

## Counting

Order:	Matters	Doesn't Matter
w/ Replacement	$n^k$	$\binom{n+k-1}{k}$
wo/ Replacement	$n P_k = \frac{n!}{(n-k)!}$	$\binom{n}{k} = \frac{n!}{k!(n-k)!}$

Binomial:  $(x + y)^n = \sum \binom{n}{k} x^k y^{n-k}$

Multinomial:

$$(x_1 + x_2 + \dots + x_r)^n = \sum_{\substack{(n_1, n_2, \dots, n_r): \\ n_1 + \dots + n_r = n}} \frac{n!}{n_1! n_2! \dots n_r!} x_1^{n_1} x_2^{n_2} \dots x_r^{n_r}$$

There are  $\binom{n+k-1}{k-1}$  distinct nonnegative integer-valued vectors  $\langle x_1, \dots, x_r \rangle$  satisfying  $x_1 + \dots + x_k = n$

## Probability

$(\bigcup_{i=1}^n E_i)^c = \bigcap_{i=1}^n E_i^c$  (DeMorgan's Law)

$(\bigcap_{i=1}^n E_i)^c = \bigcup_{i=1}^n E_i^c$  (DeMorgan's Law)

$P(E^c) = 1 - P(E)$  (complement)

$P(E \cup F) = P(E) + P(F) - P(EF)$  (double counting)

$P(E|F) = \frac{P(EF)}{P(F)}$  (conditional)

$P(E) = P(EF) + P(EF^c) = P(F)P(E|F) + P(F^c)P(E|F^c)$

$P(F^c)P(E|F^c) = P(F)P(E|F) + (1 - P(F))P(E|F^c)$

$P(E) = \sum_{i=1}^n P(E|F_i)P(F_i)$  (Law of Total Probability)

Two events are independent if  $P(EF) = P(E)P(F)$

**Bayes' Theorem:** Suppose that  $F_1, F_2, \dots, F_n$  are mutually exclusive events such that  $\bigcup_{i=1}^n F_i = S$  (sample space), then

$$P(F_j|E) = \frac{P(EF_j)}{P(E)} = \frac{P(E|F_j)P(F_j)}{\sum_{i=1}^n P(E|F_i)P(F_i)}$$

## Random Variables

PMF:  $p_X(a) = P\{X = a\}$

CDF:  $F_X(a) = P\{X \leq a\} = \sum_{x \leq a} p_X(x)$

EV:  $E[X] = \sum_x x p_X(x)$

EV of  $g(X)$ :

$E[g(X)] = \sum_x g(x) p_X(x)$

$Var(X) = E[X^2] - (E[X])^2$

$Var(aX + b) = a^2 Var(X)$

$SD(X) = \sqrt{Var(X)}$

Name	PMF	$\mu$	$Var$
<i>Bern</i> (p)	$P\{X = 1\} = p, P\{X = 0\} = 1 - p$	$p$	$p(1 - p)$
<i>Bin</i> (n, p)	$P\{X = k\} = \binom{n}{k} p^k (1 - p)^{n-k}, k = 0, \dots, n$	$np$	$np(1 - p)$
<i>Geom</i> (p)	$P\{X = n\} = p(1 - p)^{n-1}, n = 0, 1, 2, \dots$	$\frac{1}{p}$	$\frac{1-p}{p^2}$
<i>Pois</i> ( $\lambda$ )	$P\{X = n\} = e^{-\lambda} \frac{\lambda^n}{n!}, n = 0, 1, \dots$	$\lambda$	$\lambda$
<i>NBin</i> (r, p)	$P\{X = r\} = \binom{n-1}{r-1} p^r (1 - p)^{n-r}, n = r, r + 1, \dots$	$\frac{r}{p}$	$r \frac{(1-p)}{p^2}$
h.g.	$P\{X = i\} = \frac{\binom{m}{i} \binom{N-m}{n-i}}{\binom{N}{n}}$ $i = 0, 1, 2, \dots, \min(n, m)$	$\frac{nm}{N}$	$\frac{N-n}{N-1} np(1 - p)$ $p = m/N$

PDF:  $f_X(a) = \frac{1}{\epsilon} P\{a - \frac{\epsilon}{2} < X \leq a + \frac{\epsilon}{2}\}$

CDF:  $F_X(a) = P\{X \leq a\} = \int_{-\infty}^a p_X(x) dx$

EV:  $E[X] = \int_{-\infty}^{\infty} x f_X(x) dx$

EV of  $g(X)$ :

$E[g(X)] = \int_{-\infty}^{\infty} g(x) f_X(x) dx$

$P\{X \in B\} = \int B f_X(x) dx$

$f_X(x) = \frac{d}{dx} F_X(x)$

Name	PDF	CDF	$\mu$	$Var$
<i>Uni</i> (a, b)	$f(x) = \begin{cases} \frac{1}{b-a} & \text{if } a < x < b \\ 0 & \text{otherwise} \end{cases}$	$F(x) = \begin{cases} 0 & x \leq a \\ \frac{x-a}{b-a} & \text{if } a < x < b \\ 1 & x \geq b \end{cases}$	$\frac{a+b}{2}$	$\frac{(b-a)^2}{12}$
<i>Expo</i> ( $\lambda$ )	$f(x) = \begin{cases} \lambda e^{-\lambda x} & \text{if } x \geq 0 \\ 0 & \text{otherwise} \end{cases}$	$F(x) = 1 - e^{-\lambda x} \text{ if } x \geq 0$	$\frac{1}{\lambda}$	$\frac{1}{\lambda^2}$
<i>Gamm</i> (a, $\lambda$ )	$f(x) = \frac{1}{\Gamma(a)} (\lambda x)^{a-1} e^{-\lambda x} \frac{\lambda}{x}$	???	$a/\lambda$	$a/\lambda^2$
<i>Norm</i> ( $\mu, \sigma^2$ )	$f(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-(x-\mu)^2/2\sigma^2}$	$\phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^x e^{-u^2/2} du$	$\mu$	$\sigma^2$

## Continuous Random Variables

For any normal random variable  $X$  with parameters  $(\mu, \sigma^2)$ ,  $Z = \frac{X-\mu}{\sigma}$  is the standardized normal random variable.

**De Moivre-Laplace Limit Theorem:** If  $S_n$  denotes the number of successes that occur when  $n$  independent trials, each resulting in a success with probability  $p$ , are performed

then, for any  $a < b$ ,  $P\{a \leq \frac{S_n - np}{\sqrt{np(1-p)}} \leq b\} \rightarrow \phi(b) - \phi(a)$

**Memoryless RV:** We say that a nonnegative random variable  $X$  is *memoryless* if  $P\{X > s + t | X > t\} = P\{X > s\}$

**Distribution of a Function of a RV:**  $f_Y(y) = f_X[g^{-1}(y)] \left| \frac{d}{dy} g^{-1}(y) \right|$  if  $y = g(x)$  for some  $x$

## Jointly Distributed Random Variables<sup>1</sup>

Joint PDF:  $f_{XY}(a, b) = P\{X = a, Y = b\}$

Conditional PMF:  $p_{X|Y}(x|y) = P\{X = x | Y = y\} = \frac{P\{X=x, Y=y\}}{P\{Y=y\}} = \frac{p_{XY}(x, y)}{p_Y(y)}$

Joint CDF:  $F_{XY}(a, b) = P\{X \leq a, Y \leq b\}$

Conditional CDF:  $F_{X|Y}(x|y) = P\{X \leq x | Y = y\} = \sum_{a \leq x} p_{X|Y}(a|y)$

<sup>1</sup>Replace summation with integration for continuous.

Marginal PMF of X:  $p_X(a) = \sum_b p_{XY}(a, b)$   
 Marginal PMF of Y:  $p_Y(b) = \sum_a p_{XY}(a, b)$

Marginal CDF of X:  $F_{XY}(a) = F_{XY}(a, \infty)$   
 Marginal CDF of Y:  $F_{XY}(b) = F_{XY}(\infty, b)$

Conditional Jointly Distributed RV

$F_{X A}(x) = \frac{F_X(x) - F_X(a)}{F_X(b) - F_X(a)}$ if $a \leq x < b$	$f_{X A}(x) = \frac{f_X(x)}{P(A)}$ if $A$ is $a \leq X < b$	$f_{X Y}(x, y) = \frac{f_{XY}(x, y)}{f_Y(y)}$
$P(E) = \begin{cases} \sum_y P(E Y=y)P(Y=y) & \text{if } Y \text{ is discrete} \\ \int_{-\infty}^{\infty} P(E Y=y)P(Y=y)dy & \text{if } Y \text{ is cont.} \end{cases}$	$E[X A] = \int_{-\infty}^{\infty} x f_{X A}(x) dx$	$E[g(X) A] = \int_{-\infty}^{\infty} g(x) f_{X A}(x) dx$

Independent Random Variables: Two random variables  $X$  and  $Y$  are said to be independent if for every  $a$  and  $b$  we have  $f_{XY}(a, b) = f_X(a)f_Y(b)$ .

Summation of Independent Random Variables	CMF	$F_{X+Y}(a) = P\{X+Y \leq a\} = \int_{-\infty}^{\infty} F_X(a-y)f_Y(y)dy$
	PDF	$f_{X+Y}(a) = \int_{-\infty}^{\infty} f_X(a-y)f_Y(y)dy$

Sum of Normal RV: Mean and Variance are the summation of all the normal RV's mean and variance.

Jacobian Transformation (Joint Probability Distribution functions of RV)

$f_{Y_1, Y_2}(y_1, y_2) = \frac{f_{X_1, X_2}(x_1, x_2)}{ J(x_1, x_2) }$	$J(x_1, x_2) = \begin{vmatrix} \frac{\partial g_1}{\partial x_1} & \frac{\partial g_1}{\partial x_2} \\ \frac{\partial g_2}{\partial x_1} & \frac{\partial g_2}{\partial x_2} \end{vmatrix} = \frac{\partial g_1}{\partial x_1} \frac{\partial g_2}{\partial x_2} - \frac{\partial g_1}{\partial x_2} \frac{\partial g_2}{\partial x_1} \neq 0$	$y_1 = g_1(x_1, x_2)$ and $y_2 = g_2(x_1, x_2)$
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## Properties of Expectation

Joint PMF:  $E[g(X, Y)] = \sum_y \sum_x g(x, y)p(x, y)$

Joint PDF:  $E[g(X, Y)] = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g(x, y)p(x, y)dx dy$

$E[aX + b] = aE[X] + b$

$E[g(X)h(Y)] = E[g(X)]E[h(Y)]$  if  $X$  and  $Y$  are independent

**Covariance**:

$Cov(X, Y) = E[XY] - E[X]E[Y]$ ,  $Cov(X, Y) = Cov(Y, X)$

$Cov(X, X) = Var(X)$ ,  $Cov(aX, aY) = aCov(X, Y)$

$Cov(\sum_{i=1}^n X_i, \sum_{j=1}^m Y_j) = \sum_{i=1}^n \sum_{j=1}^m Cov(X_i, Y_j)$

$Var(\sum_{i=1}^n X_i) = \sum_{i=1}^n Var(X_i) + \sum_{i=1}^n \sum_{j \neq i} Cov(X_i, X_j)$

$Var(\sum_{i=1}^n X_i) = \sum_{i=1}^n Var(X_i)$  if  $X_1, \dots, X_n$  are p.w. ind.

**Correlation**:  $\rho(X, Y) = \frac{Cov(X, Y)}{\sqrt{Var(X)Var(Y)}}$ ,  $-1 \leq \rho(X, Y) \leq 1$

**Conditional Expectation**:

$E[X|Y=y] = \sum_x x p_{X|Y}(x, y)$

$E[g(X)|Y=y] = \sum_x g(x) p_{X|Y}(x, y)$

**Conditioning Property**:  $E[X] = E[E[X|Y]]$

**Expectation of X**:  $\sum_y E[X|Y=y]P\{Y=y\}$

**Conditional Variance**:  $Var(X|Y) = E[X^2|Y] - (E[X|Y])^2$

**Uncond. var. of X|Y**:  $E[Var(X|Y)] = E[X^2] - E[(E[X|Y])^2]$

$Var(E[X|Y]) = E[(E[X|Y])^2] - (E[X])^2$

$Var(X) = E[Var(X|Y)] + Var(E[X|Y])$

**Predictors**:

General form:  $E[(Y - g(X))^2] \geq E[(Y - E[Y|X])^2]$  (best case:  $g(X) = E[Y|X]$ )

Linear:  $E[(Y - (a + bX))^2] \geq E[(Y - E[Y|X])^2]$ , where  $b = \rho \frac{\sigma_Y}{\sigma_X}$  and  $a = E[Y] - bE[X]$

**Moment Generating Functions**:

$n^{\text{th}}$  moment:  $M^n(0) = E[X^n]$   $n \geq 0$

MGF of PDF:  $M(t) = E[e^{tX}] = \sum_{-\infty}^{\infty} e^{tx} p(x)$

MGF of PMF:  $M(t) = E[e^{tX}] = \sum_{-\infty}^{\infty} e^{tx} f(x)$

More than 2 RV MGF:  $M(t_1, \dots, t_n) = E[e^{t_1 X_1 + \dots + t_n X_n}]$

Individual MGF:  $M_{X_i}(t) = M(0, \dots, 0, t, 0, \dots, 0)$

Joint MGF:  $M_{XY}(t, t') = E[e^{tX + t'Y}] = \sum_y \sum_x x e^{tx + t'y} p_{XY}(x, y)$

MGF of sum of two IRVs:  $M_{X+Y}(t) = M_X(t)M_Y(t)$

Common MGFs:

Binomial	$M(t) = (pe^t + 1 - p)^n$
Poisson	$M(t) = \exp\{\lambda(e^t - 1)\}$
Exponential	$M(t) = \frac{\lambda}{\lambda - t}$ for $t < \lambda$
Standard Normal	$M(t) = e^{t^2/2}$
Normal	$M(t) = \exp\{\frac{\sigma^2 t^2}{2} + \mu t\}$

## Limit Theorems

**Markov's Inequality**: If  $X$  is a random variable that takes only nonnegative values, then for any value  $a > 0$ ,  $P\{X \geq a\} \leq \frac{E[X]}{a}$

**Chebyshev's Inequality**: If  $X$  is a random variable with finite mean  $\mu$  and variance  $\sigma^2$ , then for any value  $k > 0$ ,  $P\{|X - \mu| \geq k\} \leq \frac{\sigma^2}{k^2}$

**The Weak Law of Large Numbers**: Let  $X_1, X_2, \dots$

be a sequence of iid Random variables, each having finite mean. Then, for any  $\epsilon > 0$ ,  $P\{|\bar{X} - \mu| \geq \epsilon\} \rightarrow 0$  where  $\bar{X} = \frac{X_1 + X_2 + \dots + X_n}{n}$  as  $n \rightarrow \infty$

**The Central Limit Theorem**: Let  $X_1, X_2, \dots$  be a sequence of iid random variables, each having mean and variance. Then, the distribution of  $Z = \frac{X_1 + \dots + X_n - n\mu}{\sigma\sqrt{n}}$  tends to the standard normal as  $n \rightarrow \infty$ . That is, for  $-\infty < a < \infty$ ,  $P\{Z \leq a\} = \phi(a) \rightarrow N(0, 1)$  as  $n \rightarrow \infty$

# Random Processes

**Random Processes:** a collection of random variables usually indexed by time

**Sample Function:** the time function  $x(t, s)$  associated with the outcome  $s$  of an experiment

**Ensemble:** the set of all possible time function that can result from an experiment

**Random Sequence:** a random sequence  $X_n$  is an ordered sequence of random variables  $X_0, X_1, \dots$

**Bernoulli Process:**  $X(t)$  is a sequence of Bernoulli trials; trials are independent of each other ( $P\{X_n = 1\} = p = 1 - P\{X_n = 0\}$ )

**Counting Process:** Given a stochastic process  $N(t)$ :  
 $N(0) = 0$ ,  $N(t) \in \{0, 1, 2, \dots\}$  for all  $t \in [0, \infty)$   
for  $0 < s < t$ ,  $N(t) - N(s)$  shows the # of events in  $(s, t]$ .

**Poisson Process:** Given  $\lambda > 0$ , a counting process  $N(t), t \in [0, \infty)$  is called a Poisson process with rate  $\lambda$  if the following conditions hold:

→  $N(0) = 0$

→  $N(t)$  has independent and stationary increments

→ The # of arrivals in any interval of length  $\tau > 0$  has  $Pois(\lambda\tau)$  distribution

PMF of  $M = N(t_1) - N(t_0)$ :

$$p_M(m) = \begin{cases} \frac{[\lambda|(t_1 - t_0)|^m]}{m!} e^{-\lambda(t_1 - t_0)} & m = 0, 1, \dots \\ 0 & \text{otherwise} \end{cases}$$

Joint PMF of  $N(t_1), \dots, N(t_k), t_1 < t_2 < \dots < t_k$ :

$$p_{N(t_1), \dots, N(t_k)}(n_1, \dots, n_k) = \begin{cases} \frac{\alpha_1^{n_1} e^{-\alpha_1}}{n_1!} \frac{\alpha_1^{n_2 - n_1} e^{-\alpha_2}}{(n_2 - n_1)!} \dots \frac{\alpha_k^{n_k - n_{k-1}} e^{-\alpha_k}}{(n_k - n_{k-1})!} & m = 0, 1, \dots, \alpha_i = \lambda(t_i - t_{i-1}) \\ 0 & \text{otherwise} \end{cases}$$

For a Poisson process of rate  $\lambda$ , the inter-arrival times  $X_1, X_2, \dots$  are an iid random sequence with the exponential PDF.

$P\{X_n - x' > x | X_n > x'\} = P\{X_n > x\} = e^{-\lambda x}$  (P.P. is memoryless)

A counting process with **independent exponential inter-arrivals**  $X_1, X_2, \dots$  with mean  $E[X_i] = 1/\lambda$  is a Poisson process of rate  $\lambda$

**Brownian Motion Process:** continuous-time, continuous-value process.  $X(0) = 0$  and for  $\tau > 0$ ,  $X(t + \tau) - X(t)$  is a Gaussian random variable with mean 0 and variance  $\alpha\tau$  independent of  $X(t')$  for all  $t' \leq t$ :

$$X(t + \delta) = X(t) + [X(t + \delta) - X(t)]$$

$$\text{PDF of } Y_\delta: P_{Y_\delta}(y) = \frac{1}{\sqrt{2\pi\alpha\delta}} e^{-y^2/2\alpha\delta}$$

$$\text{Joint PDF: } f_{X(t_1), \dots, X(t_k)}(x_1, \dots, x_k) = \prod_{i=1}^k \frac{1}{\sqrt{2\pi\alpha(t_i - t_{i-1})}} e^{-(x_i - x_{i-1})^2/2\alpha(t_i - t_{i-1})}$$

**General Properties:**

Random	Process	Sequence
EV	$\mu_X(t) = E[X(t)]$	n/a
Autocov.	$C_X(t, \tau) = \text{Cov}(X(t), X(t + \tau))$	$C_X[m, k] = \text{Cov}[X_m, X_{m+k}]$
Autocor.	$R_X(t, \tau) = E[X(t)X(t + \tau)]$	$R_X[m, k] = E[X_m, X_{m+k}]$

**Stationary Properties:**

Random	Process	Sequence
EV	$\mu_X(t) = \mu_X$	$E[X_m] = \mu_X$
Autocov.	$C_X(t, \tau) = R_X(\tau) - \mu_X^2 = C_X(\tau)$	$C_X[m, k] = R_X(\tau) - \mu_X^2$
Autocor.	$R_X(t, \tau) = R_X(0, \tau) = R_X(\tau)$	$R_X[m, k] = R_X[0, k] = R_X[k]$

**WSS Properties:**

Random	Process	Sequence
EV	$E[X(t)] = \mu_X$	$E[X_m] = \mu_X$
Autocor.	$R_X(t, \tau) = R_X(0, \tau) = R_X(\tau)$	$R_X[m, k] = R_X[0, k] = R_X[k]$

**Stationary Process:** A random process  $X(t)$  is stationary if and only if for all sets of time constants  $t, \dots, t_m$  and any time difference  $\tau$ ,  $f_{X(t_1), \dots, X(t_k)}(x_1, \dots, x_k) = f_{X(t_1 + \tau), \dots, X(t_k + \tau)}(x_1, \dots, x_k)$

**Stationary Sequence:**  $f_{X_{n_1}, \dots, X_{n_m}}(x_1, \dots, x_m) = f_{X_{n_1 + k}, \dots, X_{n_m + k}}(x_1, \dots, x_m)$

If  $(X(t)/X_n)$  is WSSP/WSRS:

Processes	Sequences
$R_X(0) \geq 0$ $R_X(\tau) = R_X(-\tau)$ $ R_X(\tau)  \leq R_X(0)$	$R_X(0) \geq 0$ $R_X(k) = R_X(-k)$ $ R_X(k)  \leq R_X(0)$

$$C_X(t, \tau) = R_X(t, \tau) - \mu_X(t)\mu_X(t + \tau)$$

$$\text{Average Power of WSSP: } R_X(0) = E[X^2(t)]$$

## Random Signal Processing

**LTI Filter Output:** For  $X(t)$  input to a LTI filter with impulse response  $h(t)$ , the output  $Y(t)$  is the convolution of the input  $X(t)$  with  $h(t)$

If the input to LTI filter with impulse response  $h(t)$  is WSSP  $X(t)$ , then WSSP output  $Y(t)$  has the following:  
 $\mu_Y = \mu_X H(0)$  (mean)  
 $R_Y(\tau) = \int_{-\infty}^{\infty} h(u) \int_{-\infty}^{\infty} h(v) R_X(\tau + u - v) dv du$  (auto-correl.)

**Power Spectral Density:** For a WSSP  $X(t)$ ,  $R_X(\tau)$  and the power spectral density  $S_X(f)$  are Fourier transform pairs:  
 $S_X(f) = \mathcal{F}\{R_X(\tau)\}$   
 $R_X(\tau) = \mathcal{F}^{-1}\{S_X(f)\}$   
 $S_X(f) \geq 0$  for all  $f$

**Properties of  $S_X(f)$ :**  
 $E[X^2(t)] = \int_{-\infty}^{\infty} S_X(f) df$ ,  $S_X(f) = S_X(-f)$   
 $S_Y(f) = S_X(f) |H(f)|^2$  where  $H$  is the input response

**Gaussian Process:**  $X(t)$  is a Gaussian random process if the joint PDF of  $X(t_1), \dots, X(t_k)$  has the multivariate Gaussian density  $f_{X(t_1), \dots, X(t_k)} = \frac{1}{(2\pi)^{k/2} |C|^{1/2}} \exp\{-\frac{1}{2}(X - \mu)^T C^{-1}(X - \mu)\}$

If  $X(t)$  is WSS Gaussian process, then  $X(t)$  is a stationary Gaussian process.

$X(t)$  is a Gaussian RV if  $Y = \int_0^T g(t)X(t)dt$  is a Gaussian random variable for every  $g(t)$  such that  $E[Y^2] < \infty$ .

**Properties of Gaussian Processes:** Passing a stationary Gaussian process  $X(t)$  through a linear filter  $h(t)$  yields as the output Gaussian random process  $Y(t)$  with the following properties.

Mean:  $\mu_Y = \mu_X H(0)$

Autocorrelation:  $R_Y(\tau) = \int_{-\infty}^{\infty} h(u) \int_{-\infty}^{\infty} h(v) R_X(\tau + u - v) dv du$

**White Gaussian Noise Processes:** Noise is modeled as stationary Gaussian random process  $W(t)$ , with no DC component.

$$E[W(t_1)] = \mu_W = 0$$

$$R_W(\tau) = 0 \text{ if } \tau \neq 0$$

**Power Spectral Density of  $W(t)$ :**  $S_W(f)$  is constant. The

**Independent Processes:** If  $X(t)$   $Y(t)$  are independent for any time sample of  $t_1, \dots, t_n$ , and  $t'_1, \dots, t'_m$ , then:

$$f_{X(t_1), \dots, X(t_n), Y(t'_1), \dots, Y(t'_m)}(x_1, \dots, x_n, y_1, \dots, y_m) = f_{X(t_1), \dots, X(t_n)}(x_1, \dots, x_n) f_{Y(t'_1), \dots, Y(t'_m)}(y_1, \dots, y_m)$$

**Cross Correlation Func:**  $R_{XY}(t, \tau) = E[(X(t)Y(t + \tau))]$

**Jointly WSSP:**  $R_{XY}(t, t + \tau) = R_{XY}(\tau)$  and  $R_{XY}(\tau) = R_{XY}(-\tau)$

**Cross Spectral Density:**  $S_{XY}(f) = \mathcal{F}\{R_{XY}(\tau)\}$

**Input-Output Cross Correlation:** When a WSSP  $X(t)$  is the input to a LTI filter  $h(t)$ , the input-output cross correlation is  $R_{XY}(t, t + \tau) = R_{XY}(\tau) = \int_{-\infty}^{\infty} h(u) R_X(\tau - u) du$ . Both  $X(t)$  and  $Y(t)$  are jointly wide sense stationary.

**Output Autocorrelation:** When a wide stationary process  $X(t)$  is the input to a LTI filter  $h(t)$ , the autocorrelation of the output  $Y(t)$  is  $R_Y(\tau) = \int_{-\infty}^{\infty} h(-w) R_X(\tau - w) dw$ .

**Cross Spectral Properties:** Let  $X(t)$  be a WSS input to a LTI filter  $H(f)$ . The input  $X(t)$  and output  $Y(t)$  satisfy:

$$S_{XY}(f) = H(f) S_X(f), \text{ and}$$

$$S_Y(f) = H(f) S_X(f)$$

constant is 0 unless  $R_W(\tau) = \frac{N_0}{2} \delta(\tau)$ .  $N_0$  is the power per unit bandwidth of  $W(t)$ .

**Average Noise Power:**  $E[W^2(t)] = R_W(0) = \infty$

**Noise process output:**  $Y(t) = \int_0^t h(t - \tau) W(\tau) d\tau = \text{Constant}$

## Random Helpful Stuff

Don't panic.

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n \quad \text{for } |x| < 1$$

$$\sin \alpha \sin \beta = \frac{1}{2} [\cos(\alpha - \beta) - \cos(\alpha + \beta)]$$

$$\cos \alpha \cos \beta = \frac{1}{2} [\cos(\alpha - \beta) + \cos(\alpha + \beta)]$$

$$\sin \alpha \cos \beta = \frac{1}{2} [\sin(\alpha + \beta) + \sin(\alpha - \beta)]$$

$$\int u dv = uv - \int v du.$$

$$\Gamma(a) = \int_0^{\infty} x^{a-1} e^{-x} dx$$

$$\Gamma(n) = (n-1)! \text{ if } n \in \mathbf{Z}^+$$

## Fourier Transforms

$$X(j\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt \Leftrightarrow x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\omega) e^{j\omega t} d\omega$$

$$e^{-a|t|}, \Re\{a\} > 0 \Leftrightarrow \frac{2a}{a^2 + \omega^2}$$

$$\cos(\omega_0 t + \theta) \Leftrightarrow \pi [e^{-j\theta} \delta(\omega + \omega_0) + e^{j\theta} \delta(\omega - \omega_0)]$$

$$e^{-\alpha t} \Leftrightarrow \frac{1}{\alpha + j\omega}$$

$$1 \Leftrightarrow 2\pi \delta(\omega)$$

$$-0.5 + u(t) \Leftrightarrow \frac{1}{j\omega}$$

$$\sin(\omega_0 t + \theta) \Leftrightarrow j\pi [e^{-j\theta} \delta(\omega + \omega_0) - e^{j\theta} \delta(\omega - \omega_0)]$$