

Counting

Order:	Matters	Doesn't Matter
w/ Replacement	n^k	$\binom{n+k-1}{k}$
wo/ Replacement	$n P_k = \frac{n!}{(n-k)!}$	$\binom{n}{k} = \frac{n!}{k!(n-k)!}$

Binomial: $(x + y)^n = \sum \binom{n}{k} x^k y^{n-k}$

Multinomial:

$$(x_1 + x_2 + \dots + x_r)^n = \sum_{\substack{(n_1, n_2, \dots, n_r): \\ n_1 + \dots + n_r = n}} \frac{n!}{n_1! n_2! \dots n_r!} x_1^{n_1} x_2^{n_2} \dots x_r^{n_r}$$

There are $\binom{n+k-1}{k-1}$ distinct nonnegative integer-valued vectors $\langle x_1, \dots, x_r \rangle$ satisfying $x_1 + \dots + x_k = n$

Probability

$(\bigcup_{i=1}^n E_i)^c = \bigcap_{i=1}^n E_i$ (DeMorgan's Law)

$(\bigcap_{i=1}^n E_i)^c = \bigcup_{i=1}^n E_i^c$ (DeMorgan's Law)

$P(E^c) = 1 - P(E)$ (complement)

$P(E \cup F) = P(E) + P(F) - P(EF)$ (double counting)

$P(E|F) = \frac{P(EF)}{P(F)}$ (conditional)

$P(E) = P(EF) + P(EF^c) = P(F)P(E|F) + P(F^c)P(E|F^c)$

$P(F^c)P(E|F^c) = P(F)P(E|F) + (1 - P(F))P(E|F^c)$

$P(E) = \sum_{i=1}^n P(E|F_i)P(F_i)$ (Law of Total Probability)

Two events are independent if $P(EF) = P(E)P(F)$

Bayes' Theorem: Suppose that F_1, F_2, \dots, F_n are mutually exclusive events such that $\bigcup_{i=1}^n F_i = S$ (sample space), then

$$P(F_j|E) = \frac{P(EF_j)}{P(E)} = \frac{P(E|F_j)P(F_j)}{\sum_{i=1}^n P(E|F_i)P(F_i)}$$

Random Variables

PMF: $p_X(a) = P\{X = a\}$

CDF: $F_X(a) = P\{X \leq a\} = \sum_{x \leq a} p_X(x)$

EV: $E[X] = \sum_x x p_X(x)$

EV of $g(X)$:

$E[g(X)] = \sum_x g(x) p_X(x)$

$Var(X) = E[X^2] - (E[X])^2$

$Var(aX + b) = a^2 Var(X)$

$SD(X) = \sqrt{Var(X)}$

Name	PMF	μ	Var
<i>Bern</i> (p)	$P\{X = 1\} = p, P\{X = 0\} = 1 - p$	p	$p(1 - p)$
<i>Bin</i> (n, p)	$P\{X = k\} = \binom{n}{k} p^k (1 - p)^{n-k}, k = 0, \dots, n$	np	$np(1 - p)$
<i>Geom</i> (p)	$P\{X = n\} = p(1 - p)^{n-1}, n = 0, 1, 2, \dots$	$\frac{1}{p}$	$\frac{1-p}{p^2}$
<i>Pois</i> (λ)	$P\{X = n\} = e^{-\lambda} \frac{\lambda^n}{n!}, n = 0, 1, \dots$	λ	λ
<i>NBin</i> (r, p)	$P\{X = r\} = \binom{n-1}{r-1} p^r (1 - p)^{n-r}, n = r, r + 1, \dots$	$\frac{r}{p}$	$r \frac{(1-p)}{p^2}$
h.g.	$P\{X = i\} = \frac{\binom{m}{i} \binom{N-m}{n-i}}{\binom{N}{n}}$ $i = 0, 1, 2, \dots, \min(n, m)$	$\frac{nm}{N}$	$\frac{N-n}{N-1} np(1 - p)$ $p = m/N$

PDF: $f_X(a) = \frac{1}{\epsilon} P\{a - \frac{\epsilon}{2} < X \leq a + \frac{\epsilon}{2}\}$

CDF: $F_X(a) = P\{X \leq a\} = \int_{-\infty}^a p_X(x) dx$

EV: $E[X] = \int_{-\infty}^{\infty} x f_X(x) dx$

EV of $g(X)$:

$E[g(X)] = \int_{-\infty}^{\infty} g(x) f_X(x) dx$

$P\{X \in B\} = \int B f_X(x) dx$

$f_X(x) = \frac{d}{dx} F_X(x)$

Name	PDF	CDF	μ	Var
<i>Uni</i> (a, b)	$f(x) = \begin{cases} \frac{1}{b-a} & \text{if } a < x < b \\ 0 & \text{otherwise} \end{cases}$	$F(x) = \begin{cases} 0 & x \leq a \\ \frac{x-a}{b-a} & \text{if } a < x < b \\ 1 & x \geq b \end{cases}$	$\frac{a+b}{2}$	$\frac{(b-a)^2}{12}$
<i>Expo</i> (λ)	$f(x) = \begin{cases} \lambda e^{-\lambda x} & \text{if } x \geq 0 \\ 0 & \text{otherwise} \end{cases}$	$F(x) = 1 - e^{-\lambda x} \text{ if } x \geq 0$	$\frac{1}{\lambda}$	$\frac{1}{\lambda^2}$
<i>Gamm</i> (a, λ)	$f(x) = \frac{1}{\Gamma(a)} (\lambda x)^{a-1} e^{-\lambda x} \frac{\lambda}{x}$???	a/λ	a/λ^2
<i>Norm</i> (μ, σ^2)	$f(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-(x-\mu)^2/2\sigma^2}$	$\phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^x e^{-u^2/2} du$	μ	σ^2

Continuous Random Variables

For any normal random variable X with parameters (μ, σ^2) , $Z = \frac{X-\mu}{\sigma}$ is the standardized normal random variable.

De Moivre-Laplace Limit Theorem: If S_n denotes the number of successes that occur when n independent trials, each resulting in a success with probability p , are performed

then, for any $a < b$, $P\{a \leq \frac{S_n - np}{\sqrt{np(1-p)}} \leq b\} \rightarrow \phi(b) - \phi(a)$

Memoryless RV: We say that a nonnegative random variable X is *memoryless* if $P\{X > s + t | X > t\} = P\{X > s\}$

Distribution of a Function of a RV: $f_Y(y) = f_X[g^{-1}(y)] \left| \frac{d}{dy} g^{-1}(y) \right|$ if $y = g(x)$ for some x

Jointly Distributed Random Variables¹

Joint PDF: $f_{XY}(a, b) = P\{X = a, Y = b\}$

Conditional PMF: $p_{X|Y}(x|y) = P\{X = x | Y = y\} = \frac{P\{X=x, Y=y\}}{P\{Y=y\}} = \frac{p_{XY}(x, y)}{p_Y(y)}$

$$\frac{P\{X=x, Y=y\}}{P\{Y=y\}} = \frac{p_{XY}(x, y)}{p_Y(y)}$$

Joint CDF: $F_{XY}(a, b) = P\{X \leq a, Y \leq b\}$

Conditional CDF: $F_{X|Y}(x|y) = P\{X \leq x | Y = y\} = \sum_{a \leq x} p_{X|Y}(a|y)$

$$\sum_{a \leq x} p_{X|Y}(a|y)$$

¹Replace summation with integration for continuous.

Marginal PMF of X: $p_X(a) = \sum_b p_{XY}(a, b)$
 Marginal PMF of Y: $p_Y(b) = \sum_a p_{XY}(a, b)$

Marginal CDF of X: $F_{XY}(a) = F_{XY}(a, \infty)$
 Marginal CDF of Y: $F_{XY}(b) = F_{XY}(\infty, b)$

Conditional Jointly Distributed RV

$F_{X A}(x) = \frac{F_X(x) - F_X(a)}{F_X(b) - F_X(a)}$ if $a \leq x < b$	$f_{X A}(x) = \frac{f_X(x)}{P(A)}$	$f_{X Y}(x, y) = \frac{f_{XY}(x, y)}{f_Y(y)}$
$E[X A] = \int_{-\infty}^{\infty} x f_{X A}(x) dx$	$E[g(X) A] = \int_{-\infty}^{\infty} g(x) f_{X A}(x) dx$	

Independent Random Variables: Two random variables X and Y are said to be independent if for every a and b we have $f_{XY}(a, b) = f_X(a) f_Y(b)$.

Summation of Independent Random Variables	CMF	$F_{X+Y}(a) = P\{X + Y \leq a\} = \int_{-\infty}^{\infty} F_X(a - y) f_Y(y) dy$
	PDF	$f_{X+Y}(a) = \int_{-\infty}^{\infty} f_X(a - y) f_Y(y) dy$

Sum of Normal RV: Mean and Variance are the summation of all the normal RV's mean and variance.

Jacobian Transformation (Joint Probability Distribution functions of RV)

$f_{Y_1, Y_2}(y_1, y_2) = \frac{f_{X_1, X_2}(x_1, x_2)}{ J(x_1, x_2) }$	$J(x_1, x_2) = \begin{vmatrix} \frac{\partial g_1}{\partial x_1} & \frac{\partial g_1}{\partial x_2} \\ \frac{\partial g_2}{\partial x_1} & \frac{\partial g_2}{\partial x_2} \end{vmatrix} = \frac{\partial g_1}{\partial x_1} \frac{\partial g_2}{\partial x_2} - \frac{\partial g_1}{\partial x_2} \frac{\partial g_2}{\partial x_1} \neq 0$	$y_1 = g_1(x_1, x_2)$ and $y_2 = g_2(x_1, x_2)$
---	---	---

Properties of Expectation

Joint PMF: $E[g(X, Y)] = \sum_y \sum_x g(x, y) p(x, y)$

Joint PDF: $E[g(X, Y)] = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g(x, y) p(x, y) dx dy$

$E[aX + b] = aE[X] + b$

$E[g(X)h(Y)] = E[g(X)]E[h(Y)]$ if X and Y are independent

Covariance:

$Cov(X, Y) = E[XY] - E[X]E[Y]$, $Cov(X, Y) = Cov(Y, X)$

$Cov(X, X) = Var(X)$, $Cov(aX, aY) = aCov(X, Y)$

$Cov(\sum_{i=1}^n X_i, \sum_{j=1}^m Y_j) = \sum_{i=1}^n \sum_{j=1}^m Cov(X_i, Y_j)$

$Var(\sum_{i=1}^n X_i) = \sum_{i=1}^n Var(X_i) + \sum_{i=1}^n \sum_{j \neq i} Cov(X_i, X_j)$

$Var(\sum_{i=1}^n X_i) = \sum_{i=1}^n Var(X_i)$ if X_1, \dots, X_n are p.w. ind.

Correlation: $\rho(X, Y) = \frac{Cov(X, Y)}{\sqrt{Var(X)Var(Y)}}$, $-1 \leq \rho(X, Y) \leq 1$

Conditional Expectation:

$E[X|Y = y] = \sum_x x p_{X|Y}(x, y)$

$E[g(X)|Y = y] = \sum_x g(x) p_{X|Y}(x, y)$

Conditioning Property: $E[X] = E[E[X|Y]]$

Expectation of X: $\sum_y E[X|Y = y] P\{Y = y\}$

Conditional Variance: $Var(X|Y) = E[X^2|Y] - (E[X|Y])^2$

Uncond. var. of X|Y: $E[Var(X|Y)] = E[X^2] - E[(E[X|Y])^2]$

Limit Theorems

Markov's Inequality: If X is a random variable that takes only nonnegative values, then for any value $a > 0$, $P\{X \geq a\} \leq \frac{E[X]}{a}$

Chebyshev's Inequality: If X is a random variable with finite mean μ and variance σ^2 , then for any value $k > 0$, $P\{|X - \mu| \geq k\} \leq \frac{\sigma^2}{k^2}$

The Weak Law of Large Numbers: Let X_1, X_2, \dots

$Var(E[X|Y]) = E[(E[X|Y])^2] - (E[X])^2$

$Var(X) = E[Var(X|Y)] + Var(E[X|Y])$

Predictors:

General form: $E[(Y - g(X))^2] \geq E[(Y - E[Y|X])^2]$ (best case: $g(X) = E[Y|X]$)

Linear: $E[(Y - (a + bX))^2] \geq E[(Y - E[Y|X])^2]$, where $b = \rho \frac{\sigma_Y}{\sigma_X}$ and $a = E[Y] - bE[X]$

Moment Generating Functions:

n^{th} moment: $M^n(0) = E[X^n]$ $n \geq 0$

MGF of PDF: $M(t) = E[e^{tX}] = \sum_{-\infty}^{\infty} e^{tx} p(x)$

MGF of PMF: $M(t) = E[e^{tX}] = \sum_{-\infty}^{\infty} e^{tx} f(x) dx$

More than 2 RV MGF: $M(t_1, \dots, t_n) = E[e^{t_1 X_1 + \dots + t_n X_n}]$

Individual MGF: $M_{X_i}(t) = M(0, \dots, 0, t, 0, \dots, 0)$

Joint MGF: $M_{XY}(t, t') = E[e^{tX + t'Y}] = \sum_y \sum_x x e^{tx + t'y} p_{XY}(x, y)$

MGF of sum of two IRVs: $M_{X+Y}(t) = M_X(t) M_Y(t)$

Common MGFs:

Binomial	$M(t) = (pe^t + 1 - p)^n$
Poisson	$M(t) = \exp\{\lambda(e^t - 1)\}$
Exponential	$M(t) = \frac{\lambda}{\lambda - t}$ for $t < \lambda$
Standard Normal	$M(t) = e^{t^2/2}$
Normal	$M(t) = \exp\{\frac{\sigma^2 t^2}{2} + \mu t\}$

be a sequence of iid Random variables, each having finite mean. Then, for any $\epsilon > 0$, $P\{|\bar{X} - \mu| \geq \epsilon\} \rightarrow 0$ where $\bar{X} = \frac{X_1 + X_2 + \dots + X_n}{n}$ as $n \rightarrow \infty$

The Central Limit Theorem: Let X_1, X_2, \dots be a sequence of iid random variables, each having mean and variance. Then, the distribution of $Z = \frac{X_1 + \dots + X_n - n\mu}{\sigma\sqrt{n}}$ tends to the standard normal as $n \rightarrow \infty$. That is, for $-\infty < a < \infty$, $P\{Z \leq a\} = \phi(a) \rightarrow N(0, 1)$ as $n \rightarrow \infty$

Random Processes

Random Processes: a collection of random variables usually indexed by time

Sample Function: the time function $x(t, s)$ associated with

the outcome s of an experiment

Ensemble: the set of all possible time function that can result from an experiment

Random Sequence: a random sequence X_n is an ordered sequence of random variables X_0, X_1, \dots

Bernoulli Process: $X(t)$ is a sequence of Bernoulli trials;

Poisson Process: Given $\lambda > 0$, a counting process $N(t), t \in [0, \infty)$ is called a Poisson process with rate λ if the following conditions hold:

→ $N(0) = 0$

→ $N(t)$ has independent and stationary increments

→ The # of arrivals in any interval of length $\tau > 0$ has $Pois(\lambda\tau)$ distribution

PMF of $M = N(t_1) - N(t_0)$:

$$p_M(m) = \begin{cases} \frac{[\lambda(t_1 - t_0)]^m}{m!} e^{-\lambda(t_1 - t_0)} & m = 0, 1, \dots \\ 0 & \text{otherwise} \end{cases}$$

Joint PMF of $N(t_1), \dots, N(t_k), t_1 < t_2 < \dots < t_k$:

$$p_{N(t_1), \dots, N(t_k)}(n_1, \dots, n_k) = \begin{cases} \frac{\alpha_1^{n_1} e^{-\alpha_1}}{n_1!} \frac{\alpha_1^{n_2 - n_1} e^{-\alpha_2}}{(n_2 - n_1)!} \dots \frac{\alpha_k^{n_k - n_{k-1}} e^{-\alpha_k}}{(n_k - n_{k-1})!} & m = 0, 1, \dots, \alpha_i = \lambda(t_i - t_{i-1}) \\ 0 & \text{otherwise} \end{cases}$$

For a Poisson process of rate λ , the inter-arrival times X_1, X_2, \dots are an iid random sequence with the exponential PDF.

$P\{X_n - x' > x | X_n > x'\} = P\{X_n > x\} = e^{-\lambda x}$ (P.P. is memoryless)

A counting process with **independent exponential inter-arrivals** X_1, X_2, \dots with mean $E[X_i] = 1/\lambda$ is a Poisson process of rate λ

Brownian Motion Process: continuous-time, continuous-value process. $X(0) = 0$ and for $\tau > 0, X(t + \tau) - X(t)$ is a Gaussian random variable with mean 0 and variance $\alpha\tau$ independent of $X(t')$ for all $t' \leq t$:

$$X(t + \delta) = X(t) + [X(t + \delta) - X(t)]$$

$$\text{PDF of } Y_\delta: P_{Y_\delta}(y) = \frac{1}{\sqrt{2\pi\alpha\delta}} e^{-y^2/2\alpha\delta}$$

$$\text{Joint PDF: } f_{X(t_1), \dots, X(t_k)}(x_1, \dots, x_k) = \prod_{i=1}^k \frac{1}{\sqrt{2\pi\alpha(t_i - t_{i-1})}} e^{-(x_i - x_{i-1})^2/2\alpha(t_i - t_{i-1})}$$

General Properties:

Random	Process	Sequence
EV	$\mu_X(t) = E[X(t)]$	n/a
Autocov.	$C_X(t, \tau) = \text{Cov}(X(t), X(t + \tau))$	$C_X[m, k] = \text{Cov}[X_m, X_{m_k}]$
Autocor.	$R_X(t, \tau) = E[X(t)X(t + \tau)]$	$R_X[m, k] = E[X_m, X_{m_k}]$

Stationary Properties:

Random	Process	Sequence
EV	$\mu_X(t) = \mu_X$	$E[X_m] = \mu_X$
Autocov.	$C_X(t, \tau) = R_X(\tau) - \mu_X^2 = C_X(\tau)$	$C_X[m, k] = R_X(\tau) - \mu_X^2$
Autocor.	$R_X(t, \tau) = R_X(0, \tau) = R_X(\tau)$	$R_X[m, k] = R_X[0, k] = R_X[k]$

WSS Properties:

Random	Process	Sequence
EV	$E[X(t)] = \mu_X$	$E[X_n] = \mu_X$
Autocor.	$R_X(t, \tau) = R_X(0, \tau) = R_X(\tau)$	$R_X[n, k] = R_X[0, k] = R_X[k]$

Stationary Process: A random process $X(t)$ is stationary if and only if for all sets of time constants t, \dots, t_m and any time difference $\tau, f_{X(t_1), \dots, X(t_k)}(x_1, \dots, x_k) = f_{X(t_1 + \tau), \dots, X(t_k + \tau)}(x_1, \dots, x_k)$

Stationary Sequence: $f_{X_{n_1}, \dots, X_{n_m}}(x_1, \dots, x_m) = f_{X_{n_1 + k}, \dots, X_{n_m + k}}(x_1, \dots, x_m)$

If $(X(t)/X_n)$ is WSSP/WSRS:

Processes	Sequences
$R_X(0) \geq 0$ $R_X(\tau) = R_X(-\tau)$ $ R_X(\tau) \leq R_X(0)$	$R_X(0) \geq 0$ $R_X(k) = R_X(-k)$ $ R_X(k) \leq R_X(0)$

$$C_X(t, \tau) = R_X(t, \tau) - \mu_X(t)\mu_X(t + \tau)$$

$$\text{Average Power of WSSP: } R_X(0) = E[X^2(t)]$$

trials are independent of each other ($P\{X_n = 1\} = p = 1 - P\{X_n = 0\}$)

Counting Process: Given a stochastic process $N(t)$:

$N(0) = 0, N(t) \in \{0, 1, 2, \dots\}$ for all $t \in [0, \infty)$

for $0 < s < t, N(t) - N(s)$ shows the # of events in $(s, t]$.

Random Signal Processing

LTI Filter Output: For $X(t)$ input to a LTI filter with impulse response $h(t)$, the output $Y(t)$ is the convolution of the input $X(t)$ with $h(t)$

If the input to LTI filter with impulse response $h(t)$ is WSSP $X(t)$, then WSSP output $Y(t)$ has the following:

$$\mu_Y = \mu_X H(0) \text{ (mean)}$$

$$R_Y(\tau) = \int_{-\infty}^{\infty} h(u) \int_{-\infty}^{\infty} h(v) R_X(\tau + u - v) dv du \text{ (auto-correl.)}$$

Power Spectral Density: For a WSSP $X(t)$, $R_X(\tau)$ and the power spectral density $S_X(f)$ are Fourier transform pairs:

$$S_X(f) = \mathcal{F}\{R_X(\tau)\}$$

$$R_X(\tau) = \mathcal{F}^{-1}\{S_X(f)\}$$

$$S_X(f) \geq 0 \text{ for all } f$$

Properties of $S_X(f)$:

$$E[X^2(t)] = \int_{-\infty}^{\infty} \infty S_X(f) df, S_X(f) = S_X(-f)$$

$$S_Y(f) = S_X(f) |H(f)|^2 \text{ where } H \text{ is the input response}$$

Gaussian Process: $X(t)$ is a Gaussian random process if the joint PDF of $X(t_1), \dots, X(t_k)$ has the multivariate Gaussian density $f_{X(t_1), \dots, X(t_k)} = \frac{1}{(2\pi)^{k/2} |C|^{1/2}} \exp\{-\frac{1}{2}(X - \mu x)^T C^{-1}(X - \mu x)\}$

If $X(t)$ is WSS Gaussian process, then $X(t)$ is a stationary Gaussian process.

$X(t)$ is a Gaussian RV if $Y = \int_0^T g(t) X(t) dt$ is a Gaussian random variable for every $g(t)$ such that $E[Y^2] < \infty$.

Properties of Gaussian Processes: Passing a stationary Gaussian process $X(t)$ through a linear filter $h(t)$ yields as the output Gaussian random process $Y(t)$ with the following properties.

Mean: $\mu_Y = \mu_X H(0)$

$$\text{Autocorrelation: } R_Y(\tau) = \int_{-\infty}^{\infty} h(u) \int_{-\infty}^{\infty} h(v) R_X(\tau + u - v) dv du$$

White Gaussian Noise Processes: Noise is modeled as stationary Gaussian random process $W(t)$, with no DC component.

$$E[W(t)1] = \mu_W = 0$$

$$R_W(\tau) = 0 \text{ if } \tau \neq 0$$

Power Spectral Density of $W(t)$: $S_W(f)$ is constant. The

Independent Processes: If $X(t)$ $Y(t)$ are independent for any time sample of t_1, \dots, t_n , and t'_1, \dots, t'_m , then:

$$f_{X(t_1), \dots, X(t_n), Y(t'_1), \dots, Y(t'_m)}(x_1, \dots, x_n, y_1, \dots, y_m)$$

$$= f_{X(t_1), \dots, X(t_n)}(x_1, \dots, x_n) f_{Y(t'_1), \dots, Y(t'_m)}(y_1, \dots, y_m)$$

Cross Correlation Func: $R_{XY}(t, \tau) = E[(X(t)Y(t + \tau))]$

Jointly WSSP: $R_{XY}(t, t + \tau) = R_{XY}(\tau)$ and $R_{XY}(\tau) = R_{XY}(-\tau)$

Cross Spectral Density: $S_{XY}(f) = \mathcal{F}\{R_{XY}(\tau)\}$

Input-Output Cross Correlation: When a WSSP $X(t)$ is the input to a LTI filter $h(t)$, the input-output cross correlation is $R_{XY}(t, t + \tau) = R_{XY}(\tau) = \int_{-\infty}^{\infty} h(u) R_X(\tau - u) du$. Both $X(t)$ and $Y(t)$ are jointly wide sense stationary.

Output Autocorrelation: When a wide stationary process $X(t)$ is the input to a LTI filter $h(t)$, the autocorrelation of the output $Y(t)$ is $R_Y(\tau) = \int_{-\infty}^{\infty} h(-w) R_{XY}(\tau - w) dw$.

Cross Spectral Properties: Let $X(t)$ be a WSS input to a LTI filter $H(f)$. The input $X(t)$ and output $Y(t)$ satisfy:

$$S_{XY}(f) = H(f) S_X(f), \text{ and}$$

$$S_Y(f) = H(f) S_{XY}(f)$$

Random Helpful Stuff

Don't panic.

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n \quad \text{for } |x| < 1$$

$$\sin \alpha \sin \beta = \frac{1}{2} [\cos(\alpha - \beta) - \cos(\alpha + \beta)]$$

$$\cos \alpha \cos \beta = \frac{1}{2} [\cos(\alpha - \beta) + \cos(\alpha + \beta)]$$

$$\sin \alpha \cos \beta = \frac{1}{2} [\sin(\alpha + \beta) + \sin(\alpha - \beta)]$$

$$\int u dv = uv - \int v du.$$

$$\Gamma(a) = \int_0^{\infty} x^{a-1} e^{-x} dx$$

$$\Gamma(n) = (n-1)! \text{ if } n \in \mathbf{Z}^+$$

Fourier Transforms

$$X(j\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt \Leftrightarrow x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\omega) e^{j\omega t} d\omega$$

$$e^{-a|t|}, \Re\{a\} > 0 \Leftrightarrow \frac{2a}{a^2 + \omega^2}$$

$$\cos(\omega_0 t + \theta) \Leftrightarrow \pi [e^{-j\theta} \delta(\omega + \omega_0) + e^{j\theta} \delta(\omega - \omega_0)]$$

$$e^{-\alpha t} \Leftrightarrow \frac{1}{\alpha + j\omega}$$

$$1 \Leftrightarrow 2\pi \delta(\omega)$$

$$-0.5 + u(t) \Leftrightarrow \frac{1}{j\omega}$$

$$\sin(\omega_0 t + \theta) \Leftrightarrow j\pi [e^{-j\theta} \delta(\omega + \omega_0) - e^{j\theta} \delta(\omega - \omega_0)]$$