Differential Equation

1st order ~: $\frac{dx}{dt} + bx = 0$ $\qquad \underline{x = x(t)}$

Solution: $\frac{dx}{dt} = -bx \qquad \frac{dx}{x} = -b\,dt$

$$\int \frac{dx}{x} = \int -b\,dt \qquad \ln x = -bt + C$$

$$x(t) = e^{-bt+C} = e^{-bt} \cdot e^{C} = A e^{-bt}$$

$$\frac{dx}{dt} + p(t) \cdot x = f(t) \qquad \text{Integration factor } I$$

$$\qquad\qquad\quad I'(t)$$

Solution: $\underline{I(t) \cdot x'} + \underline{p(t)\, I(t)}\, x = f(t) \cdot I(t)$

$$\frac{d}{dt}(I(t) \cdot x) = I'(t) \cdot x + \underline{I(t) \cdot x'} \quad \text{(product rule)}$$

$$p(t)I(t) = I'(t)$$

$$\frac{d}{dt}(I(t)\cdot x) = f(t)\cdot I(t)$$

$$\frac{dI(t)}{dt} = p(t)I(t)$$

$$\frac{dI}{I} = p(t)dt \Rightarrow I(t) = e^{\int p(t)dt}$$

2nd order differential Equation:

$$a\cdot\frac{d^2x}{dt^2} + b\cdot\frac{dx}{dt} + cx = 0 \qquad \text{(homogenous)}$$

$$x(t) = e^{rt}$$

$$a\cdot r^2\cdot e^{rt} + bre^{rt} + ce^{rt} = 0$$

$$e^{rt}(ar^2+br+c) = 0 \qquad e^{rt} > 0$$

$$ar^2+br+c = 0 \qquad \text{characteristic equation}$$

$$\Delta = b^2 - 4ac$$

① $b^2-4ac>0$ $\qquad r_{1,2}=\dfrac{-b\pm\sqrt{\Delta}}{2a}$

$x(t)=Ae^{r_1 t}+Be^{r_2 t}$

② $b^2-4ac=0$ $\qquad r=-\dfrac{b}{2a}$

$x(t)=c\cdot e^{rt}$

Complex number

③ $b^2-4ac<0$ $\qquad r_1=p+iq \quad r_2=p-iq$ $\qquad i=\sqrt{-1}$

$x(t)=e^{pt}\left[A\cos(qt)+B\sin(qt)\right]$

$$a\frac{d^2x}{dt^2} + b\frac{dx}{dt} + cx = f(t) \qquad x(t) = \underline{x_h(t)} + \underline{x_p(t)}$$

$$a\frac{d^2x}{dt^2} + b\frac{dx}{dt} + cx = 0 \quad \Rightarrow \quad x_h(t)$$

$f(t)$: ☆ $\underline{C}\, e^{at}$ $\qquad$ $x_p(t)$: Ae^{at}

☆ $C_1 \sin \underline{at} + C_2 \cos \underline{at}$ $\qquad$ $A\sin at + B\cos at$

☆ polynomial of degree n $\qquad$ ~

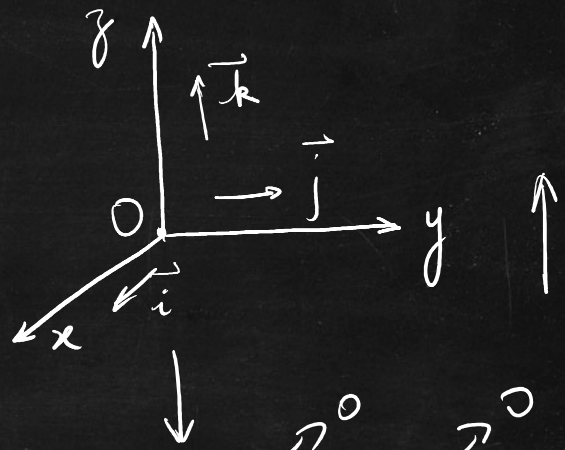
$C_1 e^{at} \sin bt + C_2 e^{at} \cos bt$ $\qquad$ $Ae^{at}\sin bt + Be^{at}\cos bt$

ex. $$\frac{d^2x}{dt^2} + 2\frac{dx}{dt} + 2x = \sin^2 t = \frac{1-\cos 2t}{2}$$

$$r^2 + 2r + 2 = 0 \qquad r = \frac{-2 \pm \sqrt{-4}}{2} = -1 \pm i \qquad x_h(t) = e^{-t}\cdot(A\sin t + B\cos t)$$

$$\underline{\vec{F}_f = -\beta\vec{v}}$$

$$\vec{F}_f = -\beta(v_x\vec{i} + v_y\vec{j} + v_z\vec{k})$$



$$\vec{F}_f = \underline{\beta v_z\vec{k}} \qquad \vec{F}_g = -mg\vec{k}$$

$$\vec{F}_{Net} = \vec{F}_f + \vec{F}_g = \underline{(\beta v_z - mg)\vec{k}} = \underline{m\vec{a}} = m(\cancelto{0}{a_x\vec{i}} + \cancelto{0}{a_y\vec{j}} - a_z\vec{k})$$

$$(\beta v_z - mg)\vec{k} = -m a_z\vec{k} \qquad \beta v_z - mg = -m a_z = -m\cdot\frac{dv_z}{dt}$$

$$\underline{\frac{dv_z}{dt} + \frac{\beta}{m}v_z = g} \qquad \boxed{\frac{dv_z}{dt} + \frac{\beta}{m}v_z = 0} \qquad \frac{dv_z}{dt} = -\frac{\beta}{m}v_z$$

$$\boxed{\frac{dv_z}{v_z} = -\frac{\beta}{m}dt} \qquad \ln v_z = -\frac{\beta}{m}t + C_1 \qquad v_z = \exp\left\{-\frac{\beta}{m}\cdot t\right\}\cdot C$$

$$v_z = C \cdot \exp\left\{-\frac{\beta}{m} t\right\} \qquad t=0 \qquad v_z = u_z \qquad v_z = u_z \exp\left\{-\frac{\beta}{m} t\right\}$$

$$v_z = \frac{dz}{dt} \qquad z = \int v_z dt = \int u_z \exp\left\{-\frac{\beta}{m} t\right\} = -\frac{um}{\beta} e^{-\frac{\beta}{m} t} + C_2$$

$$t=0 \qquad z=0 \qquad 0 = -\frac{um}{\beta} e^{0} + C_2 \Rightarrow C_2 = \frac{um}{\beta}$$

$$\boxed{\begin{aligned} mg &= \beta v \\ v &= \frac{mg}{\beta} \end{aligned}}$$

$$z = \frac{um}{\beta} - \frac{um}{\beta} \cdot e^{-\frac{\beta}{m} t} = \frac{um}{\beta}\left(1 - e^{-\frac{\beta}{m} t}\right)$$

$$-m \frac{dv_z}{dt} = \beta v_z - mg \qquad m \frac{dv_z}{dt} = mg - \beta v_z \qquad \frac{dv_z}{dt} = \boxed{g - \frac{\beta}{m} v_z}$$

$$\frac{dv_z}{dt} = -\frac{\beta}{m}\left(\frac{-gm}{\beta} + v_z\right) \qquad \frac{dv_z}{v_z - \frac{gm}{\beta}} = -\frac{\beta}{m} dt$$

$$v_z = \frac{mg}{\beta}\left(1 - \exp\left(-\frac{\beta}{m} t\right)\right) \qquad t=0 \quad v_z = 0 \qquad t \to \infty \quad v_z \to \frac{mg}{\beta}$$

$$y = e^{x} \qquad \frac{dy}{dx} = e^{x} \qquad \int e^{x}\,dx = e^{x} + C$$

$$y = e^{ax} \qquad \frac{dy}{dx} = a \cdot e^{ax} \qquad \int e^{ax}\,dx = \frac{1}{a} e^{ax} + C$$

$$\vec{f} = m\vec{a} = m \cdot \frac{d\vec{v}}{dt} \qquad \vec{f}\,dt = m\frac{d\vec{v}}{dt}\,dt = m\,d\vec{v} = d\vec{p} \qquad \boxed{\vec{f} = ma}$$

$$\vec{p} := m\vec{v} \qquad y = uv \qquad \frac{dy}{dx} = \frac{du}{dx}v + u \cdot \frac{dv}{dx} \qquad dy = v\,du + u\,dv$$

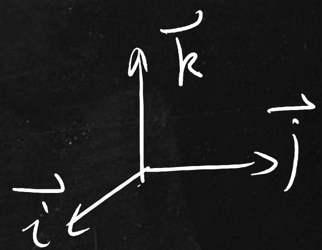
$$d\vec{p} = dm \cdot \vec{v} + m\,d\vec{v} \qquad \frac{d\vec{p}}{dt} = \frac{dm}{dt}\vec{v} + m \cdot \frac{d\vec{v}}{dt} \qquad m = \frac{m_0}{\sqrt{1 - \frac{v^2}{c^2}}} \ (S.R)$$

$$\vec{f} = \frac{d\vec{p}}{dt} \qquad v \sim 10^{-1} c \qquad p \quad \text{electron} \sim 0.99c$$

$$3 \times 10^{7}$$

$$\int \vec{f}\,dt = \int d\vec{p} \qquad \int_{t_1}^{t_2} \vec{f}\,dt = \int_{\vec{p}_1}^{\vec{p}_2} d\vec{p} = (\vec{p}_2 - \vec{p}_1) = \Delta\vec{p}$$

Energy Definition Conservation Law of Energy



(the ablity to do work) Scalar

Do some work

work done by some force: $W = \int dW = \int \vec F \cdot d\vec r$ $W = \underline{F d \cos\theta}$

$\vec F = F_x \vec i + F_y \vec j + F_z \vec k$ $\vec r = x\vec i + y\vec j + z\vec k$ $d(x\vec i) = dx\vec i + x \cdot \overset{=0}{d\vec i}$

$d\vec r = dx\vec i + dy\vec j + dz\vec k$

$W = \int_{\vec r_A}^{\vec r_B} (F_x dx + F_y dy + F_z dz)$ $(F_x\vec i + F_y\vec j + F_z\vec k) \cdot (dx\vec i + dy\vec j + dz\vec k)$

$\vec i \cdot \vec i = \vec j \cdot \vec j = \vec k \cdot \vec k = 1$

$\vec i \cdot \vec j = \vec i \cdot \vec k = \vec j \cdot \vec k = 0$

ex. $\vec F = 2xy\vec i + e^z \vec j + x^2 y z^3 \vec k$ $(0,0,0) \downarrow (1,1,1)$

$$W_F = \int_{(0,0,0)}^{(1,1,1)} \left(2xy\,dx + e^z\,dy + x^2yz^3\,dz\right)$$

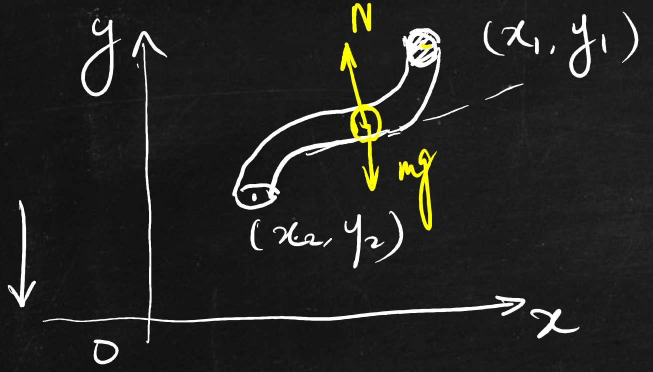
$$= \int_{(0,0,0)}^{(1,1,1)} 2xy\,dx + \int_{(0,0,0)}^{(1,1,1)} e^z\,dy + \int_{(0,0,0)}^{(1,1,1)} x^2yz^3\,dz$$

$$= \left(x^2y + ye^z + \frac{1}{4}x^2yz^4\right)\Bigg|_{(0,0,0)}^{(1,1,1)} = \left(1+e+\frac{1}{4}-0-0-0\right) = e+\frac{5}{4}\ (J)$$

$$\iiint xyz\,dx\,dy\,dz = \int\left(\int\left(\int xyz\,dx\right)dy\right)dz = \int\left(\int \frac{1}{2}x^2yz + C_1\right)dy\Big)\,dz$$

$$= \int\left(\frac{1}{4}x^2y^2z + C_1y + C_2\right)dz = \frac{1}{8}x^2y^2z^2 + C_1yz + C_2z + C_3$$

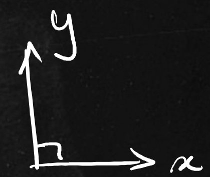
Example:



$\vec{F}_g = -mg\vec{j}$

$$dw = (\vec{N} + \vec{F}_g)\cdot d\vec{r}$$

$$= \vec{N}\cdot d\vec{r} + \vec{F}_g\cdot d\vec{r}$$



$$\vec{N}\cdot \frac{d\vec{r}}{dx} = 0 \qquad \vec{N}\cdot d\vec{r} = 0$$

$$dw = \vec{F}_g\cdot d\vec{r} = mg(y_1 - y_2)$$

Potential Energy

- gravitational ~
- elastic ~
- electrical ~

$\boxed{\text{System}}$

Conservative Force 保守力 friction Dispated Force 耗散力

Entropy 熵