2}\Big) = f_i h.
\end{aligned}
$$

Por otro lado,

$$
\begin{aligned}
b_1 &= \int_a^b \Big(f(x)\varphi_1 - \alpha\varphi_0'\varphi_1' - \alpha c(x)\varphi_0'\varphi_1 - \alpha d(x)\varphi_0\varphi_1\Big)\,dx\\
&\approx f_1 h - \alpha\int_{x_0}^{x_1}\varphi_0'\varphi_1'\,dx - \alpha c_1\int_{x_0}^{x_1}\varphi_0'\varphi_1\,dx - \alpha d_1\int_{x_0}^{x_1}\varphi_0\varphi_1\,dx\\
&= f_1 h - \alpha\int_{x_0}^{x_1} -\frac{1}{h^2}\,dx - \alpha c_1\int_{x_0}^{x_1} -\frac{x - x_0}{h^2}\,dx - \alpha d_1\int_{x_0}^{x_1}\frac{(x_1 - x)(x - x_0)}{h^2}\,dx\\
&= f_1 h + \frac{\alpha h}{h^2} + \frac{\alpha c_1}{h^2}\Big[\frac{(x - x_0)^2}{2}\Big]_{x=x_0}^{x=x_1} - \frac{\alpha d_1}{h^2}\Big(x_1\int_{x_0}^{x_1} x\,dx - x_0x_1h - \int_{x_0}^{x_1} x^2\,dx + x_0\int_{x_0}^{x_1} x\,dx\Big)\\
&= f_1 h + \frac{\alpha}{h} + \frac{\alpha c_1}{h^2}\frac{h^2}{2} - \frac{\alpha d_1}{h^2}\Big(x_1\frac{x_1^2 - x_0^2}{2} - x_0x_1h - \frac{x_1^3 - x_0^3}{3} + x_0\frac{x_1^2 - x_0^2}{2}\Big).\\
&= f_1 h + \frac{\alpha}{h} + \frac{\alpha c_1}{2} - \frac{\alpha d_1}{h^2}\Big(x_1\frac{x_1^2 - x_0^2}{2} - x_0x_1h - \frac{x_1^3 - x_0^3}{3} + x_0\frac{x_1^2 - x_0^2}{2}\Big).\\
b_{n-1} &= \int_a^b \Big(f(x)\varphi_{n-1} - \beta\varphi_n'\varphi_{n-1}' - \beta c(x)\varphi_n'\varphi_{n-1} - \beta d(x)\varphi_n\varphi_{n-1}\Big)\,dx\\
&\approx f_{n-1}h - \beta\int_{x_{n-1}}^{x_n}\varphi_n'\varphi_{n-1}'\,dx - \beta c_{n-1}\int_{x_{n-1}}^{x_n}\varphi_n'\varphi_{n-1}\,dx - \beta d_{n-1}\int_{x_{n-1}}^{x_n}\varphi_n\varphi_{n-1}\,dx\\
&= f_{n-1}h - \beta\int_{x_{n-1}}^{x_n} -\frac{1}{h^2}\,dx - \beta c_{n-1}\int_{x_{n-1}}^{x_n}\frac{x_n - x}{h^2}\,dx - \beta d_{n-1}\int_{x_{n-1}}^{x_n}\frac{(x - x_{n-1})(x_n - x)}{h^2}\,dx\\
&= f_{n-1}h + \frac{\beta h}{h^2} - \frac{\beta c_{n-1}}{h^2}\Big[-\frac{(x_n - x)^2}{2}\Big]_{x=x_{n-1}}^{x=x_n}\\
&\quad - \frac{\beta d_{n-1}}{h^2}\Big(x_n\int_{x_{n-1}}^{x_n} x\,dx - x_{n-1}x_nh - \int_{x_{n-1}}^{x_n} x^2\,dx + x_{n-1}\int_{x_{n-1}}^{x_n} x\,dx\Big)\\
&= f_{n-1}h + \frac{\beta}{h} - \frac{\beta c_{n-1}}{h^2}\frac{h^2}{2} - \frac{\beta d_{n-1}}{h^2}\Big(x_n\frac{x_n^2 - x_{n-1}^2}{2} - x_{n-1}x_nh - \frac{x_n^3 - x_{n-1}^3}{3} + x_{n-1}\frac{x_n^2 - x_{n-1}^2}{2}\Big)\\
&= f_{n-1}h + \frac{\beta}{h} - \frac{\beta c_{n-1}}{2} - \frac{\beta d_{n-1}}{h^2}\Big(x_n\frac{x_n^2 - x_{n-1}^2}{2} - x_{n-1}x_nh - \frac{x_n^3 - x_{n-1}^3}{3} + x_{n-1}\frac{x_n^2 - x_{n-1}^2}{2}\Big).
\end{aligned}
$$