Q1

(i) During the fall, the total force acting on the baking cup has got an intensity of $F = W - F_f$, where W is the weight of the baking cup and F_f is air friction.

If air friction scales up with velocity, during the fall the intensity of the force will decrease until it will be neglectable so that one could assume $F_f = W$. In such a set-up, we can consider the motion to be uniform linear motion with speed $v_L = 1.4\ \mathrm{ms}^{-1}$.

During the fall, air friction is negligible and the motion occurs as a result of weight producing an acceleration equal to g.

We know apply to the function $v(t)$ the two conditions described in the problem. The first condition is satisfied if

$$\lim_{t\to+\infty} \alpha\frac{e^{\beta t}-1}{e^{\beta t}+1} = v_L,$$

which can be found by applying De L'Hopital, i.e.,

$$\lim_{t\to+\infty} \alpha\frac{e^{\beta t}-1}{e^{\beta t}+1} = \lim_{t\to+\infty} \alpha\frac{\beta e^{\beta t}}{\beta e^{\beta t}} = \alpha.$$

Thus,

$$\alpha = v_L = 1.4\ \mathrm{ms}^{-1}.$$

The acceleration is described by the function

$$a(t) = \frac{\mathrm{d}v}{\mathrm{d}t} = \alpha\frac{\beta e^{\beta t}(e^{\beta t}+1) - \beta e^{\beta t}(e^{\beta t}-1)}{(e^{\beta t}+1)^2} = 2\alpha\beta\frac{e^{\beta t}}{(e^{\beta t}+1)^2}.$$

For the acceleration of the baking cup to be equal to the gravitational acceleration g, one must have

$$\alpha(t=0) = \frac{2\alpha\beta}{4} = g$$

whence

$$\alpha\beta = 2g \implies \beta = \frac{2g}{\alpha} = \frac{2g}{v_L} = \frac{2\times 9.8\ \mathrm{ms}^{-2}}{1.4\ \mathrm{ms}^{-1}} = 14\ \mathrm{s}^{-1}.$$

The function $v(t)$ satifying the conditions enumerated in the task is the following:

$$v(t) = 1.4\frac{e^{14t}-1}{e^{14t}+1}.$$

(ii) During the stage in which the fall occurs at constant velocity v_L, from the principle of inertia (Newton's First Law), the sum of the forces acting on the baking cup must be zero.

$$F_f = W \implies kv_L^2 = mg \implies v_L = \sqrt{\frac{mg}{k}}$$

whence

$$\alpha = v_L = \sqrt{\frac{mg}{k}}.$$

From the condition that initial acceleration must be equal to g, one has:

$$\beta = \frac{2g}{\alpha} = \frac{2g}{V_L} = 2g\sqrt{\frac{k}{mg}} = 2\sqrt{\frac{kg}{m}}.$$

Therefore, the equation of $v(t)$ can be written as

$$v(t) = \sqrt{\frac{mg}{k}} \frac{e^{2t\sqrt{kg/m}} - 1}{e^{2t\sqrt{kg/m}} + 1}.$$

(iii) We have set

$$z = \sqrt{k} \quad \text{and} \quad b = 2t\sqrt{\frac{g}{m}}.$$

Let us calculate the limit for z approaching zero of the given function. By De l'Hopital's theorem, one has:

$$v = \sqrt{mg} \lim_{z \to 0} \frac{e^{bz} - 1}{z(e^{bz} + 1)} = \sqrt{mg} \lim_{z \to 0} \frac{be^{bz}}{e^{bz} + 1 + zbe^{bz}} = \frac{\sqrt{mg}}{2} b.$$

From position $b = 2t\sqrt{g/m}$, it follows that

$$v = \frac{\sqrt{mg}}{2} \cdot 2t\sqrt{\frac{g}{m}} = gt.$$

The significance of this result can be interpreted in the following way: in absence of friction, the baking cup falls with constant acceleration equal to g.

(iv) In the expression for $v(t)$, the fraction turns out to be adimensional. Therefore, we need the 1.4 factor to be homogeneous with velocity, and, thus, to be expressed in metres per second.

The derivative of the function $F(t)$ is

$$F'(t) = A\frac{2}{e^{14t}+1}\frac{14e^{14t}\cdot 2}{4} + B$$

which, after some algebraic manipulations, can be rewritten in the form

$$F'(t) = \frac{(14A+B)e^{14t}+B}{e^{14t}+1}.$$

Hence, the function

$$F'(t) = 0.2\ln\left(\frac{e^{14t}+1}{2}\right) - 1.4t$$

is a primitive of the function $v(t)$.

The function $v(t)$ is continuous in the time interval (in seconds) $[0,1]$. The average value in the time interval (in seconds) $[0,1]$ is given by

$$v_{\text{avg}} = \frac{\int_0^1 v(t)\,\mathrm{d}t}{1-0} = F(1) - F(0) = 0.2\ln\left(\frac{e^{14}+1}{2}\right) - 1.4 = 1.26\ \mathrm{ms}^{-1}.$$

The integral $\displaystyle\int_0^1 v(t)\,\mathrm{d}t$ is the area of the surface subtended by the graph of $v(t)$ in the time interval (in seconds) $[0,1]$. This area corresponds to the distance travelled by the baking cup in one second. Therefore, the value of v_{avg} represents the average velocity of the baking cup during the first second of its fall.

Q2

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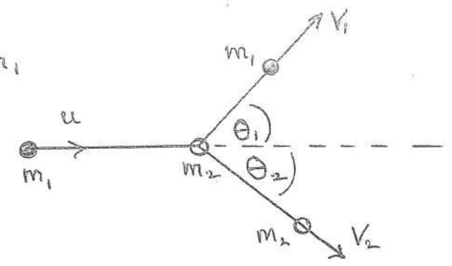
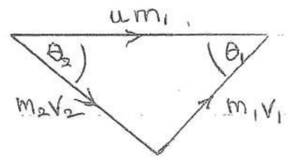
Q8 (a) $m_2 = 4m_1$



TRIANGLE OF MOMENTA



Vector Diagram 1

Using sin rule,

$$\frac{m_1 V_1}{\sin\theta_2} = \frac{m_2 V_2}{\sin\theta_1} \quad 1$$

Thus

$$\frac{V_1}{V_2} = \frac{m_2 \sin\theta_2}{m_1 \sin\theta_1} = \frac{4\sin\theta_2}{\sin\theta_1} \quad (1) \quad 1$$

Also

$$\frac{m_1 V_1}{\sin\theta_2} = \frac{m_1 u}{\sin(180-\theta_1-\theta_2)} \quad 1$$

Thus

$$\frac{V_1}{u} = \frac{\sin\theta_2}{\sin(\theta_1+\theta_2)} \quad (2) \quad 1$$

5

(b) Final K.E., $T_F = \frac{1}{2}m_1 V_1^2 + \frac{1}{2}m_2 V_2^2$ (3) 1

Initial K.E. $T_I = \frac{1}{2}m_1 u^2$

From (3) & (1)

$$T_F = \frac{1}{2}m_1 V_1^2 + \frac{1}{2}m_2 V_1^2\left(\frac{m_1 \sin\theta_1}{m_2 \sin\theta_2}\right)^2$$

$$= \frac{1}{2}m_1 V_1^2\left[1 + \frac{m_1 \sin^2\theta_1}{m_2 \sin^2\theta_2}\right]$$

Q8 (b) Substituting from ②

$$T_F = \frac{1}{2} m_1 u^2 \frac{\sin^2\theta_2}{\sin^2(\theta_1+\theta_2)} \left[1 + \frac{m_1 \sin^2\theta_1}{m_2 \sin^2\theta_2} \right]$$

$$= T_I \frac{\sin^2\theta_2}{\sin^2(\theta_1+\theta_2)} \left[1 + \frac{m_1 \sin^2\theta_1}{m_2 \sin^2\theta_2} \right]$$ 2

As $m_2 = 4m_1$

$$T_F = T_I \frac{\sin^2\theta_2}{\sin^2(\theta_1+\theta_2)} \left[1 + \frac{\sin^2\theta_1}{4\sin^2\theta_2} \right]$$ 4

(i) Substituting $\theta_1 = \theta_2 = 60^\circ$

$$T_F = T_I \frac{\sin^2 60^\circ}{\sin^2(120^\circ)} \left[1 + \frac{\sin^2 60^\circ}{4\sin^2 60^\circ} \right]$$ 1

As $\sin 60^\circ = \sin 120^\circ$ and $\sin 60 = \sqrt{3}/2$

$$T_F = T_I \left[1 + \frac{1}{4} \right] = \frac{5}{4} T_I$$ 1

Energy not conserved.

Q8 (b)

$$T_F = T_I \frac{\sin^2\theta_2}{\sin^2(\theta_1+\theta_2)} \left[1 + \frac{\sin^2\theta_1}{4\sin^2\theta_2} \right]$$

(ii) $\theta_1 = \theta_2 = 56^\circ$

$$T_F = T_I \frac{\sin^2 56^\circ}{\sin^2 112^\circ} \left[1 + \frac{1}{4} \right]$$ 1

$T_F = T_I$ to accuracy of 2×10^{-3} 1

Energy conserved to within accuracy of 2×10^{-3}. 1

Q(8)(b)(iii) $\frac{v_1}{v_2} = 1 = \frac{4\sin\theta_2}{\sin 90^\circ}$

$\therefore \sin\theta_2 = \frac{1}{4}$ $(\sin 90^\circ = 1)$

From (4)

$$T_F = T_I \frac{\sin^2\theta_2}{\sin^2(90^\circ+\theta_2)}\left[1 + \frac{1}{4}\frac{1}{\sin^2\theta_2}\right]$$

Substituting,

$$T_F = T_I \frac{(\frac{1}{4})^2}{\sin^2(90^\circ+\theta_2)}\left[1 + \left(\frac{1}{4}\right)\frac{1}{(1/16)}\right]$$

$$= T_I \frac{5/16}{\sin^2(104.48^\circ)}$$ 2

$$\underline{T_F = 0.333\ T_I}$$ 1

Thus energy lost

12

(c)

$3\theta_1 = -8\theta_2$

Substituting into (4) with $\sin\theta_2 \sim \theta_2$, $\sin\theta_1 \sim \theta_1$

$$T_F = T_I\left(\frac{\theta_2}{\theta_1+\theta_2}\right)^2\left[1 + \frac{\theta_1^2}{4\theta_2^2}\right]$$ 1

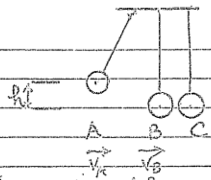
$$= T_I \frac{(-3/8)^2}{[1+(-3/8)]^2}\left[1 + \frac{1}{4(-3/8)^2}\right]$$ 1

$\underline{T_F = T_I}$ Energy conserved 1

3

Q3

Q6



<u>First Collision between A and B</u>

<u>Let V_A and V_B be velocities of A and B after collision</u>

<u>Conservation of momentum before and after A hits B with velocity $\sqrt{2gh}$</u>

$$M\sqrt{2gh} = M(V_A + V_B)$$ 1

<u>Conservation of energy</u>

$$\tfrac{1}{2}M(2gh) = \tfrac{1}{2}M(V_A^2 + V_B^2)$$ 1

<u>Simplifying the equations</u>

$$\sqrt{2gh} = V_A + V_B \qquad (1)$$

$$2gh = V_A^2 + V_B^2 \qquad (2)$$

$$= (\sqrt{2gh} - V_B)^2 + V_B^2 \quad \text{from } (1)$$

$$= 2gh - 2V_B\sqrt{2gh} + V_B^2 + V_B^2$$

$$V_B(2\sqrt{2gh}) = 2V_B^2$$

1

As $V_B \neq 0$, $\underline{V_B = \sqrt{2gh}}$ 1

Substituting into (1) $\underline{V_A = 0}$ 1

<u>A is at rest and B has velocity $\sqrt{2gh}$ after the collision</u>

<u>Collision of B with C</u>

Collision of B with C has same conservation equations as A with B as B initially has velocity $\sqrt{2gh}$.
Thus after collision B at rest and C has velocity $\sqrt{2gh}$ and consequently C rises to height h.
Now C starts at height h, with A and B at rest, and so the collisions are repeated, in reverse order, with the cycle repeated indefinitely. 3

8

Q26

(ii) Collision between A and B

Conservation of momentum (A and B have velocities V_A and V_B after collision)

$$M\sqrt{2gh} = MV_A + 2MV_B$$ 1

Conservation of energy

$$\tfrac{1}{2}M(2gh) = \tfrac{1}{2}MV_A^2 + \tfrac{1}{2}(2M)V_B^2$$ 1

Simplifying,

$$\sqrt{2gh} = V_A + 2V_B \qquad (3)$$

$$2gh = V_A^2 + 2V_B^2 \qquad (4)$$

Substituting V_A from (3) into (4)

$$2gh = (\sqrt{2gh} - 2V_B)^2 + 2V_B^2$$

Giving

$$2gh = 2gh - 4V_B\sqrt{2gh} + 4V_B^2 + 2V_B^2$$ 1

As $V_B \neq 0$,

$$4\sqrt{2gh} = 6V_B$$

$$V_B = \tfrac{2}{3}\sqrt{2gh}$$ 1

Substituting into (3),

$$V_A = \sqrt{2gh} - 2V_B$$

$$= \sqrt{2gh} - 2\left(\tfrac{2}{3}\right)\sqrt{2gh}$$

$$= \sqrt{2gh}\left(1 - \tfrac{4}{3}\right)$$

$$V_A = -\tfrac{1}{3}\sqrt{2gh}$$ 1

Collision of B with C (B and C have velocities V_B' and V_C' after collision)

Momentum conservation $\quad 2M\left(\tfrac{2}{3}\sqrt{2gh}\right) = 2MV_B' + 3MV_C'$ 1

Energy conservation $\quad \tfrac{1}{2}(2M)\tfrac{4}{9}(2gh) = \tfrac{1}{2}(2M)V_B'^2 + \tfrac{1}{2}(3M)V_C'^2$ 1

Simplifying these equations

$$\tfrac{4}{3}\sqrt{2gh} = 2V_B' + 3V_C' \qquad (5)$$

$$\tfrac{8}{9}(2gh) = 2V_B'^2 + 3V_C'^2 \qquad (6)$$

Subs for B' from (5) into (6)

$$\tfrac{8}{9}(2gh) = \tfrac{2}{4}\left(\tfrac{4}{3}\sqrt{2gh} - 3V_C'\right)^2 + 3V_C'^2$$

Q6

(ii)
$$\frac{16}{9}(2gh) = \frac{1}{2}\left(\frac{16}{9}(2gh) - 8\sqrt{2gh}\,V_c' + 9V_c'^2\right) + 3V_c'^2$$

$$= \frac{16}{9}(2gh) - 4V_c'\sqrt{2gh} + \frac{9}{2}V_c'^2 + 3V_c'^2$$

Giving
$$4\sqrt{2gh}\,V_c' = V_c'^2\left(\frac{9}{2}+3\right) = \frac{15}{2}V_c'^2$$

As $V_c' \neq 0$
$$V_c' = \frac{8}{15}\sqrt{2gh} \quad (7)$$

3

So C rises to a height H given by, using conservation of energy

$$3MgH = \frac{1}{2}3MV_c'^2$$

From (7)
$$= \frac{3}{2}M\left(\frac{64}{225}(2gh)\right)$$

1

$$H = \left(\frac{64}{225}\right)h$$

1

12

Q4

(a) Threshold Condition: A moves L relatie to C and just stops at B on C. A, B, C have the same speed.

$v_A = v_B = v_C = v$

momentum conservation for system (A, B, C)

$m \cdot v_0 = 3mv \quad v = \frac{1}{3} v_0$

Energy consumed only by friction

$\mu m g L = \Delta E_K = \frac{1}{2} m v_0^2 - \frac{1}{2} \cdot 3m \cdot (\frac{1}{3} v_0)^2$

$= \frac{1}{3} m v_0^2 \quad v_0 = \sqrt{3 \mu g L}$

$v_0 \geq \sqrt{3 \mu g L}$ for A collides with B.

(b) Same as A, consider B just collides with C as threshold condition

Energy consumed only by friction

$\mu m g \cdot 2L = \Delta E_K = \frac{1}{3} m v_0^2 \quad v_0 = \sqrt{6 \mu g L}$

$v_0 \geq \sqrt{6 \mu g L}$

(c) Couldn't happen. A, B are relatively stationary after B collids with C.

(Same speed for A, B. same acceleration)

A, C relative stationary before B collids with C.

$\sqrt{\mu g}$

(d) Same as Before

$\mu mg \cdot 4L = \frac{1}{3} m v_0^2 \quad v_0 = \sqrt{12\mu g L}$

$v_0 \geq \sqrt{12\mu g L}$

(e) Consider A, B, C as a whole system

when A falls. $v_A = v_B = v_1$, v_c

$$\begin{cases} m v_0 = 2m v_1 + m v_c \\ \frac{1}{2} m v_0^2 - \left(\frac{1}{2} m v_1^2 + \frac{1}{2} m v_1^2 + \frac{1}{2} m v_c^2\right) = \mu mg \cdot 4L \end{cases}$$

then consider B, C as a system

when B falls $v_B = v_c = v_2$ (threshold)

$$\begin{cases} m v_1 + m v_c = 2 m v_2 \\ \left(\frac{1}{2} m v_1^2 + \frac{1}{2} m v_c^2\right) - \left(\frac{1}{2} m \cdot v_2^2 + \frac{1}{2} m v_2^2\right) = \mu mg L \end{cases}$$

solve above 4 equations: $v_0 = 4\sqrt{\mu g L}$

so $v_0 > 4\sqrt{\mu g L}$