Lagrangian Method (Conservative force)

$$K = \frac{1}{2} m v^2 = \frac{1}{2} m (v_x^2 + v_y^2 + v_z^2) = \frac{1}{2} m (\dot{x}^2 + \dot{y}^2 + \dot{z}^2)$$

$$= \frac{1}{2} m \dot{\vec{r}}^{\,2} = \frac{1}{2} m \cdot (\dot{r}\hat{r} + r\dot{\theta}\hat{\theta}) \cdot (\dot{r}\hat{r} + r\dot{\theta}\hat{\theta}) \qquad \hat{r}\cdot\hat{\theta} = 0$$

$$= \frac{1}{2} m \dot{r}^2 + \frac{1}{2} r^2 \dot{\theta}^2$$

$$\vec{r} = r\hat{r}$$

$$\dot{\vec{r}} = \frac{d}{dt}(r\hat{r})$$

$$= \dot{r}\hat{r} + r\dot{\hat{r}} = \dot{r}\hat{r} + r\dot{\theta}\hat{\theta}$$

$$\hat{r} = \cos\theta\,\vec{i} + \sin\theta\,\vec{j}$$

$$\dot{\hat{r}} = -\sin\theta\,\dot{\theta}\,\vec{i} + \cos\theta\,\dot{\theta}\,\vec{j}$$

$$= \dot{\theta}\hat{\theta}$$

$V(\vec{r}) \qquad V(x, y, z) \qquad \vec{F} = -\vec{\nabla} V \qquad V = -\int \vec{F} \cdot d\vec{r}$

Lagrangian : $L = K - V$ (Definition) $\qquad E = K + V$

Euler - Lagrange Equation, $\dfrac{d}{dt}\left(\dfrac{\partial L}{\partial \dot{x}_i}\right) = \dfrac{\partial L}{\partial x_i}$

$$\vec{F} = F_x \hat{x} + F_y \hat{y} + F_z \cdot \hat{z} = -\left(\frac{\partial}{\partial x}\hat{x} + \frac{\partial}{\partial y}\hat{y} + \frac{\partial}{\partial z}\hat{z}\right) V = -\frac{\partial V}{\partial x}\hat{x} - \frac{\partial V}{\partial y}\hat{y} - \frac{\partial V}{\partial z}\hat{z}$$

$$F_x = -\frac{\partial V(x,y,z)}{\partial x} \qquad F_y = -\frac{\partial V(x,y,z)}{\partial y} \qquad F_z = -\frac{\partial V(x,y,z)}{\partial z}$$

$$\vec{F} = m\ddot{\vec{r}} = m\ddot{x}\,\hat{x} + m\ddot{y}\,\hat{y} + m\ddot{z}\,\hat{z} \qquad \dot{\vec{x}} = \dot{x}\,\hat{x}$$

$$\underline{m\ddot{x}} = -\frac{\partial V(x,y,z)}{\partial x} \qquad K = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}m\dot{y}^2 + \frac{1}{2}m\dot{z}^2$$

$$\frac{\partial K}{\partial \dot{x}} = m\dot{x} \qquad \frac{d}{dt}\left(\frac{\partial K}{\partial \dot{x}}\right) = \underline{m\ddot{x}}$$

$$\frac{d}{dt}\left(\frac{\partial K}{\partial \dot{x}}\right) = -\frac{\partial V}{\partial x} \qquad \frac{d}{dt}\left(\frac{\partial K}{\partial \dot{y}}\right) = -\frac{\partial V}{\partial y} \qquad \frac{d}{dt}\left(\frac{\partial K}{\partial \dot{z}}\right) = -\frac{\partial V}{\partial z}$$

$$L(x_1, x_2, x_3, \ldots x_N, \dot{x}_1, \dot{x}_2, \ldots \dot{x}_N, t)$$

$$\frac{d}{dt}\left(\frac{\partial K}{\partial \dot{x}_i}\right) = -\frac{\partial V}{\partial x_i} \Rightarrow \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_i}\right) = \frac{\partial L}{\partial x_i} \qquad \dot{x}_i \;,\; V(x_i)$$

$$L = K - V \qquad \frac{\partial L}{\partial \dot{x}_i} = \frac{\partial K}{\partial \dot{x}_i} - \cancelto{0}{\frac{\partial V(x_i)}{\partial \dot{x}_i}} \qquad \frac{\partial V}{\partial x_i} = \cancelto{0}{\frac{\partial K}{\partial x_i}} - \frac{\partial L}{\partial x_i}$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_i}\right) = \frac{\partial L}{\partial x_i}$$