Image Reconstruction in Magnetic Resonance Imaging

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A phenomenological model of Magnetic Resonance is derived from the bulk motion of hydrogen nuclei ($\mathbb{Z}$-$\frac{1}{2}$ spin) in a constant magnetic field, which results in a paramagnetic polarisation. In a macroscopic scale, only 'like' spins are considered, i.e., spins with the same orientation. When a radiofrequency field is applied at an angle normal to the initial magnetic field, the nuclei are forced to precess before relaxing to the equilibrium state. This has an interesting and very rich geometric structure, which will be discussed here. In the case of medical diagnostics, this precession allows an image to be generated, which can be reconstructed via a Fourier transform in order to highlight abnormalities so that a diagnosis may be placed. Hence we can use this to estimate the amount of time needed to run an MRI scan.

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I. INTRODUCTION

Nuclear Magnetic Resonance (NMR) is a physical phenomenon in which nuclei are excited by applying varying magnetic fields. Discovered in the mid-20$^{\text{th}}$ century, it was initially used as a method for measuring magnetic fields and nuclear magnetic moments. NMR eventually found applications across all areas of science in engineering, such as; NMR Spectroscopy, crystallography, determining the structure of atoms, acquiring data in petroleum exploration, and in medical imaging.

In medicine, NMR is used as a non-intrusive diagnostic tool known as Magnetic Resonance Imaging (MRI). MRI is used for detection of ailments in the brain, internal organs, muscles, and bones, as well analysis of the human body for pedagogical studies of the nervous system. It is particularly revered for its safety, since it does not emit any ionising radiation but harmless radio waves, which other human body imaging methods such as X-ray Computed Tomography do. Despite this, any ferromagnetic materials present in the sample (patient) may harm both the MRI machine and the sample.

Nowadays, MRI has been studied extensively and many techniques exist for reading and reconstructing images acquired. Most model-based techniques make use of the Fast Fourier Transform (FFT) algorithm, which reconstructs raw data in k-space in order to highlight any defects, although in rare cases, FFT may not be the optimal technique to choose, in which case one must seek other methods to improve the quality of the image.[6] What we seek to achieve is an optimal resolution so that the scan may be run in the shortest amount of time while still preserving the integrity of the image.

From an academic point of view, it is interesting to see how the geometry of the motion of nuclei in a magnetic field may be described using abstract mathematics. We will find that the equations that govern the motion of the nuclei possess a structure very similar to that of a rigid body precessing about the vertical axis, i.e., a spinning top. This will allow us to better understand how the motion may be visualised.

This paper is loosely based on a paper by Charles Epstein on the mathematics of Magnetic Resonance Imaging [5]. It begins with a brief overview of the physics of magnetic resonance and a derivation of the Bloch phenomenological equation. Then it presents a Hamiltonian formulation of the geometric structure of the motion by considering a Lie-Poisson bracket, and an analogy with the precessing motion of a rigid body. After that, we will look to constructing a method of reconstructing the density of nuclei using a Fourier transform generated by the solution to the Bloch Equation, in particular, using a technique known as spin-warp imaging. This is then implemented in Matlab code, which is a starting point for estimating the amount of time required to run an MRI scan.

II. FOUNDATIONS AND BASIC PHYSICS OF MRI

The underlying concepts of Magnetic Resonance are described quantum-mechanically, since nuclear spins in a magnetic field have quantised angular momenta, a phenomenon referred to as the Zeeman effect [4][10]. Using this, we may derive a phenomenological, macroscopic model which is in the form of a linear ODE. From the Zeeman effect, since the energy levels of nuclei are quantised, we suppose that a nucleus (which we can think of as a loop) possesses a magnetic dipole moment, defined $\mu = I\vec{A}$ where I is a current going through a loop of area $\vec{A}$.

In this model, we may define the **magnetisation** as the sum of all the moments μ with respect to their position in space in a volume V, which is small relative to the sample but large relative to the extent of the nuclei

$$\boldsymbol{M}(\boldsymbol{x}) = \frac{1}{V}\sum_{\boldsymbol{x}_i \in V} \mu(\boldsymbol{x}_i)$$

This quantity is normalised by rewriting the above equation with respect to the total number of nuclei in the space, namely

$$\boldsymbol{M}(\boldsymbol{x}) = \frac{N}{V}\left[\frac{1}{N}\sum_{\boldsymbol{x}_i \in V} \mu(\boldsymbol{x}_i)\right] = n(\boldsymbol{x})\langle\mu\rangle \tag{1}$$

where $n(\boldsymbol{x})$ is the number density of nuclei, and $\langle\mu\rangle$ is the average nuclear magnetic moment, which is equivalent to the ensemble average and independent of $\boldsymbol{x}$. The effect of applying an external magnetic field $B_0 = b_0\hat{\boldsymbol{z}}$ on a particle is to force the magnetic moment to precess along it. Then for the **measured magentic field** B there is also a contribution from the bulk magnetisation

$$\boldsymbol{B} = B_0 + \mu_0 \boldsymbol{M} \tag{2}$$

where μ_0 is the magnetic permeability in a vacuum. If we introduce a new quantity, $H = B_0/\mu_0$, as the **magnetic field strength**, then we can determine a dimensionless proportionality constant between $\boldsymbol{M}$ and H, known as the magnetic susceptibility [3]

$$\chi = \frac{\boldsymbol{M}}{H}$$

Then, from (1), $\boldsymbol{M}$ is averaged over the entire sample such that $\boldsymbol{M} = (N/V)|\langle\mu\rangle|$, so that equating this with $\boldsymbol{M} = \chi H = \chi B_0/\mu_0$, the total magnetic dipole moment may be written in terms of the susceptibility as

$$|\langle\mu\rangle| = \chi\left(\frac{V}{N}\right)\left(\frac{B_0}{\mu_0}\right)$$

The total dipole moment can then be used to get an expression for the equilibrium magnetisation. We may define this equilibrium magnetisation $M_0(\boldsymbol{x}) \equiv |\boldsymbol{M}(\boldsymbol{x})|$, then we may use our previous results to get

$$\begin{aligned} M_0(x) &= n(\boldsymbol{x})|\langle\mu\rangle| \\ &= n(\boldsymbol{x})\chi\left(\frac{V}{N}\right)\left(\frac{B_0}{\mu_0}\right) \\ &= \frac{\chi}{\mu_0}\rho(\boldsymbol{x})B_0 \end{aligned}$$

where $\rho(\boldsymbol{x})$ is the density of nuclei distributed in V.
Using the above results, we are ready to derive the differential equation for the motion of nuclei in a magnetic field. When a magnetic field $\boldsymbol{B}$ is applied at an angle to I, a torque is exerted on μ, such that $\tau = \mu \times \boldsymbol{B}$.

The torque on a magnetic dipole moment is defined

$$\frac{d\mu}{dt} = \frac{ge}{2m}J$$

The factor $ge/2m$ is the gyromagnetic ratio for a nucleus. Here, e denotes the charge on the nucleus and m is the mass of the nucleus. g is the so-called g-factor, which is a dimensionless quantity that characterises the magnetic moment of a nucleus [8]. We will label this ratio γ, and we have an exact value for it,

$$\gamma = 42.5764 \times 10^6 \frac{\text{rad}}{\text{s} \cdot \text{Tesla}}$$

Since torque is by definition the rate of change of angular momentum, i.e. $\tau = dJ/dt$, the equation for precession (known as Larmor equation) may be written as (with $\boldsymbol{B}$ as some magnetic field)

$$\frac{d\mu}{dt} = \gamma\mu \times \boldsymbol{B} \tag{3}$$

or, for the bulk magnetisation,

$$\frac{d\boldsymbol{M}}{dt} = \gamma\boldsymbol{M} \times \boldsymbol{B}.$$

which is the leading term in the precession equation, which has solution (derived in Appendix 1),

$$\boldsymbol{M}(\boldsymbol{x}, t) = \begin{pmatrix} \cos\omega_0 t & -\sin\omega_0 t & 0 \\ \sin\omega_0 t & \cos\omega_0 t & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

where ω_0 is the angular frequency, known as the Larmor frequency.

Under the effect of a static magnetic field $b_0\hat{\boldsymbol{z}}$ the magnetisation is in thermal equilibrium when it is oriented along the z-axis. However, in a real-life application we would be working with a variable magnetic field, which would force the magnetisation away from equilibrium. This implies that we would require some timeframe for the magnetisation to approach equilibrium after applying a magnetic field along some direction perpendicular to $\hat{\boldsymbol{z}}$[8], which gives

$$\frac{dM_z}{dt} = \frac{M_0 - \boldsymbol{M}^{\|}}{T_1}$$

where T_1 is known as the spin-lattice, or longitudinal, relaxation time and $M^{\|} = M_z$ is the magnetisation along the $\hat{\boldsymbol{z}}$ axis.

If the components transverse to $b_0\hat{\boldsymbol{z}}$, M_x and M_y, are non-zero, then they will decay to zero in order to return to thermal equilibrium. This implies that there is an extra term for the components in the x and y directions

$$\begin{aligned} \frac{dM_x}{dt} &= \gamma(\boldsymbol{M} \times \boldsymbol{B})_x - \frac{M_x}{T_2} \\ \frac{dM_y}{dt} &= \gamma(\boldsymbol{M} \times \boldsymbol{B})_y - \frac{M_y}{T_2} \end{aligned}$$

where T_2 is the spin-spin, or transverse, relaxation time. Generally T_2 is rapid compared to $T1$ and both of these times depend on the properties of the sample being scanned; greater for fluids and smaller for solids. Labelling $M^{\perp} = (M_x, M_y)$, the three equations above reduce to just one

$$\frac{d\boldsymbol{M}}{dt}(\boldsymbol{x}, t) = \gamma \boldsymbol{M}(\boldsymbol{x}, t) \times \boldsymbol{B} - \frac{1}{T_2}\boldsymbol{M}^{\perp}(\boldsymbol{x}, t) + \frac{1}{T_1}(M_0(\boldsymbol{x}) - \boldsymbol{M}^{\|}(\boldsymbol{x}, t)) \tag{4}$$

which is called the **Bloch phenomenological equation**. The solution to this equation is found in the same way as the Larmor equation, and it is

$$\begin{aligned} \boldsymbol{M}(\boldsymbol{x}, t) &= M_0(\boldsymbol{x})[e^{-t/T_2}\cos\omega_0 t, -e^{-t/T_2}\sin\omega_0 t, 1 - e^{-t/T_1}] \\ &= \frac{\chi b_0 \rho(\boldsymbol{x})}{\mu_0}[e^{-t/T_2}\cos\omega_0 t, -e^{-t/T_2}\sin\omega_0 t, 1 - e^{-t/T_1}] \end{aligned} \tag{5}$$

III. MATHEMATICAL DESCRIPTION OF PRECESSION

In this section, we will analyse the geometric structure of the Larmor precession, and we will show that the magnetisation is a conserved quantity under the effect of a magnetic field.
We start with the equation for the magnetisation $\boldsymbol{M}$ precessing about some magnetic field $\boldsymbol{B}$,

$$\frac{d\boldsymbol{M}}{dt} = \boldsymbol{M} \times \boldsymbol{B}$$

Indeed these are, in general, some arbitrary vectors which depend on space and possibly time. Since the magnetisation is some vector in $\mathbb{R}^3$ we will think of it as some unit vector $\boldsymbol{e}$. There are several important theorems which may be proved about this motion.

Theorem 1. *Under the evolution equation*

$$\frac{d\boldsymbol{e}}{dt} = \gamma \boldsymbol{e} \times B, \qquad \boldsymbol{e}_0(\mathbf{x}) = \boldsymbol{e}(t = 0)$$

the norm of $\boldsymbol{e}$ *is preserved, i.e.,* $|\boldsymbol{e}(t)| = |\boldsymbol{e}_0|, \ \forall \ t > 0$

Proof. Take the dot product of the equation as

$$\begin{aligned} \tfrac{1}{2}\frac{d}{dt}|\boldsymbol{e}|^2 &= \boldsymbol{e} \cdot \frac{d\boldsymbol{M}}{dt} \\ &= \boldsymbol{e} \cdot (\gamma \boldsymbol{e} \times B_0) \\ &= 0 \end{aligned}$$

From which we can see that $|\boldsymbol{e}(t)| = |\boldsymbol{e}_0| = \text{Const.}$

We will focus on a unidirectional constant magnetic field $B = b_0 \hat{\boldsymbol{z}}$. Under this condition the precession have solutions as a matrix $U(t)$,

Theorem 2. *The equation*

$$\frac{d\boldsymbol{e}}{dt} = \gamma b_0 \boldsymbol{e} \times \hat{\boldsymbol{z}} \tag{6}$$

has solution

$$\boldsymbol{e}(t) = U(t)\boldsymbol{e}_0$$

where we call

$$U(t) = \begin{pmatrix} \cos\omega_0 t & -\sin\omega_0 t & 0 \\ \sin\omega_0 t & \cos\omega_0 t & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

a ***solution operator*** *with* $\omega_0 = \gamma b_0$

Proof. We prove this by differentiating $\boldsymbol{e}(t)$

$$
\begin{aligned}
\frac{d\boldsymbol{e}_j}{dt} &= \dot{U}_{ij}\,\boldsymbol{e}_i(0) \\
&= \dot{U}_{ij}U^{-1}_{jk}\,\boldsymbol{e}_k && \boldsymbol{e}(0) = U^{-1}\boldsymbol{e}(t) \\
&= \dot{U}_{ij}U^{T}_{jk}\boldsymbol{e}_k && \text{orthogonality} \Rightarrow U^{-1} = U^T \\
&= \omega_0\epsilon_{ikm}(\hat{\boldsymbol{z}} \cdot \boldsymbol{e}_m\,\boldsymbol{e}_k) \\
&= \frac{1}{2}\omega_0\epsilon_{ikm}(\hat{\boldsymbol{z}} \cdot \boldsymbol{e}_m)\boldsymbol{e}_k - \frac{1}{2}\omega_0\epsilon_{ikm}(\hat{\boldsymbol{z}} \cdot \boldsymbol{e}_k)\boldsymbol{e}_m \\
&= -\frac{1}{2}\omega_0\epsilon_{ikm}(\hat{\boldsymbol{z}} \times (\boldsymbol{e}_k \times \boldsymbol{e}_m)) && (\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = \mathbf{b}(\mathbf{a} \cdot \mathbf{c}) - \mathbf{c}(\mathbf{a} \cdot \mathbf{b})) \\
&= -\omega_0\hat{\boldsymbol{z}} \times M_i \\
&= \gamma b_0\,\boldsymbol{e} \times \hat{\boldsymbol{z}}
\end{aligned}
$$

This theorem bears some interesting consequences, which delve further into the geometry of the motion of precession. Firstly, there are some important definitions that we must recall

Definition. A **topological space** is a set X consisting of a family of subsets S called open sets, which satisfy the following conditions:

1. The empty set $\emptyset$ and X itself are open.
2. $\bigcup S$ is open
3. $\bigcap S_i$ is open for $i = 1, \ldots, n$

Definition. A **smooth manifold** is a topological space which inherits its structure from an infinitely differentiable Euclidean space C^∞.

Definition. A **group** is a set G paired with some operation $\circ$ which assigns the following conditions to some pair of elements $g, h \in G$:

1. $f \circ (g \circ h) = (f \circ g) \circ h$, for $f \in G$ (**associative**)
2. $\exists e \in G$ such that $e \circ g = g \circ e = g$, (**identity**)
3. $\forall g \in G\ \exists\ g^{-1} \in G$ such that $g \circ g^{-1} = g^{-1} \circ g = e$, (**inverse**)

Moreover, a **Lie group** is a smooth manifold which is compatible with group operations.

An example of a Lie group is the group of orthogonal 3×3 matrices with determinant one. It is known as the **special orthogonal** group $SO(3)$ [9].

Now are now ready to state the results that follow from Theorem 2.

Corollary 3. *The solution operator in Theorem 2 is a subgroup of the Lie group of rotation matrices $U(t) \in SO(3)$ in one parameter t. Hence if $\widehat{u} = (dU/dt)_{t=0}$ denotes an element in the Lie algebra $so(3)$ then (6) may be written as*

$$\frac{d\boldsymbol{e}}{dt} = \widehat{u}\boldsymbol{e}_0$$

Corollary 4. *Solutions to Equation (6) with initial position $\boldsymbol{e}(t=0) = \boldsymbol{e}_0$ are trajectories on the sphere in $\mathbb{R}^3$, centred at zero, with radius $|\boldsymbol{e}_0|$*

Theorem 5. *Corollary 4 suggests that the geometric structure of this precessing motion may be described by the following Hamiltonian*

$$\mathcal{H} = -\gamma \boldsymbol{x} \cdot b_0 \widehat{\boldsymbol{z}}$$

and a Lie-Poisson bracket,

$$\frac{d\boldsymbol{e}}{dt} = \{\mathcal{H}, \boldsymbol{e}\}$$

where the bracket is defined, for all scalar fields $\mathcal{F}$ on $\mathbb{R}^3$

$$\begin{aligned} \{\cdot,\cdot\} : \mathcal{F}(\mathbb{R}^3) \times \mathcal{F}(\mathbb{R}^3) &\to \mathbb{R}, \\ (F, G) &\mapsto F, G := -\boldsymbol{x} \cdot (\nabla_x F \times \nabla_x G), \end{aligned}$$

The Hamiltonian in this case depends only on the phase space defined by the dimensions of $\mathbb{R}^3$. For the above bracket to be well-defined, some important algebraic identities [2] must hold for some functions in the phase space $F, G, H \in \mathbb{R}^3$

$$\{G, F\} = -\{F, G\} \qquad \textbf{(antisymmetric)}$$
$$\{F + H, G\} = \{F, G\} + \{H, G\} \qquad \textbf{(bilinear)}$$
$$\{F, GH\} = \{F, G\}H + G\{F, H\} \qquad \textbf{(Lebniz)}$$
$$\{F, \{G, H\}\} + \{G, \{H, F\}\} + \{H, \{F, G\}\} = 0 \qquad \textbf{(Jacobi)}$$

which are proven in Appendix C.

What we will show now is that there are some conserved quantities in this motion, specifically that the angular momentum is conserved. For this, we must define a notion of commutation for the Poisson bracket, or more precisely, a structure which forces the Poisson bracket to commute

Definition. On a manifold $\mathcal{P}$ with a Poisson bracket $\{\cdot,\cdot\}$ defined, two functions $F, G : \mathcal{P} \to \mathbb{R}$ are said to **Poisson commute** if

$$\{F, G\} = 0.$$

A function that Poisson commutes with every function in $\mathcal{P}$ is called a **Casimir** (function), which will be denoted $c(\boldsymbol{x})$, so that

$$\{c, F\} = \{F, c\} = 0 \qquad \forall F \in \mathcal{P}$$

For the motion here, there are two Casimirs that we may identify. The first of which is, trivially, $\mathcal{H}$ itself. The second is $|\boldsymbol{e}|^2$, i.e.,

$$\frac{d}{dt}|\boldsymbol{e}|^2 = \frac{d}{dt}c(\boldsymbol{e}) = \{c(\boldsymbol{e}), \mathcal{H}(\boldsymbol{e})\} = 0$$

which shows that the squared angular momentum is a conserved quantity. Hence, the motion takes place on the intersection between a plane $\mathcal{H} = \text{Const.}$ with normal vector $\hat{\boldsymbol{z}}$ and a sphere. So the motion is exactly the precession.

An analogy with rigid body motion

The motion of a nuclear magnetic dipole moment precessing about a magnetic field is analogous with the motion of a rigid body rotating about a vertical axis. Here, instead of being given by a gyromagnetic ratio with a magnetic field, the energy is described by a construct known as the **moment of inertia matrix** and an angular frequency which are defined

$$\mathcal{I}_O = \begin{pmatrix} I_{xx} & -I_{xy} & -I_{xz} \\ -I_{yx} & I_{yy} & -I_{yz} \\ -I_{zx} & -I_{zy} & I_{zz} \end{pmatrix}$$

and $\boldsymbol{\omega}$ respectively. Then the angular momentum about the origin O is defined

$$\mathcal{L}_O = [\mathcal{I}_O]\boldsymbol{\omega}.$$

And the equation of motion for a rigid body is

$$\mathcal{I}_O \dot{\boldsymbol{e}} = \mathcal{I}_O \boldsymbol{e} \times \boldsymbol{e}$$

where $\boldsymbol{e}$ is the body angular velocity vector and the dot denotes differentiation with respect to time, or equivalently

$$\begin{aligned} I_{xx}\dot{e}_1 &= (I_{yy} - I_{zz})e_2e_3 \\ I_{yy}\dot{e}_2 &= (I_{zz} - I_{xx})e_3e_1 \\ I_{zz}\dot{e}_3 &= (I_{xx} - I_{zz})e_1e_2 \end{aligned}$$

which are the Euler equations of motion for a rigid body.

From this, we can re-express the equation of motion as a dynamical system

$$\dot{\boldsymbol{e}}(t) = \mathbf{F}(\boldsymbol{x})$$

or, in Hamiltonian form

$$\dot{\boldsymbol{e}}(t) = \{\boldsymbol{e}, \mathcal{H}\}$$

in terms of a Poisson bracket

$$\{\cdot,\cdot\} : \mathcal{F}(\mathbb{R}^3) \times \mathcal{F}(\mathbb{R}^3) \to \mathbb{R}, \\ (F, G) \mapsto F, G := -\boldsymbol{x} \cdot (\nabla_{\boldsymbol{x}} F \times \nabla_{\boldsymbol{x}} G),$$

and Hamiltonian

$$\mathcal{H} = \frac{1}{2}\boldsymbol{x} \cdot \mathcal{I}^{-1}\boldsymbol{x}.$$

This is very much the same as nuclear precession, only this time the motion takes place on the intersection between an ellipsoid and a sphere. [7]

IV. MAGNETIC RESONANCE IMAGING, STEP-BY-STEP

In this section, we describe a basic imaging experiment which will set the system to equilibrium, then apply a transverse magnetic field in a rotated reference frame, and finally apply an elegant technique known as spin-warp imaging. So far, we have explained NMR only in the case of a constant magnetic field pointing in one direction. However, in real MRI the applied magnetic field is variable in time and space. We may define the magnetic field in the following way:

$$\boldsymbol{B} = B_0 + B_1 + \widehat{\mathrm{G}}$$

where B_0 is the same constant magnetic field $b_0\hat{\boldsymbol{z}}$ as in the previous sections,

$$B_1 = U(t)\begin{pmatrix}\alpha(t)\\ \beta(t)\\ 0\end{pmatrix}.$$

is the radiofrequency (RF) frequency where $U(t)$ is the rotation matrix that was found previously and $\alpha(t)$ and $\beta(t)$ define an envelope that modulates the harmonic field $[\cos\omega_0 t, \sin\omega_0 t, 0]$, and

$$\widehat{\mathrm{G}} = (0, 0, \mathrm{g}\cdot\boldsymbol{x})\hat{\boldsymbol{z}}$$

is a gradient field along the initial magnetic field B_0 with g a constant vector. These two quantities are useful in the different steps during an MRI scan.

Equilibration and rotation

At the beginning of the experiment, a uniform magnetic field $B_0 = b_0\hat{\boldsymbol{z}}$ is applied to the sample for a time from $t = -\infty$ to $t = 0$. The relaxation to an equilibrium state $M_0\hat{\boldsymbol{z}}$ is described by the previously derived Bloch equation

$$\frac{d\boldsymbol{M}}{dt}(\boldsymbol{x}, t) = \gamma\boldsymbol{M}(\boldsymbol{x}, t) \times \boldsymbol{B} - \frac{1}{T_2}\boldsymbol{M}^{\perp}(\boldsymbol{x}, t) + \frac{1}{T_1}(M_0(\boldsymbol{x}) - \boldsymbol{M}^{\parallel}(\boldsymbol{x}, t)).$$

In the absence of dissipation, the solution to this is, to reiterate,

$$M(\boldsymbol{x}, t) = U(t)M(\boldsymbol{x}, 0)$$

where $U(t) = \mathrm{Rot}_z(\omega_0 t) \in SO(3)$ is a rotation matrix with the Larmor frequency $\omega_0 = \gamma b_0$.

At the end of this step, the magnetisation vector is purely along the z-axis. After this step, a different magnetic field is applied from time $t = 0$ until some time $t = T_{\mathrm{rf}}$ so that the magnetisation will lie on the xy-plane. This new magnetic field is

$$B_{\mathrm{app}} = B_0 + B_1$$

with terms as defined in the previous section.

After the system is in equilibrium, the RF field B_1 is applied normal to the incident magnetic field B_0 which transforms the magnetisation into the rotating frame of reference by setting $\boldsymbol{m}(\boldsymbol{x}, t) = U(t)^{-1}\boldsymbol{M}(\boldsymbol{x}, t)$ and modifies the equation of motion as

$$\frac{d\boldsymbol{m}}{dt}(\boldsymbol{x}, t) = \gamma[\boldsymbol{m}(\boldsymbol{x}, t) \times \boldsymbol{B}_{\mathrm{eff}}] - \frac{1}{T_2}\boldsymbol{m}(\boldsymbol{x}, t)^{\perp} + \frac{1}{T_1}[M_0(\boldsymbol{x}) - \boldsymbol{m}^{\parallel}(\boldsymbol{x}, t)]$$

where

$$\boldsymbol{B}_{\mathrm{eff}} = U(t)^{-1}\boldsymbol{B} - (0, 0, B_0) = U(t)^{-1}B_1.$$

For the purpose of this experiment, we will set $\beta(t) = 0$ and $T_1 = T_2 = \infty$ so then the solution to this transformed equation is

$$\boldsymbol{m}(\boldsymbol{x}, t) = V(t)\boldsymbol{m}(\boldsymbol{x}, t), \qquad V(t) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta(t) & \sin\theta(t) \\ 0 & -\sin\theta(t) & \cos\theta(t) \end{pmatrix}$$

where $\theta(t) = \int_0^t \alpha(s)ds$ for $t \in [0, T_{\mathrm{rf}}]$ is the angle by which the magnetisation is rotated during the RF pulse.
The information found here may be used to determine both α and T_{rf}. First we assume that $\alpha =$ Const. and that the magnetisation lies only along the x-axis at the end of the RF step, i.e.

$$\begin{aligned} \boldsymbol{M}(\boldsymbol{x}, t = T_{\mathrm{rf}}) &= \boldsymbol{M}(\boldsymbol{x}, t = 0)\hat{\boldsymbol{x}} = (1, 0, 0)^T \\ &= U(T_{\mathrm{rf}})V(T_{\mathrm{rf}})\boldsymbol{m}(\boldsymbol{x}, t = 0) \\ &= U(T_{\mathrm{rf}})V(T_{\mathrm{rf}})(0, 0, 1)^T. \end{aligned}$$

Then if we set $\omega T_{\mathrm{rf}} = \varphi_0$ and $\alpha_0 T_{\mathrm{rf}} = \theta_0$ both constant, the matrix equation above then becomes

$$\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} \cos\varphi_0 & -\sin\varphi_0 & 0 \\ \sin\varphi_0 & \cos\varphi_0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta_0 & \sin\theta_0 \\ 0 & -\sin\theta_0 & \cos\theta_0 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

and through matrix multiplication, there remain two equations

$$\cos\theta_0 = 0, \qquad -\sin\varphi_0 \sin\theta_0 = 1$$

from which we find that

$$\theta_0 = (\tfrac{1}{2} + p)\pi, \qquad \varphi_0 = (2q - p - \tfrac{1}{2})\pi, \qquad \forall p, q \in \mathbb{Z}.$$

The values for p and q we are free to choose as long as they satisfy $p > 0$ and $(2q-p-\frac{1}{2}) > 0$. From these two values, an expression for α_0 can be determined if we isolate T_{rf}:

$$\begin{aligned}
\frac{\theta_0}{\alpha_0} &= \frac{\varphi_0}{\omega_0} \\
\alpha_0 &= \omega_0 \left(\frac{\theta_0}{\varphi_0}\right) \\
&= \omega_0 \left(\frac{2p+1}{4q-2p-1}\right).
\end{aligned}$$

Then using this, and $\alpha_0 T_{\text{rf}} = \theta_0$, we may compute an expression for the RF pulse operation time:

$$T_{\text{rf}} = \frac{2\pi}{\omega_0}(4q - 2p - 1)$$

These quantities are useful when building a computational model of an MRI scan since they allow us to set more precise numerical values.
Now that the system has relaxed to equilibrium and rotated, we will now see how the effect of a gradient field will generate a signal as a Fourier transform.

Spin-warp imaging

After applying the RF pulse, the magnetisation will be aligned in the x-direction,

$$\boldsymbol{M}(\boldsymbol{x}, T_{\text{rf}}) = (1, 0, 0)^T$$

To acquire the signal for an MRI scan, the applied magnetic field is taken to be $B_{\text{ap}} = (b_0 + \mathrm{g}\cdot\boldsymbol{x})\hat{\boldsymbol{z}}$. Here, g is a constant vector and we take $|\mathrm{g}\cdot\boldsymbol{x}| \ll b_0$. In the laboratory frame, the Bloch equations may then be written as:

$$\begin{aligned}
\frac{dM_x}{dt} &= \gamma(b_0 + \mathrm{g}\cdot\boldsymbol{x})M_y - \frac{1}{T_2}M_x \\
\frac{dM_y}{dt} &= \gamma(b_0 + \mathrm{g}\cdot\boldsymbol{x})M_x - \frac{1}{T_2}M_y \\
\frac{dM_z}{dt} &= \frac{1}{T_2}(M_0(\boldsymbol{x}) - M_z)
\end{aligned}$$

where

$$M_0(\boldsymbol{x}) = \frac{\rho(\boldsymbol{x})\chi(b_0 + \mathrm{g}\cdot\boldsymbol{x})}{\mu_0}$$

is the magnetisation aligned along the direction of an externally imposed magnetic field. The equations above are solved after resetting the clock, starting at $t = 0$, and using the condition that $|\mathrm{g}\cdot\boldsymbol{x}| \ll b_0$ the solution comes out as

$$\boldsymbol{M}(\boldsymbol{x},t) = \frac{\rho(\boldsymbol{x})\chi b_0}{\mu_0}[e^{-t/T2}\cos(\omega_0 t + \gamma\ell t), -e^{-t/T_2}\sin(\omega_0 t + \gamma\ell t), 1 - e^{-t/T_1}], \qquad \ell(\boldsymbol{x}) = \mathrm{g}\cdot\boldsymbol{x}$$

with the frame rotating at the Larmor frequency $\omega_0 = \gamma b_0$. In the rotated reference frame, the magnetisation reads

$$\begin{aligned}\boldsymbol{m}(\boldsymbol{x},t) &= U(t)^{-1}\boldsymbol{M}(\boldsymbol{x},t)\\ &= \frac{\rho(\boldsymbol{x})\chi b_0}{\mu_0}e^{-t/T_2}\begin{pmatrix}\cos\omega_0 t & \sin\omega_0 t & 0\\ -\sin\omega_0 t & \cos\omega_0 t & 0\\ 0 & 0 & 1\end{pmatrix}\begin{pmatrix}\cos(\omega_0 t + \gamma\ell t)\\ -\sin(\omega_0 t + \gamma\ell t)\\ 0\end{pmatrix}\\ &\qquad + \frac{\rho(\boldsymbol{x})\chi b_0}{\mu_0}(1 - e^{-t/T_1})U(t)^{-1}\begin{pmatrix}0\\0\\1\end{pmatrix}.\end{aligned}$$

In the rotated frame, the focus is solely on the xy plane, so we can ignore the terms aligned along $\hat{\boldsymbol{z}}$, so then we get

$$\begin{aligned}\boldsymbol{m}^{\parallel}(\boldsymbol{x},t) &:= (\mathbb{I}_3 - \hat{\boldsymbol{z}}\otimes\hat{\boldsymbol{z}})\boldsymbol{m}(\boldsymbol{x},t)\\ &= \frac{\rho(\boldsymbol{x})\chi b_0}{\mu_0}e^{-t/T_2}\begin{pmatrix}\cos\omega_0 t & \sin\omega_0 t\\ -\sin\omega_0 t & \cos\omega_0 t\end{pmatrix}\begin{pmatrix}\cos(\omega_0 t + \gamma\ell t)\\ -\sin(\omega_0 t + \gamma\ell t)\end{pmatrix}\\ &= \frac{\rho(\boldsymbol{x})\chi b_0}{\mu_0}e^{-t/T_2}\begin{pmatrix}\cos\omega_0 t & \sin\omega_0 t\\ -\sin\omega_0 t & \cos\omega_0 t\end{pmatrix}\begin{pmatrix}\cos\omega_0 t & -\sin\omega_0 t\\ \sin\omega_0 t & \cos\omega_0 t\end{pmatrix}\begin{pmatrix}\cos\gamma\ell t\\ \sin\gamma\ell t\end{pmatrix}\\ &= \frac{\rho(\boldsymbol{x})\chi b_0}{\mu_0}e^{-t/T_2}\begin{pmatrix}\cos\gamma\ell t\\ \sin\gamma\ell t\end{pmatrix}\end{aligned}$$

What we hope to achieve is to measure a two-dimensional Fourier transform along an infinitesimal 'slice' of the density profile $\rho(\boldsymbol{x})$, which will allow us to reconstruct the image using an inverse Fourier transform.

At time $t = 0$ a gradient $\mathrm{g} = (g_1, -g_2)$ is switched on, and it is left in effect until a time $t = T_{\mathrm{ph}}$. During this time, the transverse components of the magnetisation read

$$\boldsymbol{m}^{\parallel}(\boldsymbol{x},t) = \frac{\rho(\boldsymbol{x})\chi b_0}{\mu_0}e^{-t/T_2}\begin{pmatrix}\cos(\gamma(-g_2 y + g_1 x)t)\\ \sin(\gamma(-g_2 y + g_1 x)t)\end{pmatrix}, \qquad t\in[0, T_{\mathrm{ph}}]$$

and when a measurement is taken at $t = T_{\text{ph}}$, we identify $(k_x, k_y) = \gamma T_{\text{ph}}(g_1, -g_2)$ and the magnetisation is written

$$\begin{aligned}\boldsymbol{m}^{\|}(\boldsymbol{x}, T_{\text{ph}}) &\propto \begin{pmatrix}\cos(k_x x + k_y y)\\ \sin(k_x x + k_y y)\end{pmatrix}\\ &\propto \begin{pmatrix}\cos(\mathbf{k}\cdot\boldsymbol{x})\\ \sin(\mathbf{k}\cdot\boldsymbol{x})\end{pmatrix}.\end{aligned}$$

This gradient in this step is called a **phase encoding** gradient and as it is applied over the entire plane in order to rotate the magnetisation.

At time $t = T_{\text{ph}}$ the phase encoding gradient is switched off and a different gradient, $\mathrm{G}_{\text{fr}} = (0, 0, -g_1 x)$ is applied so that the measured magnetic field reads

$$\boldsymbol{B} = (b_0 + (-g_1 x))\hat{\boldsymbol{z}}.$$

This is called a **frequency encoding** gradient and it is applied to the plane line-by-line with the y-component fixed. During this time, the magnetisation reads

$$\boldsymbol{m}^{\|}(\boldsymbol{x}, t) \propto e^{-t/T_2}\begin{pmatrix}\cos\gamma\ell' t & -\sin\gamma\ell' t\\ \sin\gamma\ell' t & \cos\gamma\ell' t\end{pmatrix}\boldsymbol{m}^{\|}(\boldsymbol{x}, T_{\text{ph}}), \qquad t > T_{\text{ph}}$$

where $\ell' = -g_1 x$. Then we can rewrite the above as

$$\begin{aligned}\boldsymbol{m}^{\|}(\boldsymbol{x}, t) &\propto e^{-t/T_2}\begin{pmatrix}\cos\gamma\ell' t & -\sin\gamma\ell' t\\ \sin\gamma\ell' t & \cos\gamma\ell' t\end{pmatrix}\begin{pmatrix}\cos(\mathbf{k}\cdot\boldsymbol{x})\\ \sin(\mathbf{k}\cdot\boldsymbol{x})\end{pmatrix}\\ &= e^{-t/T_2}\begin{pmatrix}\cos(\gamma\ell' t)\cos(\mathbf{k}\cdot\boldsymbol{x}) - \sin(\gamma\ell' t)\sin(\mathbf{k}\cdot\boldsymbol{x})\\ \sin(\gamma\ell' t)\cos(\mathbf{k}\cdot\boldsymbol{x}) + \cos(\gamma\ell' t)\sin(\mathbf{k}\cdot\boldsymbol{x})\end{pmatrix}\\ &= e^{-t/T_2}\begin{pmatrix}\cos(\gamma\ell' t + \mathbf{k}\cdot\boldsymbol{x})\\ \sin(\gamma\ell' t + \mathbf{k}\cdot\boldsymbol{x})\end{pmatrix}\end{aligned}$$

and the argument in the trigonometric functions above can be computed explicitly as

$$\begin{aligned}\gamma\ell' t + \mathbf{k}\cdot\boldsymbol{x} &= \gamma(-g_1 x t) + \gamma(-g_2 y + g_1 x)T_{\text{ph}},\\ &= \gamma g_1 x(t - T_{\text{ph}}) - \gamma g_2 y T_{\text{ph}},\\ &= \gamma g_1 x\Delta t + k_y y, \qquad \Delta t = t - T_{\text{ph}}, \qquad t > T_{\text{ph}}\\ &= (\gamma g_1 \Delta t, k_y)\cdot\boldsymbol{x}.\end{aligned}$$

Now, we can collect these results and rewrite the magnetisation in the laboratory frame as

$$\boldsymbol{M}^{\|}(\boldsymbol{x}, t) = \frac{\rho(\boldsymbol{x})\chi b_0}{\mu_0}e^{-t/T_2}\begin{pmatrix}\cos\omega_0 t & -\sin\omega_0 t\\ \sin\omega_0 t & \cos\omega_0 t\end{pmatrix}\begin{pmatrix}\cos((\gamma g_1\Delta t, k_y)\cdot\boldsymbol{x})\\ \sin((\gamma g_1\Delta t, k_y)\cdot\boldsymbol{x})\end{pmatrix}$$

To measure the MRI signal, we take a sample enclosed in a two-dimensional loop in the yz plane of area A. The width of the loop is assumed infinitesimal in x and has an outward pointing unit normal $\boldsymbol{x}$. Then the measured magnetic field passing through the loop is

$$\boldsymbol{B} = B_{\text{ap}} + \mu_0 \boldsymbol{M}$$

with B_{ap} constant, then

$$\frac{d\boldsymbol{B}}{dt} = \mu_0 \frac{d\boldsymbol{M}}{dt}.$$

Using this, an expression for the signal $\mathcal{E}$ may be computed using E.M.F. defined by Faraday's law.

$$\begin{aligned}\mathcal{E} &= -\frac{d}{dt}\int_A \boldsymbol{B}\cdot \mathrm{d}A \\ &= -\mu_0 \int_A \frac{d\boldsymbol{M}}{dt}\cdot \mathrm{d}A \\ &= -\mu_0 \int_A \mathrm{d}y\mathrm{d}z \frac{dM_x}{dt}\end{aligned}$$

Using the definition of the Larmor frequency, the applied frequency can be defined

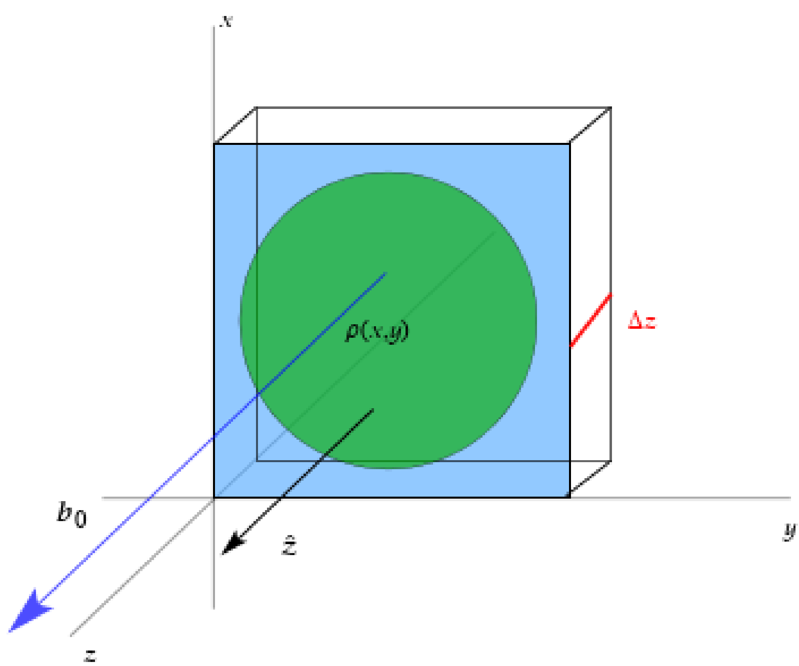

FIG. 1. A current loop is in the yz plane, with an outward unit normal $\hat{\boldsymbol{x}}$

$$\omega = \gamma B_{\text{ap}} = \gamma(b_0 + \mathrm{g}\cdot\boldsymbol{x})\hat{\boldsymbol{z}} = \omega_0 + \gamma\mathrm{g}\cdot\boldsymbol{x}.$$

Then, from the solution to the Bloch equation (5), we can write the E.M.F. as

$$\begin{aligned}\mathcal{E} &= -\int_A [\chi\rho(\boldsymbol{x})(b_0 + \mathrm{g}\cdot\boldsymbol{x}]\,\frac{d}{dt}[\cos(\omega t)]\mathrm{d}y\mathrm{d}x \\ &= -\chi\omega_0(b_0 + \mathrm{g}\cdot\boldsymbol{x})\int_A \rho(\boldsymbol{x})\sin(\omega_0 t + \mathrm{g}\cdot\boldsymbol{x}t)\mathrm{d}y\mathrm{d}z.\end{aligned}$$

Since we are considering $\mathrm{g}\cdot\boldsymbol{x} \ll b_0$, the above expression can be written as

$$\mathcal{E} = -\frac{\chi\omega_0^2}{\gamma}\iint_{-\infty}^{\infty}\mathrm{d}y\mathrm{d}z\rho(\boldsymbol{x})\sin(\omega_0 t + \mathrm{g}\cdot\boldsymbol{x}t)$$

which is in a form of the imaginary part of a Fourier transform. With $\rho(\boldsymbol{x}) = \rho(y, z)$ in this case, the above may be rewritten as

$$\mathcal{E} = \frac{\chi\omega_0^2}{\gamma}\Im\left\{FT[\rho(y,z)e^{\mathrm{i}\omega_0 t}]_{\mathbf{k}}\right\}, \qquad \mathbf{k} = -\mathrm{g}t$$

where the Fourier transform is defined

$$FT[\rho(y,z)]_k = \iint_{-\infty}^{\infty}\mathrm{d}y\mathrm{d}z\rho(y,z)e^{-\mathrm{i}\mathbf{k}\cdot\boldsymbol{x}}. \qquad \mathbf{k} = (k_x, k_y).$$

Using this, we can find an expression for the signal generated after a round of spin warp, which is

$$\mathcal{E}_{\mathrm{sw}} = \frac{\chi\omega_0^2}{\gamma}\int_A \rho(\boldsymbol{x})\cos(\omega_0 t + (\gamma g_1\Delta t, k_y)\cdot\boldsymbol{x})\mathrm{d}x\mathrm{d}y.$$

This step goes on until some time $t = T_{\mathrm{sw}}$ after which the clock is reset to $t = 0$ and the system is returned to equilibrium state.

V. A NUMERICAL MODEL FOR SPIN-WARP IMAGING

The tools used in the previous section allow us to build a computational model for the spin-warp imaging technique, the code for which may be found in Appendix D. For this code, a discrete $N_x \times N_y$ lattice of length (L_x, L_y) in k-space is considered with N_x and N_y even, such that the lattice is defined such that it is centred at the origin, as well as to account for the zeroth row and column:

$$\mathcal{K} = \left\{(k_x, k_y) = \left(\frac{\pi}{L_x}i, \frac{\pi}{L_y}j\right)\middle| i = -\tfrac{1}{2}N_x, \cdots, \tfrac{1}{2}N_x - 1, j = -\tfrac{1}{2}N_y, \cdots, \tfrac{1}{2}N_x - 1\right\}$$

First, a wavenumber is selected from $\mathcal{K}$, so that

$$k_y = -\frac{\pi}{L_y}\frac{N_y}{2},$$

and since $k_y = -\gamma g_2 T_{\text{ph}}$, we get that

$$g_2 = \frac{\pi N_y}{2L_y \gamma T_{\text{ph}}}$$

as an expression for the gradient. And for the wavenumber on x, we get

$$k_x = \gamma g_1 \Delta t, \qquad \Delta t = t - T_{\text{ph}}, \qquad t > T_{\text{ph}}$$

This gradient would traverse along the points N_x at discrete times $t_1, \cdots, t_j, \cdots$ as

$$\gamma g_1 t_1 = 1\Delta k_x, \qquad \cdots \qquad \gamma g_1 t_j = j\Delta k_x, \qquad \cdots$$

where $\Delta k_x = \pi / L_x$ up until the loop reaches the edge of the sublattice $j = (N_x/2) - 1$. Then the signal is acquired via a Fourier transform as a time series through the lattice, keeping only the real part of the Fourier transform

$$\begin{aligned}
&FT[\rho(\boldsymbol{x})e^{\mathrm{i}\omega_0 t_1}](\Delta k_x, -(N_y/2)\Delta k_y), \\
&\qquad\qquad FT[\rho(\boldsymbol{x})e^{\mathrm{i}\omega_0 t_2}](2\Delta k_x, -(N_y/2)\Delta k_y), \cdots \\
&\qquad\qquad FT[\rho(\boldsymbol{x})e^{\mathrm{i}\omega_0 t_j}](j\Delta k_x, -(N_y/2)\Delta k_y), \cdots \\
&\qquad\qquad\quad FT[\rho(\boldsymbol{x})e^{\mathrm{i}\omega_0 t_{(N_x/2)-1}}]((((N_x/2) - 1)\Delta k_x, -(N_y/2)\Delta k_y)
\end{aligned}$$

After this is finished, the gradient fields are switched off, the system is relaxed back to equilibrium state and is rotated by another RF pulse. Then spin warp starts over again for the next point $k_y = (\pi/L_y)[-(N_y/2) + 1]$. This cycle is repeated until the end of the lattice at $k_y = (\pi/L_y)[(N_y/2) - 1]$, which gives us knowledge of the right-half plane of k-space. The left-half plane is described by the imaginary part of the Fourier transform, so as the system keeps cycling through the spin-warp process, we will eventually have determined the Fourier transform for the entirety of $\mathcal{K}$ and hence the density $\rho(\boldsymbol{x})$ may be reconstructed by using the inverse FFT algorithm.

In order to determine the time taken to run a full MRI scan, the clock must start at $t = 0$ when the system is first relaxed to equilibrium. Then, the RF field is applied for some time $t \in [0, T_{\text{rf}}]$, for which we have found that

$$T_{\text{rf}} = \frac{2\pi}{\omega_0}(4q - 2p - 1)$$

where we are free to choose $p, q \in \mathbb{Z}$ as long as they satisfy the conditions mentioned when we derived them. Then the time for the spin warp step is added to T_{rf}, which we will call T_{sw}. This is repeated for each sublattice in N_y until the entirety of $\rho(\boldsymbol{x})$ is transformed.

In the toy model, an attempt is made to implement spin-warp imaging for one iteration. The code uses a sine density profile

$$\rho(x, y) = 1 + 0.5 \sin(k_x x) \sin(k_y y)$$

defined on a 50×50 grid of length $(L_x, L_y) = (1, 1)$ where $(k_x, k_y) = (2\pi/L_x, 3\pi/L_y)$ are two wave numbers. Then the gradient field is applied such that the final time T_{sw} may be written:

$$\begin{aligned} g_1 &= 1, \qquad \Rightarrow T_{\text{sw}} = \frac{\pi}{2L_x \gamma g_1} \\ g_2 &= -\frac{3\pi}{L_y \gamma T_{\text{sw}}}. \end{aligned}$$

Then we implement the code as described in the last part of Section IV. The signal generated from this is then compared with the analytical solution

$$\mathcal{E} = e^{-t/T_2} \int_A \rho(\boldsymbol{x}) \cos(\omega_0 t + \mathbf{k} \cdot \boldsymbol{x}) \mathrm{d}x \mathrm{d}y$$

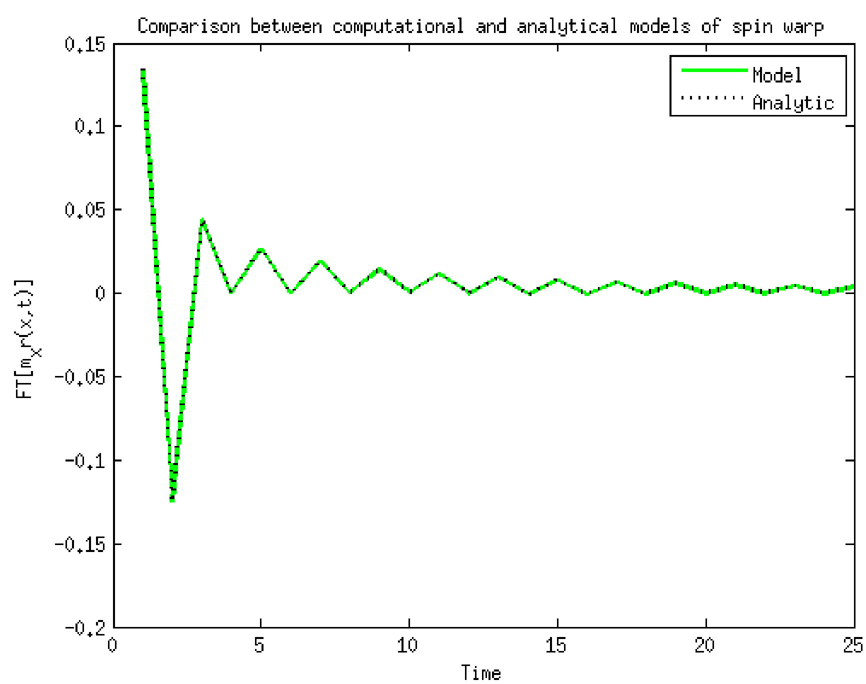

FIG. 2. A comparison between the analytical and computational models of spin-warp imaging along one line of the density profile

VI. CONCLUSIONS

The mathematical description of the motion of nuclei in a magnetic field allows us to further examine the geometric structure and symmetries of the motion without long calculations, only using the axioms of groups and manifolds.

On looking through the calculation of the sequencing in an MRI scan gives us a good idea of how we may calculate the time taken to run such a scan, and thus use this to find an optimal resolution so that the scan may be run in the shortest amount of time while preserving the integrity of the raw data.

The computational model seems to coincide with the analytical solution, although some further investigation may be required in order to implement it to a more complete model of an MRI scan, that part would be suitable for implementation.

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Appendix A: Derivation of Rotation Matrix solution

$$\frac{d\boldsymbol{M}}{dt} = \gamma \boldsymbol{M} \times B \text{ where } \gamma = \frac{ge}{2m}$$
$$\frac{d\boldsymbol{M}}{dt} = \gamma \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ M_x & M_y & M_z \\ B_x & B_y & B_z \end{vmatrix}$$

In our case, the magnetic field is purely along the z-axis, so we define $B = \left(\begin{smallmatrix} 0 \\ 0 \\ b_0 \end{smallmatrix}\right)$ so our equations become

$$\frac{d\boldsymbol{M}}{dt} = \gamma \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ M_x & M_y & M_z \\ 0 & 0 & b_0 \end{vmatrix} = \gamma(M_y b_0 - M_x b_0)$$
$$\Rightarrow \frac{d}{dt}\begin{pmatrix} M_x \\ M_y \\ M_z \end{pmatrix} = -\omega_0 \begin{pmatrix} M_y \\ -M_x \\ 0 \end{pmatrix} \text{ where } \omega_0 = -\gamma b_0$$

Mapping the $\mathbb{R}^2$-plane to a $\mathbb{C}$-plane, we can label $\widehat{\boldsymbol{M}} = M_x + \mathrm{i} M_y$ and rewrite our equation as

$$\begin{aligned} \frac{d\widehat{\boldsymbol{M}}}{dt} &= -\omega_0(M_y - \mathrm{i}M_x) \\ &= -\mathrm{i}\omega_0(\frac{M_y}{\mathrm{i}} - M_x) \\ &= -\mathrm{i}\omega_0(-\mathrm{i}M_y - M_x) \\ &= \mathrm{i}\omega_0(M_x + \mathrm{i}M_y) = \mathrm{i}\omega_0 \widehat{\boldsymbol{M}} \end{aligned}$$

We can now solve this equation by integrating.

$$\begin{aligned} \int \frac{d\widehat{\boldsymbol{M}}(t)}{dt} dt &= \mathrm{i}\omega_0 \int \widehat{\boldsymbol{M}}(t) dt \\ \ln(\widehat{\boldsymbol{M}}(t)) &= \mathrm{i}\omega_0 t + \widehat{\boldsymbol{M}}(0) \\ \Rightarrow M_\dagger(t) &= M_\dagger(0) e^{\mathrm{i}\omega_0 t} \\ &= (M_x + \mathrm{i}M_y)(\cos\omega_0 t + \mathrm{i}\sin\omega_0 t) \\ \Rightarrow M_x + iM_y &= (M_x \cos\omega_0 t - M_y \sin\omega_0 t + \mathrm{i}(M_x \sin\omega_0 t + M_y \cos\omega_0 t) \end{aligned}$$

Equating the real and imaginary parts, and mapping back to $\mathbb{R}^2$, we get that

$$\begin{aligned} \begin{pmatrix} M_x \\ M_y \\ M_z \end{pmatrix} &= \begin{pmatrix} M_x(0)\cos\omega_0 t - M_y(0)\cos\omega_0 t \\ M_x(0)\sin\omega_0 t + M_y(0)\sin\omega_0 t \\ M_z \end{pmatrix} \\ \Rightarrow M &= U(t) M_0 \end{aligned}$$

where

$$U(t) = \begin{pmatrix} \cos\omega_0 t & -\sin\omega_0 t & 0 \\ \sin\omega_0 t & \cos\omega_0 t & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

For a general, variable magnetic field

Starting with the solution to the Bloch phenomenological equation

$$\frac{\boldsymbol{M}(\boldsymbol{x},t)}{dt} = \gamma \boldsymbol{M}(\boldsymbol{x},t) \times B(\boldsymbol{x},t) - \frac{1}{T_2}\boldsymbol{M}^{\perp}(\boldsymbol{x},t) + \frac{1}{T_1}(M_0(\boldsymbol{x}) - \boldsymbol{M}^{\parallel}(\boldsymbol{x},t))$$

with the magnetic field B oriented along the z-axis. With $U(t)$ from A

$$\boldsymbol{M}(\boldsymbol{x},t) = U(t)\boldsymbol{M}(\boldsymbol{x},0)$$

We identified

$$\boldsymbol{B}(\boldsymbol{x},t) = B_0 + \widehat{\mathrm{G}} + B_1$$

in the previous derivation. Here we have $\widehat{\mathrm{G}} = (0, 0, \mathrm{g} \cdot \boldsymbol{x})$, $B_0 = (0, 0, B_0)^\top$ and $B_1(t) = U(t)(\alpha(t), \beta(t), 0)^\top$. In our specific case, we set $\mathbf{G} = (0, 0, l(x, y, z))$ where $l(\cdot)$ is a function. From this, following the same procedure as the first derivation, we get

$$\widehat{M}(t) = \boldsymbol{M}(0)e^{\gamma l(x,y,z)t}$$
$$\Rightarrow \boldsymbol{M}(\boldsymbol{x},t) = U_l(t)\boldsymbol{M}_0(\boldsymbol{x})$$

where

$$U_l(t) = \begin{pmatrix} \cos\gamma l(x,y,z)t & -\sin\gamma l(x,y,z)t & 0 \\ \sin\gamma l(x,y,z)t & \cos\gamma l(x,y,z)t & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Appendix B: The radiofrequency field

We start with a solution $\boldsymbol{m}(\boldsymbol{x},t) = U^{-1}\boldsymbol{M}(\boldsymbol{x},t)$, or more specifically $\boldsymbol{M}(\boldsymbol{x},t) = U\boldsymbol{m}(\boldsymbol{x},t)$ and the solution for B as B_0+B_1 where $B_0 = \left(\begin{smallmatrix} 0 \\ 0 \\ b_0 \end{smallmatrix}\right)$ and $B_1 = \left(\begin{smallmatrix} \alpha(t) \\ \beta(t) \\ 0 \end{smallmatrix}\right)$. Subbing the solution for $M(\boldsymbol{x},t)$ into Bloch's equation, we get that

$$\frac{d}{dt}(U\boldsymbol{m}(\boldsymbol{x},t)) = \gamma[(U\boldsymbol{m}(\boldsymbol{x},t)) \times B] - \frac{1}{T_2}(U\boldsymbol{m}(\boldsymbol{x},t)^{\perp}) + \frac{1}{T_1}(M_0(\boldsymbol{x}) - (U\boldsymbol{m}^{\parallel}(\boldsymbol{x},t))$$

is the form of the equation on a rotating frame. To eliminate the U terms, we multiply across by U^{-1}.

$$U^{-1}\frac{d}{dt}(U\boldsymbol{m}(\boldsymbol{x},t)) = \gamma U^{-1}[(U\boldsymbol{m}(\boldsymbol{x},t)) \times B] - \frac{1}{T_2}U^{-1}(U\boldsymbol{m}(\boldsymbol{x},t)^{\perp}) + \frac{1}{T_1}U^{-1}(M_0(\boldsymbol{x}) - (U\boldsymbol{m}^{\parallel}(\boldsymbol{x},t))$$

Recall that for rotation matrices $U^{-1}U = U^{\top}U = \mathbb{I}_3$ [1]. The last two terms are trivial. For the remaining terms we get

$$\begin{aligned}
U^{-1}U\frac{d\boldsymbol{m}}{dt} + U^{-1}\frac{dU}{dt}\boldsymbol{m} &= \gamma[\boldsymbol{m} \times (U^{-1}B)] \\
\frac{dm}{dt} &= \gamma[\boldsymbol{m} \times (U^{-1}B)] - U^{-1}\frac{dU}{dt}\boldsymbol{m} \\
&= \gamma[\boldsymbol{m} \times (U^{-1}B)] - \omega_0 U^{-1}U\dot{U}\boldsymbol{m} \text{ where } \dot{U} = \begin{pmatrix} -\sin(\omega_0 t) & -\cos(\omega_0 t) & 0 \\ \cos(\omega_0 t) & -\sin(\omega_0 t) & 0 \\ 0 & 0 & 0 \end{pmatrix} \\
&= \gamma[\boldsymbol{m} \times (U^{-1}B)] - \omega_0 \mathcal{V}\boldsymbol{m} \text{ where } \mathcal{V} = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \\
&= \gamma[\boldsymbol{m} \times (U^{-1}B)] - \omega_0 \begin{pmatrix} -m_x \\ m_y \\ 0 \end{pmatrix} \\
&= \gamma[\boldsymbol{m} \times (U^{-1}B)] - [\boldsymbol{m} \times (0, 0, \omega_0)] \\
&= [\boldsymbol{m} \times (U^{-1}B - (0, 0, \frac{\omega_0}{\gamma})] \\
\Rightarrow \frac{d\boldsymbol{m}}{dt} &= \boldsymbol{m} \times B_{\text{eff}} \text{ where } B_{\text{eff}} = U^{-1}(t)(0, 0, \frac{\omega_0}{\gamma})
\end{aligned}$$

Now we can write our original Bloch equation as

$$\frac{d\boldsymbol{m}(\boldsymbol{x},t)}{dt} = \gamma[\boldsymbol{m}(\boldsymbol{x},t) \times (B_{\text{eff}})] - \frac{1}{T_2}\boldsymbol{m}(\boldsymbol{x},t)^{\perp} + \frac{1}{T_1}[M_0(\boldsymbol{x}) - (\boldsymbol{m}^{\parallel}(\boldsymbol{x},t))]$$

Now we solve it with $T_1 = T_2 = \infty$ and $B_0 = 0$ and $\beta(t) = 0$. So now we solve for

$$\begin{aligned}
\frac{d\boldsymbol{m}}{dt} &= \boldsymbol{m} \times B_{\text{eff}} \\
&= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ m_x & m_y & m_z \\ \alpha(t) & 0 & 0 \end{vmatrix} \\
&= -m_z\alpha(t)\hat{j} + m_y\alpha(t)\hat{k} \\
\Rightarrow \frac{d}{dt}\begin{pmatrix} m_x \\ m_y \\ m_z \end{pmatrix} &= \alpha(t)\begin{pmatrix} 0 \\ -m_z \\ m_y \end{pmatrix}
\end{aligned}$$

Now, mapping the yz $\mathbb{R}^2$-plane to the $\mathbb{C}$-plane, we can identify $\boldsymbol{m}(t) = \widehat{m}(t) = m_y + \mathrm{i}m_z$ and substituting into the above equation, we will get

$$\begin{aligned}
\frac{d\widehat{m}(t)}{dt} &= \alpha(t)(-m_z + \mathrm{i}m_y) \\
&= \mathrm{i}\alpha(t)(-\frac{m_z}{\mathrm{i}} + m_y) \\
&= \mathrm{i}\alpha(t)(m_y + \mathrm{i}m_z) \\
&= \mathrm{i}\alpha(t)\widehat{m}(t)
\end{aligned}$$

Then, integrating the equation, we get

$$\begin{aligned}
\int_0^t \frac{d\widehat{m}(s)}{dt} ds &= \mathrm{i}\int_0^t \alpha(s)\widehat{m}(s)ds \\
&= \mathrm{i}\int_0^t \alpha(s)ds \int_0^t \widehat{m}(s)ds \\
&= \mathrm{i}\theta(t)\int_0^t \widehat{m}(s)ds \text{ where } \theta(t) = \int_0^t \alpha(s)ds \text{ is the rotation about the x-axis} \\
\ln(\widehat{m}(t)) &= \mathrm{i}\theta(t)t \\
\Rightarrow \widehat{m}(t) &= \widehat{m}(0)e^{\mathrm{i}\theta(t)t} \\
&= (m_y + \mathrm{i}m_z)(\cos\theta(t) + \mathrm{i}\sin\theta(t))) \\
m_y + \mathrm{i}m_z &= m_y\cos\theta(t) - m_z\sin\theta(t) + \mathrm{i}(m_y\sin\theta(t) + m_z\cos\theta(t))
\end{aligned}$$

Equating real and imaginary parts and mapping back to $\mathbb{R}^3$ we get

$$\begin{aligned}
\begin{pmatrix} m_x \\ m_y \\ m_z \end{pmatrix} &= V(t)\begin{pmatrix} m_x \\ m_y \\ m_z \end{pmatrix} \\
\Rightarrow \boldsymbol{m}(\mathbf{r}, t) &= V(t)\boldsymbol{m}(0)
\end{aligned}$$

where

$$V(t) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta(t) & \sin\theta(t) \\ 0 & -\sin\theta(t) & \cos\theta(t) \end{pmatrix}$$

Appendix C: The Lie Bracket

The following is the proof of Theorem 5. For any two functions in the phase space $F(\boldsymbol{x}), G(\boldsymbol{x}) \in \mathbb{R}^3$

$$
\begin{aligned}
\{\cdot,\cdot\} : \mathcal{F}(\mathbb{R}^3) \times \mathcal{F}(\mathbb{R}^3) &\to \mathbb{R} \\
(F,G) \mapsto \{F,G\} &= -\boldsymbol{x} \cdot (\nabla_x F \times \nabla_x G)
\end{aligned}
$$

1. **Antisymmetric:** We want to show that $\{G,F\} = -\{F,G\}$

$$
\begin{aligned}
\{G,F\} &= -\boldsymbol{x} \cdot (\nabla_x G \times \nabla_x F) \\
&= -\boldsymbol{x} \cdot \det \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial G}{\partial x} & \frac{\partial G}{\partial y} & \frac{\partial G}{\partial z} \\ \frac{\partial F}{\partial x} & \frac{\partial F}{\partial y} & \frac{\partial F}{\partial z} \end{vmatrix} \\
&= -\boldsymbol{x} \cdot \Bigg[\left(\frac{\partial G}{\partial y}\frac{\partial F}{\partial z} - \frac{\partial G}{\partial z}\frac{\partial F}{\partial y} \right) \hat{i} + \left(\frac{\partial G}{\partial z}\frac{\partial F}{\partial x} - \frac{\partial G}{\partial x}\frac{\partial F}{\partial z} \right) \hat{j} \\
&\qquad\qquad + \left(\frac{\partial G}{\partial x}\frac{\partial F}{\partial y} - \frac{\partial G}{\partial y}\frac{\partial F}{\partial x} \right) \hat{k} \Bigg] \\
&= \boldsymbol{x} \cdot \Bigg[\left(\frac{\partial G}{\partial z}\frac{\partial F}{\partial y} - \frac{\partial G}{\partial y}\frac{\partial F}{\partial z} \right) \hat{i} + \left(\frac{\partial G}{\partial x}\frac{\partial F}{\partial z} - \frac{\partial G}{\partial z}\frac{\partial F}{\partial x} \right) \hat{j} \\
&\qquad\qquad + \left(\frac{\partial G}{\partial y}\frac{\partial F}{\partial x} - \frac{\partial G}{\partial x}\frac{\partial F}{\partial y} \right) \hat{k} \Bigg] \\
&= \boldsymbol{x} \cdot (\nabla_x F \times \nabla_x G) \\
&= -\{F,G\}
\end{aligned}
$$

2. **Bilinear:** For any function H, we want to show that $\{F+H,G\} = \{F,G\} + \{H,G\}$

$$
\begin{aligned}
\{F+H,G\} &= -\boldsymbol{x} \cdot (\nabla_x (F+H) \times \nabla_x G) \\
&= -\boldsymbol{x} \cdot \det \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial (F+H)}{\partial x} & \frac{\partial (F+H)}{\partial y} & \frac{\partial (F+H)}{\partial z} \\ \frac{\partial G}{\partial x} & \frac{\partial G}{\partial y} & \frac{\partial G}{\partial z} \end{vmatrix}
\end{aligned}
$$

$$
\begin{aligned}
&= -\boldsymbol{x}\cdot\Bigg[\left(\frac{\partial(F+H)}{\partial y}\frac{\partial G}{\partial z}-\frac{\partial(F+H)}{\partial z}\frac{\partial G}{\partial y}\right)\hat{i}+\left(\frac{\partial(F+H)}{\partial z}\frac{\partial G}{\partial x}-\frac{\partial(F+H)}{\partial x}\frac{\partial G}{\partial z}\right)\hat{j}\\
&\qquad\qquad+\left(\frac{\partial(F+H)}{\partial x}\frac{\partial G}{\partial y}-\frac{\partial(F+H)}{\partial y}\frac{\partial G}{\partial x}\right)\hat{k}\Bigg]\\
&= -\boldsymbol{x}\cdot\Bigg[\left(\frac{\partial F}{\partial y}\frac{\partial G}{\partial z}-\frac{\partial F}{\partial z}\frac{\partial G}{\partial y}\right)\hat{i}+\left(\frac{\partial F}{\partial z}\frac{\partial G}{\partial x}-\frac{\partial F}{\partial x}\frac{\partial G}{\partial z}\right)\hat{j}\\
&\qquad\qquad+\left(\frac{\partial F}{\partial x}\frac{\partial G}{\partial y}-\frac{\partial F}{\partial y}\frac{\partial G}{\partial x}\right)\hat{k}\Bigg]\\
&- \boldsymbol{x}\cdot\Bigg[\left(\frac{\partial H}{\partial y}\frac{\partial G}{\partial z}-\frac{\partial H}{\partial z}\frac{\partial G}{\partial y}\right)\hat{i}+\left(\frac{\partial H}{\partial z}\frac{\partial G}{\partial x}-\frac{\partial H}{\partial x}\frac{\partial G}{\partial z}\right)\hat{j}\\
&\qquad\qquad+\left(\frac{\partial H}{\partial x}\frac{\partial G}{\partial y}-\frac{\partial H}{\partial y}\frac{\partial G}{\partial x}\right)\hat{k}\Bigg]\\
&= -\boldsymbol{x}\cdot(\nabla_x F\times\nabla_x G)-\boldsymbol{x}\cdot(\nabla_x H\times\nabla_x G)\\
&= \{F,G\}+\{H,G\}
\end{aligned}
$$

3. **Leibniz:** We want to show $\{F,GH\}=\{F,G\}H+G\{F,H\}$ for functions F,G and H

$$
\begin{aligned}
\{F,GH\} &= -\boldsymbol{x}\cdot(\nabla_x(F)\times\nabla_x GH)\\
&= -\boldsymbol{x}\cdot\det\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ \frac{\partial F}{\partial x} & \frac{\partial F}{\partial y} & \frac{\partial F}{\partial z}\\ \frac{\partial(GH)}{\partial x} & \frac{\partial(GH)}{\partial y} & \frac{\partial(GH)}{\partial z}\end{vmatrix}\\
&= -\boldsymbol{x}\cdot\Bigg[\left(\frac{\partial F}{\partial y}\frac{\partial(GH)}{\partial z}-\frac{\partial F}{\partial z}\frac{\partial(GH)}{\partial y}\right)\hat{i}+\left(\frac{\partial F}{\partial z}\frac{\partial(GH)}{\partial x}-\frac{\partial F}{\partial x}\frac{\partial(GH)}{\partial z}\right)\hat{j}\\
&\qquad\qquad+\left(\frac{\partial F}{\partial x}\frac{\partial(GH)}{\partial y}-\frac{\partial F}{\partial y}\frac{\partial(GH)}{\partial x}\right)\hat{k}\Bigg]\\
&= -\boldsymbol{x}\cdot\Bigg[\left(\frac{\partial F}{\partial y}\frac{\partial G}{\partial z}-\frac{\partial F}{\partial z}\frac{\partial G}{\partial y}\right)H\hat{i}+\left(\frac{\partial F}{\partial z}\frac{\partial G}{\partial x}-\frac{\partial F}{\partial x}\frac{\partial G}{\partial z}\right)H\hat{j}\\
&\qquad\qquad+\left(\frac{\partial F}{\partial x}\frac{\partial G}{\partial y}-\frac{\partial F}{\partial y}\frac{\partial G}{\partial x}\right)H\hat{k}\Bigg]\\
&- \boldsymbol{x}\cdot\Bigg[G\left(\frac{\partial F}{\partial y}\frac{\partial G}{\partial z}-\frac{\partial F}{\partial z}\frac{\partial G}{\partial y}\right)\hat{i}+G\left(\frac{\partial F}{\partial z}\frac{\partial G}{\partial x}-\frac{\partial F}{\partial x}\frac{\partial G}{\partial z}\right)\hat{j}\\
&\qquad\qquad+G\left(\frac{\partial}{\partial x}\frac{\partial G}{\partial y}-\frac{\partial}{\partial y}\frac{\partial G}{\partial x}\right)\hat{k}\Bigg]\\
&= -\boldsymbol{x}\cdot(\nabla_x F\times\nabla_x G)H-\boldsymbol{x}\cdot G(\nabla_x F\times\nabla_x G)\\
&= \{F,G\}H+G\{F,H\}
\end{aligned}
$$

4. **Jacobi:** We want to show $\{F,\{G,H\}\}+\{G,\{H,F\}\}+\{H,\{F,G\}\}=0$ for some functions F,G and H We have

$$\{G,H\} = -\boldsymbol{x}\cdot\left[\left(\frac{\partial G}{\partial y}\frac{\partial H}{\partial z}-\frac{\partial G}{\partial z}\frac{\partial H}{\partial y}\right)\hat{\imath}+\left(\frac{\partial G}{\partial z}\frac{\partial H}{\partial x}-\frac{\partial G}{\partial x}\frac{\partial H}{\partial z}\right)\hat{\jmath}\right.$$
$$\left.+\left(\frac{\partial G}{\partial x}\frac{\partial H}{\partial y}-\frac{\partial G}{\partial y}\frac{\partial H}{\partial x}\right)\hat{k}\right]$$

$$\{H,F\} = -\boldsymbol{x}\cdot\left[\left(\frac{\partial H}{\partial y}\frac{\partial F}{\partial z}-\frac{\partial H}{\partial z}\frac{\partial F}{\partial y}\right)\hat{\imath}+\left(\frac{\partial H}{\partial z}\frac{\partial F}{\partial x}-\frac{\partial H}{\partial x}\frac{\partial F}{\partial z}\right)\hat{\jmath}\right.$$
$$\left.+\left(\frac{\partial H}{\partial x}\frac{\partial F}{\partial y}-\frac{\partial H}{\partial y}\frac{\partial F}{\partial x}\right)\hat{k}\right]$$

$$\{F,G\} = -\boldsymbol{x}\cdot\left[\left(\frac{\partial F}{\partial y}\frac{\partial G}{\partial z}-\frac{\partial F}{\partial z}\frac{\partial G}{\partial y}\right)\hat{\imath}+\left(\frac{\partial F}{\partial z}\frac{\partial G}{\partial x}-\frac{\partial F}{\partial x}\frac{\partial G}{\partial z}\right)\hat{\jmath}\right.$$
$$\left.+\left(\frac{\partial F}{\partial x}\frac{\partial G}{\partial y}-\frac{\partial F}{\partial y}\frac{\partial G}{\partial x}\right)\hat{k}\right]$$

Then,

$$\{F,\{G,H\}\} = |\boldsymbol{x}|^2\cdot\left[\frac{\partial F}{\partial y}\left(\frac{\partial G}{\partial x}\frac{\partial H}{\partial y}-\frac{\partial G}{\partial y}\frac{\partial H}{\partial x}\right)+\frac{\partial F}{\partial z}\left(\frac{\partial G}{\partial z}\frac{\partial H}{\partial x}-\frac{\partial G}{\partial x}\frac{\partial H}{\partial z}\right)\right.$$
$$+\frac{\partial F}{\partial z}\left(\frac{\partial G}{\partial y}\frac{\partial H}{\partial z}-\frac{\partial G}{\partial z}\frac{\partial H}{\partial y}\right)+\frac{\partial F}{\partial x}\left(\frac{\partial G}{\partial x}\frac{\partial H}{\partial y}-\frac{\partial G}{\partial y}\frac{\partial H}{\partial x}\right)$$
$$\left.+\frac{\partial F}{\partial x}\left(\frac{\partial G}{\partial z}\frac{\partial H}{\partial x}-\frac{\partial G}{\partial x}\frac{\partial H}{\partial z}\right)+\frac{\partial F}{\partial y}\left(\frac{\partial G}{\partial y}\frac{\partial H}{\partial z}-\frac{\partial G}{\partial z}\frac{\partial H}{\partial y}\right)\right]$$

$$\{G,\{H,F\}\} = |\boldsymbol{x}|^2\cdot\left[\frac{\partial G}{\partial y}\left(\frac{\partial H}{\partial x}\frac{\partial F}{\partial y}-\frac{\partial H}{\partial y}\frac{\partial F}{\partial x}\right)+\frac{\partial G}{\partial z}\left(\frac{\partial H}{\partial z}\frac{\partial F}{\partial x}-\frac{\partial H}{\partial x}\frac{\partial F}{\partial z}\right)\right.$$
$$+\frac{\partial G}{\partial z}\left(\frac{\partial H}{\partial y}\frac{\partial F}{\partial z}-\frac{\partial H}{\partial z}\frac{\partial F}{\partial y}\right)+\frac{\partial G}{\partial x}\left(\frac{\partial H}{\partial x}\frac{\partial F}{\partial y}-\frac{\partial H}{\partial y}\frac{\partial F}{\partial x}\right)$$
$$\left.+\frac{\partial G}{\partial x}\left(\frac{\partial H}{\partial z}\frac{\partial F}{\partial x}-\frac{\partial H}{\partial x}\frac{\partial F}{\partial z}\right)+\frac{\partial G}{\partial y}\left(\frac{\partial H}{\partial y}\frac{\partial F}{\partial z}-\frac{\partial H}{\partial z}\frac{\partial F}{\partial y}\right)\right]$$

$$\{H,\{F,G\}\} = |\boldsymbol{x}|^2\cdot\left[\frac{\partial H}{\partial y}\left(\frac{\partial F}{\partial x}\frac{\partial G}{\partial y}-\frac{\partial F}{\partial y}\frac{\partial G}{\partial x}\right)+\frac{\partial H}{\partial z}\left(\frac{\partial F}{\partial z}\frac{\partial G}{\partial x}-\frac{\partial F}{\partial x}\frac{\partial G}{\partial z}\right)\right.$$
$$+\frac{\partial H}{\partial z}\left(\frac{\partial F}{\partial y}\frac{\partial G}{\partial z}-\frac{\partial F}{\partial z}\frac{\partial G}{\partial y}\right)+\frac{\partial H}{\partial x}\left(\frac{\partial F}{\partial x}\frac{\partial G}{\partial y}-\frac{\partial F}{\partial y}\frac{\partial G}{\partial x}\right)$$
$$\left.+\frac{\partial H}{\partial x}\left(\frac{\partial F}{\partial z}\frac{\partial G}{\partial x}-\frac{\partial F}{\partial x}\frac{\partial G}{\partial z}\right)+\frac{\partial H}{\partial y}\left(\frac{\partial F}{\partial y}\frac{\partial G}{\partial z}-\frac{\partial F}{\partial z}\frac{\partial G}{\partial y}\right)\right]$$

We then add the terms up, and find that every term cancels.

$$\Rightarrow \{F,\{G,H\}\}+\{G,\{H,F\}\}+\{H,\{F,G\}\}=0$$

Appendix D: Matlab code for spin-warp

```
function [my_fourier,my_fourier_analytic]=spin_warp_test()

% B0=1;
omega0=1;
gamma=1;

Nx=50;
Ny=50;

Lx=1;
Ly=1;

dx=Lx/Nx;
dy=Ly/Ny;

% T1=42576400;
T2=4257640;

% ************************************************************

Mx=ones(Nx,Ny);
My=zeros(Nx,Ny);

kx1=(pi/Lx)*2;
ky1=(pi/Ly)*3;

rho_in=zeros(Nx,Ny);
X_mx=zeros(Nx,Ny);
Y_mx=zeros(Nx,Ny);

xx=0*(1:Nx);
yy=0*(1:Ny);

% *************************************************************
% Initialise density profile

for i=1:Nx
    for j=1:Ny
        x_val=(i-1)*dx;
```

```
        y_val=(j-1)*dy;

        rho_in(i,j)=1+0.5*sin(kx1*x_val)*sin(ky1*y_val);

        X_mx(i,j)=x_val;
        Y_mx(i,j)=y_val;

        xx(i)=x_val;
        yy(j)=y_val;
    end
end

% ***************************************************************
% Declare values for gradients

g1=1;
t1_phase_encoding=(pi/Lx)*(1/(gamma*g1));
T_sw=0.5*t1_phase_encoding;
g2=-(3*pi/Ly)/(gamma*T_sw);

% ****************************************************************
% Phase encoding

U_sw_inv=zeros(2,2);
U_sw_inv(1,1)=cos(omega0*T_sw);
U_sw_inv(1,2)=sin(omega0*T_sw);
U_sw_inv(2,1)=-sin(omega0*T_sw);
U_sw_inv(2,2)=cos(omega0*T_sw);

mx_sw=zeros(Nx,Ny);
my_sw=zeros(Nx,Ny);

for i=1:Nx
    for j=1:Ny

    x_val=(i-1)*dx;
    y_val=(j-1)*dy;

    V_sw=zeros(2,2);

    omega_loc=omega0+gamma*(g1*x_val-g2*y_val);
```

```

    V_sw(1,1)=cos(omega_loc*T_sw);
    V_sw(1,2)=-sin(omega_loc*T_sw);
    V_sw(2,1)=sin(omega_loc*T_sw);
    V_sw(2,2)=cos(omega_loc*T_sw);

    tempv_twod=exp(-T_sw/T2)*U_sw_inv*V_sw*[Mx(i,j);My(i,j)];
    % tempv_z=M0(i,j)+(Mz(i,j)-M0(i,j))*exp(-T_sw/T1_over);

    mx_sw(i,j)=tempv_twod(1);
    my_sw(i,j)=tempv_twod(2);

    end
end

% Frequency encoding

t_vec=2*T_sw*(1: (Nx/2) );
my_fourier=0*(1:length(t_vec));

for ii=1:length(t_vec)
    for i=1:Nx
        for j=1:Ny

        x_val=(i-1)*dx;
        % y_val=(j-1)*dy;

        omega_loc=gamma*g1*x_val;

            V_sw(1,1)=cos(omega_loc*(t_vec(ii)-T_sw));
            V_sw(1,2)=-sin(omega_loc*(t_vec(ii)-T_sw));
            V_sw(2,1)=sin(omega_loc*(t_vec(ii)-T_sw));
            V_sw(2,2)=cos(omega_loc*(t_vec(ii)-T_sw));

            tempv_twod=exp(-t_vec(ii)/T2)*V_sw...
                *[mx_sw(i,j);my_sw(i,j)];

            U_sw=zeros(2,2);
            U_sw(1,1)= cos(omega0*t_vec(ii));
            U_sw(1,2)=-sin(omega0*t_vec(ii));
            U_sw(2,1)=sin(omega0*t_vec(ii));
```

```
            U_sw(2,2)=cos(omega0*t_vec(ii));

            tempv_twod=U_sw*tempv_twod;
            Mx(i,j)=tempv_twod(1);
            My(i,j)=tempv_twod(2);
        end
    end

    my_fourier(ii)=sum(sum(Mx.*rho_in))*dx*dy;

end


% ***********************************************************
% Analytic solution

my_fourier_analytic=0*(1:length(t_vec));

for i=1:length(t_vec)
    kx_val=gamma*(pi/Lx)*i;
    ky_val=-(3*gamma*pi/Ly);
    my_fourier_analytic(i)=exp(-t_vec(i)/T2)*sum(sum( cos(kx_val*X_mx...
        -ky_val*Y_mx+omega0*t_vec(i)).*rho_in))*dx*dy;

end
Xx=1:length(my_fourier);
plot(Xx,my_fourier,'g',Xx,my_fourier_analytic,'k:','LineWidth',2)
title('Comparison between computational and analytical models of spin warp');
xlabel('Time'); ylabel('FT[m_{x}\rho(x,t)]')
legend Model Analytic
end
```