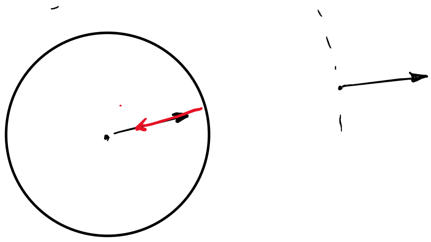
$$\begin{cases}\vec\nabla\cdot\vec E = \frac{\rho}{\varepsilon_0}\\ \vec\nabla\times\vec E = -\frac{\partial B}{\partial t}\\ \vec\nabla\cdot\vec B=0\\ \vec\nabla\times\vec B = \mu_0\vec J+\mu_0\varepsilon_0\cdot\frac{\partial E}{\partial t}\end{cases}$$

$$d\vec B = \frac{\mu_0}{4\pi}\frac{I\,d\vec l\times\vec r}{r^3}$$

A spherically symmetric (and constant) current density flows radially inward to a spherical shell, causing the charge on the shell to increase at the constant rate $dQ/dt$. Verify that Maxwell's equation, $\nabla\times\mathbf{B}=\mu_0\mathbf{J}+\mu_0\epsilon_0\,\partial\mathbf{E}/\partial t$, is satisfied at points outside the shell.



Symmetry $\Rightarrow$ $\vec B=0$ $\quad \vec\nabla\times\vec B=0$

$$\cancel{\mu_0}\vec J = -\cancel{\mu_0}\varepsilon_0\frac{\partial\vec E}{\partial t} \qquad \vec J = -\varepsilon_0\frac{\partial\vec E}{\partial t}$$

$$\frac{\partial\rho}{\partial t}+\vec\nabla\cdot\vec J=0$$

$$\vec E=\frac{Q}{4\pi r^2\cdot\varepsilon_0} \qquad \frac{\partial\vec E}{\partial t}=\frac{1}{4\pi r^2\varepsilon_0}\cdot\frac{\partial Q}{\partial t}\cdot\hat r$$

$$Q=\int\rho\,dV$$

$$\vec J = -\cancel{\varepsilon_0}\frac{1}{4\pi r^2\cancel{\varepsilon_0}}\cdot\frac{\partial Q}{\partial t}\hat r = -\frac{1}{4\pi r^2}\frac{\partial Q}{\partial t}\hat r$$

$$\frac{\partial Q}{\partial t}+\vec J\cdot4\pi r^2=0$$

A half-infinite wire carries current $I$ from negative infinity to the origin, where it builds up at a point charge with increasing $q$ (so $dq/dt = I$). Consider the circle shown in Fig. 9.12, which has radius $b$ and subtends an angle $2\theta$ with respect to the charge. Calculate the integral $\int \mathbf{B}\cdot d\mathbf{s}$ around this circle. Do this in three ways.

(a) Find the $\mathbf{B}$ field at a given point on the circle by using the Biot–Savart law to add up the contributions from the different parts of the wire.

(b) Use the integrated form of Maxwell's equation (that is, the generalized form of Ampère's law including the displacement current),

$$\int_C \mathbf{B}\cdot d\mathbf{s} = \mu_0 I + \mu_0\epsilon_0\int_S \frac{\partial\mathbf{E}}{\partial t}\cdot d\mathbf{a}, \qquad (9.59)$$

with $S$ chosen to be a surface that is bounded by the circle and doesn't intersect the wire, but is otherwise arbitrary. (You can invoke the result from Problem 1.15.)

(c) Use the same strategy as in (b), but now let $S$ intersect the wire.

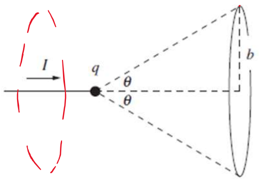
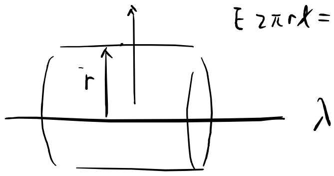



Figure 9.12.

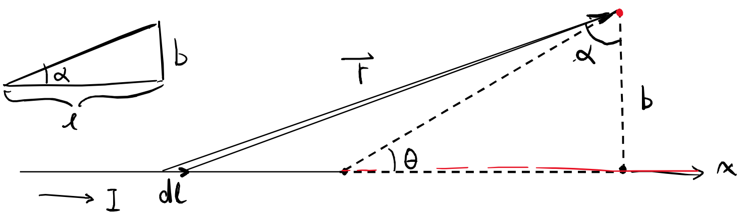



$$E\,2\pi r\ell = \frac{\ell\lambda}{\varepsilon_0}$$

$$\vec\nabla\times\vec B=\mu_0\vec J+\mu_0\varepsilon_0\cdot\frac{\partial\vec E}{\partial t}$$

$$d\vec B=\frac{\mu_0}{4\pi}\frac{I\,d\vec l\times\vec r}{r^3} = \frac{\mu_0}{4\pi}\frac{I\,dl\,\cancel{r}\cos\alpha}{r^{\cancel{3}\,2}} = \frac{\mu_0}{4\pi}\frac{I\,dl\cos\alpha}{r^2}$$

$$\vec B=\int\frac{\mu_0}{4\pi}\frac{I\,dl\cos\alpha}{r^2} = \frac{\mu_0 I}{4\pi}\int\frac{dl\cos\alpha}{r^2}$$




$$r=\frac{b}{\cos\alpha} \qquad l=b\tan\alpha \qquad dl=\frac{b}{\cos^2\alpha}d\alpha \qquad \frac{dl}{d\alpha}=b\sec^2\alpha$$

$$\vec B=\frac{\mu_0 I}{4\pi}\int_{\frac\pi2-\theta}^{\frac\pi2}\frac{\frac{\cancel{b}}{\cancel{\cos^2\alpha}}d\alpha\cdot\cos\alpha}{\frac{b^{\cancel{2}}}{\cancel{\cos^2\alpha}}} = \frac{\mu_0 I}{4\pi}\int_{\frac\pi2-\theta}^{\frac\pi2}\frac{\cos\alpha}{b}d\alpha$$

$$=\frac{\mu_0 I}{4\pi b}\left(\sin\alpha\Big|_{\frac\pi2-\theta}^{\frac\pi2}\right)=\frac{\mu_0 I}{4\pi b}(1-\cos\theta)$$

$$\int\vec B\cdot d\vec s = B\cdot2\pi b=\frac{\mu_0 I}{2}(1-\cos\theta)$$

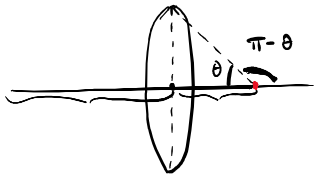
$\underline{\mu_0 I = 0}$ $\quad \mu_0 \varepsilon_0 \int_s \frac{\partial \vec{E}}{\partial t} \cdot d\vec{a} = \mu_0 \varepsilon_0 \cdot \frac{\partial}{\partial t} \int \vec{E} \cdot d\vec{a} = \mu_0 \varepsilon_0 \cdot \frac{\partial \varphi_{\vec{E}}}{\partial t}$

$$\boxed{\varphi_{\vec{E}} = \frac{q}{2\varepsilon_0}(1-\cos\theta)} \qquad \frac{\partial \varphi_E}{\partial t} = \frac{1-\cos\theta}{2\varepsilon_0} \cdot \frac{\partial q}{\partial t} = \frac{1-\cos\theta}{2\varepsilon_0} I$$

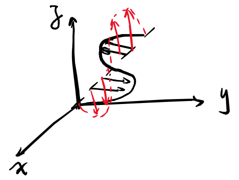
$$\mu_0 \varepsilon_0 \int \frac{\partial \vec{E}}{\partial t} \cdot d\vec{a} = \mu_0 \cancel{\varepsilon_0} \cdot \frac{1-\cos\theta}{2\cancel{\varepsilon_0}} \cdot I = \frac{\mu_0 I}{2}(1-\cos\theta)$$

(c) $\int_c \vec{B} \cdot d\vec{s} = \mu_0 I + \varepsilon_0 \mu_0 \int \frac{\partial \vec{E}}{\partial t} \cdot d\vec{a} = \mu_0 I - \frac{\mu_0 I}{2}(1+\cos\theta)$

$$= \frac{\mu_0 I}{2}(1-\cos\theta)$$




$$\vec{\nabla} \times \vec{B} = \mu_0 \vec{J} + \mu_0 \varepsilon_0 \int \frac{\partial \vec{E}}{\partial t} \cdot d\vec{a}$$ displacement current




$$\begin{cases} E_x = 0,\ E_y = E_0 \sin(kx+\omega t),\ E_z = 0 \\ B_x = 0,\ B_y = 0,\ B_z = -\frac{E_0}{c}\sin(kx+\omega t) \end{cases}$$

prove: $\omega \sim k$ $\quad \vec{B}, \vec{E}$ satisfy Maxwell Equation.

① $\vec{\nabla} \cdot \vec{E} = \left(\frac{\partial}{\partial x}\hat{x} + \frac{\partial}{\partial y}\hat{y} + \frac{\partial}{\partial z}\hat{z}\right) \cdot \left(\cancel{E_x \hat{x}} + E_y \hat{y} + \cancel{E_z \hat{z}}\right) = \left|\frac{\partial E_y}{\partial y} = \frac{\rho}{\varepsilon_0}\right| = 0$

② $\vec{\nabla} \times \vec{E} = -\frac{\partial B}{\partial t}$

$$\vec{\nabla} \times \vec{E} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ E_x & E_y & E_z \end{vmatrix} = \hat{x}\cancel{\left(\frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z}\right)} - \hat{y}\cancel{\left(\frac{\partial E_z}{\partial x} - \frac{\partial E_x}{\partial z}\right)} + \hat{z}\left(\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y}\right)$$

$$= \cancel{E_0} \cdot \cos(\cancel{kx+\omega t}) \cdot k \cdot \hat{z} = + \frac{\cancel{E_0}}{c} \cdot \omega \cdot \cos(\cancel{kx+\omega t})\,\hat{z}$$

$\star$ $k = \frac{\omega}{c}$

③ $\vec{\nabla} \cdot \vec{B} = \frac{\partial B_z}{\partial z} = 0$

④ $\underline{\vec{\nabla} \times \vec{B} = \mu_0 \vec{J} + \mu_0 \varepsilon_0 \frac{\partial \vec{E}}{\partial t}}$

$$\vec{\nabla} \times \vec{B} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ B_x & B_y & B_z \end{vmatrix} = \hat{x}\cancel{\left(\frac{\partial B_z}{\partial y} - \frac{\partial B_y}{\partial z}\right)} - \hat{y}\left(\frac{\partial B_z}{\partial x} - \frac{\partial B_x}{\partial z}\right) + \hat{z}\cancel{\left(\frac{\partial B_y}{\partial x} - \frac{\partial B_x}{\partial y}\right)}$$

$$= -\frac{\partial B_z}{\partial x}\hat{y} = \frac{E_0}{c} \cdot k \cdot \underline{\cos(kx+\omega t)}\,\hat{y}$$

$\mu_0 \vec{J} = 0$ $\quad \mu_0 \varepsilon_0 \cdot \frac{\partial \vec{E}}{\partial t} \cdot d\vec{a} = \mu_0 \varepsilon_0 \left(\cancel{\frac{\partial E_x}{\partial t}}\hat{x} + \frac{\partial E_y}{\partial t}\hat{y} + \cancel{\frac{\partial E_z}{\partial t}}\hat{z}\right)$

$$= \mu_0 \varepsilon_0 \ \underline{E_0} \, \underline{\cos(kx+\omega t)} \cdot \omega \hat{y}.$$

✓ $\underline{k = \frac{\omega}{c}}$ $\quad \omega \cdot \mu_0 \varepsilon_0 = \frac{k}{c}$ $\quad k = \omega \cdot c \cdot \mu_0 \varepsilon_0$ $\quad c^2 = \frac{1}{\mu_0 \varepsilon_0}$ $\quad k = \omega \cdot c \cdot \frac{1}{c^2} = \frac{\omega}{c}$

$E = E_0 \sin(kx - \omega t)$ $\quad B = B_0 \sin(kx - \omega t)$ $\quad B_0 = \frac{E_0}{c}$