Driven and Damped Harmonic Motion

driving agular frequency ωd $\quad F = F_0 \sin(\omega_d t + \varphi)$ $\quad \alpha . \beta . \gamma . \omega . \Omega . \rho$

$\delta , \Delta , \theta . \varphi . \psi , \pi ,$

angular frequency ω $\quad \longrightarrow x$ $\quad \tau . \eta . \lambda . \varepsilon . \sigma . \Sigma . \mu$

natural frequency ω_0

$$\frac{d^2 x}{dt^2} + 2\gamma \frac{dx}{dt} + \omega_0^2 x = C_0 e^{i\omega_d t} \qquad i = \sqrt{-1}$$

$$\ddot{x} + 2\gamma \dot{x} + \omega_0^2 x = C_0 e^{i\omega_d t}$$ (2nd inhomogenous ODE)

$$x(t) = x_h(t) + x_p(t)$$

$$r = -\gamma \pm \sqrt{\gamma^2 - \omega_0^2}$$

$$x_h(t) = A e^{(-\gamma + \sqrt{\gamma^2 - \omega_0^2})t} + B \cdot e^{(-\gamma - \sqrt{\gamma^2 - \omega_0^2})t}$$

$$x_h(t) = A e^{\gamma_1 t} + B e^{\gamma t}$$

$$x_p(t) = C e^{i\omega_d t}$$

$$= e^{-\gamma t} \left(A e^{\sqrt{\gamma^2 - \omega_0^2}\, t} + B \cdot e^{-\sqrt{\gamma^2 - \omega_0^2}\, t} \right).$$

$$\ddot{x} + 2\gamma \cdot \dot{x} + \omega_0^2 x = \underline{C_0 e^{i\omega_d t}}$$

$$x_p(t) = C e^{i\omega_d t} \qquad \dot{x} = C \cdot i\omega_d e^{i\omega_d t} \qquad \ddot{x} = -C\omega_d^2 e^{i\omega_d t}$$

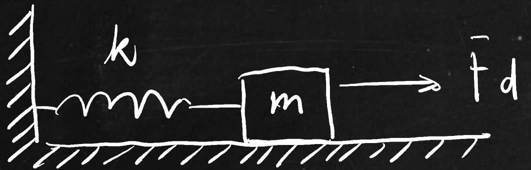
$$-C \cdot \omega_d^2 \cancel{e^{i\omega_d t}} + 2\gamma \cdot C \cdot i\omega_d \cancel{e^{i\omega_d t}} + \omega_0^2 \cdot C \cdot \cancel{e^{i\omega_d t}} = C_0 \cancel{e^{i\omega_d t}}$$

$$-\omega_d^2 C + 2i\omega_d \gamma C + \omega_0^2 C = C_0 \qquad C(\omega_0^2 + 2i\gamma\omega_d - \omega_d^2) = C_0$$

$$C = \frac{C_0}{\omega_0^2 + 2i\gamma\omega_d - \omega_d^2} \qquad x_p(t) = \frac{C_0}{\omega_0^2 + 2i\gamma\omega_d - \omega_d^2} e^{i\omega_d t}$$

$$\underline{x(t) = e^{-\gamma t}\left(A e^{\sqrt{\gamma^2 - \omega_0^2}\, t} + B e^{-\sqrt{\gamma^2 - \omega_0^2}\, t}\right) + \frac{C_0}{\omega_0^2 + 2i\gamma\omega_d - \omega_d^2} \cdot e^{i\omega_d t}}$$

physical driving forces.



$$F_d = F_0 \cos\omega_d t \longrightarrow C_0 e^{-i\omega_d t}$$

$$\boxed{e^{i\alpha} = \cos\alpha + i\sin\alpha}$$

$$F_0\cos\omega_d t = F_0 \cdot \frac{e^{i\omega_d t} + e^{-i\omega_d t}}{2}$$

$$\Rightarrow \ddot{x} + 2\gamma\dot{x} + \omega_0^2 x = \frac{F}{2}\left(e^{i\omega_d t} + e^{-i\omega_d t}\right)$$

$$F = \frac{F_0}{m} \qquad \vec{f} = m\vec{a}$$

$$m\frac{d^2x}{dt^2} = -\beta\frac{dx}{dt} + F_d - \omega_0^2 x$$

$$\vec{F} = -\beta\vec{v}$$

Principle of Superposition

$$x(t) = \underbrace{\left(\frac{F/2}{\omega_0^2 + 2i\gamma\omega_d - \omega_d^2}\right)e^{i\omega_d t} + \left(\frac{F/2}{\omega_0^2 - 2i\gamma\omega_d - \omega_d^2}\right)e^{-i\omega_d t}}_{x_p(t)} + \underbrace{e^{-\gamma t}\left(\_\_\_\_\_\_\right)}_{x_h(t)}$$

Transforming

complex $a+ib$

$\cos\alpha = \frac{1}{2}(e^{i\alpha}+e^{-i\alpha})$

$\sin\alpha = \frac{1}{2i}(e^{i\alpha}-e^{-i\alpha})$

$$\frac{F/2}{(\omega_0^2-\omega_d^2)+i(2\gamma\omega_d)} = \frac{F/2}{(\omega_0^2-\omega_d^2)+i(2\gamma\omega_d)} \times \frac{(\omega_0^2-\omega_d^2)-i(2\gamma\omega_d)}{(\omega_0^2-\omega_d^2)-i(2\gamma\omega_d)}$$

$$= \frac{(F/2)\left((\omega_0^2-\omega_d^2)-i(2\gamma\omega_d)\right)}{(\omega_0^2-\omega_d^2)^2+4\gamma^2\omega_d^2}$$

$i^2=-1$

$$= \left(\frac{\frac{F}{2}(\omega_0^2-\omega_d^2)}{(\qquad)} - i\frac{\frac{F}{2}\cdot 2\gamma\omega_d}{(\qquad)}\right)$$

$e^{i\omega_d t}$

$$\frac{F/2}{(\omega_0^2-\omega_d^2)-i(2\gamma\omega_d)} = \left(\frac{\frac{F}{2}(\omega_0^2-\omega_d^2)}{(\qquad)} + i\frac{\frac{F}{2}\cdot 2\gamma\omega_d}{(\qquad)}\right)$$

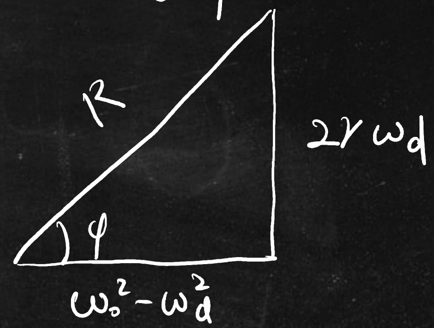
$e^{-i\omega_d t}$

$$x_p(t) = \frac{F(\omega_0^2-\omega_d^2)}{(\omega_0^2-\omega_d^2)^2+4\gamma^2\omega_d^2}\cdot\underbrace{\frac{1}{2}(e^{i\omega_d t}+e^{-\omega_d t})}_{\cos\omega_d t} + \frac{F\cdot 2\gamma\omega_d}{(\omega_0^2-\omega_d^2)^2+4\gamma^2\omega_d^2}\cdot\underbrace{\frac{1}{2i}(e^{i\omega_d t}-e^{-i\omega_d t})}_{\sin\omega_d t}$$

$$x_p(t) = \left(\frac{F(\omega_0^2-\omega_d^2)}{(\omega_0^2-\omega_d^2)^2+4\gamma^2\omega_d^2}\right)\cos\omega_d t + \left(\frac{F\cdot 2\gamma\omega_d}{(\omega_0^2-\omega_d^2)^2+4\gamma^2\omega_d^2}\right)\cdot\sin\omega_d t$$

$$R = \sqrt{(\omega_0^2-\omega_d^2)^2+(2\gamma\omega_d)^2}$$

$$x_p(t) = \frac{F}{R}\left(\underbrace{\frac{\omega_0^2-\omega_d^2}{R}}_{\cos\varphi}\cos\omega_d t + \underbrace{\frac{2\gamma\omega_d}{R}}_{\sin\varphi}\sin\omega_d t\right)$$



$$x_p(t) = \frac{F}{R}\cos(\omega_d t-\varphi)$$

phase 相位

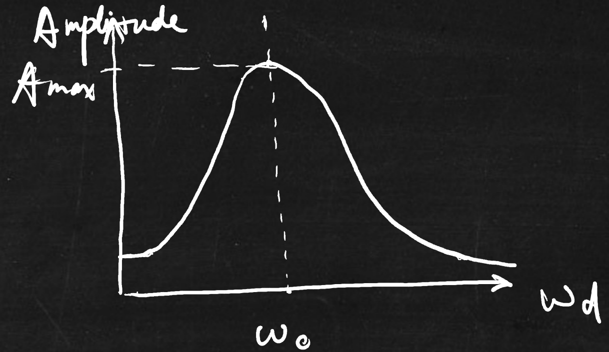
$$x(t) = \overbrace{\frac{F}{R}}^{\text{Amplitude}}\cos(\omega_d t-\varphi) + \underbrace{e^{-\gamma t}\left(Ae^{\sqrt{\gamma^2-\omega_0^2}\,t}+Be^{-\sqrt{\gamma^2-\omega_0^2}\,t}\right)}_{\downarrow 0}$$

stationary state solution $t\to\infty$

$\tilde{\omega}$ angular motion of damped motion

$$A = \frac{F}{K} = \frac{F_0}{m\sqrt{(\omega_0^2-\omega_d^2)^2+(2\gamma\omega_d)^2}}$$

$$\tan\varphi = \frac{2\gamma\omega_d}{\omega_0^2-\omega_d^2}$$



1° $\omega_d \approx 0$ $\quad \gamma\omega_d \approx 0$ $\quad \varphi \approx 0$

2° $\omega_d \approx \omega_0$ Resonance 共振 $\tan\varphi = \frac{2\gamma\cdot\omega_0}{\omega_0^2-\omega_0^2} \approx \infty$ $\quad \varphi = \frac{\pi}{2}$

3° $\omega_d \to \infty$ $\quad \gamma\omega_d \ll \omega_d^2-\omega_0^2$ $\quad \varphi \approx \pi$

Quality Factor

Work done by driving Force $\vec{F}$

$$W = \int_{\vec{r}_1}^{\vec{r}_2} \vec{F} \cdot d\vec{r} \qquad V(r) = -\int_{\vec{r}_1}^{\vec{r}_2} \vec{F} \cdot d\vec{r}$$

$$W = \int F dx \qquad T_d = \frac{2\pi}{\omega_d} \qquad F_0 = \frac{F_d}{m}$$

$$= \int_0^T F dx = \int_0^T F \cdot v dt \qquad F = F_0 \cos \omega_d t \qquad x = \frac{\bar{F}_d}{mR} \cdot \cos(\omega_d t - \varphi)$$

$$v = \frac{dx}{dt} = -\sin(\omega_d t - \varphi) \cdot \omega_d \cdot \frac{F_d}{R}$$

$$= -\int_0^T \frac{F_d}{m} \cos \omega_d t \, (F_d / R) \cdot \omega_d \sin(\omega_d t - \varphi) \, dt$$

$$= -\frac{F_d^2}{mR} \omega_d \int_0^T \cos \underbrace{\omega_d t}_{\theta} \sin(\underbrace{\omega_d t}_{\theta} - \varphi) dt \qquad \theta = \omega_d t \qquad d\theta = \omega_d dt$$

$$= -\frac{F_d^2}{mR} \int_0^{2\pi} \cos\theta \sin(\theta - \varphi) d\theta = -\frac{F_d^2}{mR} \int_0^{2\pi} (\cancel{\cos\theta \sin\theta \cos\varphi} - \cos\theta \cos\theta \sin\varphi) d\theta$$

$$\int_0^{2\pi}\sin\theta\cos\theta\,d\theta=0 \qquad \int_0^{2\pi}\cos^2\theta\,d\theta=\pi$$

$$W=\frac{\bar F_d^2}{mR}\pi\cdot\sin\varphi \qquad \bar P=\frac{W}{T_d}=\frac{\omega_d}{2\cancel{\pi}}\cdot\frac{\bar F_d^2\cancel{\pi}\sin\varphi}{mR}$$

$$=\frac{\bar F_d^2\,\omega_d}{2mR}\sin\varphi.$$

$$\tan\varphi=\frac{2\gamma\omega_d}{\omega_0^2-\omega_d^2} \qquad \sin\varphi=\frac{2\gamma\omega_d}{R}$$

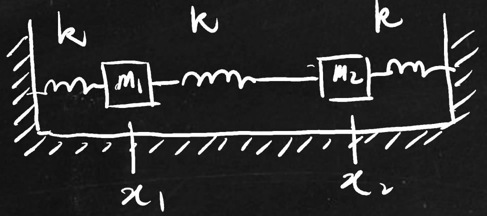
$$\bar P=\frac{\bar F_d^2\,\omega_d}{\cancel{2}mR}\cdot\frac{\cancel{2}\gamma\omega_d}{R}=\frac{\omega_d^2F_d^2}{mR^2}\gamma$$

$$x=\frac{F_d}{mR}\cos(\omega_d t-\varphi)$$

$$v=-\underbrace{\frac{F_d}{mR}\omega_d}_{v_{max}}\sin(\omega_d t-\varphi)$$

$$v_{max}=\frac{\bar F_d}{mR}\omega_d \qquad \bar P=m\gamma\cdot v_{max}^2$$

$$K=\frac{1}{2}mv^2 \qquad K_{max}=\frac{1}{2}mv_{max}^2 \qquad \bar P=2\gamma K_{max}.$$



$\longrightarrow \hat{x}$

x_1, x_2 displacement w.r.t. equilibrium

$$F_1 = \underline{-kx_1} - k(x_1 - x_2) = m_1 a_1 = m_1 \ddot{x}_1$$

$$F_2 = -kx_2 - k(x_2 - x_1) = -kx_2 - k(x_2 - x_1) = m_2 a_2 = m_2 \ddot{x}_2$$

$$\begin{cases} m_1\ddot{x}_1 + 2kx_1 - kx_2 = 0 \\ m_2\ddot{x}_2 + 2kx_2 - kx_1 = 0 \end{cases} \qquad \begin{aligned} \ddot{x}_1 + 2\frac{k}{m_1}x_1 - \frac{k}{m_1}x_2 = 0 \\ \ddot{x}_2 + 2\frac{k}{m_2}x_2 - \frac{k}{m_2}x_1 = 0 \end{aligned}$$

$$m_1 = m_2 = m \qquad \omega = \sqrt{\frac{k}{m}}$$

$$\begin{cases} \ddot{x}_1 + \underline{2\omega^2 x_1} \, \underline{- \omega^2 x_2} = 0 & ① \\ \ddot{x}_2 + \underline{2\omega^2 x_2} \, \underline{- \omega^2 x_1} = 0 & ② \end{cases}$$

$① + ②$: $\underline{(\ddot{x}_1 + \ddot{x}_2) + \omega^2 (x_1 + x_2) = 0}$ $\qquad ① - ②$: $\underline{(\ddot{x}_1 - \ddot{x}_2) + 3\omega^2 (x_1 - x_2) = 0}$

normal modes of coupled oscillations.

$$\begin{cases} x_1 + x_2 = \alpha \\ x_1 - x_2 = \beta \end{cases} \Rightarrow \begin{cases} \ddot{\alpha} + \omega^2 \alpha = 0 \\ \ddot{\beta} + 3\omega^2 \beta = 0 \end{cases}$$

$$\alpha = x_1 + x_2 = A_+ \cos(\omega t + \varphi_+)$$

$$\beta = x_1 - x_2 = A_- \cos(\sqrt{3}\,\omega t + \varphi_-)$$

$$\begin{cases} x_1(t) = \frac{A_+}{2}\cos(\omega t + \varphi_+) + \frac{A_-}{2}\cos(\sqrt{3}\,\omega t + \varphi_-) \\ x_2(t) = \frac{A_+}{2}\cos(\omega t + \varphi_+) - \frac{A_-}{2}\cos(\sqrt{3}\,\omega t + \varphi_-) \end{cases}$$