Project 2

author: Alexander Krasovskiy

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0.0.1 Task 2.1 Documentation

1. Data Source:

- The data source for the Task 2.1 is a generated random uniformly distributed sample of size n (scipy.stats.uniform module is used)
- Initial assumptions: X is a uniformly distributed sample with values in range $[0, \theta]$ (where θ is a model parameter)

2. Research purpose:

1. Given samples:

- $X \in$ { X1, X2 , ..., X100 }
- Sample size: |X1| = 1000, |X2| = 2000, ... , |X100| = 100000

2. Given three statistics over a sample X:

- Min(X) = $ X_{1:n} - $ Sample X minimum
- Mean(X) = $(X_{1:n}, X_{n:n}) - - - || - - -$ mean
- Max(X) = $X_{n:n} - - - || - - -$ maximum

3. $\theta = 10$ (or any natural number)

4. Null hypothesis, H0: $X \quad \mathit{Unif}\,[0, \theta]$

5. Null hypothesis, H0: **Max(X)** is a sufficient statistics for the parameter θ

6. Alternative hypothesis, H1: **Mean(X)** is a sufficient statistics for the parameter θ

- Research purpose: Prove / decline the Null Hypothesis H0

[48]:
```python
theta = 10
# Define sample (X) generator
def T(theta, sample_size):
    return uniform.rvs(0,theta,sample_size)
```

[49]:
```python
#Define statistics functions
def Mean(sample):
    return sample.mean()

def Min(sample):
    return sample.min()

def Max(sample):
    return sample.max()
```

3. Classifier & numeric variables:

1. Classifiers:

- None

2. Numeric variables:

- $X \quad Unif\,[0, \theta]$

4. Introduction (Probability distribution analysis)

1. The **Kolmogorov-Smirnov** test is used to confirm / reject the Null hypothesis H0 (which assumes, that $X \quad Unif\,[0, \theta]$)
2. In the example below I used a function scipy.stats.kstest that implements the **Kolmogorov-Smirnov** test.
3. A brief description of the solution:

- method: scipy.stats.kstest(rvs, cdf, args=(), N=20, alternative='two-sided', method='auto')
- reference: https://docs.scipy.org/doc/scipy/reference/generated/scipy.stats.kstest.html
- Performs the (one-sample or two-sample) Kolmogorov-Smirnov test for goodness of fit. The one-sample test compares the underlying distribution F(x) of a sample against a given distribution G(x) **(this option is used)**. The two-sample test compares the underlying distributions of two independent samples. Both tests are valid only for continuous distributions.
- parameters:
  - rvs - random values $\equiv X$
  - cdf - expected cumulative distribution function $\equiv$ Unif$(0, \theta)$
  - N - sample size $\equiv$ size of X
  - If the sample comes from distribution $F(x) \equiv cdf$, then D_n converges to 0 almost surely in the limit when n goes to infinity.
  - H0: $\sqrt{n}D_n$ converges to the Kolmogorov distribution, which does not depend on $F(x)$
- $D_n = \sup_x |F_n(x) - F(x)|$

[50]:
```python
from scipy.stats import kstest
Lambda = 0.05

n = 100000
test = kstest(T(theta,n),uniform.cdf, N=n, args=(0,theta))
s, p_value = test
print("sample size = %d"%n)
print()
print(test)
if (p_value>Lambda):
    print()
    print("H0 for the Kolmogorov-Smirnov test is correct, lambda = %.3f (%d%%␣
 ↪percentile)"%(Lambda,100-100*Lambda))
    print()
    print("The sample X distribution is likely to be equivalent to␣
 ↪Unif[0,theta]")
else:
    print("H0 for the Kolmogorov-Smirnov test is not correct, lambda = %.
 ↪3f"%Lambda)
    print("X distribution is unlikely to be equivalent to Unif[0,theta]")
```

```
sample size = 100000
KstestResult(statistic=0.0026661944797430337, pvalue=0.4750146065722749)

H0 for the Kolmogorov-Smirnov test is correct, lambda = 0.050 (95% percentile)

The sample X distribution is likely to be equivalent to Unif[0,theta]
```

4. The **Kolmogorov-Smirnov** test is positive (95 percentile) =>

- H0 is correct =>

- X is a random sample with the distribution $\equiv \text{Unif}(0, \theta)$

5. Additionally, it is necessary to check, that the sample generated by T is simple.

- Therefore we define a function isSimpleRandomSample(T, theta, expected_range, eps $= 10^{-5}$) - check if a uniformly distributed random variable T is a simple random sample given the expected range [a,b].
  1. Calculate the expected mean and standard deviation of the Uniform[a,b] distribution
  2. Iterate for each sample size =n min: 10, max: 10^7
  3. For each sample size (=n): - Generate a random sample of size n - Calculate [a_iter,b_iter] == [min(sample),max(sample)] - Compute p == P(x<a_iter or x>b_iter) - Calculate the sample mean and standard deviation - In order to prove that T(n) is a simple random sample, we need to check that the random variable T(n) is not limited for the population [0,theta] - Therefore, it is necessary and sufficient to check that lim(p)=0, n->inf for a given eps. - => If abs(p) < eps: T generates a simple random sample.
     - return True
  4. Else: - return False

[57]:
```python
from math import sqrt

def isSimpleRandomSample(T, theta, expected_range, eps):
    a_iter=None
    b_iter=None
    diff = abs(expected_range[0]-expected_range[1])
    sample_sizes = [int(x) for x in [10,1e2,1e3,1e4,1e5,1e6, 1e7]]
    print("expected range:",expected_range)
    expected_mean = (expected_range[1]-expected_range[0])/2
    expected_std = sqrt(1/12 * (expected_range[1]-expected_range[0])**2)
    print("expected mean:",expected_mean)
    print("expected standard deviation:",expected_std)
    print()
    for size in sample_sizes:
        samp = T(theta, size)
        a_iter = samp.min()
        b_iter = samp.max()

        p = abs(a_iter-expected_range[0])/diff + abs(b_iter-expected_range[1])/
 ↪diff
        print("for sample size = %d:"%size)
        print("p_value == P(x<a_iter or x>b_iter) = %.8f"%p)
        print("sample mean = E(X):",samp.mean())
        print("sample standard deviation = sqrt(Var(X)):",samp.std())
        print("actual range: ",[a_iter,b_iter])
        print()

        if p < eps:
            print("Random sample is simple, eps=%e"%eps)
            return True
    return False

eps=1e-5
population=[0,theta]

if isSimpleRandomSample(T,theta,population, eps=eps):
    success("\nT(n) generates a simple random sample for the expected␣
 ↪population=%s, eps=%e"%(population,eps))
else:
    fail("\nT(n) does not generate a simple random sample for the expected␣
 ↪population=%s, eps=%e"%(population,eps))
```

```
expected range: [0, 10]
expected mean: 5.0
expected standard deviation: 2.8867513459481287

for sample size = 10:
p_value == P(x<a_iter or x>b_iter) = 0.10368617
```

```
sample mean = E(X): 5.643999061499329
sample standard deviation = sqrt(Var(X)): 2.871822291800391
actual range:  [0.9269957357854164, 9.890134010618748]

for sample size = 100:
p_value == P(x<a_iter or x>b_iter) = 0.00762663
sample mean = E(X): 4.67245060952827
sample standard deviation = sqrt(Var(X)): 3.058244597728054
actual range:  [0.028974161437055335, 9.952707822710888]

for sample size = 1000:
p_value == P(x<a_iter or x>b_iter) = 0.00154257
sample mean = E(X): 4.990070772551295
sample standard deviation = sqrt(Var(X)): 2.9470834430016635
actual range:  [0.009418768815103729, 9.993993082600863]

for sample size = 10000:
p_value == P(x<a_iter or x>b_iter) = 0.00020790
sample mean = E(X): 5.0136269329335414
sample standard deviation = sqrt(Var(X)): 2.9008435888900697
actual range:  [0.001960486448134846, 9.999881494744043]

for sample size = 100000:
p_value == P(x<a_iter or x>b_iter) = 0.00000881
sample mean = E(X): 4.99221187269252
sample standard deviation = sqrt(Var(X)): 2.8887436061886724
actual range:  [2.3191380660314564e-05, 9.999935093523947]

Random sample is simple, eps=1.000000e-05
```

```
T(n) generates a simple random sample for the expected population=[0, 10],
eps=1.000000e-05
```

5. Model definition: $(X, \{P_\theta, \theta \in \Theta\})$:
- X $= [0, \theta]$
- $\theta = 10$; $\Theta = \mathbb{R}$
- $P_\theta \quad Unif(0, \theta)$

Null hypothesis, H0: **Max(X)** is a sufficient statistics for the parameter θ

Alternative hypothesis, H1: **Mean(X)** or **Min(X)** is a sufficient statistics for the parameter θ

6. Analysis:

1. First we define function for computing the mean squared error of $|\theta - S(X)|$

- Where $S(X)$ is a statistics function
- The squared error is computer for every sample $X \in$ { X1, X2 , ..., X100 }: $E_i = |\theta - T(X_i)|$
- => The mean squared error is equal to $\frac{1}{n}(E_1 + E_2 + ... + E_{100})$ = MSE

Brief description of the function **error(theta, stat, T)**:

1. Arg: θ - estimated parameter

2. Arg: $T == T(\theta, n)$ - random sample generator

3. Arg: stat - a reference to the statistics function over a sample T(theta, sample_size)

4. for each sample size in range(1000,100000,1000):
- Approximate θ based on the random sample stat(T(theta,sample_size))
- Compute the squared error between theta and theta approximation
- Compute the mean squared error MSE

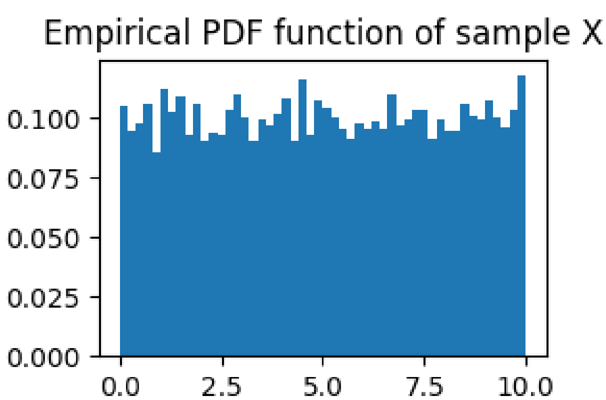
5. Print results & draw the error plot

[67]:
```
def error(theta,stat, T):
    print("Theta =",theta)
    print("Theta estimator:",stat)
    mse = 0
    error_arr=[]
    sample_sizes=range(1000,101000,1000)
    for sample_size in sample_sizes:
        theta_appr = stat(T(theta,sample_size))
        sqe=(theta-theta_appr)**2
        mse+=sqe
        error_arr.append(sqe)
    mse/=len(sample_sizes)
    print("MSE:",mse)
    plt.plot(sample_sizes,error_arr, color="b", label="squared error")
    plt.axhline(mse,color="r", label="mean squared error")
    return mse
```

Empirical PDF Preview

- A method ${**plt.hist()**} $ is used for plotting the empirical PDF
- ${**density=True**} $ indicates that the histogram represents a probability density function

[68]:
```
plt.hist(T(theta,10000),bins=50,density=True)
plt.title("Empirical PDF function of sample X")
plt.gcf().set_size_inches(3,2)
plt.show()
```

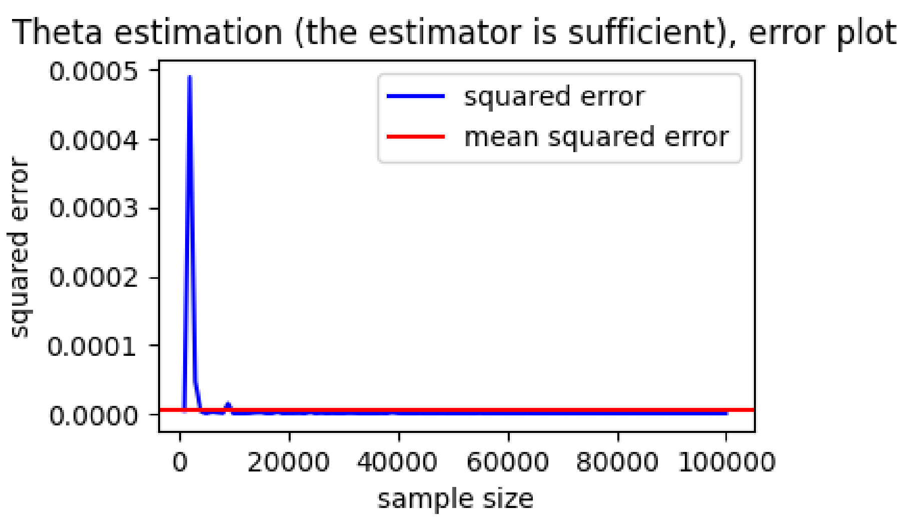


Theta estimation

- In the example below we use the **error()** function defined earlier for the ϑestimation process:

[71]:
```
plt.gcf().set_size_inches(4,2.5)
plt.xlabel("sample size")
plt.ylabel("squared error")
mse = error(theta,Max, T)
plt.legend(loc=1)
plt.title("Theta estimation (the estimator is sufficient), error plot")
plt.show()
if (mse<Lambda):
    success("The estimator %s is sufficient."%Max)
else:
    fail("The estimator %s is not sufficient."%Max)
```

```
Theta = 10
Theta estimator: <function Max at 0x000001F5E2675510>
MSE: 5.81649238563254e-06
```

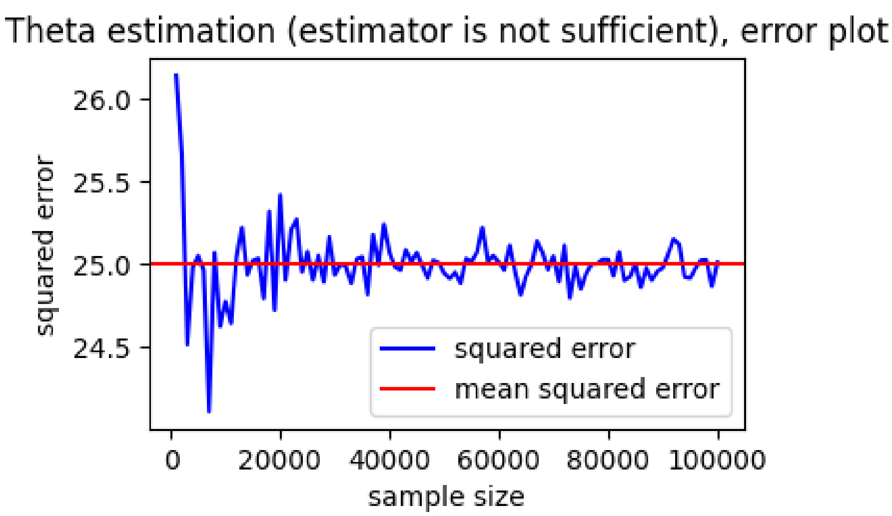



```
The estimator <function Max at 0x000001F5E2675510> is sufficient.
```

- In the example below, θ is estimated using a different statistics function $-$ **Mean(X)**

[72]:
```
plt.gcf().set_size_inches(4,2.5)
plt.xlabel("sample size")
plt.ylabel("squared error")
mse = error(theta,Mean, T)
plt.legend(loc=4)
plt.title("Theta estimation (estimator is not sufficient), error plot")
plt.show()
if (mse<Lambda):
    success("The estimator %s is sufficient."%Mean)
else:
    fail("The estimator %s is not sufficient."%Mean)
```

```
Theta = 10
Theta estimator: <function Mean at 0x000001F5E1EF15A0>
MSE: 25.000415469126352
```




```
The estimator <function Mean at 0x000001F5E1EF15A0> is not sufficient.
```

7. Conclusions

1. The Null hypothesis is true:

- H0: $X \quad Unif\,[0,\theta]$
- H0: $\mathbf{Max(X)} \equiv X_{n:n}$ is a sufficient statistics for the parameter θ

2. The **Kolmogorov-Smirnov** test confirmed, that the distribution of sample X is equivalent to $Unif\,[0,\theta]$ with p-value=0.05 $<=>$ 95 percentile

3. $\mathbf{Max(X)} \equiv X_{n:n} => X_{n:n}$ is a sufficient statistics function for the parameter θ

4. Random sample generator $T(\theta, n)$ generates simple random samples for the distribution $Unif\,[0,\theta]$

5. Statistical Model:

- $(X, \{P_\theta, \theta \in \Theta\})$
  - X $=[0,\theta]$
  - $\theta = 10$; $\Theta = \mathbb{R}$
  - $P_\theta \quad Unif(0,\theta)$