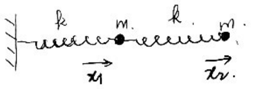
BPHO & PUPC Class No.20210612

Mechanics Test

汪光远, 29/07/2021

1. You are allowed to use calculator.
2. The test lasts for 3 hours from 19:15-22:15. You are allowed to read the questions from 19:00 to 19:15.
3. You are encouraged to try every question.
4. Total mark equals 100 points. Try your best to solve the physics part if mathematics is complicated.
5. Open your camera during the whole exam. You are allowed to go to the restroom or drink water if you would like.
6. This test is closed book. Dictionary is allowed.
7. After the exam, please take a photo of all your answers with a .zip file or .pdf file and send it to the study group. You will have 15mins to submit your answers. Make sure your solutions is submitted before 22:30



1. $\begin{cases} m\ddot{x}_1 = k(x_2-x_1) - kx_1. \\ m\ddot{x}_2 = -kx_2 + kx_1. \end{cases} \Rightarrow \begin{cases} \ddot{x}_1 = \frac{k}{m}(x_2-x_1) - \frac{k}{m}x_1. \\ \ddot{x}_2 = -\frac{k}{m}(x_2-x_1). \end{cases}$

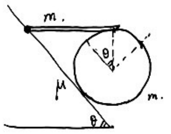
$(\omega = \sqrt{\frac{k}{m}})$ $\Rightarrow \begin{cases} \ddot{x}_1 - \omega^2 x_2 + 2\omega^2 x_1 = 0 \quad ① \\ \ddot{x}_2 + \omega^2 x_2 - \omega^2 x_1 = 0. \quad ② \end{cases}$

① + ②:

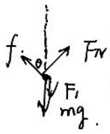
$(\ddot{x}_1 + \ddot{x}_2) + \omega^2 x_1 = 0.$ (N.M.1)

① − ②:

$(\ddot{x}_1 - \ddot{x}_2) - 2\omega^2 x_2 + 3\omega^2 x_1 = 0$ (N.M.2)

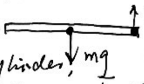


2. (a) For the cylinder, it exerts 4 force;
For stick, in and of itself, we apply torque equations on it, so as to get



(Cylinder's Free Body diagram).

$2 \cdot F_1 = 1 \cdot mg \Rightarrow mg = 2F_1,\ F_1 = \frac{1}{2}mg.$ ①



(For stick's sake).

In seek to decompose forces acted on cylinder, we get $\begin{cases} f\cos\theta + F_N\sin\theta = -F_1 + mg. \quad ② \\ f\sin\theta + F_N\cos\theta = 0. \quad ③ \end{cases} \Rightarrow f = |-F_N \,\mathrm{ctg}\,\theta|.$ ④ $= F_N\,\mathrm{ctg}\,\theta$

According to the definition of friction(s), it cannot be directly derived from $f = \mu N$. Thus, substituting ④ into ②, we get

$$F_N = \left|\frac{3}{2}mg \Big/ \left(\sin\theta - \frac{\cos^2\theta}{\sin\theta}\right)\right| = \left|\frac{3}{2}mg\cdot\left(-\frac{\sin\theta}{\cos 2\theta}\right)\right| = \frac{3}{2}\frac{\sin\theta}{\cos 2\theta}mg.$$

(b) When the system is about to slip, $f_d = \mu N = \mu F_N = |-F_N \,\mathrm{ctg}\,\theta| = F_N\,\mathrm{ctg}\,\theta.$

$\Rightarrow$ if the system stays still, the smallest value of μ yields to be $\mu = \mathrm{ctg}\,\theta$.

4. $\begin{cases} x = r\cos\theta. \\ y = r\sin\theta \end{cases}$ $\quad \hat{r} = \cos\theta\,\vec{i} + \sin\theta\,\vec{j}.$

$\hat{\theta} = \frac{d\hat{r}}{d\theta} = -\sin\theta\,\vec{i} + \cos\theta\,\vec{j}.$

$\vec{v} = \frac{d\vec{r}}{dt} = \frac{d}{dt}(r\cdot\hat{r}) = \frac{dr}{dt}\cdot\hat{r} + r\cdot\frac{d\hat{r}}{dt} = \dot{r}\hat{r} + r\dot{\hat{r}}$, where

$\dot{\hat{r}} = \frac{d}{dt}(\cos\theta\,\vec{i} + \sin\theta\,\vec{j}) = -\sin\theta\,\dot{\theta}\,\vec{i} + \cos\theta\,\dot{\theta}\,\vec{j} = \dot{\theta}\hat{\theta},$

thus $\vec{v} = \dot{r}\hat{r} + r\dot{\theta}\hat{\theta}.$

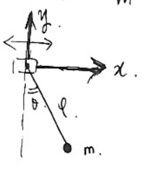
$\vec{a} = \frac{d\vec{v}}{dt} = \ddot{r}\hat{r} + \dot{r}\dot{\hat{r}} + \dot{r}\dot{\theta}\hat{\theta} + r\ddot{\theta}\hat{\theta} + r\dot{\theta}\dot{\hat{\theta}}$, where $\dot{\hat{\theta}} = \frac{d\hat{\theta}}{dt} = -\cos\theta\,\dot{\theta}\,\vec{i} - \sin\theta\,\dot{\theta}\,\vec{j}$

thus $\vec{a} = \ddot{r}\hat{r} + \dot{r}\dot{\theta}\hat{\theta} + \dot{r}\dot{\theta}\hat{\theta} + r\ddot{\theta}\hat{\theta} - r\dot{\theta}\dot{\theta}\hat{r} = (\ddot{r} - \dot{\theta}^2 r)\hat{r} + (2\dot{r}\dot{\theta} + r\ddot{\theta})\hat{\theta}$ $\Big|$ $= -\dot{\theta}\hat{r}$,

When a single centripetal force yields, $m\vec{a} = -m\frac{(\vec{v_t})^2}{r}\hat{r} = -m\cdot\omega^2 r\hat{r} = \vec{F}_{net} = \vec{F_c}.$

$\Rightarrow \vec{a} = \frac{\vec{F_c}}{m} = -\omega^2 r\hat{r}.$

5.



$(X, Y)_m = (x + l\sin\theta,\ -l\cos\theta).$ $\quad (v_x, v_y)_m = (\dot{X}, \dot{Y})_m = (\dot{x} + l\cos\theta\dot{\theta},\ l\sin\theta\dot{\theta}).$

$K = \frac{1}{2}mv_m^2 = \frac{1}{2}m(v_x^2 + v_y^2) = \frac{1}{2}m(l^2\cos^2\theta\dot{\theta}^2 + l^2\sin^2\theta\dot{\theta}^2 + \dot{x}^2 + 2\dot{x}l\cos\theta\dot{\theta})$

$= \frac{1}{2}m(\dot{x}^2 + l^2\dot{\theta}^2 + 2\dot{x}l\cos\theta\dot{\theta}).$

$V = -l\cos\theta\cdot mg.$

$\mathcal{L} = K - V = \frac{1}{2}m(\dot{x}^2 + l^2\dot{\theta}^2 + 2\dot{x}l\cos\theta\dot{\theta}) + mgl\cos\theta$

By applying Lagrangian method, $\frac{d}{dt}\left(\frac{\partial\mathcal{L}}{\partial\dot{\theta}}\right) = \frac{\partial\mathcal{L}}{\partial\theta} \Rightarrow \cancel{m}l^{\cancel{2}}\ddot{\theta} + \cancel{ml}\ddot{x}\cos\theta - \cancel{ml}\dot{x}\sin\theta\,\dot{\theta} = -\cancel{ml\dot{x}}\sin\theta\,\dot{\theta} - \cancel{mgl}\sin\theta$

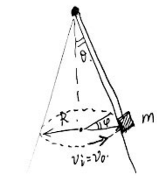
$l\ddot{\theta} + \ddot{x}\cos\theta + g\sin\theta = 0.$ $\quad (\ddot{x} = -A\omega^2\cos(\omega t))$

When $\theta \to 0$, $\sin\theta \to \theta$, $\cos\theta \to 1 - \frac{\theta^2}{2} \to 1.$

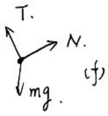
$\Rightarrow l\ddot{\theta} - A\omega^2\cos(\omega t) + g\theta = 0.$ $\quad$ Let $\omega_0^2 = \frac{g}{l}$, $C_0 = \frac{A\omega^2}{l}.$ $\quad \theta_p(t)$ is also in the form of $C_0\cos(\omega t).$

$\ddot{\theta} + \frac{g}{l}\theta = \frac{A\omega^2}{l}\cdot\cos(\omega t).$ $\quad \ddot{\theta} + \omega_0^2\theta = C_0\cos(\omega t).$

$r^2 + \omega_0^2 = 0,\quad \theta_h(t) = Ae^{\omega_0 t} + Be^{-\omega_0 t}$ $\quad \Rightarrow \theta(t) = Ae^{\omega_0 t} + Be^{-\omega_0 t} + C\cos(\omega t)$



3.



As for the block's 2-dimensional motion, we derive the polar-coordinate expressions.

(f).

$$\hat{R} = \cos\varphi\,\vec{i} + \sin\varphi\,\vec{j}.$$

$$\hat{\varphi} = \frac{d\hat{R}}{d\varphi} = -\sin\varphi\,\vec{i} + \cos\varphi\,\vec{j}.$$

$$\vec{v}_m = \frac{d(R\hat{R})}{dt} = \dot{R}\hat{R} + R\dot{\hat{R}} = \dot{R}\hat{R} + R\dot{\varphi}\hat{\varphi}.$$

$$\vec{a}_m = \frac{d\vec{v}_m}{dt} = (\ddot{R} - R\dot{\varphi}^2)\hat{R} + (2\dot{R}\dot{\varphi} + R\ddot{\varphi})\hat{\varphi}.$$

$$\vec{F}_{net} = m\vec{a}_m \Rightarrow m\left[(\ddot{R} - R\dot{\varphi}^2)\hat{R} + (2\dot{R}\dot{\varphi} + R\ddot{\varphi})\hat{\varphi}\right] = -mR\dot{\varphi}^2\hat{R}$$

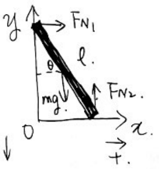
$$\Rightarrow \begin{cases} 2\dot{R}\dot{\varphi} + R\ddot{\varphi} = 0. \\ \ddot{R} - R\dot{\varphi}^2 = -R\dot{\varphi}^2 \Rightarrow \ddot{R} = 0 \end{cases}$$

$$\Rightarrow 2v_c\omega + R\frac{d\omega}{dt} = 0. \Rightarrow \int 2v_c\,dt = \int -\frac{R}{\omega}\,d\omega.$$

$$\Rightarrow \int_0^{T} 2a_c t\,dt = \int_{\omega_i}^{\omega_f} -\frac{R}{\omega}\,d\omega. \Rightarrow a_c T^2 = -R\ln(\omega_i)$$

$$\omega_i = \frac{v_i}{R}.$$

6.



$$(X, Y)_{cm} = \left(\tfrac{1}{2}l\cos\theta, \tfrac{1}{2}l(1-\sin\theta)\right).$$

$$(v_x, v_y)_{cm} = \left(-\tfrac{1}{2}l\sin\theta\,\dot{\theta}, -\tfrac{1}{2}l\cos\theta\,\dot{\theta}\right).$$

$$(a_x, a_y)_{cm} = \left(-\tfrac{1}{2}l\cos\theta\,\dot{\theta}^2, \tfrac{1}{2}l\sin\theta\,\dot{\theta}^2\right).$$

For horizonral componeats, $F_{N_1} = ma_x = -\frac{1}{2}ml\cos\theta\,\dot{\theta}^2\,\hat{x}$.

When the ladder falls down,

$$mg\frac{l}{2}\left(\sin\frac{\pi}{2} - \sin\theta\right) = \frac{1}{2}m(v_{cx}^2 + v_{cy}^2) + \frac{1}{2}I_c\omega^2 \quad ①$$

where I_c is the rotational inertia with respect to point O.

$I_c = \frac{1}{12}ml^2$. $\begin{cases} v_{cx} = \frac{1}{2}\sin\theta\cdot\omega \\ v_{cy} = -\frac{1}{2}\cos\theta\cdot\omega. \end{cases}$, substitute them into ①,

$$\omega = \sqrt{\frac{3g}{l}\cdot(1-\sin\theta)}. \Rightarrow v_c = \sqrt{\frac{1}{4}\cdot\frac{3g}{l}(1-\sin\theta)\cdot\sin^2\theta + \frac{1}{4}\cdot\frac{3g}{l}(1-\sin\theta)\cdot\cos^2\theta}$$

$$= \sqrt{\frac{3g}{2l}}\cdot\sqrt{1-\sin\theta}.$$

(End)