Q 1. Solutions.

(i)

$$F_1 = -kx_1 - k(x_1 - x_2) = m_1\ddot{x}_1$$

$$F_2 = -kx_2 - k(x_2 - x_1) = m_2\ddot{x}_2$$

$$\begin{cases} m_1\ddot{x}_1 + 2kx_1 - kx_2 = 0 \\ m_2\ddot{x}_2 + 2kx_2 - kx_1 = 0 \end{cases} \Rightarrow \begin{cases} \ddot{x}_1 + 2\frac{k}{m_1}x_1 - \frac{k}{m_1}x_2 = 0 & ① \\ \ddot{x}_2 + 2\frac{k}{m_2}x_2 - \frac{k}{m_2}x_1 = 0 & ② \end{cases}$$

(ii) ① + ② :

$$(\ddot{x}_1 + \ddot{x}_2) + \left(2\frac{k}{m_1} - \frac{k}{m_2}\right)x_1 + \left(2\frac{k}{m_2} - \frac{k}{m_1}\right)x_2 = 0.$$

$$\ddot{x}_1 + \ddot{x}_2 + \frac{2m_2 - m_1}{m_1 m_2}kx_1 + \frac{2km_1 - m_2}{m_1 m_2}kx_2 = 0$$

① − ②

$$(\ddot{x}_1 - \ddot{x}_2) + \left(2\frac{k}{m_1} + \frac{k}{m_2}\right)x_1 + \left(2\frac{k}{m_2} + \frac{k}{m_1}\right)x_2 = 0$$

for $m_1 = m_2 = m$

$$\ddot{x}_1 + \ddot{x}_2 + \frac{k}{m}x_1 + \frac{k}{m}x_2 = 0 \Rightarrow \ddot{y}_1 + w^2 y_1 = 0$$

$$\ddot{x}_1 - \ddot{x}_2 + 3\frac{k}{m}x_1 - 3\frac{k}{m}x_2 = 0 \Rightarrow \ddot{y}_2 - 3w^2 y_2 = 0$$

$$\omega_1^2 = w^2 = \frac{k}{m} \qquad \omega_2^2 = 3w^2 = 3\frac{k}{m}$$

$$\frac{\omega_1}{\omega_2} = \frac{1}{\sqrt{3}}.$$

(iii)

$$y_1 = A\sin(wt)$$

$$\dot{y}_1 = Aw\cos(wt) \qquad t=0 \quad \dot{y}_1 = \dot{x}_1 + \dot{x}_2 = 0 + v\hat{x} = v\hat{x}$$

$$V = Aw \qquad A = \frac{v}{w}$$

$$y_1 = v\frac{1}{w}\sin wt = v\sqrt{\frac{m}{k}}\sin\left(t\sqrt{\frac{k}{m}}\right)$$

(iv)

$$\begin{cases} \ddot{x}_1 + 2\gamma\dot{x}_1 + 2w^2x_1 - w^2x_2 = 0 \\ \ddot{x}_2 + 2\gamma\dot{x}_2 + 2w^2x_2 - w^2x_1 = 0 \end{cases} \qquad 2\gamma = \frac{\beta}{m}$$

(v)

$$\ddot{y}_1 + 2\gamma y_1 + w^2 y_1 = 0 \qquad \ddot{y}_2 + 2\gamma y_2 + 3w^2 y_2 = 0$$

$y_2 = A e^{qt}$

$A q^2 \cancel{e^{qt}} + 2\gamma \cdot \cancel{A} q \cancel{e^{qt}} + 3\omega^2 \cancel{A} \cancel{e^{qt}} = 0$

$q^2 + 2\gamma q + 3\omega^2 = 0 \qquad q = \frac{-2\gamma \pm \sqrt{4\gamma^2 - 12\omega^2}}{2} \qquad 2\gamma = \frac{\beta}{m}$

$q < 0 \quad \text{so} \quad t \to \infty \quad y_2 \to 0 \qquad y_2 = x_1 - x_2 = 0 \qquad x_1 = x_2$

$\ddot{y}_1 + 2\gamma \dot{y}_1 + \omega^2 y_1 = 0 \qquad y_1 = B e^{\alpha_1 t} + C \cdot e^{\alpha_2 t} = 2x_1 = 2x_2$

$\alpha^2 + 2\gamma\alpha + \omega^2 = 0 \qquad \alpha_1 = \frac{-\gamma + \sqrt{4\gamma^2 - 4\omega^2}}{2} \quad \text{or} \quad \alpha_2 = \frac{-\gamma - \sqrt{4\gamma^2 - 4\omega^2}}{2}$

$y_1 = B \cdot e^{\alpha_1 t} + C e^{\alpha_2 t}$

$x_1 = x_2 = \frac{1}{2} B e^{\alpha_1 t} + \frac{1}{2} C e^{\alpha_2 t}$

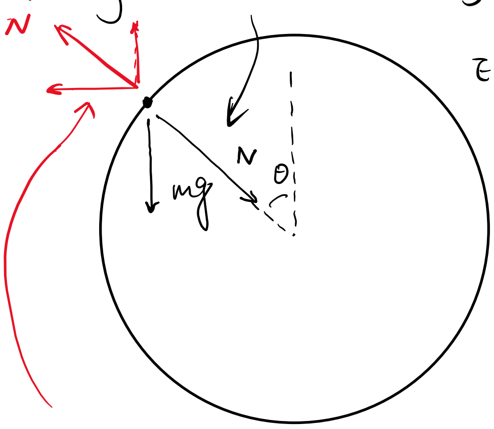
Q2. $(m_2 + \frac{m_1}{3}) L^2 \ddot{\theta} + (m_2 + \frac{m_1}{2}) g L \theta = 0$

$\ddot{\theta} + \frac{g}{L} \cdot \frac{m_2 + \frac{m_1}{2}}{m_2 + \frac{m_1}{3}} \theta = 0$

$\omega = \sqrt{\frac{g}{L} \cdot \frac{6m_2 + 3m_1}{6m_2 + 2m_1}} \qquad m_1 \to 0 \qquad \omega \approx \sqrt{\frac{g}{L}}$

Simple pendulum.

Q3. Analyse the beads: force: $mg\cos\theta + N = m\frac{v^2}{R}$



Energy: $\frac{1}{2}mv^2 = mgR - mgR\cos\theta$

$v^2 = \sqrt{2gR(1-\cos\theta)}$

$N = m\frac{v^2}{R} - mg\cos\theta = mg(2 - 3\cos\theta)$

Analyse the hoop: $F_y = 2N\cos\theta = 2mg(2-3\cos\theta)\cos\theta$

$F_{y\,max} \geq Mg \Rightarrow \frac{m}{M} \leq \frac{3}{2}$

Q4.

(a) Label two diametrically opposite points as the equilibrium positions. Let the positions of the masses relative to these points be x_1 and x_2, measured counterclockwise. Then the $F = ma$ equations are

$$\begin{aligned} m\ddot{x}_1 + 2k(x_1 - x_2) &= 0, \\ m\ddot{x}_2 + 2k(x_2 - x_1) &= 0. \end{aligned} \tag{4.107}$$

The determinant method works here, but let's just do it the easy way. Adding the equations gives

$$\ddot{x}_1 + \ddot{x}_2 = 0, \tag{4.108}$$

and subtracting them gives

$$(\ddot{x}_1 - \ddot{x}_2) + 4\omega^2(x_1 - x_2) = 0. \tag{4.109}$$

The normal coordinates are therefore

$$\begin{aligned} x_1 + x_2 &= At + B, \\ x_1 - x_2 &= C\cos(2\omega t + \phi). \end{aligned} \tag{4.110}$$

Solving these two equations for x_1 and x_2, and writing the results in vector form, gives

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}(At + B) + C\begin{pmatrix} 1 \\ -1 \end{pmatrix}\cos(2\omega t + \phi), \tag{4.111}$$

where the constants A, B, and C are defined to be half of what they were in Eq. (4.110). The normal modes are therefore

$$\begin{aligned} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} &= \begin{pmatrix} 1 \\ 1 \end{pmatrix}(At + B), \quad \text{and} \\ \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} &= C\begin{pmatrix} 1 \\ -1 \end{pmatrix}\cos(2\omega t + \phi). \end{aligned} \tag{4.112}$$

The first mode has frequency zero. It corresponds to the masses sliding around the circle, equally spaced, at constant speed. The second mode has both masses moving to the left, then both to the right, back and forth. Each mass feels a force of $4kx$ (because there are two springs, and each one stretches by $2x$), hence the $\sqrt{4} = 2$ in the frequency.

(b) Label three equally spaced points as the equilibrium positions. Let the positions of the masses relative to these points be x_1, x_2, and x_3, measured counterclockwise. Then the $F = ma$ equations are, as you can show,

$$\begin{aligned} m\ddot{x}_1 + k(x_1 - x_2) + k(x_1 - x_3) &= 0, \\ m\ddot{x}_2 + k(x_2 - x_3) + k(x_2 - x_1) &= 0, \\ m\ddot{x}_3 + k(x_3 - x_1) + k(x_3 - x_2) &= 0. \end{aligned} \tag{4.113}$$

The sum of all three of these equations definitely gives something nice. Also, differences between any two of the equations give something useful. But let's use the determinant method to get some practice. Trying solutions of the form $x_1 = A_1 e^{i\alpha t}$, $x_2 = A_2 e^{i\alpha t}$, and $x_3 = A_3 e^{i\alpha t}$, we obtain the matrix equation,

$$\begin{pmatrix} -\alpha^2 + 2\omega^2 & -\omega^2 & -\omega^2 \\ -\omega^2 & -\alpha^2 + 2\omega^2 & -\omega^2 \\ -\omega^2 & -\omega^2 & -\alpha^2 + 2\omega^2 \end{pmatrix} \begin{pmatrix} A_1 \\ A_2 \\ A_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}. \tag{4.114}$$

Setting the determinant equal to zero yields a cubic equation in α^2. But it's a nice cubic equation, with $\alpha^2 = 0$ as a solution. The other solution is the double root $\alpha^2 = 3\omega^2$.

The $\alpha = 0$ root corresponds to $A_1 = A_2 = A_3$. That is, it corresponds to the vector $(1, 1, 1)$. This $\alpha = 0$ case is the one case where our exponential solution isn't really an exponential. But α^2 equalling zero in Eq. (4.114) basically tells us that we're dealing with a function whose second derivative is zero, that is, a linear function $At + B$. Therefore, the normal mode is

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} (At + B). \tag{4.115}$$

This mode has frequency zero. It corresponds to the masses sliding around the circle, equally spaced, at constant speed.

The two $\alpha^2 = 3\omega^2$ roots correspond to a two-dimensional subspace of normal modes. You can show that any vector of the form (a, b, c) with $a + b + c = 0$ is a normal mode with frequency $\sqrt{3}\omega$. We will arbitrarily pick the vectors $(0, 1, -1)$ and $(1, 0, -1)$ as basis vectors for this space. We can then write the normal modes as linear combinations of the vectors

$$\begin{aligned} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} &= C_1 \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} \cos(\sqrt{3}\omega t + \phi_1), \\ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} &= C_2 \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \cos(\sqrt{3}\omega t + \phi_2). \end{aligned} \tag{4.116}$$

The normal coordinates in this problem are $x_1+x_2+x_3$ (obtained by adding the three equations in (4.113)), and also any combination of the form $ax_1+bx_2+cx_3$, where $a+b+c = 0$ (obtained by taking a times the first equation in Eq. (4.113), plus b times the second, plus c times the third). The three normal coordinates that correspond to the mode in Eq. (4.115) and the two modes we chose in Eq. (4.116) are, respectively, $x_1 + x_2 + x_3$, $x_1 - 2x_2 + x_3$, and $-2x_1 + x_2 + x_3$, because each of these combinations gets no contribution from the other two modes (demanding this is how you can derive the coefficients of the x_i's, up to an overall constant). ♣

(c) In part (b), when we set the determinant of the matrix in Eq. (4.114) equal to zero, we were essentially finding the eigenvectors and eigenvalues[9] of the matrix,

$$\begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix} = 3I - \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}, \tag{4.117}$$

where I is the identity matrix. We haven't bothered writing the common factor ω^2 here, because it doesn't affect the eigenvectors. As an exercise, you can show that for the general case of N springs and N masses on a circle, the above matrix becomes the $N \times N$ matrix,

$$3I - \begin{pmatrix} 1 & 1 & 0 & 0 & & 1 \\ 1 & 1 & 1 & 0 & \cdots & 0 \\ 0 & 1 & 1 & 1 & & 0 \\ 0 & 0 & 1 & 1 & & 0 \\ & \vdots & & & \ddots & \vdots \\ 1 & 0 & 0 & 0 & \cdots & 1 \end{pmatrix} \equiv 3I - M. \tag{4.118}$$

In the matrix M, the three consecutive 1's keep shifting to the right, and they wrap around cyclicly. We must now find the eigenvectors of M, which will require being a little clever.

We can guess the eigenvectors and eigenvalues of M if we take a hint from its cyclic nature. A particular set of things that are rather cyclic are the Nth roots of 1. If β is an Nth root of 1, you can verify that $(1, \beta, \beta^2, \ldots, \beta^{N-1})$ is an eigenvector of M with eigenvalue $\beta^{-1} + 1 + \beta$. (This general method works for any matrix where the entries keep shifting to the right. The entries don't have to be equal.) The eigenvalues of the entire matrix in Eq. (4.118) are therefore $3 - (\beta^{-1} + 1 + \beta) = 2 - \beta^{-1} - \beta$. There are N different Nth roots of 1, namely $\beta_n = e^{2\pi i n/N}$, for $0 \le n \le N - 1$. So the N eigenvalues are

$$\begin{aligned} \lambda_n = 2 - \left(e^{-2\pi i n/N} + e^{2\pi i n/N}\right) &= 2 - 2\cos(2\pi n/N) \\ &= 4\sin^2(\pi n/N). \end{aligned} \tag{4.119}$$

The corresponding eigenvectors are

$$V_n = \left(1, \beta_n, \beta_n^2, \ldots, \beta_n^{N-1}\right). \tag{4.120}$$

Since the numbers n and $N - n$ yield the same value for λ_n in Eq. (4.119), the eigenvalues come in pairs (except for $n = 0$, and $n = N/2$ if N is even). This is fortunate, because we can then form real linear combinations of the two corresponding complex eigenvectors given in Eq. (4.120). We see that the vectors

$$V_n^+ \equiv \frac{1}{2}(V_n + V_{N-n}) = \begin{pmatrix} 1 \\ \cos(2\pi n/N) \\ \cos(4\pi n/N) \\ \vdots \\ \cos\big(2(N-1)\pi n/N\big) \end{pmatrix} \tag{4.121}$$

and

$$V_n^- \equiv \frac{1}{2i}(V_n - V_{N-n}) = \begin{pmatrix} 0 \\ \sin(2\pi n/N) \\ \sin(4\pi n/N) \\ \vdots \\ \sin\big(2(N-1)\pi n/N\big) \end{pmatrix} \tag{4.122}$$

both have eigenvalue $\lambda_n = \lambda_{N-n}$ (as does any linear combination of these vectors). For the special case of $n = 0$, the eigenvector is $V_0 = (1, 1, 1, \ldots, 1)$ with eigenvalue $\lambda_0 = 0$. And for the special case of $n = N/2$ if N is even, the eigenvector is $V_{N/2} = (1, -1, 1, \ldots, -1)$ with eigenvalue $\lambda_{N/2} = 4$.

Referring back to the $N = 3$ case in Eq. (4.114), we see that we must take the square root of the eigenvalues and then multiply by ω to obtain the frequencies (because it was an α^2 that appeared in the matrix, and because we dropped the factor of ω^2). The frequency corresponding to the above two normal modes is therefore, using Eq. (4.119),

$$\omega_n = \omega\sqrt{\lambda_n} = 2\omega\sin(\pi n/N). \tag{4.123}$$

For even N, the largest value of the frequency is 2ω, with the masses moving in alternating equal positive and negative displacements. But for odd N, it is slightly less than 2ω.

To sum everything up, the N normal modes are the vectors in Eqs. (4.121) and (4.122), where n runs from 1 up to the greatest integer less than $N/2$. And then we have to add on the V_0 vector, and also the $V_{N/2}$ vector if N is even.[10] The frequencies are given in Eq. (4.123). Each frequency is associated with two modes, except the V_0 mode and the $V_{N/2}$ mode if N is even.

Remark: Let's check our results for $N = 2$ and $N = 3$. For $N = 2$: The values of n are the two "special" cases of $n = 0$ and $n = N/2 = 1$. If $n = 0$, we have $\omega_0 = 0$ and $V_0 = (1, 1)$. If $n = 1$, we have $\omega_1 = 2\omega$ and $V_1 = (1, -1)$. These results agree with the two modes in Eq. (4.112).

For $N = 3$: If $n = 0$, we have $\omega_0 = 0$ and $V_0 = (1, 1, 1)$, in agreement with Eq. (4.115). If $n = 1$, we have $\omega_1 = \sqrt{3}\omega$, and $V_1^+ = (1, -1/2, -1/2)$ and $V_1^- = (0, 1/2, -1/2)$. These two vectors span the same space we found in Eq. (4.116). And they have the same frequency as in Eq. (4.116). You can also find the vectors for $N = 4$. These are fairly intuitive, so try to write them down first without using the above results. ♣