Angluar Momentum and Central Forces

hat

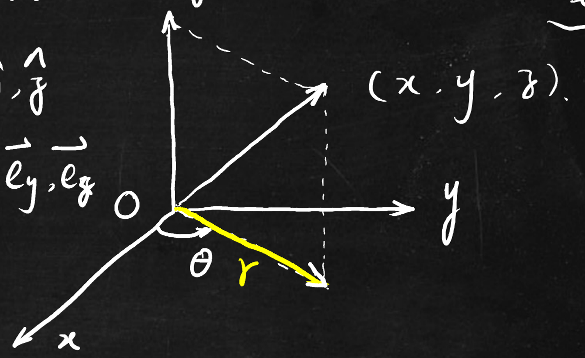
element 要素

Polar Coordinates (r, θ, z)

Spherical coordinate

$\vec{i}\quad \vec{j}\quad \vec{k}\qquad \hat{x}, \hat{y}, \hat{z}$

$\vec{e}_1\ \vec{e}_2\ \vec{e}_3\qquad \vec{e}_x, \vec{e}_y, \vec{e}_z$

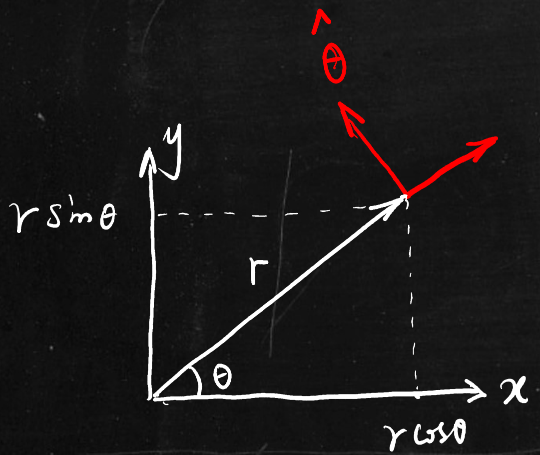


$x = r\cos\theta \qquad y = r\sin\theta$

$$\hat{r} = \cos\theta\,\vec{i} + \sin\theta\,\vec{j}$$

$$\hat{\theta} = \frac{d\hat{r}}{d\theta} = -\sin\theta\,\vec{i} + \cos\theta\,\vec{j}$$

$$\frac{d\vec{i}}{dt} = \frac{d\vec{j}}{dt} = \frac{d\vec{k}}{dt} = 0$$



$$\vec{v} = \frac{d\vec{r}}{dt} = \frac{d}{dt}(r\cdot\hat{r}) \qquad d\vec{r} = dr\cdot\hat{r}$$

$$= \frac{dr}{dt}\hat{r} + r\cdot\frac{d\hat{r}}{dt} = \dot{r}\hat{r} + r\dot{\hat{r}}$$

$\vec{a}\qquad \hat{a}$

$$\frac{d\hat{r}}{dt} = \frac{d}{dt}(\cos\theta\,\vec{i} + \sin\theta\,\vec{j}) = \frac{d}{dt}(\cos\theta\,\vec{i}) + \frac{d}{dt}(\sin\theta\,\vec{j})$$

$$= \frac{d(\cos\theta)}{d\theta}\cdot\frac{d\theta}{dt}\vec{i} + \frac{d(\sin\theta)}{d\theta}\cdot\frac{d\theta}{dt}\vec{j}$$

$$\frac{d\hat{r}}{dt} = -\sin\theta \cdot \dot{\theta} \cdot \vec{i} + \cos\theta \cdot \dot{\theta}\vec{j} = \dot{\theta}\left(\underline{-\sin\theta\vec{i} + \cos\theta\vec{j}}\right) = \dot{\theta} \cdot \hat{\theta}$$

$$\underline{\vec{v} = \dot{r}\hat{r} + r\dot{\theta}\hat{\theta}} \qquad (r, \theta) \qquad \vec{v} = \dot{x}\vec{i} + \dot{y}\vec{j}$$

$$\vec{a} = \frac{d\vec{v}}{dt} = \frac{d}{dt}(\dot{r}\hat{r} + r\dot{\theta}\hat{\theta}) = \frac{d}{dt}(\dot{r}\hat{r}) + \frac{d}{dt}(r\dot{\theta}\hat{\theta})$$

$$= \underline{\ddot{r}\hat{r} + \dot{r}\dot{\hat{r}}} + \dot{r}\dot{\theta}\hat{\theta} + r\underline{\frac{d}{dt}(\dot{\theta}\hat{\theta})} \qquad \ddot{\theta}\hat{\theta} + \dot{\theta}\dot{\hat{\theta}}$$

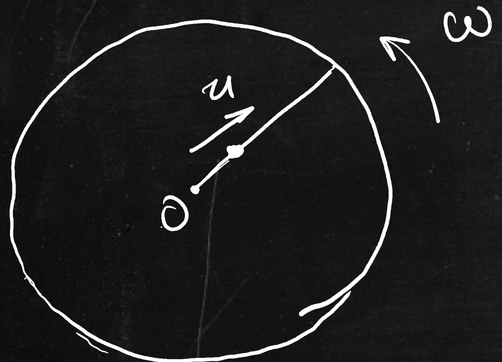
$$= \ddot{r}\hat{r} + \underline{\dot{r}\dot{\hat{r}} + \dot{r}\dot{\theta}\hat{\theta}} + r\ddot{\theta}\hat{\theta} + r\dot{\theta}\dot{\hat{\theta}} \qquad \vec{a} = \ddot{x}\vec{i} + \ddot{y}\vec{j}$$

$$\hat{\theta} = -\sin\theta\vec{i} + \cos\theta\vec{j} \qquad \hat{r}$$

$$\frac{d\hat{\theta}}{dt} = -\cos\theta\frac{d\theta}{dt}\vec{i} - \sin\theta\frac{d\theta}{dt}\vec{j} = -\frac{d\theta}{dt}\left(\underline{\cos\theta\vec{i} + \sin\theta\vec{j}}\right) = -\dot{\theta}\hat{r}$$

$$\vec{a} = \ddot{r}\hat{r} + \dot{r}\dot{\theta}\hat{\theta} + \dot{r}\dot{\theta}\hat{\theta} + r\ddot{\theta}\hat{\theta} - r\dot{\theta}\dot{\theta}\hat{r} = (\ddot{r} - r\dot{\theta}^2)\hat{r} + (2\dot{r}\dot{\theta} + r\ddot{\theta})\hat{\theta}$$

Ex. A bead moves outwards along a frictionless spoke of a bicycle wheel at a constant speed u. The bead is at the centre of the weel at $t=0$. The bead has a constant angular speed ω. Find the velocity and acceleration of the bead.



$\underline{u=\dot{r}}$ $\quad \dot{\theta}=\omega$

$r=\underline{ut}$ $\quad \vec{v}=\dot{r}\hat{r}+r\dot{\theta}\hat{\theta}$

$\vec{v}=u\hat{r}+r\omega\hat{\theta}=u\hat{r}+ut\omega\hat{\theta}$ $\qquad \vec{i}, \vec{j}$

$\vec{a}=(\cancel{\ddot{r}}-r\dot{\theta}^2)\hat{r}+(2\dot{r}\dot{\theta}+\cancel{r\ddot{\theta}})\hat{\theta}$

$\vec{a}=-ut\cdot\omega^2\hat{r}+2u\omega\hat{\theta}$

practice: $\underline{\omega=\omega_0 t}$ $\quad \vec{v}=?\quad \vec{a}=?$

$v=u\hat{r}+u\omega_0 t^2\hat{\theta}$

$a=-u\omega_0^2 t^3\hat{r}+3u\omega_0 t\hat{\theta}$

Uniform circular motion (r, θ)

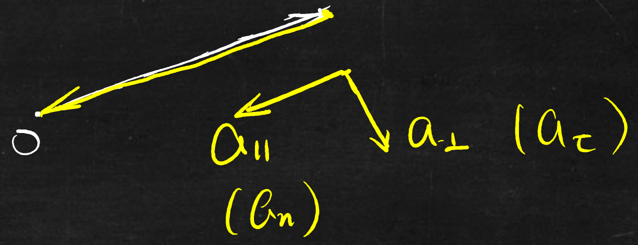
$r = \text{constant}$ $\quad \dot{r} = 0 \quad \ddot{r} = 0$

解析几何

$$\vec{v} = \cancel{\dot{r}\hat{r}} + r\dot{\theta}\hat{\theta} = r\dot{\theta}\hat{\theta}$$

$$\dot{\theta} = \omega \qquad \vec{v} = r\dot{\theta}\hat{\theta} = \omega r\hat{\theta}$$

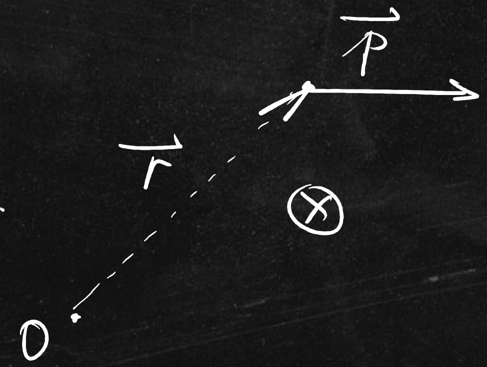
$$v = \omega r$$



$$\vec{a} = (\cancel{\ddot{r}} - r\dot{\theta}^2)\hat{r} + (\cancel{2\dot{r}\dot{\theta}} + \cancel{r\ddot{\theta}})\hat{\theta} = -\omega^2 r\hat{r} \qquad a = \omega^2 r = \frac{v^2}{r}$$

$$\vec{F} = m\vec{a} = -m\omega^2 r\hat{r}$$ Centripetal force

Angular Momentum $\quad \vec{p} = m\vec{v}$ momentum.



$$\vec{L} = \vec{r} \times \vec{p} = \vec{r} \times (m\vec{v}) = m\vec{r} \times \vec{v}$$

$$L = mrv\sin\theta$$

Torque: $\vec{T} = \vec{r} \times \vec{F}$

$\vec{F}$ is resultant force

$$\vec{T} = \vec{r} \times \vec{F}_{Net} = \vec{r} \times (m\vec{a}) = m \cdot \vec{r} \times \frac{d\vec{v}}{dt}$$

$$r \times \frac{d\vec{v}}{dt} = \frac{d(\vec{r} \times \vec{v})}{dt}$$

$$\frac{d(\vec{r} \times \vec{v})}{dt} = \frac{d\vec{r}}{dt} \times \vec{v} + \vec{r} \times \frac{d\vec{v}}{dt}$$

$$= \cancelto{0}{\vec{v} \times \vec{v}} + \underline{\vec{r} \times \frac{d\vec{v}}{dt}}$$

$$\vec{a} \times \vec{b} = -\vec{b} \times \vec{a}$$

$$\vec{a} \cdot \vec{b} = \vec{b} \cdot \vec{a}$$

$$\vec{a} \times \vec{a} = 0 \qquad \vec{L}$$

$$\vec{r} \times \frac{d\vec{v}}{dt} = \frac{d(\vec{r} \times \vec{v})}{dt}$$

$$\vec{T} = m \cdot \vec{r} \times \frac{d\vec{v}}{dt} = m \frac{d(\vec{r} \times \vec{v})}{dt} = \frac{d(m\vec{r} \times \vec{v})}{dt}$$

$$\vec{T} = \frac{d\vec{L}}{dt} = \dot{\vec{L}}$$

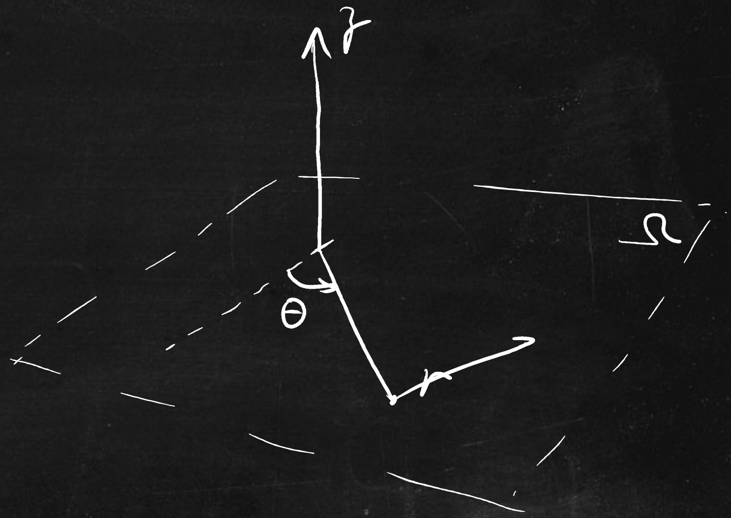
$$\begin{cases} \vec{f} = \dfrac{d\vec{p}}{dt} \\ \vec{T} = \dfrac{d\vec{L}}{dt} \end{cases}$$

$\vec{L} = ?$ (Polar System).

$$\vec{L} = m\vec{r}\times\vec{v} = m\vec{r}\times(\dot{r}\hat{r} + r\dot{\theta}\hat{\theta})$$

$$= \cancel{m\vec{r}\times(\dot{r}\hat{r})} + m\vec{r}\times r\dot{\theta}\hat{\theta}$$

$$= mr\overset{=1}{|\hat{\theta}|}\cdot r\dot{\theta} = mr^2\dot{\theta}\,\hat{z}$$



$$\underline{L = mr^2\dot{\theta}} \qquad v_t = \omega r = r\frac{d\theta}{dt}$$

$$L = mrv_t$$

Central Force 有心力

$$F_s(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\cdot\frac{q_1q_2}{r^2}\hat{r} \qquad F_g(\vec{r}) = -G\frac{M_1M_2}{r^2}\hat{r}$$

$$\vec{F}(\vec{r}) = F(r)\hat{r} \qquad \vec{F} = m\vec{a}$$

$$F(r)\hat{r} = m\vec{a} = m\left[(\ddot{r} - r\dot{\theta}^2)\hat{r} + (2\dot{r}\dot{\theta} + r\ddot{\theta})\hat{\theta}\right]$$

径向,法向　　　　切向

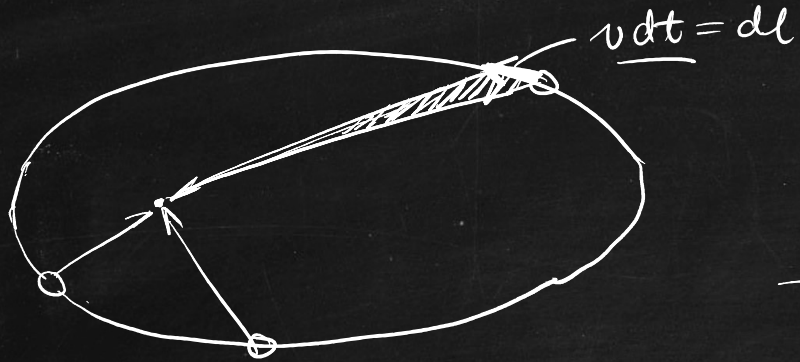
$$\begin{cases} F(r) = m(\ddot{r} - r\dot{\theta}^2) \\ 2\dot{r}\dot{\theta} + r\ddot{\theta} = 0 \end{cases} \qquad L = mr^2\dot{\theta}$$

$$\frac{dL}{dt} = \frac{d}{dt}(mr^2\dot{\theta}) = m(2r\dot{r}\dot{\theta} + r^2\ddot{\theta}) = mr(2\dot{r}\dot{\theta} + r\ddot{\theta}) = 0$$

$\frac{dL}{dt} = 0 \qquad L = \text{constant} \qquad$ (conservation Law of angular momentum)

$$\frac{d\vec{L}}{dt} = \vec{T} \qquad \frac{d\vec{L}}{dt} = 0 \qquad \vec{T} = 0$$

O



$$\frac{dL}{dt} = 0 \qquad L = \text{constant}$$

$$\underline{m\vec{r} \times \vec{v} = \text{constant}}$$

$$V(\vec{r}) = -\int \vec{F}(r) \cdot d\vec{r}$$

$$F(\vec{r}) = -\frac{dV}{dr}$$

$$\vec{F}(\vec{r}) = F_x \vec{i} + F_y \vec{j} + F_z \vec{k}$$

$$d\vec{r} = dx\vec{i} + dy\vec{j} + dz\vec{k}$$

$$\vec{\nabla} = \frac{\partial}{\partial x}\vec{i} + \frac{\partial}{\partial y}\vec{j} + \frac{\partial}{\partial z}\vec{k}$$