Tensor operations

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Lesson summary:

- Denote vectors $\boldsymbol{u}(x, y, z, t), \boldsymbol{v}(x, y, z, t)$ and scalar $f(x, y, z, t)$
- Inner product
  $$\begin{aligned}\boldsymbol{u} \cdot \boldsymbol{v} &= |\boldsymbol{u}||\boldsymbol{v}| \cos\theta \\ &= (u_i \boldsymbol{e}_i) \cdot \left(v_j \boldsymbol{e}_j\right) \\ &= u_i v_j \delta_{ij} \\ &= u_i v_i\end{aligned}$$
- Cross product
  $$\begin{aligned}\boldsymbol{u} \times \boldsymbol{v} &= |\boldsymbol{u}||\boldsymbol{v}| \sin\theta \\ &= (u_i \boldsymbol{e}_i) \times \left(v_j \boldsymbol{e}_j\right) \\ &= (u_2 v_3 - u_3 v_2)\boldsymbol{e}_1 + (u_3 v_1 - u_1 v_3)\boldsymbol{e}_2 + (u_1 v_2 - u_2 v_1)\boldsymbol{e}_3\end{aligned}$$
- Gradient ∇
  $$\nabla f = \left(\frac{\partial}{\partial x}\boldsymbol{e}_1 + \frac{\partial}{\partial y}\boldsymbol{e}_2 + \frac{\partial}{\partial z}\boldsymbol{e}_3\right) f$$
- Divergence $\nabla \cdot$
  $$\nabla \cdot \boldsymbol{v} = \frac{\partial v_1}{\partial x} + \frac{\partial v_2}{\partial y} + \frac{\partial v_3}{\partial z}$$
- Laplacian ∇^2
  $$\nabla^2 f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}$$
- Curl $\nabla \times$
  $$\nabla \times \boldsymbol{v} = \left(\frac{\partial v_3}{\partial y} - \frac{\partial v_2}{\partial z}\right)\boldsymbol{e}_1 + \left(\frac{\partial v_1}{\partial z} - \frac{\partial v_3}{\partial x}\right)\boldsymbol{e}_2 + \left(\frac{\partial v_2}{\partial x} - \frac{\partial v_1}{\partial y}\right)\boldsymbol{e}_3$$

Worked examples:

1. (E/M, USAYPT22 P2) Given $\nabla \cdot \boldsymbol{B} = \boldsymbol{0}$, $\nabla \cdot (\nabla \times \boldsymbol{A}) = \boldsymbol{0}$, find the expression for $\boldsymbol{B}$ in terms of $\mu_0, \boldsymbol{m}, \boldsymbol{R}$.
   Solution: $\boldsymbol{B}(\boldsymbol{m}, \boldsymbol{R}) = \frac{\mu_0}{4\pi}\left(\frac{3(\boldsymbol{m}\cdot\boldsymbol{R})\boldsymbol{R}}{R^5} - \frac{\boldsymbol{m}}{R^3}\right)$
2. (P2 extension) In 3D space, consider Euler angle ϕ, θ, find $\boldsymbol{B}(\boldsymbol{r}, \boldsymbol{\theta}, \boldsymbol{\phi}, \boldsymbol{M})$.
   Solution: Note that $\boldsymbol{B}$ is independent of angle ϕ, then $\boldsymbol{B}(\boldsymbol{r}, \boldsymbol{\theta}, \boldsymbol{\phi}, \boldsymbol{M}) = \sum_i \frac{\mu_0 \boldsymbol{M}}{4\pi}\frac{1}{r_i^3}\left(\hat{\boldsymbol{r}}\,(2\cos\theta) + \hat{\boldsymbol{\theta}}\,(\sin\theta_i)\right)$