Theory and Modeling

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Rotation的进一步计算

- Given the coordinate of a point with respect to some origin (定点) , we have

$$\nabla \cdot \left[\hat{\boldsymbol{n}} f(r)\right] = \frac{2}{r} f + \frac{\partial f}{\partial r}$$

$$(\boldsymbol{a} \cdot \nabla) \hat{\boldsymbol{n}} f(r) = \frac{f(r)}{r}\left[\boldsymbol{a} - \hat{\boldsymbol{n}}\left(\boldsymbol{a} \cdot \hat{\boldsymbol{n}}\right)\right] + \hat{\boldsymbol{n}}\left(\boldsymbol{a} \cdot \hat{\boldsymbol{n}}\right) \frac{\partial f}{\partial r}$$

- With further derivations,

Outside of sphere i (for $|\mathbf{r}-\mathbf{r}_i| > a_i$), its magnetic field is given by [14, 15]

$$\mathbf{B}_i(\mathbf{r}) = \mathbf{B}(\mathbf{m}_i; \mathbf{r} - \mathbf{r}_i), \tag{2}$$

where $\mathbf{r} - \mathbf{r}_i$ is the position vector relative to the sphere center, and where

$$\mathbf{B}(\mathbf{m}; \mathbf{r}) = \frac{\mu_0}{4\pi}\left(\frac{3\mathbf{m} \cdot \mathbf{r}}{r^5}\mathbf{r} - \frac{\mathbf{m}}{r^3}\right) \tag{3}$$

Rotation的进一步计算

- Do dot product with the small magnetic moment m_i in dV, we get the magnetic potential U_1to2, namely,
- Then according to F-U relation, F_1to2
- Note: 磁铁的磁矩方向是从磁铁的指南极指向指北极，磁矩的大小取决于磁铁的磁性与量值。

$$
\begin{aligned}
U_{12} &= -\mathbf{m}_2 \cdot \mathbf{B}_1(\mathbf{r}_2) && (4)\\
&= \frac{\mu_0}{4\pi}\left[\frac{\mathbf{m}_1 \cdot \mathbf{m}_2}{{r_{12}}^3} - 3\frac{(\mathbf{m}_1 \cdot \mathbf{r}_{12})(\mathbf{m}_2 \cdot \mathbf{r}_{12})}{{r_{12}}^5}\right], && (5)
\end{aligned}
$$

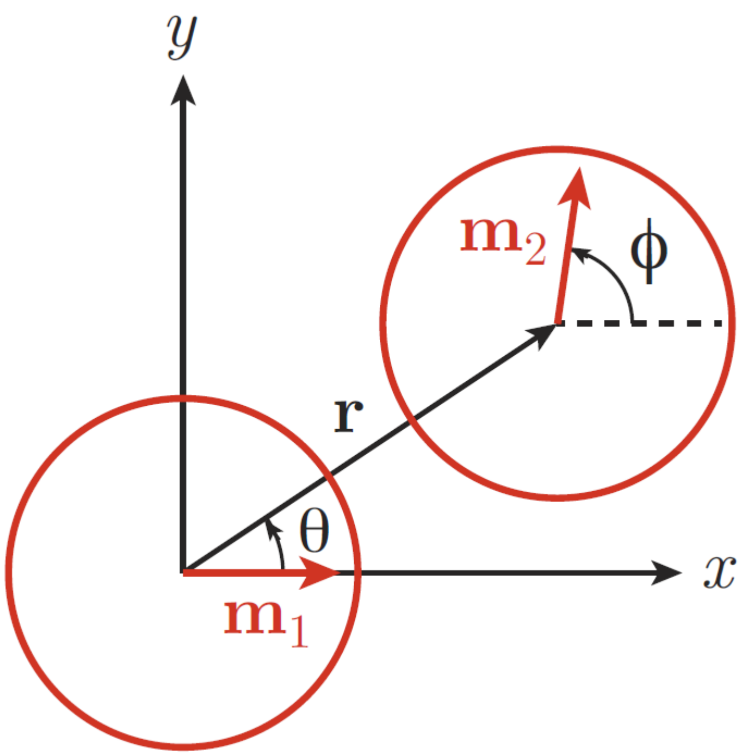
The force of sphere 1 on sphere 2 follows as

$$
\begin{aligned}
\mathbf{F}_{12} =& -\boldsymbol{\nabla}_2 U_{12} && (6)\\
=& \frac{3\mu_0}{4\pi {r_{12}}^5}\Big[(\mathbf{m}_1 \cdot \mathbf{r}_{12})\,\mathbf{m}_2 + (\mathbf{m}_2 \cdot \mathbf{r}_{12})\,\mathbf{m}_1\\
&+ (\mathbf{m}_1 \cdot \mathbf{m}_2)\,\mathbf{r}_{12} - 5\frac{(\mathbf{m}_1 \cdot \mathbf{r}_{12})\,(\mathbf{m}_2 \cdot \mathbf{r}_{12})}{{r_{12}}^2}\mathbf{r}_{12}\Big]. && (7)
\end{aligned}
$$

Rotation的进一步计算

- In a tangent polar coordinate system, set one of the permanent magnet spheres still, and use (r, θ) to describe an arbitrary position of (**r_2, θ_2**) in the x-y plane.

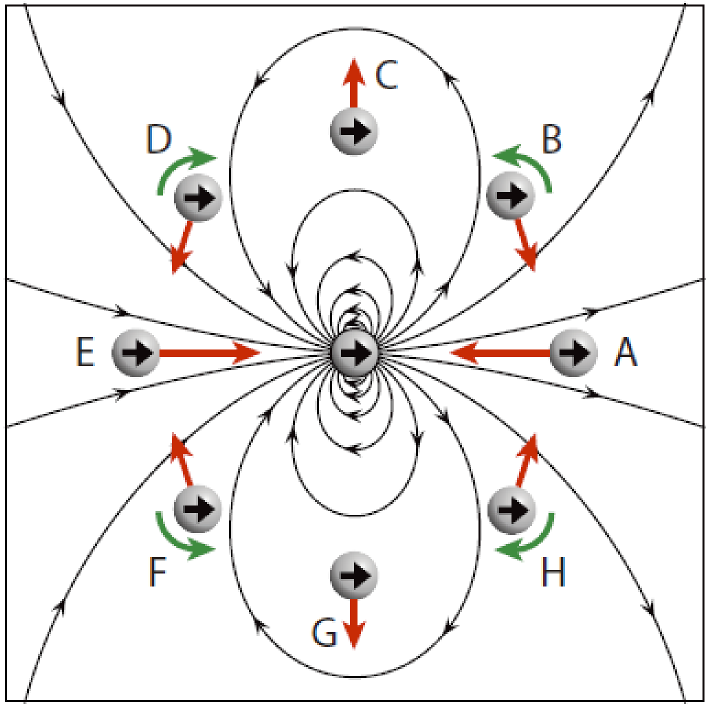
- Thus, T $= \frac{p_r^2}{2\tilde{m}} + \frac{p_\theta^2}{2\tilde{m}r^2} + \frac{p_\phi^2}{2I}. \qquad (15)$



Rotation的进一步计算

1. *Attractive, Central* ($\beta = 0$): The force between two spheres is attractive and central when $\mathbf{m}_1$ and $\mathbf{m}_2$ are parallel and collinear [Fig. 3(a) A, E], leading to a stable equilibrium state with the north pole of one magnet contacting the south pole of the other. The force is also attractive and central when $\mathbf{m}_1$ and $\mathbf{m}_2$ are antiparallel to each other and perpendicular to the line through their centers [Fig. 3(e) C, G]. In both cases, $\mathbf{m}_2$ is parallel to $\mathbf{B}_1$ ($\beta = 0$), giving $\boldsymbol{\tau}_{12} = 0$ and $\mathbf{m}_2 \cdot \mathbf{B}_1 = mB_1$. To increase this positive product, $\mathbf{m}_1$ attracts $\mathbf{m}_2$ into its vicinity, where its field is stronger.

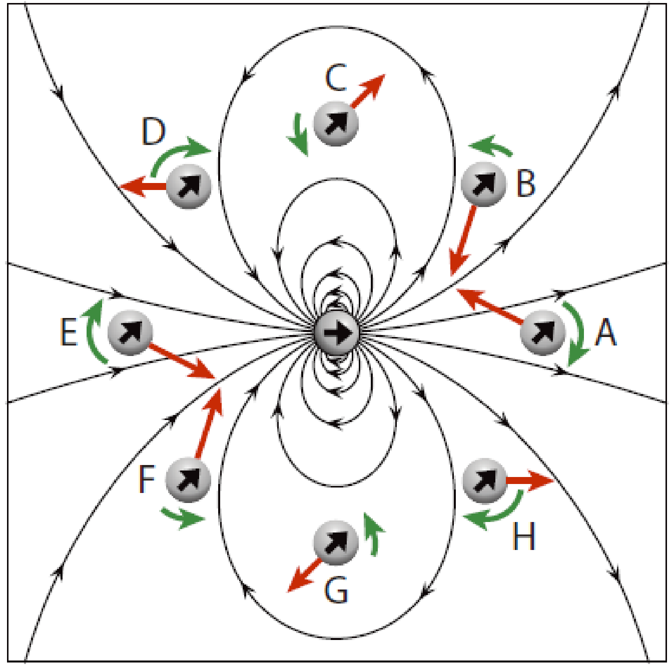
(a)



Rotation的进一步计算

2. *Attractive, Oblique* $(0 < \beta < \pi/2)$: For acute values of β, the force is attractive and non-central [Fig. 3(a) B, D, F, and H; Fig. 3(b) A, B, E, and F; Fig. 3(c) B, F; Fig. 3(d) B, C, F, and G]. The force acts to increase $\mathbf{m}_2 \cdot \mathbf{B}_1 = mB_1 \cos\beta$ by moving $\mathbf{m}_2$ into regions where the field is stronger (larger B_1) and better aligned with $\mathbf{m}_2$ (smaller β, larger $\cos\beta$). The torque also acts to increase $\mathbf{m}_2 \cdot \mathbf{B}_1$ by rotating $\mathbf{m}_2$ into alignment with $\mathbf{B}_1$.

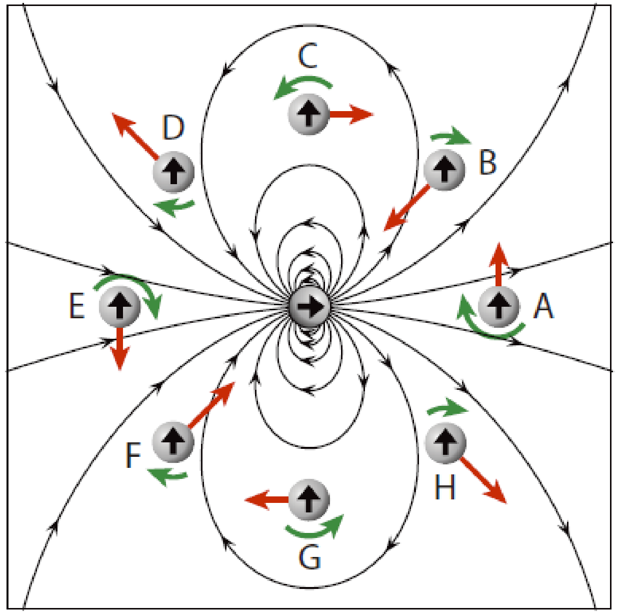
(b)



Rotation的进一步计算

3. *Perpendicular* ($\beta = \pi/2$): When $\mathbf{m}_2$ is perpendicular to $\mathbf{B}_1$, the force of $\mathbf{m}_1$ on $\mathbf{m}_2$ is perpendicular to the line joining these spheres – it is neither attractive nor repulsive [Fig. 3(c) A, C, E, and G]. This force acts to increase $\mathbf{m}_2 \cdot \mathbf{B}_1$ by moving $\mathbf{m}_2$ toward a region where it is better aligned with the field. The torque again acts to rotate $\mathbf{m}_2$ into alignment with $\mathbf{B}_1$.

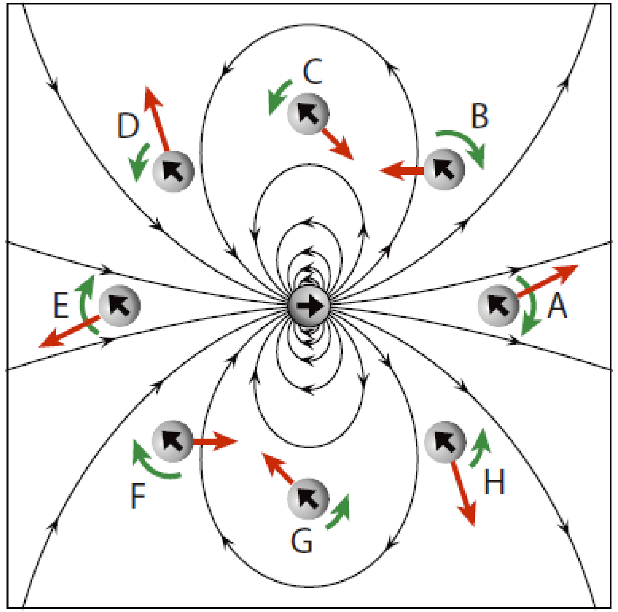
(c)



Rotation的进一步计算

4. *Repulsive, Oblique* ($\pi/2 < \beta < \pi$): For obtuse values of β, the force is repulsive and non-central [Fig. 3(b) C, D, G, and H; Fig. 3(c) D, H; Fig. 3(d) A, D, E, and H; Fig. 3(e) B, D, F, and H]. The force acts to increase the negative product $\mathbf{m}_2 \cdot \mathbf{B}_1 = -mB_1|\cos\beta|$ (that is, to bring it closer to zero) by moving $\mathbf{m}_2$ into regions where the field is weaker (smaller B_1) and better aligned with $\mathbf{m}_2$ (β approaching $\pi/2$ from above, smaller $|\cos\beta|$). The torque acts to increase $\mathbf{m}_2 \cdot \mathbf{B}_1$ by rotating $\mathbf{m}_2$ into alignment with the local field.

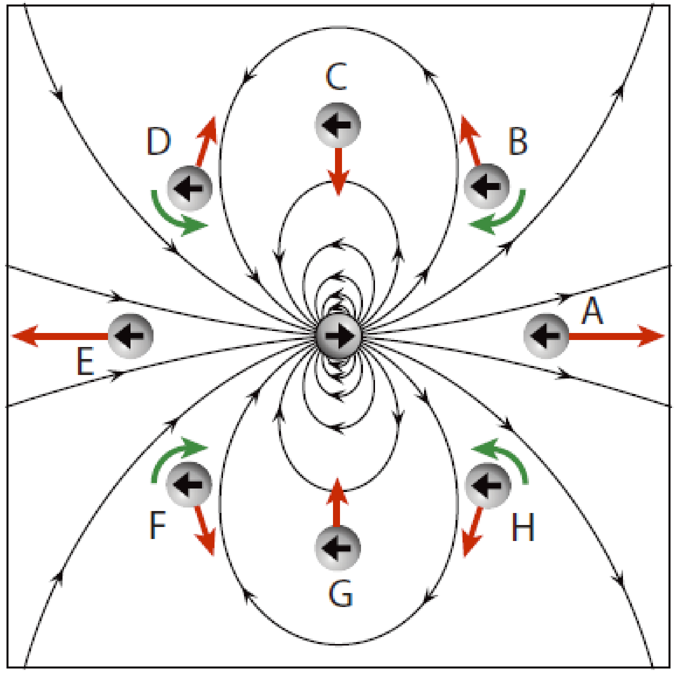
(d)



Rotation的进一步计算

5. *Repulsive, Central* ($\beta = \pi$): The force is repulsive and central when $\mathbf{m}_1$ and $\mathbf{m}_2$ are parallel to each other and perpendicular to the line through their centers [Fig. 3(a) C, G], and when $\mathbf{m}_1$ and $\mathbf{m}_2$ are antiparallel and collinear [Fig. 3(e) A, E]. In both cases, $\mathbf{m}_2$ is antiparallel to $\mathbf{B}_1$ ($\beta = \pi$), giving $\boldsymbol{\tau}_{12} = 0$ and $\mathbf{m}_2 \cdot \mathbf{B}_1 = -mB_1$. To increase this negative product, $\mathbf{m}_1$ repels $\mathbf{m}_2$ into regions where the field is weaker.

(e)



Rotation的进一步计算

在此基础上，考虑non-conservative forces，即桌面上滑动的磁铁受到的动摩擦力f_t，磁场中移动导电磁铁受到的涡流力F^{eddy}，磁球间摩擦力f_m，另一磁球所给的正压力F_N。

$$\overrightarrow{f_t} = -\mu m g \hat{v},\ \overrightarrow{f_m} = -\mu m F_N \hat{v}_t.$$

$$\Rightarrow \overrightarrow{F_{eddy}} = -\overrightarrow{v} B^2 \int \sigma \mathrm{d}V = -\frac{4}{3}\pi a^3 \bar{\sigma} B^2 \overrightarrow{v}.$$

$$\text{对于正压力(用力矩解)}, F_N = -3U - \dot{\phi}/10.$$

Rotation的进一步计算 2D

- Expands the previous magnetic vector expression with the expression of the polar coordinates

$$\mathbf{B}_1 = \frac{1}{12r^3}\left[(1 + 3\cos 2\theta)\,\hat{\mathbf{x}} + 3\sin 2\theta\,\hat{\mathbf{y}}\right], \tag{34}$$

which has magnitude

$$B_1(r,\theta) = \frac{1}{12r^3}\left(10 + 6\cos 2\theta\right)^{1/2} \tag{35}$$

and direction ϕ_m (measured counterclockwise from the $+x$ direction) given by

$$\tan\phi_m = \frac{3\sin 2\theta}{1 + 3\cos 2\theta}. \tag{36}$$

Rotation的进一步计算 3D

$$\vec{B} = \nabla \times \vec{A} = \frac{1}{r^2 \sin\theta}\begin{vmatrix} \hat{r} & r\hat{\theta} & r\sin\theta\hat{\phi} \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial \phi} \\ A_r & rA_\theta & r\sin\theta A_\phi \end{vmatrix} = \frac{1}{r^2 \sin\theta}\begin{vmatrix} \hat{r} & r\hat{\theta} & r\sin\theta\hat{\phi} \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial \phi} \\ 0 & 0 & r\sin\theta\frac{\mu_0 M}{4\pi}\frac{\sin\theta}{r^2} \end{vmatrix}$$

$$= \frac{\mu_0 M}{4\pi}\frac{1}{r^2 \sin\theta}\begin{vmatrix} \hat{r} & r\hat{\theta} & r\sin\theta\hat{\phi} \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial \phi} \\ 0 & 0 & \frac{\sin^2\theta}{r} \end{vmatrix}$$

$$= \frac{\mu_0 M}{4\pi}\frac{1}{r^2 \sin\theta}\left[\hat{r}\frac{\partial}{\partial\theta}\left(\frac{\sin^2\theta}{r}\right) - r\hat{\theta}\frac{\partial}{\partial r}\left(\frac{\sin^2\theta}{r}\right)\right]$$

$$= \frac{\mu_0 M}{4\pi}\frac{1}{r^2 \sin\theta}\left[\hat{r}\left(\frac{2\sin\theta\cos\theta}{r}\right) + r\hat{\theta}\left(\frac{\sin^2\theta}{r^2}\right)\right]$$

$$= \frac{\mu_0 M}{4\pi}\frac{1}{r^3}[\hat{r}(2\cos\theta) + \hat{\theta}(\sin\theta)]$$

$$\vec{B}(r,\theta) = \sum_i \frac{\mu_0 M}{4\pi}\frac{1}{r_i^3}[\hat{r}(2\cos\theta_i) + \hat{\theta}(\sin\theta_i)] \qquad \square.$$

Rotation的进一步计算

分别求$\overrightarrow{F}_{12}$，则不难得到$\overrightarrow{F}_{net} = \overrightarrow{F}_{12} + \overrightarrow{f}_t + \overrightarrow{f}_m + \overrightarrow{F}_{eddy} + \overrightarrow{F}_N$

$$= -\nabla(-\overrightarrow{m}_2 \cdot \overrightarrow{B}) - (\mu m g + \frac{4}{3}\pi a^3 \bar{\sigma} B^2 v)\hat{v} - \mu m(-3U - \dot{\phi}/10)\hat{v}_t - (3U + \dot{\phi}/10)\hat{v}_n.$$

建模

现有的永磁模型（Comsol官网）：

$1.tip\ defelction\ in\ degrees\ \mathbf{vs.}\ magnetic\ field\ in\ milliTesla\ graph;$

$2.F\ \mathbf{vs.}\ R,\theta,\phi\ graph;$

$3.r\theta\phi\ frame\ of\ the\ normal\ magnetic\ field;$

$4.xyz\ frame\ of\ the\ normal\ magnetic\ field;$

$5.free\ body\ diagram\ of\ magnet\ spheres;$

$6.etc.$

建模

- Comsol, Matlab:

1. 两软件都有装机（comsol: wgy; matlab: lxy);
2. 微分方程较少或简单，直接上Comsol，更形象更易上手;
3. 如果涉及较多的方程或空间位置函数，建议用Matlab（暴算小能手）；

物理现象：这个对工程师来说是直观的物理现象和物理量，温度多少度，载荷是多大等等。通常来说，用户界面中呈现的、用户对工程问题进行设置时输入的都是此类信息。

数学方程：将物理现象翻译成相应的数学方程，例如流体对应的是NS方程，传热对应的是传热方程等等；大部分描述这些现象的方程在空间上都是偏微分方程，偶尔也有ODE（如粒子轨迹、化学反应等）。在这个层面，软件把物理现象“翻译”为以解析式表示的数学模型。

数值模型：在定义了数学模型，并执行了网格剖分后，商业软件会将数学模型离散化，利用有限元方法、边界元法、有限差分法、不连续伽辽金法等方法生成数值模型。软件会组装并计算方程组雅可比矩阵，并利用求解器求解方程组。这个层面的计算通常是隐藏在后台的，用户只能通过一些求解器的参数来干预求解。

（Matlab 主要集中在3.的求解过程，以计算为主，不包括上面提到的方程的翻译、准备、离散化等等；Comsol是商业有限元软件，包含所有三个层面的工具）