# ARML Handbook

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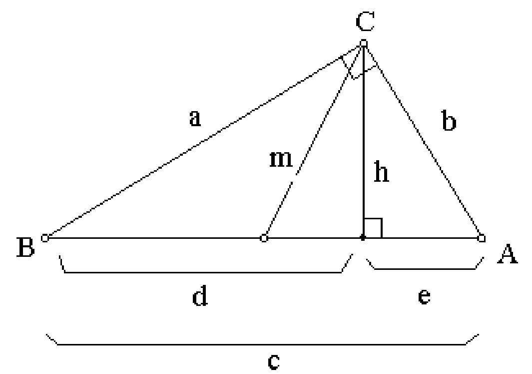
# Triangles

## Right Triangles




Pythagorean Theorem: $a^2 + b^2 = c^2$

Geometric relationships:

$h^2 = de$

$a^2 = dc$

Median to hypotenuse:

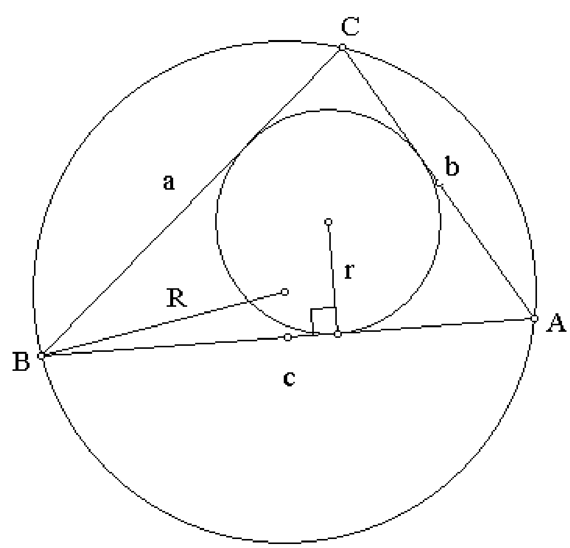
$m = \frac{c}{2}$

## General Triangles

Law of Sines: $\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$

Law of Cosines: $c^2 = a^2 + b^2 - 2ab\cos C$

In the following formulas, the semiperimeter is $s = \frac{a+b+c}{2}$, $K$ is the area of the triangle, $r$ is the radius of the inscribed circle, and $R$ is the radius of the circumscribed circle.




Area: $K = \frac{1}{2}ab\sin C = \frac{1}{2}h_c c$

Heron's Formula: $K = \sqrt{s(s-a)(s-b)(s-c)}$

Inscribed radius: $r = \frac{K}{s}$

Circumscribed radius: $R = \frac{abc}{4K}$, $R = \frac{c}{2\sin C}$

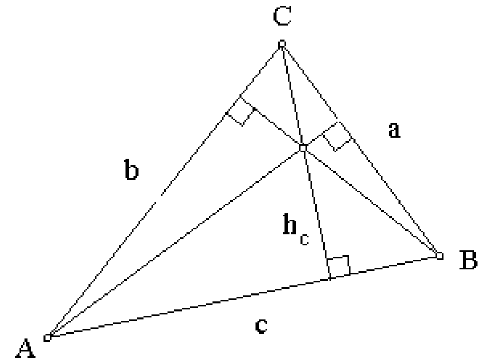
# Cevians

A cevian is any segment drawn from the vertex of a triangle to the opposite side. Cevians with special properties include altitudes, angle bisectors, and medians. Let $h_c$, $t_c$, and $m_c$ represent the altitude, angle bisector, and median to side c, respectively.

**Altitudes**:

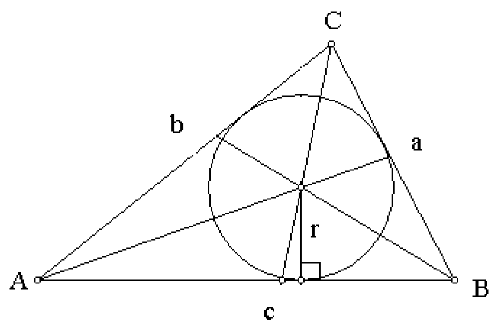
The altitudes of a triangle intersect at the *orthocenter*.

$$h_c = a\sin B \qquad\qquad h_c = \frac{2K}{c}$$



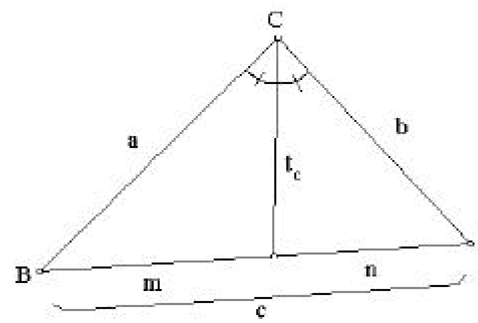

**Angle Bisectors:**

The angle bisectors of a triangle intersect at the *incenter*, the center of the triangle's inscribed circle.



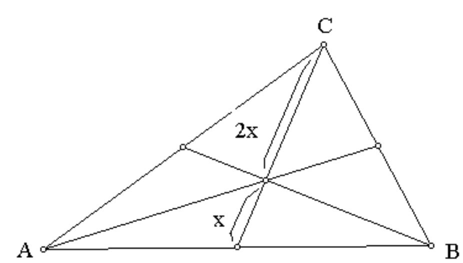

Angle Bisector Theorem: $\frac{a}{m} = \frac{b}{n}$

Length of an Angle Bisector: $t_c = \sqrt{ab\left(1 - \frac{c^2}{a^2+b^2}\right)}$

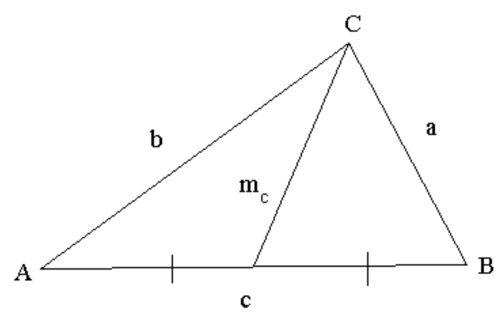



**Medians:**

The medians of a triangle intersect at the *centroid*. Along the median, the distance from a vertex to the centroid is twice the distance from the centroid to the opposite side.



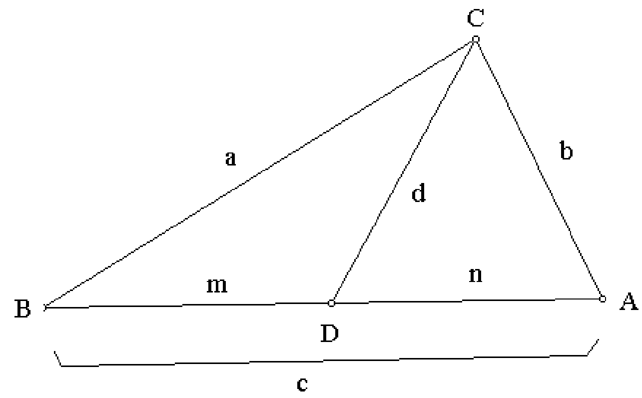

Length of a Median: $m_c = \sqrt{\frac{a^2}{2} + \frac{b^2}{2} - \frac{c^2}{4}}$



**Stewart's Theorem**

If a cevian of length $d$ is drawn and divides side $c$ into segments $m$ and $n$, then

$$a^2 n + b^2 m = c(d^2 + mn)$$

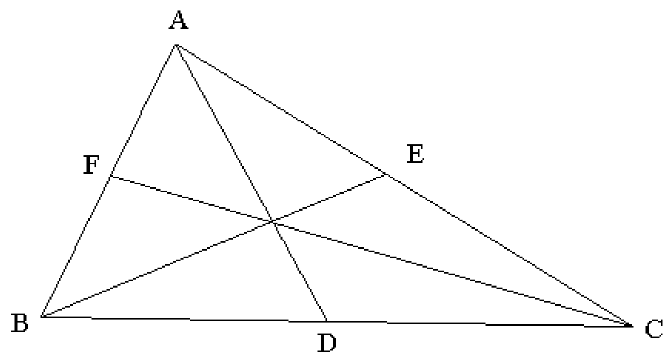



**Ceva's Theorem**

A necessary and sufficient condition for AD, BE, CF, where D, E, and F are points on the respective side lines BC, CA, AB of a triangle ABC, to be concurrent is that

$$BD \cdot CE \cdot AF = +DC \cdot EA \cdot FB$$

where all segments in the formula are directed segments.



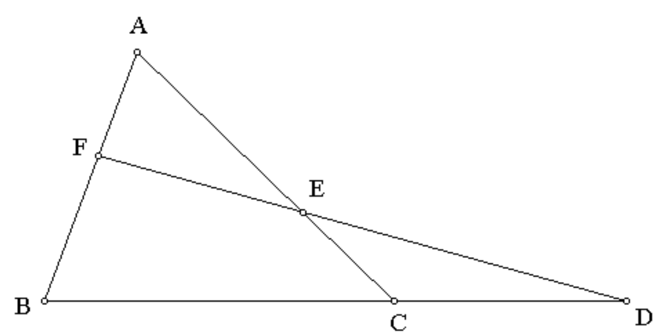

Ex. Suppose AB, AC, and BC have lengths 13, 14, and 15. If AF:FB = 2:5 and CE:EA = 5:8. If BD = $x$ and DC = $y$, then $10x = 40y$, and $x + y = 15$. Solving, we have $x = 12$ and $y = 3$.

**Menelaus' Theorem**

A necessary and sufficient condition for points D, E, F on the respective side lines BC, CA, AB of a triangle ABC to be collinear is that

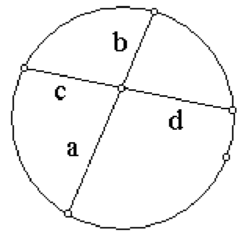
$$BD \cdot CE \cdot AF = -DC \cdot EA \cdot FB$$

where all segments in the formula are directed segments.



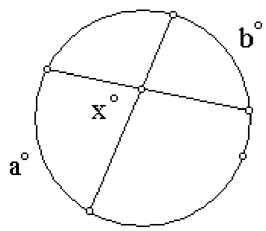
# Circles

**Two intersecting chords**: $ab = cd$

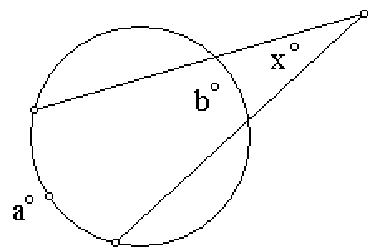



**Angle measurements:**

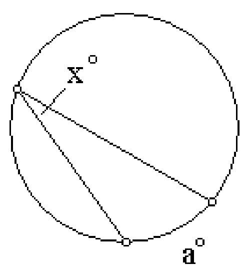
Intersecting chords: $x = \dfrac{a+b}{2}$




Two secants: $x = \dfrac{a-b}{2}$

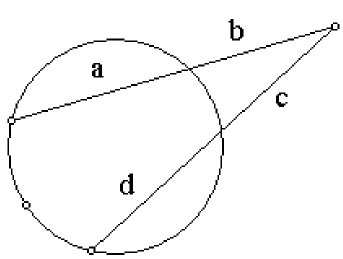



**Inscribed angles:** $x = \dfrac{a}{2}$

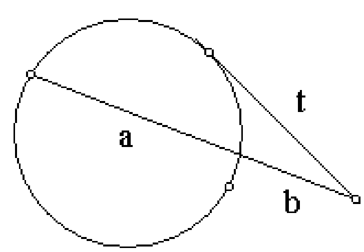



**Secants and Tangents**:

Two secants: $b(a+b) = c(c+d)$

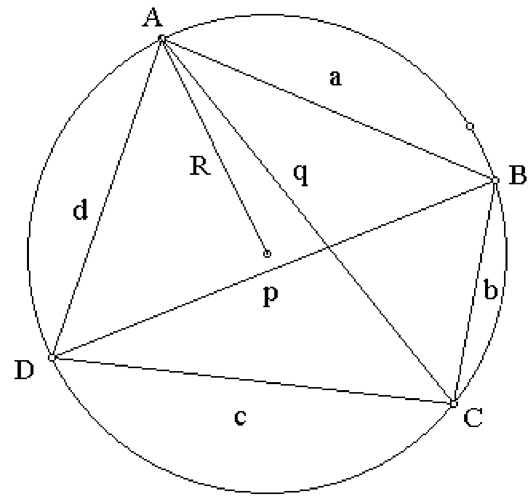



Secant and tangent: $t^2 = b(a+b)$



# Cyclic Quadrilaterals

A quadrilateral is cyclic if the quadrilateral can be inscribed in a circle.




$$A + C = B + D = 180°$$

If the quadrilateral has sides $a$, $b$, $c$, $d$, the semiperimeter $s = \frac{a+b+c+d}{2}$. Let $R$ be the radius of the circumscribed circle and let the diagonals be $p$ and $q$.

**Brahmagupta's formula:** $K = \sqrt{(s-a)(s-b)(s-c)(s-d)}$

**Radius of circumscribed circle:** $R = \frac{\sqrt{(ac+bd)(ad+bc)(ab+cd)}}{4K}$

**Ptolemy's Theorem:** A convex quadrilateral with consecutive sides $a$, $b$, $c$, $d$ and diagonals $p$, $q$ is cyclic if and only if

$$ac + bd = pq.$$

# Regular Polygons

For the following formulas, $n$ is the number of sides in the polygon and $s$ is the length of each side. $q$ is the measure of one of the interior angles. The radius of the inscribed circle is $r$, and the radius of the circumscribed radius is $R$.

**Sum of interior angles:** 180(n-2)°

**Interior angle measure:** $q = \frac{180(n-2)}{n}$

**Area:** $K = \frac{rns}{2}$

| **Polygon** | ***n*** | ***K*** | ***r*** | ***R*** |
|---|---|---|---|---|
| Triangle | 3 | $\frac{s^2\sqrt{3}}{4}$ | $\frac{s\sqrt{3}}{6}$ | $\frac{s\sqrt{3}}{3}$ |
| Square | 4 | $s^2$ | $\frac{s}{2}$ | $\frac{s}{2}\sqrt{2}$ |
| Hexagon | 6 | $\frac{3s^2\sqrt{3}}{2}$ | $\frac{s\sqrt{3}}{2}$ | $s$ |
| Octagon | 8 | $2s^2(1+\sqrt{2})$ | $s\sqrt{\left(1+\frac{\sqrt{2}}{2}\right)}$ | $\frac{s\sqrt{3-2\sqrt{2}}}{2}$ |

# Solid Geometry

**Spheres**

Surface area: $4pr^2$

Volume: $\frac{4}{3}pr^3$

**Cylinders**

Lateral area: $2pRh$

Total surface area: $2pR(R+h)$

Volume: $pr^2h$

**Right Circular Cone**

Slant height: $s=\sqrt{R^2+h^2}$

Lateral Area: $pRs$

Volume: $\frac{pR^2h}{3}$

**Frustrums**

For a frustrum with height $h$ and base areas $B_1$ and $B_2$,

Volume: $V=\frac{1}{3}h\left(B_1+B_2+\sqrt{B_1B_2}\right)$

**Regular Polyhedra**

Let $v$ = number of vertices, $e$ = number of edges, $f$ = number of faces, $a$ = length of each edge, $A$ = area of each face, $r$ and $R$ the radii of the inscribed and circumscribed spheres, respectively, and $V$ = volume.

| **Name** | ***v*** | ***e*** | ***f*** | ***A*** | ***r*** | ***R*** | ***V*** |
|---|---|---|---|---|---|---|---|
| Tetrahedron | 4 | 6 | 4 | $\frac{a^2\sqrt{3}}{4}$ | $\frac{a\sqrt{6}}{12}$ | $\frac{a\sqrt{6}}{4}$ | $\frac{a^3\sqrt{2}}{12}$ |
| Hexahedron | 8 | 12 | 6 | $a^2$ | $\frac{a}{2}$ | $\frac{a\sqrt{3}}{2}$ | $a^3$ |
| Octahedron | 6 | 12 | 8 | $\frac{a^2\sqrt{3}}{4}$ | $\frac{a\sqrt{6}}{6}$ | $\frac{a\sqrt{2}}{2}$ | $\frac{a^3\sqrt{2}}{3}$ |

| **Name** | ***v*** | ***e*** | ***f*** | ***A*** | ***r*** |
|---|---|---|---|---|---|
| Dodecahedron | 20 | 30 | 12 | $\frac{1}{4}a^2\sqrt{25+10\sqrt{5}}$ | $\frac{1}{20}a\sqrt{250+110\sqrt{5}}$ |

| ***R*** | ***V*** |
|---|---|
| $\frac{1}{4}a\left(\sqrt{15}+\sqrt{3}\right)$ | $\frac{1}{4}a^3\left(15+7\sqrt{5}\right)$ |

| **Name** | ***v*** | ***e*** | ***f*** | ***A*** | ***r*** |
|---|---|---|---|---|---|
| Icosahedron | 12 | 30 | 20 | $\frac{a^2\sqrt{3}}{4}$ | $\frac{1}{12}a\sqrt{42+18\sqrt{5}}$ |

| ***R*** | ***V*** |
|---|---|
| $\frac{1}{4}a\sqrt{10+2\sqrt{5}}$ | $\frac{5}{12}a^3\left(3+\sqrt{5}\right)$ |

For any convex polyhedron:

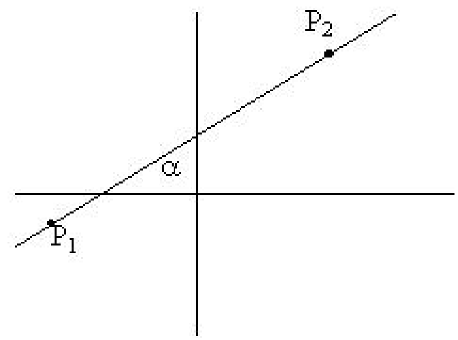
**Euler's Formula:** $v - e + f = 2$

# Analytic Geometry

**Points and Lines:**

For any points $P_1$ $(x_1, y_1)$ and $P_2$ $(x_2, y_2)$ in a rectangular coordinate plane,

Distance between $P_1$ and $P_2$ : $\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$




Slope $m$ of $P_1$ and $P_2$ : $m = \dfrac{y_2 - y_1}{x_2 - x_1} = \tan a$

Angle $q$ between two lines of slopes $m_1$ and $m_2$ : $\tan q = \dfrac{m_2 - m_1}{1 + m_1 m_2}$

Distance from $Ax + By + C = 0$ to $P_1$ : $\left| \dfrac{Ax_1 + Bx_2 + C}{\sqrt{A^2 + B^2}} \right|$

**Triangles:**

For a triangle with vertices $P_1$ $(x_1, y_1)$ , $P_2$ $(x_2, y_2)$ and $P_3$ $(x_3, y_3)$,

Area: $\dfrac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$

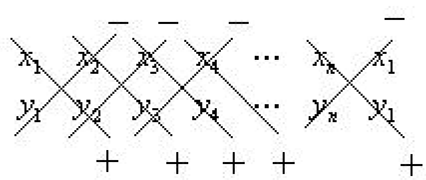
Coordinates of Centroid: $\left( \dfrac{x_1 + x_2 + x_3}{3}, \dfrac{y_1 + y_2 + y_3}{3} \right)$

**Polygons:**

Area of Polygon $P_1\ P_2 \ldots P_n$ :

$$\frac{1}{2}\left(x_1y_2 + x_2y_3 + \cdots x_{n-1}y_n + x_ny_1 - y_1x_2 - y_2x_3 - \ldots - y_{n-1}x_n - y_nx_1\right)$$

This is the sums of the products of the coordinates on lines slanting downward minus the products of the coordinates on lines slanting upwards (like a 3 by 3 determinant).




**Pick’s Theorem:**

For any polygon whose vertices are lattice points, the area is given by $K = \frac{1}{2}B + I - 1$, where $B$ is the number of lattice points on the boundary of the polygon and $I$ is the number of lattice points in the interior.

# Algebra

## Series

### Arithmetic series:

$a_1$ = 1st term, $a_n$ = $n$th term, $d$ = common difference

$n$th term: $a_n = a_1 + (n-1)d$

Sum of 1st $n$ terms: $\sum_{k=1}^{n} a_k = \frac{n}{2}\left(2a_1 + (n-1)d\right) = \frac{n}{2}\left(a_1 + a_n\right)$

Arithmetic mean of $a$ and $b$: $\frac{a+b}{2}$

### Geometric series:

$a_1$ = 1st term, $a_n$ = $n$th term, $r$ = common ratio

$n$th term: $a_n = ar^{n-1}$

Sum of 1st $n$ terms: $\sum_{k=1}^{n} a_k = \frac{a_1(r^n - 1)}{r-1}$

Sum of infinite geometric sequences ($|r| < 1$): $\sum_{k=1}^{\infty} a_k \frac{a}{1-r}$

Geometric mean of $a$ and $b$: $\sqrt{ab}$

**Harmonic series:** A sequence of numbers is harmonic if the reciprocals of the numbers form an arithmetic progression.

Harmonic mean of $a$ and $b$: $\left(\frac{\frac{1}{a}+\frac{1}{b}}{2}\right)^{-1} = \frac{2ab}{a+b}$

If $A$, $G$, and $H$ respectively represent the arithmetic, geometric, and harmonic mean between $a$ and $b$, then $G^2 = AH$.

**Binomial Theorem:** $(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k$ , where $\binom{n}{k} = \frac{n!}{k!(n-k)!}$

When $x$ and $y$ are replaced by probabilities $p$ and $q$, with $p+q=1$, the term at $k$ represents the probability of getting $k$ successful events with probability $q$ and the other $n$-$k$ events of probability $p$.

**Sum of binomial coefficients:** replace $x$ and $y$ both with 1, $\sum_{k=0}^{n} \binom{n}{k} = 2^n$ .

**Multinomial expansions:**

For an expansion $(x_1 + x_2 + \cdots x_k)^n$ , the coefficient of the term $x_1^{e_1} x_2^{e_2} \cdots x_k^{e_k}$ is $\frac{n!}{e_1!e_2!\cdots e_k!}$ , where $\sum_{i=1}^{k} e^i = n$ .

**Other series:**

$$1+2+3+\cdots n-1+n = \frac{n(n+1)}{2}$$

$$1+2^2+3^2+\cdots(n-1)^2+n^2 = \frac{n(n+1)(2n+1)}{6}$$

$$1+2^3+3^3+\cdots(n-1)^3+n^3 = \left(\frac{n(n+1)}{2}\right)^2$$

## Functions

**Even Functions:** A function is even if for all $x$, $f(x)=f(-x)$. This is the same as symmetry about the y-axis.

**Odd Functions:** A function is odd if for all $x$, $f(x) = -f(-x)$. This is the same as symmetry about the origin.

**Inverse Trig Functions:**

| Function | Domain | Range |
|---|---|---|
| $Sin^{-1}(x)$ | $-1 \le x \le 1$ | $-\frac{p}{2} \le Sin^{-1}(x) \le \frac{p}{2}$ |
| $Cos^{-1}(x)$ | $-1 \le x \le 1$ | $0 \le Cos^{-1}(x) \le p$ |
| $Tan^{-1}(x)$ | $-\infty \le x \le \infty$ | $-\frac{p}{2} \le Tan^{-1}(x) \le \frac{p}{2}$ |

**Logarithmic Functions:**

$$\log_b x + \log_b y = \log_b xy \qquad\qquad \log_b x - \log_b y = \log_b \frac{x}{y}$$

$$\log_b x^y = y \log_b x \qquad\qquad \log_b x = \frac{\log_n x}{\log_n b}$$

**Trig Functions:** For the general sine or cosine function,

$A \sin B(x + C) + D$, $|A|$ is the amplitude, $\dfrac{2p}{B}$ is the period, -*C* is the phase shift, and *D* is the vertical shift.

**Greatest Integer Function:** *[x]* is the greatest integer that does not exceed the real number *x*.

$$[x] \leq x < [x] + 1 \qquad\qquad x - 1 < [x] \leq x$$

$$0 \leq x - [x] < 1 \qquad\qquad -x - 1 < [-x] \leq -x$$

If n is an integer,

$$[x + n] = [x] + n \qquad\qquad \left[\frac{[x]}{n}\right] = \left[\frac{x}{n}\right]$$

$$[x] + [y] \leq [x + y] \leq [x] + [y] + 1$$

## Complex Numbers

Complex numbers $x + yi$ can be written in the form $r(\cos q + i \sin q)$, where $r = \sqrt{x^2 + y^2}$ and $q = \tan^{-1}\left(\dfrac{y}{x}\right)$.

**Multiplication and Division:** For two complex numbers $z_1$ and $z_2$:

$$z_1 z_2 = r_1 r_2[\cos(q_1 + q_2) + i \sin(q_1 + q_2)] \qquad\qquad \frac{z_1}{z_2} = \frac{r_1}{r_2}[\cos(q_1 - q_2) + i \sin(q_1 - q_2)]$$

**DeMoivre's Theorem**: $z^n = [r(\cos q + i \sin q)]^n = r^n(\cos nq + i \sin nq)$

This can be used to find the roots of complex numbers by making *n* the appropriate fraction.

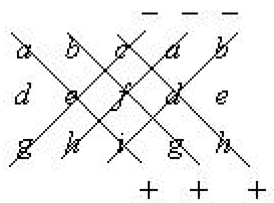
The *n*th roots of 1 when graphed in the complex plane form the *n* vertices of a regular polygon with *n* sides, inscribed the circle with $r = 1$.

## Matrices

For a 2 by 2 matrix, $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$,

Determinant: $\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$ Inverse: $\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \dfrac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

For a 3 by 3 matrix $\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$,

Determinant: 

$= aei + bfg + cdh - bdi - afh - ceg$

Inverses of all $n$ by $n$ matrices can be found using row reduction. Write the identity matrix to the right of the original matrix, and multiply and/or add rows to obtain the identity matrix on the left side.

$$\begin{bmatrix} a & b & c & 1 & 0 & 0 \\ d & e & f & 0 & 1 & 0 \\ g & h & i & 0 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & a' & b' & c' \\ 0 & 1 & 0 & d' & e' & f' \\ 0 & 0 & 1 & g' & h' & i' \end{bmatrix}$$

# Polynomials

For all polynomials of the form $P(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots a_1 x + a_0$, where $a_i \in R$:

**Fundamental Theorem of Algebra**: *P(x)* has *n* roots

**Sum of roots:** $-\frac{a_{n-1}}{a_n}$

**Product of roots:** $\frac{a_0}{a_n}(-1)^n$

For any $a_k$, $\frac{a_k}{a_n}(-1)^{n+k}$ represents the sum of the product of the roots, taken $(n-k)$ at a time.

Ex. when $n=3$, $\frac{a_1}{a_3}(-1)^{3+1}$ is the sum of product of the roots, taken *3-1* or *2* at a time. $\frac{a_1}{a_3} = (r_1 r_2 + r_2 r_3 + r_3 r_1)$, where $r_1$, $r_2$, and $r_3$ are the roots of the polynomial.

**Remainder Theorem:**

The remainder when *P(x)* is divided by *(x – w)* is *P(w)*.

**Descartes' Rule of Signs:**

The number of positive real roots of *P(x)* is *z* decreased by some multiple of two, *(z, z-2, z-4, etc…)*. *z* is the number of sign changes in the coefficients of *P(x)*, counting from $a_n$ to $a_0$. The number of negative real roots is found similarly by finding *z* for *P(-x)*.

Ex. For the polynomial $x^5 - 4x^4 + 3x^2 - 6x + 1$, there are possibly *4, 2,* or *0* positive roots and *1* negative root.

**Rational Root Theorem:**

If all $a_i$ are integers, then the only possible rational roots of *P(x)* are of the form $\pm\frac{k}{a_n}$, where *k* is a factor of $a_0$.

# Factors and Expansions

$$(a \pm b)^2 = a^2 \pm 2ab + b^2$$

$$(a \pm b)^3 = a^3 \pm 3a^2b + 3ab^2 \pm b^3$$

$$(a \pm b)^4 = a^4 \pm 4a^3b + 6a^2b^2 \pm 4ab^3 + b^4$$

$$a^2 - b^2 = (a-b)(a+b)$$

$$a^3 - b^3 = (a-b)(a^2 + ab + b^2)$$

$$a^3 + b^3 = (a+b)(a^2 - ab + b^2)$$

$$a^4 + b^4 = (a^2 + ab\sqrt{2} + b^2)(a^2 - ab\sqrt{2} + b^2)$$

$$a^n - b^n = (a-b)(a^{n-1} + a^{n-2}b + \ldots + b^{n-1})$$

$$a^n - b^n = (a+b)(a^{n-1} - a^{n-2}b + \ldots - b^{n-1}) \quad \text{for even values of } n$$

$$a^n + b^n = (a+b)(a^{n-1} - a^{n-2}b + \ldots + b^{n-1}) \text{ for odd values of } n$$

$$a^4 + a^2b^2 + b^4 = (a^2 + ab + b^2)(a^2 - ab + b^2)$$

$$(a+b+c)^2 = a^2 + b^2 + c^2 + 2ab + 2bc + 2ac$$

$$(a+b+c)^3 = a^3 + b^3 + c^3 + 3a^2(b+c) + 3b^2(a+c) + 3c^2(a+b) + 6abc$$

$$a^3 + b^3 + c^3 = (a+b+c)^3 - 3(a+b)(b+c)(a+c)$$

# Discrete Mathematics

## Combinatorics

**Counting principle:** If a choice consists of $k$ steps, of which the first can be made in $n_1$ ways, the second in $n_2$ ways, … , and the $k^{\text{th}}$ in $n_k$ ways, then the whole choice can be made in $n_1 n_2 \ldots n_k$ ways.

**Factorials:** $n! = 1 \cdot 2 \cdot 3 \cdots (n-1) \cdot n$

**Permutations:** A permutation is an arrangement of objects where order matters. (123 and 213 are considered different permutations of the digits 1, 2, and 3).

${}_nP_r$ is the number of permutations of $r$ objects chosen from $n$ objects.

$$ {}_nP_r = \frac{n!}{(n-r)!} $$

Special cases: there are $n!$ ways of arranging all $n$ objects.

Repeated objects: In an arrangement of $n$ objects, if there are $r_1$ objects of type $1$, $r_2$ objects of type $2$, … $r_k$ objects of type $k$, where objects of the same type are indistinguishable, then there are $\frac{n!}{r_1!r_2!\cdots r_k!}$ ways to arrange the $n$ objects.

Circular Permutations: If $n$ objects are arranged in a circle, there are $(n-1)!$ possible arrangements.

"Key-ring" permutations: If $n$ objects are arranged on a key ring, there are $\frac{(n-1)!}{2}$ possible arrangements.

**Combinations:** In a combination, the order of objects does not matter (123 is the same as 213).

${}_nC_r$ is the number of combinations of $r$ objects chosen from $n$ objects.

$$ {}_nC_r = \binom{n}{r} = \frac{n!}{r!(n-r)!} $$

## Sets:

For sets $A$ and $B$,

**Union**: $A \bigcup B$ is the set that contains the elements in either $A$, $B$, or both.
**Intersection**: $A \bigcap B$ is the set that contains only elements that are in both $A$ and $B$.
**Complement:** $A'$ is the set of all elements not in $A$.
**Inclusion-Exclusion principle:** If $n(S)$ is the number of elements in set $S$, then

$$n(A \bigcup B) = n(A) + n(B) - n(A \bigcap B).$$

This can be extended for more than two sets. (ex. For sets $A$, $B$, and $C$, $n(A \bigcup B \bigcup C) = n(A) + n(B) + n(C) - n(A \bigcap B) - n(B \bigcap C) - n(A \bigcap C) + n(A \bigcap B \bigcap C)$.

## Probability:

If an experiment can occur in exactly $n$ ways, and if $m$ of these correspond to an event $E$, then the probability of $E$ is given by

$$P(E) = \frac{m}{n}$$

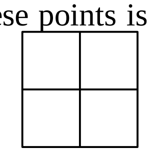
$P(A$ and $B)$ = $P(A \cap B) = P(A)P(B)$ if A and B are independent events.
$P(A$ or $B)$ = $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

Conditional Probability: the conditional probability of an event $E$, given an event $F$, is denoted by $P(E/F)$ and is defined as $P(E / F) = \dfrac{P(E \cap F)}{P(F)}$.

Pigeonhole principle: If there are more than $k$ times as many pigeons as pigeonholes, then some pigeonhole must contain at least $k+1$ pigeons. Or, if there are $m$ pigeons and $n$ pigeonholes, then at least one pigeonhole contains at least $\left\lfloor \dfrac{m-1}{n} \right\rfloor + 1$ pigeons.

Ex. Consider any five points $P_1$, $P_2$, $P_3$, $P_4$, and $P_5$ in the interior of a square $S$ with side length $1$. Denote by $d_{ij}$ the distance between points $P_I$ and $P_j$. Prove that at least one of the distances between these points is less than $\dfrac{\sqrt{2}}{2}$.

*Solution:* Divide $S$ into four congruent squares. By the pigeonhole principle, two points belong to one of these squares (a point on the boundary can be claimed by both squares). The distance between these points is less than $\dfrac{\sqrt{2}}{2}$. (Problem and solution from Larson, number 2.6.2).



# Number Theory

**Figurate Numbers:**

Triangular: 1, 3, 6, 10, … $\frac{1}{2}n(n+1)$

Square: 1, 4, 9, 16, … $n^2$

Pentagonal: 1, 5, 12, 22, 35, … $\frac{1}{2}(3n^2 - n)$

K-gonal: 1, k … $\frac{1}{2}k(n^2 - n) - n^2 + 2n$

**Pythagorean triples:** These take the form of $M^2 - N^2$, $2MN$, and $M^2 + N^2$. The product of the sides is always divisible by 60.

**Primes:**

Mersenne: primes of the form $2^p - 1$, where *p* is a known prime. Not all numbers of this form are prime.

Fermat: primes of the form $2^{2^n} + 1$. The only primes of this form found so far are for $n = 0$ through *4*.

Gauss: A regular polygon can only be constructed if the number of vertices is a Fermat prime or the product of distinct Fermat primes.
(Ex. n = 3, 5, or 15 = 3*5). Note: once any n-gon has been constructed, one can easily construct the 2n-gon.

Neighbors of Six: All primes must be in the form *6n+1* or *6n-1* (after 2 and 3)

**Composite Numbers:**

**Fundamental Theorem of Arithmetic:** every integer greater than 1 has a unique factorization into prime factors.

For an integer *n* greater than 1, let the prime factorization be $n = p_1^{e_1} p_2^{e_2} \cdots p_k^{e_k}$

Number of divisors: $d(n) = (e_1 + 1)(e_2 + 1) \cdots (e_k + 1)$

Sum of divisors: $s(n) = \left(\frac{p_1^{e_1+1} - 1}{p_1 - 1}\right)\left(\frac{p_2^{e_2+1} - 1}{p_2 - 1}\right)\cdots\left(\frac{p_n^{e_n+1} - 1}{p_n - 1}\right)$

Any number *n* such that *d(n)* is odd is a perfect square.
If *s(n)=2n*, then *n* is a perfect number.
If $2^p - 1$ is a prime (Mersenne), then $2^{p-1}(2^p - 1)$ is a perfect number.

**Congruences:**

For any integers a, b, and positive integer m, a is congruent to b modulo m if a – b is divisible by m. This is represented by

$$a \equiv b \pmod{m}$$

This is equivalent to saying $a - b = mk$ for some integer k.

For any integers a, b, c, and positive integers m,

Reflexive property: $a \equiv a \pmod{m}$

Symmetric property: If $a \equiv b \pmod{m}$, then $b \equiv a \pmod{m}$.

Transitive property: If $a \equiv b \pmod{m}$ and $b \equiv c \pmod{m}$, then $a \equiv c \pmod{m}$

For any integers a, b, c, d, k, and m, with m > 0, if $a \equiv b \pmod{m}$ and $c \equiv d \pmod{m}$:

(i) $a \pm k \equiv b \pm k \pmod{m}$

(ii) $ak \equiv bk \pmod{m}$

(iii) $a \pm c \equiv b \pm d \pmod{m}$

(iv) $ac \equiv bd \pmod{m}$

(v) $a^k \equiv b^k \pmod{m}$

If $ac \equiv bc \pmod{m}$, then $a \equiv b \pmod{m}$ only if $m$ and c are relatively prime.

**Fermat's Little Theorem:** For any integer $a$ and prime $p$, where $a$ and $p$ are relatively prime, $a^{p-1} \equiv 1 \pmod{p}$.

**Wilson's Theorem:** An integer $p$ is prime if and only if $(p-1)! \equiv -1 \pmod{p}$.

**Linear Diophantine Equations:** The equation $ax + by = c$ has infinitely many solutions for integral $x$ and $y$ if the greatest common divisor of $a$ and $b$ divides $c$. If this condition is not satisfied there are no possible solutions.

**Divisibility Rules**

Let n be represented by the digits $\bar{d}_n\bar{d}_{n-1}\cdots\bar{d}_2\bar{d}_1$. $a \mid b$ means that $a$ divides into $b$, or that $a$ is a factor of $b$.

**3:** A number is divisible by 3 if the sum of its digits is divisible by 3. $3 \mid n$ if $3 \mid \sum_{k=1}^{n} d_k$.

**4:** A number is divisible by 4 if the number represented by the last two digits is divisible by 4. $4 \mid n$ if $4 \mid 10d_2 + d_1$. This can be reduced to $4 \mid n$ if $4 \mid 2d_2 + d_1$.

**6:** check for divisibility by both 2 and 3.

**8:** A number is divisible by 8 if the number represented by the last three digits is divisible by 8. $8 \mid n$ if $8 \mid 100d_3 + 10d_2 + d_1$. More specifically, $8 \mid n$ if $8 \mid 4d_3 + 2d_2 + d_1$.

**9:** A number is divisble by 9 if the sum of its digits is divisible by 9. $9 \mid n$ if $9 \mid \sum_{k=1}^{n} d_k$.

**$2^k$:** A number is divisible by $2^k$ if the number represented by the last $k$ digits is divisible by $2^k$.

**7:**

Rule 1: Partition $n$ into 3 digit numbers starting from the right ($\bar{d}_3\bar{d}_2\bar{d}_1, \bar{d}_6\bar{d}_5\bar{d}_4, \bar{d}_9\bar{d}_8\bar{d}_7$, etc…) If the alternating sum ($\bar{d}_3\bar{d}_2\bar{d}_1$ - $\bar{d}_6\bar{d}_5\bar{d}_4$ + $\bar{d}_9\bar{d}_8\bar{d}_7$ - …) is divisible by 7, then $n$ is divisible by 7.

Rule 2: Truncate the last digit of $n$, and subtract twice that digit from the remaining number. If the result is divisible by 7, then $n$ was divisible by 7. This process can be repeated for large numbers.

Ex. $n = 228865 \rightarrow 22886 - 2(5) = 22876 \rightarrow 2287 - 2(6) = 2275$
$\rightarrow 227 - 2(5) = 217 \rightarrow 7 \mid 217$, so $7 \mid 228865$ ($228865 = 7*32695$)

Rule 3: Partition the number into groups of 6 digits, $d_1$ through $d_6$, $d_7$ through $d_{12}$, etc. For a 6 digit number $n$, $n$ is divisible by 7 if $(d_1 + 3d_2 + 2d_2 - d_4 - 3d_5 - 2d_6)$ is divisible by $7$. For larger numbers, just add the similar sum from the next cycle. The coefficients counting from $d_1$ are (1, 3, 2, -1, -3, -2, 1, 3, 2, -1, -3, -2, …)

**11:** A number $n$ is divisible by 11 if the alternating sum of the digits is divisible by 11

$11 \mid n$ if $11 \mid (d_1 - d_2 + d_3 - d_4 + d_5 - \ldots - d_n (-1)^n)$.

**13:**

Rule 1: See rule 1 for divisibility by 7, $n$ is divisible by 13 if the same specified sum is divisible by 13.

Rule 2: Same process as in rule 3 for 7, the cycle of the coefficients is (1, -3, -4, -1, 3, 4, … )

# Trigonometric Identities

$$\sin^2 x + \cos^2 x = 1$$
$$1 + \tan^2 x = \sec^2 x$$
$$1 + \cot^2 x = \csc^2 x$$
$$\sin x = \cos(90 - x) = \sin(180 - x)$$
$$\cos x = \sin(90 - x) = -\cos(180 - x)$$
$$\tan x = \cot(90 - x) = -\tan(180 - x)$$

**Angle-sum and angle-difference formulas**

$$\sin(a \pm b) = \sin a \cos b \pm \cos a \sin b$$
$$\cos(a \pm b) = \cos a \cos b \mp \sin a \sin b$$
$$\tan(a \pm b) = \frac{\tan a \pm \tan b}{1 \mp \tan a \tan b}$$
$$\cot(a \pm b) = \frac{\cot a \cot b \mp 1}{\cot b \pm \cot a}$$
$$\sin(a + b)\sin(a - b) = \sin^2 a - \sin^2 b = \cos^2 b - \cos^2 a$$
$$\cos(a + b)\cos(a - b) = \cos^2 a - \sin^2 b = \cos^2 b - \sin^2 a$$

**Double-angle relations**

$$\sin 2a = 2\sin a \cos a = \frac{2\tan a}{1 + \tan^2 a}$$
$$\cos 2a = \cos^2 a - \sin^2 a = 2\cos^2 a - 1 = 1 - 2\sin^2 a = \frac{1 - \tan^2 a}{1 + \tan^2 a}$$
$$\tan 2a = \frac{2\tan a}{1 - \tan^2 a} \qquad \cot 2a = \frac{\cot^2 a - 1}{2\cot a}$$

**Multiple-angle relations**

$$\sin 3a = 3\sin a - 4\sin^3 a$$
$$\cos 3a = 4\cos^3 a - 3\cos a \qquad \tan 3a = \frac{3\tan a - \tan^3 a}{1 - 3\tan^2 a}$$
$$\sin 4a = 4\sin a \cos a - 8\sin^3 a \cos a$$
$$\cos 4a = 8\cos^4 a - 8\cos^2 a + 1 \qquad \tan 4a = \frac{4\tan a - 4\tan^3 a}{1 - 6\tan^2 a + \tan^4 a}$$
$$\sin 5a = 5\sin a - 20\sin^3 a + 16\sin^5 a$$
$$\cos 5a = 16\cos^5 a - 20\cos^3 a + 5\cos a$$
$$\sin 6a = 32\cos^5 a \sin a - 32\cos^3 \sin a + 6\cos a \sin a$$
$$\cos 6a = 32\cos^6 a - 48\cos^4 a + 18\cos^2 a - 1$$

$$\sin na = 2\sin(n-1)a\cos a - \sin(n-2)a$$

$$\tan na = \frac{\tan(n-1)a + \tan a}{1-\tan(n-1)a\tan a}$$

$$\cos na = 2\cos(n-1)\cos a - \cos(n-2)a$$

**Function-product relations**

$$\sin a\sin b = \frac{1}{2}\left(\cos(a-b)-\cos(a+b)\right)$$

$$\cos a\cos b = \frac{1}{2}\left(\cos(a-b)+\cos(a+b)\right)$$

$$\sin a\cos b = \frac{1}{2}\left(\sin(a+b)+\sin(a-b)\right)$$

$$\cos a\sin b = \frac{1}{2}\left(\sin(a+b)-\sin(a-b)\right)$$

**Function-sum and function-difference relations**

$$\sin a + \sin b = 2\sin\left(\frac{a+b}{2}\right)\cos\left(\frac{a-b}{2}\right)$$

$$\sin a - \sin b = 2\cos\left(\frac{a+b}{2}\right)\sin\left(\frac{a-b}{2}\right)$$

$$\cos a + \cos b = 2\cos\left(\frac{a+b}{2}\right)\cos\left(\frac{a-b}{2}\right)$$

$$\cos a - \cos b = -2\sin\left(\frac{a+b}{2}\right)\sin\left(\frac{a-b}{2}\right)$$

$$\tan a + \tan b = \frac{\sin(a+b)}{\cos a\cos b}$$

$$\tan a - \tan b = \frac{\sin(a-b)}{\cos a\cos b}$$

**Half-angle relations**

$$\sin\frac{a}{2} = \pm\sqrt{\frac{1-\cos a}{2}} \qquad \cos\frac{a}{2} = \pm\sqrt{\frac{1+\cos a}{2}}$$

$$\tan\frac{a}{2} = \pm\sqrt{\frac{1-\cos a}{1+\cos a}} = \frac{1-\cos a}{\sin a} = \frac{\sin a}{1+\cos a}$$

$$\cot\frac{a}{2} = \pm\sqrt{\frac{1+\cos a}{1-\cos a}} = \frac{1+\cos a}{\sin a} = \frac{\sin a}{1-\cos a}$$

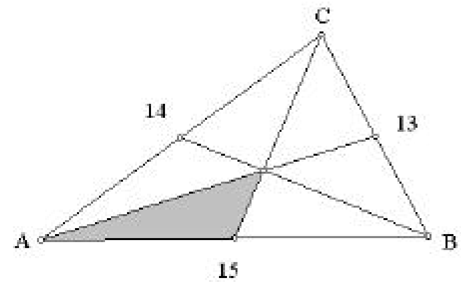
**Trig functions of special angles**

| Angle | sin | cos | tan |
|---|---|---|---|
| 0 | 0 | 1 | 0 |
| 15 | $\frac{\sqrt{2}}{4}\left(\sqrt{3}-1\right)$ | $\frac{\sqrt{2}}{4}\left(\sqrt{3}+1\right)$ | $2-\sqrt{3}$ |
| 18 | $\frac{\sqrt{5}-1}{4}$ | $\frac{\sqrt{5+\sqrt{5}}}{2\sqrt{2}}$ | $\frac{\sqrt{2}\left(\sqrt{5}-1\right)}{2\sqrt{5+\sqrt{5}}}$ |
| 30 | $\frac{1}{2}$ | $\frac{\sqrt{3}}{2}$ | $\frac{\sqrt{3}}{3}$ |
| 36 | $\frac{\sqrt{5-\sqrt{5}}}{2\sqrt{2}}$ | $\frac{\sqrt{5}+1}{4}$ | $\frac{\left(\sqrt{5}-1\right)\sqrt{5-\sqrt{5}}}{2\sqrt{2}}$ |
| 45 | $\frac{\sqrt{2}}{2}$ | $\frac{\sqrt{2}}{2}$ | 1 |
| 54 | $\frac{\sqrt{5}+1}{4}$ | $\frac{\sqrt{5-\sqrt{5}}}{2\sqrt{2}}$ | $\frac{\left(\sqrt{5}+1\right)\sqrt{2}}{2\sqrt{5-\sqrt{5}}}$ |
| 60 | $\frac{\sqrt{3}}{2}$ | $\frac{1}{2}$ | $\sqrt{3}$ |
| 72 | $\frac{\sqrt{5+\sqrt{5}}}{2\sqrt{2}}$ | $\frac{\sqrt{5}-1}{4}$ | $\frac{\left(\sqrt{5}+1\right)\sqrt{5-\sqrt{5}}}{2\sqrt{2}}$ |
| 75 | $\frac{\sqrt{2}}{4}\left(\sqrt{3}+1\right)$ | $\frac{\sqrt{2}}{4}\left(\sqrt{3}-1\right)$ | $2+\sqrt{3}$ |
| 90 | 1 | 0 | … |

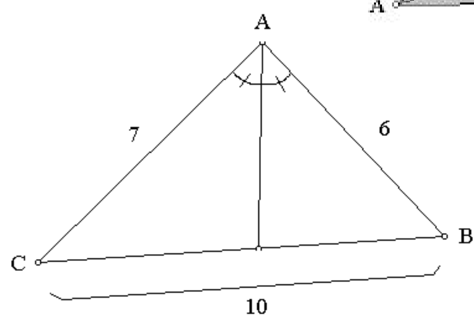
## Problems

Solutions are left as an exercise for the reader. All answers must be simplified and exact answers unless otherwise specified (irrational decimal answers require infinitely many decimal places.)

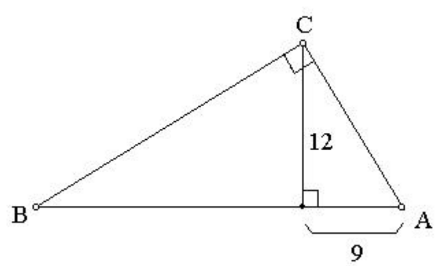
1. What is the ratio of the side of a regular octahedron with equal volume and surface area to the side of a tetrahedron with equal volume and surface area?
2. What is the sum of the product and the sum of the roots of the equation $x^2+7x-1$? What about $x^3-3x^2+7x-25$?
3. The medians of a triangle with sides *13*, *14*, and *15* intersect as shown below. What is the area of the shaded region?



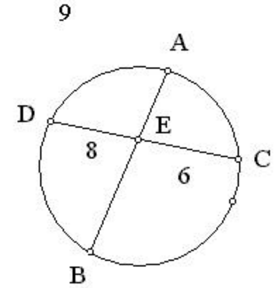

4. What is the ratio of the area of $\Delta ADC$ to the area of $\Delta ADB$?



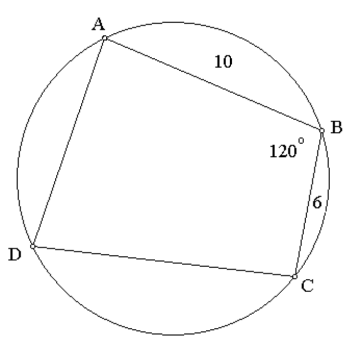

5. What is the length of the median to side $\overline{AB}$?



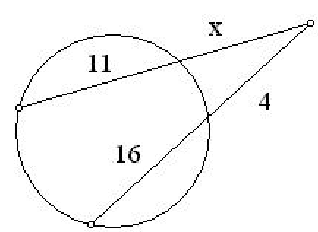

6. $\overline{AB}$ and $\overline{CD}$ are intersecting chords in the circle. The radius of the circle is 10, and the distance from the center to $\overline{AB}$ is 6. What are the lengths of the segments $\overline{AE}$ and $\overline{BE}$?



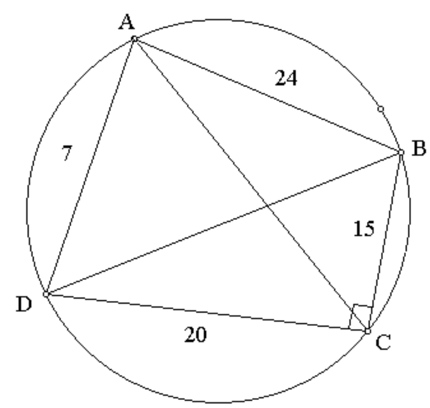

7. $\overline{AB} \cong \overline{CD}$. What is the area of *ABCD*?




8. What is *x*?

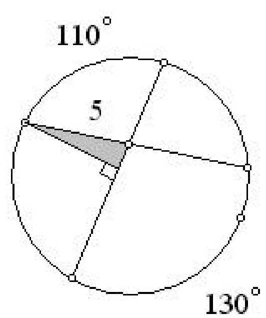


9. $\bar{x}34\bar{y}73$ is a 6 digit number that is divisible by 7. What is $y - x$?
10. Find an even number that has 7 factors. Is this the only such number?
11. Compute $\sin\left(3\cos^{-1}\left(\frac{3}{5}\right)\right)$.
12. A regular polygon has an exterior angle whose measure is equal to $\frac{1}{8}$ of its interior angle. How many sides does the polygon have?
13. What is the sum of the factors of *1572*?
14. There are 4 different types of monitors, 5 different CPU's, and 3 different types of printers that can be purchased. Two of the CPU's are not compatible with one of the monitors. How many different systems can be purchased?
15. A committee of 3 people must be chosen from a group of 10 individuals. One must be appointed the leader and another the secretary. How many different ways can a committee be chosen?
16. How many 5 digit numbers exist whose digits are all in descending order?
17. Out of a group of 100 people, 70 people are taking math, 60 are taking science, and 50 are taking history. 40 are taking both math and science, 25 are taking both math and history, and 35 are taking both science and history. How many are taking all three subjects?
18. What is the probability that if an integer between 1 and 1000 is chosen, that it is divisible by either 2 or 5?



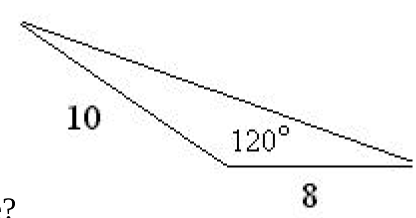

19. What is the area of *ABCD*? What are the lengths of the diagonals?

20. What is the area of the shaded region?




21. If *r, s,* and *t* are the roots of the equation $x^3 - 3x^2 + 8x - 5$, what is the value of $r^3 + s^3 + t^3$?

22. What is the radius of the circumscribed circle of the triangle?




23. How many negative roots does $x^{11} - 5x^6 + 4x^3 + 2x^2 - x + 1 = 0$ have?
24. If *q* is the angle the line *3x + 5y = 10* makes with the x-axis, then what is *cos q* ?
25. What is the distance from the point *(7, 12)* to the line *x = -y* ?

26. A triangle has vertices at *(2, 5)*, *(0, 8)*, and *(4, 12)*. What is its area?

27. What is the inverse of the matrix $\begin{bmatrix} 5 & -2 \\ -7 & 1 \end{bmatrix}$?

28. What is the remainder when $13^8$ is divided by $7$?

29. What is the lateral area of a cone with radius *4* and height *3*?

30. Find all solutions to the equation $8x + 28y = 34$, where $x$ and $y$ are integers.

31. What is the sum of the first $n$ triangular numbers?

32. $34\bar{a}743164$ is divisible by 11. What is the value of $a$?

33. A biased coin is flipped 10 times. If the probability of getting heads is $\frac{2}{3}$, then what is the probability of getting exactly 6 tails?

34. The 2nd term of an arithmetic sequence is 2, and the 10th term is 26. What is the sum of the first 15 terms of the sequence?

35. What is $\sin(22.5°)$ ?

36. A sphere is inscribed inside a regular tetrahedron with side length 6. What is its surface area?

37. In the expansion of $(3a - b + c + d - 2e)^{10}$, what is the coefficient of the $a^2b^3cd^3e$ term?

38. $\cos 2x + \cos 4x$ is equivalent to $2\cos ax \cos bx$. What is $a + b$?

39. What is $\cos^2 18°$ ?

40. $n$ is the sum of the first 25 digits in the decimal approximation of $e^p$. It is also a factor of 703987968917520. A sphere is inscribed inside a regular icosahedron with side length 108. Its radius has a length of $z\sqrt{a + b\sqrt{c}}$, where $z$, $a$, $b$, and $c$ are all integers, and the expression is simplified as much as possible. Let $q$ equal the sum of the first 5 decimal digits in $p$. The 6th triangular number is equal to $k + 1$. If $n = z + a + b + c + q + k$, what is the value of $n$?

## Strategies

1. **Search for a pattern:**

Ex. Compute $\sqrt{(31)(30)(29)(28)+1}$ (no calculators) (1989 AIME, #1)

Starting at 1 instead of 28, we see that

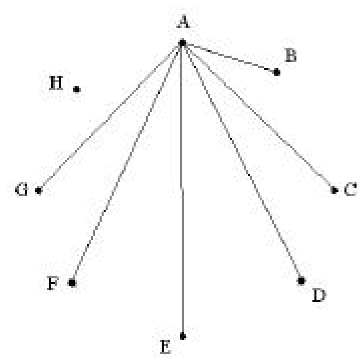
$$\sqrt{(3)(2)(1)(0)+1} = \sqrt{1} = 1 = 1^2 + 0$$
$$\sqrt{(4)(3)(2)(1)+1} = \sqrt{25} = 5 = 2^2 + 1$$
$$\sqrt{(5)(4)(3)(2)+1} = \sqrt{121} = 11 = 3^2 + 2$$
$$\sqrt{(6)(5)(4)(3)+1} = \sqrt{361} = 19 = 4^2 + 3$$

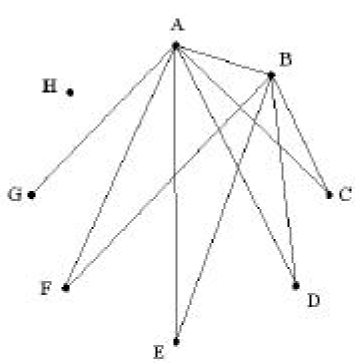
Then it appears $\sqrt{(n+3)(n+2)(n+1)n+1} = (n+1)^2 + n$, so the solution to our problem would be $29^2 + 28 = 869$. Multiplying out the polynomials will show that the formula is accurate.

2. **Draw a Figure:**

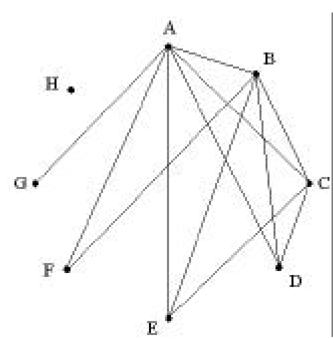
Ex. Mr. and Mrs. Adams recently attended a party at which there were three other couples. Various handshakes took place. No one shook hands with his/her own spouse, no one shook hands with the same person twice, and no one shook his/her own hand. After all the handshaking was finished, Mr. Adams asked each person, including his wife, how many hands he or she had shaken. To his surprise, each gave a different answer. How many hands did Mrs. Adams shake? (Larson 1.2.4)

A diagram with dots representing people is helpful. The numbers that Mr. Adams received must have been 0, 1, 2, 3, 4, 5, and 6. Suppose A shook hands with 6 other people (B through G, for example). This is represented by the diagram to the right. H must be the person who shook 0 hands, and A and H must be a couple since A shook hands with everyone else. Now suppose B shook 5 hands (A, C, D, E, and F, for example). This diagram is shown below.




G must be the person who shook 1 hand, and B and G must be spouses. If C is the person who shook 4 hands, we find similarly that F shook 2 hands. Completing the diagram, we see that D and E both shook 3 hands. They must be Mr. and Mrs. Adams.



3. **Formulate an Equivalent Problem:**
Sometimes when the problem or the calculations are complicated, the problem can often be rewritten or manipulated into a different form that is easier to solve. Ways to do this include algebraic or trig manipulation or substitution, use of a one-to-one correspondence, or reinterpreting the problem into a different subject.

Ex. On a circle *n* points are selected and the chords joining them in pairs are drawn. Assuming that no three of these chords are concurrent, (except at the endpoints), how many points of intersection are there? (Larson 1.3.5)

When four points are selected, connecting all the points together produces a quadrilateral with two intersecting diagonals. Therefore with any selection of 4 points, there is exactly one point of intersection. The problem is equivalent to the number of ways to chose 4 points from *n* points, which is just $\binom{n}{4}$.

4. **Modify the Problem:**
This method is closely related to number 3. It is very general, and many types of problems could potentially fall under this category.

5. **Choose Effective Notation:**
Problems can often be simpler depending on the notation used.
Ex. The sum of 5 consecutive terms is 195. Of these terms, what is the largest one given a common difference of 13.

If $a = 13$, one might call the largest term $x$ and the other terms $x - a$, $x- 2a$, $x-3a$, and $x - 4a$. However, letting $x$ be the middle term produces the other terms $x - 2a$, $x - a$, $x + a$, and $x + 2a$. The $a$'s cancel out nicely when added together, so $5x = 195$, or $x$ is $39$. Then the largest term is $39 + 26$ or $65$.

6. **Exploit Symmetry:**
Using symmetry in certain problems often reduces the amount of work that must be done. For example, when multiplying out a polynomial such as $(a+b+c)(a^2+b^2+c^2-ab-ac-bc)$, all the variables can be interchanged, so if there is an $a^3$ term, there must be a $b^3$ and a $c^3$ term with the same coefficient. The terms $a^2b, a^2c, ab^2, b^2c, ac^2, bc^2$ will all have the same coefficients as well. Also, in another example, when graphing a function like $|x|+|y|=4$, there is symmetry across both axes, so only one quadrant must be plotted before reflecting across the axes.

7. **Divide into Cases:**
Some problems can be divided into smaller sub-problems that can be solved individually.
Ex. When finding the probability of getting at least 7 heads if a coin is flipped 10 times, the problem is usually split into finding the probability of exactly 7, 8, 9 and exactly 10 heads.

**8. Work Backwards**

Many proofs and some problems are easiest if worked backwards and then reversing the steps to obtain the desired result.

Ex. Prove that the arithmetic mean of a number is always greater than or equal to the geometric mean.

Suppose that this is true. Then $\frac{x+y}{2} \geq \sqrt{xy}$. Squaring, we have $x^2 + 2xy + y^2 \geq 4xy$

This simplifies to $(x-y)^2 \geq 0$, which is obviously always true. We can reverse our steps so the proof is valid.

**9. Argue by Contradiction:**

Some proofs are done by assuming the opposite of what you want is true, and then working until a contradiction is reached.

Ex. Prove the harmonic series $1+\frac{1}{2}+\frac{1}{3}+\cdots\frac{1}{n}$ diverges. (Larson 1.9)

Suppose the series converges, and the sum is *r*. Then

$$r = 1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{8}\cdots+\frac{1}{n-1}+\frac{1}{n}$$

$$r > \frac{1}{2}+\frac{1}{2}+\frac{1}{4}+\frac{1}{4}+\frac{1}{6}+\frac{1}{6}+\frac{1}{8}+\frac{1}{8}+\cdots\frac{1}{n}+\frac{1}{n}$$

$$r > 1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\cdots$$

But this implies that *r* > *r*, which is a contradiction. Therefore the series must diverge.

**10. Pursue Parity**

Whether a number is even or odd can help solve problems that otherwise seem unrelated.

Ex. Place a knight on each square of a 7-by-7 chess board. Is it possible for each knight to simultaneously make a legal move? (Larson 1.10.2)

Assume a chessboard is colored in the usual checkered pattern. The board has 49 squares; suppose 24 of them are white and 25 are black. Consider 25 knights which rest on the black squares. If they were to make a legal move, they must move onto 25 white squares. But this is impossible, since there are only 24 white squares.

**11. Consider Extreme Cases:**

Often if a problem says that something works for all cases, it must work in specialized cases. For example, a theorem that works for all triangles must work for equilateral or right triangles. Testing extreme cases can either provide counterexamples or help to determine a pattern for general cases.

**12. Generalize:**

Sometimes a more general case is easier to solve than a specific case. Replacing a specific number with a variable may make a solution more visible.

# Sources

Beyer, William H. CRC Standard Mathematical Tables. Florida: CRC Press, 1981.

Larson, Loren C. Problem-Solving Through Problems. New York: Springer-Verlag, 1983.

Many useful formulas were compiled in the CRC book. The list of problem-solving strategies and many of the examples in that section came from this second source. A few problems were taken from past AIME exams.