Laboratory frame

Centre of mass frame

$$\vec{R}_{CM} = \frac{\sum m_i \vec{r}_i}{M}$$

$\vec{r}_i$ position vector of ith particle

$$v = \frac{d\vec{r}}{dt}$$

$$M = \sum m_i$$

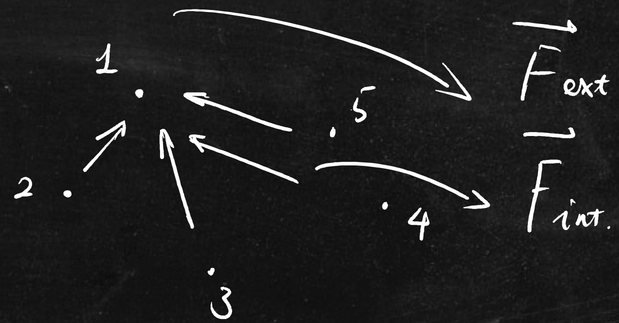
$$\vec{V}_{CM} = \frac{d\vec{R}_{CM}}{dt} = \frac{d}{dt}\sum m_i \vec{r}_i / M = \frac{\sum m_i \boxed{\frac{d\vec{r}_i}{dt}}}{M} = \frac{\sum m_i \vec{v}_i}{M}$$

$$\frac{d\vec{r}_i}{dt} \rightarrow \vec{v}_i$$

$$\vec{P}_{CM} = M \cdot \vec{V}_{CM} = \cancel{M} \cdot \frac{\sum m_i \vec{v}_i}{\cancel{M}} = \sum m_i \vec{v}_i$$

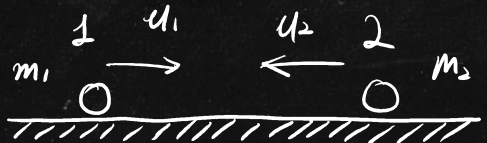
$$\vec{F} = \frac{dP}{dt}$$

$$\frac{d\vec{P}_{CM}}{dt} = M \cdot \frac{d\vec{V}_{CM}}{dt} = M \cdot \frac{d^2 \vec{R}_{CM}}{dt^2} = \vec{F}_{ext}$$



if $\vec{F}_{ext} = 0$ $\quad \frac{d\vec{P}_{CM}}{dt} = 0 \quad \vec{P}_{CM}$ = constant.

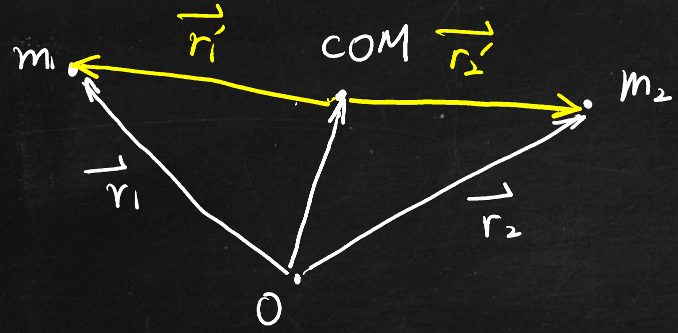
$$\vec{p}_1 + \vec{p}_2 + \vec{p}_3 + \dots + \vec{p}_n = \text{constant}$$



$$m_1 \vec{u}_1 + m_2 \vec{u}_2 = m_1 \vec{v}_1 + m_2 \vec{v}_2$$

elastic / inelastic

COM frame:



$$\vec{R}_{CM} = \frac{m_1\vec{r}_1 + m_2\vec{r}_2}{m_1+m_2}$$

$$\underline{\vec{r}_1 = \vec{r}_1' + \vec{R}_{CM}} \qquad \underline{\vec{r}_2 = \vec{r}_2' + \vec{R}_{CM}}$$

$$\vec{r}_i' = \vec{r}_i - \vec{R}_{CM} \qquad \frac{d\vec{r}_i'}{dt} = \frac{d\vec{r}_i}{dt} - \frac{d\vec{R}_{CM}}{dt} \Rightarrow \vec{v}_i' = \vec{v}_i - \vec{V}_{CM}$$

$$\vec{p}_i' = m_i\vec{v}_i' = m_i(\vec{v}_i - \vec{V}_{CM}).$$

$$\sum \vec{P}_i' = \sum m_i(\vec{v}_i - \vec{V}_{CM}) = \sum m_i\vec{v}_i - \sum m_i \vec{V}_{CM} = 0$$

$$K = \frac{1}{2}mv^2 \qquad v^2 = \vec{v}\cdot\vec{v}$$

能量 (laboratory frame). $\quad K = \sum \frac{1}{2} m_i v_i^2$

$$K = \frac{1}{2} \sum m_i v_i^2 = \frac{1}{2} \sum m_i \vec{v}_i \cdot \vec{v}_i$$

$$\vec{a} \cdot \vec{b} = \vec{b} \cdot \vec{a}$$

$$\vec{a} \times \vec{b} = -\vec{b} \times \vec{a}$$

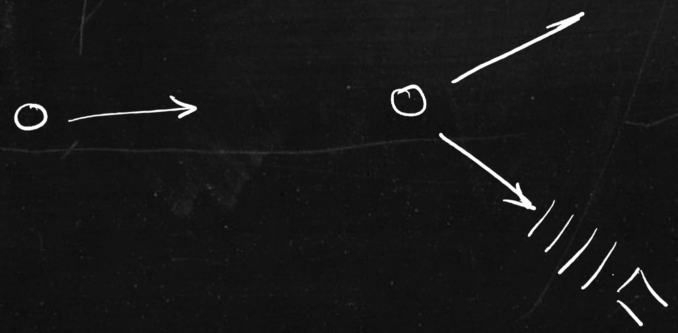
$$\underline{\vec{v}_i'} = \vec{v}_i - \vec{V}_{CM}$$

$$K = \frac{1}{2} \sum m_i (\underline{\vec{v}_i - \vec{V}_{CM}} + \vec{V}_{CM}) \cdot (\underline{\vec{v}_i - \vec{V}_{CM}} + \vec{V}_{CM}).$$

$$= \frac{1}{2} \sum m_i (\vec{v}_i' + \vec{V}_{CM})(\vec{v}_i' + \vec{V}_{CM}) = \frac{1}{2} \sum m_i (v_i'^2 + V_{CM}^2 + 2\vec{v}_i' \cdot \vec{V}_{CM}).$$

$$= \underline{\frac{1}{2} \sum m_i v_i'^2} + \frac{1}{2} \underline{\sum m_i} V_{CM}^2 + \underline{\cancel{\left(\sum m_i \vec{v}_i'\right) \cdot \vec{V}_{CM}}} \rightarrow 0$$

$$= \frac{1}{2} \sum m_i v_i'^2 + \boxed{\frac{1}{2} M \cdot V_{CM}^2} = K_{rel} + K_{CM}$$



α- particle

$$\begin{cases} MV = mv'\cos\gamma + MV'\cos\varphi \\ MV'\sin\varphi = mv'\sin\gamma \end{cases} \Rightarrow \begin{aligned} &(mv'\cos\gamma)^2 = (MV - MV'\cos\varphi)^2 \quad ① \\ &(mv'\sin\gamma)^2 = (MV'\sin\varphi)^2 \quad ② \end{aligned}$$

①+②: $$\underline{m^2v'^2}\cos^2\gamma + \underline{m^2v'^2}\sin^2\gamma = M^2V^2 + \underline{M^2V'^2\cos^2\varphi} - 2M^2VV'\cos\varphi + \underline{M^2V'^2\sin^2\varphi}$$

$$m^2v'^2 = M^2V^2 + M^2V'^2 - 2M^2VV'\cos\varphi \quad ③$$

Elastic: $$\cancel{\tfrac{1}{2}}MV^2 = \cancel{\tfrac{1}{2}}MV'^2 + \cancel{\tfrac{1}{2}}mv'^2 \qquad v'^2 = \frac{MV^2 - MV'^2}{m} \quad ④$$

④→③ $$m^{\cancel{2}} \cdot \frac{MV^2 - MV'^2}{\cancel{m}} = M^2V^2 + M^2V'^2 - 2M^2VV'\cos\varphi.$$

$$m\cancel{M}(V^2 - V'^2) = M^{\cancel{2}}V^2 + M^{\cancel{2}}V'^2 - 2M^{\cancel{2}}VV'\cos\varphi.$$

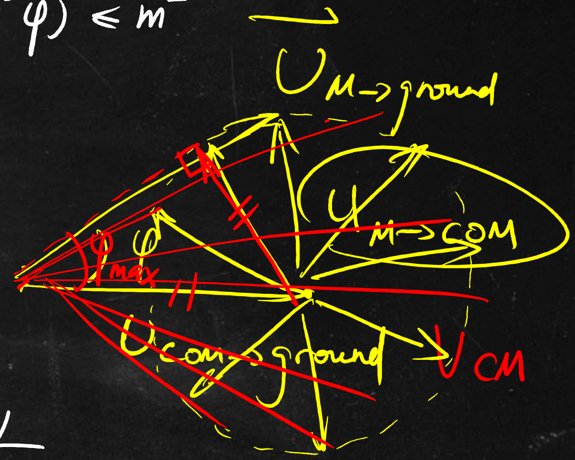
$$\underline{m(V^2 - V'^2) = MV^2 + MV'^2 - 2MVV'\cos\varphi}$$

$$(m+M)V'^2 - 2MV\cos\varphi\, V' + (M-m)V^2 = 0$$

$$\Delta = \cancel{4}M^2\cancel{V^2}\cos^2\varphi - \cancel{4}\cdot\underline{(M+m)(M-m)}\cancel{V^2} \geq 0$$

$$M^2\cos^2\varphi - (M^2 - m^2) \geq 0 \qquad M^2 - M^2\cos^2\varphi \leq m^2 \qquad M^2(1-\cos^2\varphi) \leq m^2$$

$$\sin^2\varphi \leq \frac{m^2}{M^2} \qquad \sin\varphi \leq \frac{m}{M} \qquad \underline{\varphi_{max} = \sin^{-1}\frac{m}{M}.}$$



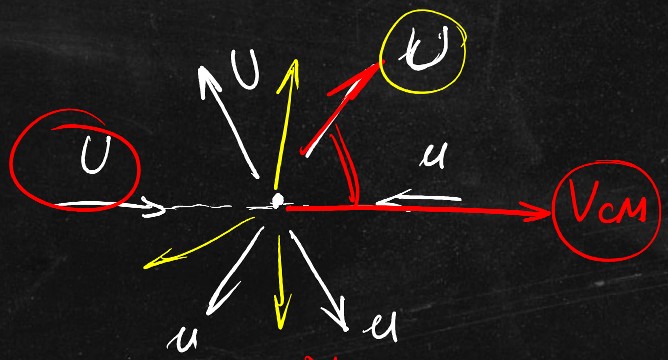
Method 2: COM Frame

$M \;\bullet\!\xrightarrow{V} \qquad \bullet\!\xrightarrow{V_{CM}} \qquad \bullet\, m$

$$V_{CM} = \frac{MV + m0}{M+m} = \frac{MV}{M+m}$$

$$\underline{U} = V - V_{CM} = V - \frac{MV}{M+m} = \frac{mV}{M+m}$$

$$u = 0 - V_{CM} = -\frac{MV}{M+m}$$



$$\underline{\overrightarrow{U_{M\to ground}}} = \overrightarrow{U_{M\to COM}} + \overrightarrow{U_{COM\to ground}}$$

$$\sin\varphi_{max} = \frac{\frac{mV}{m+M}}{\frac{MV}{m+M}} = \frac{m}{M} \qquad \varphi_{max} = \sin^{-1}\frac{m}{M}.$$

Orbits

$$\begin{cases} F(r) = m(\ddot{r} - r\dot{\theta}^2) \\ \frac{dL}{dt}(2mr\dot{r}\dot{\theta} + mr^2\ddot{\theta}) = 0 \end{cases}$$

$$L = mr^2\dot{\theta}$$

$$\dot{\theta} = \frac{L}{mr^2}$$

$$\vec{F} = -\frac{GMm}{r^2}\hat{r}$$

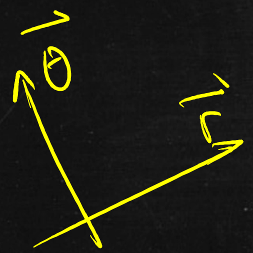
$$F(r) = m\left(\ddot{r} - r\left(\frac{L}{mr^2}\right)^2\right) = m\ddot{r} - \frac{L^2}{mr^3}$$

$$m\ddot{r} = F(r) + \frac{L^2}{mr^3}$$

$$F_{eff} = F(r) + F_c$$ Centrifugal Force

effective force

$\ddot{r} = 0$



r = constant $\quad \ddot{r} = 0$ (u.c.m)

Conservative Force → Potential Energy

Centrifugal Potential

$$V_c = \frac{1}{2} \cdot \frac{L^2}{mr^2} \qquad V_c = -\int \vec{F_c} \cdot d\vec{r} = -\int \frac{L^2}{mr^3} dr = \frac{L^2}{2mr^2}$$

Effective Potential $\quad V_{eff} = V(r) + V_c(r)$

Minimum Effective potential $\Rightarrow$ <u>Stable orbit</u>