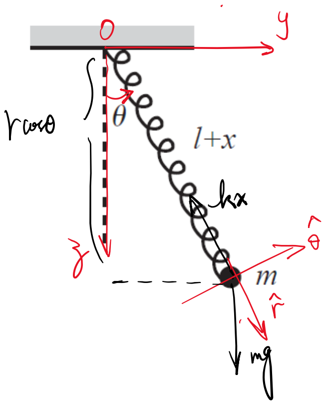
Example (Spring pendulum): Consider a pendulum made of a spring with a mass m on the end (see Fig. 6.1). The spring is arranged to lie in a straight line (which we can arrange by, say, wrapping the spring around a rigid massless rod). The equilibrium length of the spring is ℓ. Let the spring have length $\ell + x(t)$, and let its angle with the vertical be $\theta(t)$. Assuming that the motion takes place in a vertical plane, find the equations of motion for x and θ.



Spring Pendulum:

$x(t)$, $\theta(t)$ → E.O.M.

$$K = \frac{1}{2} m(\dot{r}^2 + r^2\dot{\theta}^2) \qquad r = l + x \qquad \dot{r} = \dot{x}$$

$$= \frac{1}{2} m(\dot{x}^2 + (l+x)^2\dot{\theta}^2).$$

$$V_1(r,\theta) = -mgr\cos\theta = -mg(l+x)\cos\theta$$

$$V_2(k) = \frac{1}{2}kx^2$$

$$V = V_1 + V_2 = -mg(l+x)\cos\theta + \frac{1}{2}kx^2$$

$$L = K - V = \frac{1}{2}m(\dot{x}^2 + (l+x)^2\dot{\theta}^2) - (mg(l+x)\cos\theta + \frac{1}{2}kx^2)$$

$$= \frac{1}{2}m(\dot{x}^2 + (l+x)^2\dot{\theta}^2) + mg(l+x)\cos\theta - \frac{1}{2}kx^2$$

$$L = L(x, \dot{x}, \theta, \dot{\theta})$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) = \frac{\partial L}{\partial x}. \qquad \frac{d}{dt}(m\dot{x}) = m(l+x)\dot{\theta}^2 + mg\cos\theta - kx$$

$$m\ddot{x} = m(l+x)\dot{\theta}^2 + mg\cos\theta - kx \quad ①$$

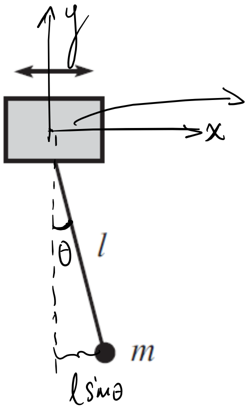
$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) = \frac{\partial L}{\partial \theta}. \qquad \frac{d}{dt}(m(l+x)^2\cdot\dot{\theta}) = -mg(l+x)\sin\theta$$

$$\cancel{m}2\cdot\cancel{(l+x)}\cdot\dot{x}\dot{\theta} + \cancel{m}(l+x)^{\cancel{2}}\ddot{\theta} = -\cancel{m}g\,\cancel{(l+x)}\sin\theta$$

Coriolis force → $2\dot{x}\dot{\theta} + (l+x)\ddot{\theta} = -g\sin\theta$ ②

$x(t), \theta(t) \Rightarrow$ ①, ②

A pendulum consists of a mass m and a massless stick of length ℓ. The pendulum support oscillates horizontally with a position given by $x(t) = A\cos(\omega t)$; see Fig. 6.10. What is the general solution for the angle of the pendulum as a function of time?



$$x(t) = A\cos\omega t \Rightarrow \theta(t)$$

$$(X, Y)_m = (x + l\sin\theta,\ -l\cos\theta)$$

$$V_m^2 = \dot{X}^2 + \dot{Y}^2 = (\dot{x} + l\cos\theta\,\dot\theta)^2 + l^2\sin^2\theta\,\dot\theta^2$$

$$= \dot{x}^2 + 2\dot{x}l\dot\theta\cos\theta + \underline{l^2\cos^2\theta\cdot\dot\theta^2} + \underline{l^2\sin^2\theta\,\dot\theta^2}$$

$$= \dot{x}^2 + l^2\dot\theta^2 + 2l\dot{x}\dot\theta\cos\theta$$

$$L = \frac{1}{2}mV_m^2 - (-mgl\cos\theta) = \frac{1}{2}mV_m^2 + mgl\cos\theta$$

$$= \frac{1}{2}m(\dot{x}^2 + l^2\dot\theta^2 + \underline{2l\dot{x}\dot\theta\cos\theta}) + mgl\cos\theta \qquad \dot{x}, \dot\theta, x, \theta$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial\dot\theta}\right) = \frac{\partial L}{\partial\theta} : \frac{d}{dt}\left(\cancel{2}\dot\theta\cdot\frac{1}{\cancel{2}}\cancel{m}l^{\cancel{2}} + \cancel{m}\cancel{l}\dot{x}\cos\theta\right) = -\cancel{ml}\dot{x}\dot\theta\sin\theta - \cancel{m}g\cancel{l}\sin\theta$$

$$l\ddot\theta + \ddot{x}\cos\theta - \cancel{\dot{x}\sin\theta\cdot\dot\theta} = -\cancel{\dot{x}\dot\theta\sin\theta} - g\sin\theta$$

$$l\ddot\theta + \ddot{x}\cos\theta = -g\sin\theta \qquad \underline{l\ddot\theta - A\omega^2\cos\omega t\cos\theta + g\sin\theta = 0}$$

For small angle θ: $\quad \sin\theta \approx \theta \quad \cos\theta \approx 1 - \frac{\theta^2}{2} \to 0 \quad = 1$

$$l\ddot\theta - A\omega^2\cos\omega t + g\theta = 0$$

$$\ddot\theta + \frac{g}{l}\theta = \frac{A\omega^2}{l}\cos\omega t \qquad \omega_0 = \sqrt{\frac{g}{l}} \qquad a = \frac{A}{l}$$

$$\underline{\ddot\theta + \omega_0^2\theta = a\omega^2\cos\omega t}$$

$$\underline{\ddot\theta + \omega_0^2\theta = 0} \Rightarrow \quad x^2 + \omega_0^2 = 0 \quad x = \omega_0 \quad x = -\omega_0$$

$$\theta_h(t) = Ae^{\omega_0 t} + Be^{-\omega_0 t}$$

$$\theta_p(t) = C\cos\omega t$$

$$\theta(t) = \theta_h(t) + \theta_p(t) = Ae^{\omega_0 t} + Be^{-\omega_0 t} + C\cos\omega t.$$