Experiments with Two Crossed Treatment Factors

Two-Way Complete Models

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Introduction

Two factors are crossed if all combinations of the two factors are used in the experiment. A two-way model can investigate the interaction effects.

Notations:

- A: Factor with a levels;
- B: Factor with b levels;
- treatments are denotes by ij, $\overbrace{i = 1, 2, \ldots, a}^{\text{Factor A}}$, $\overbrace{j = 1, 2, \ldots, b}^{\text{Factor B}}$;
- observations: Y_{ijt} denotes the tth observation for treatment ij.

Example (Battery Experiment) Two treatment factors: Duty (two levels: alkaline and heavy duty, coded as 1 and 2, respectively) and Brand (two levels: name brand and store brand, coded as 1 and 2, respectively). The two factors are crossed. The four treatment combinations were coded as 1, 2,3, and 4, and one-way model was used in previous chapters to analyze the effects of the treatments. $2\times 2 = 4$ $A\times B = \text{Total}$

If we code the treatment combinations as $\overset{AB}{11}, \overset{AB}{12}, \overset{AB}{21}$ and $\overset{AB}{22}$, the same one-way model can be equivalently written as

$$Y_{ijt} = \mu_{ij} + \varepsilon_{ijt}, i = 1, 2, j = 1, 2, t = 1, \ldots, r_{ij}$$

where ε_{ij} are i.i.d $N\left(0, \sigma^2\right)$. This is also called a cell-mean model.

In this chapter, we investigate the contributions that each of the factors individually makes to the response. We could either have a 2-way complete model or a 2-way main effects model (if it is assured that this is no interaction).

Two-Way Complete Models

Given any numbers $\mu_{ij}, i = 1, \cdots, a, j = 1, \cdots, b$, we can always write

$$\mu_{ij} = \overset{\text{intercept}}{\mu} + \overset{A}{\alpha_i} + \overset{B}{\beta_j} + \overset{\text{interaction}}{(\alpha\beta)_{ij}}, \overset{a\text{ levels}}{i = 1, \cdots, a}, \overset{b\text{ levels}}{j = 1, \cdots, b}$$

where $\sum_i^a \alpha_i = 0$, $\sum_j^b \beta_j = 0$, and $\sum_{i=1}^a (\alpha\beta)_{ij} = 0$ for any j, $\sum_{j=1}^b (\alpha\beta)_{ij} = 0$ for any i.

Then the model becomes

$$Y_{ijt} = \mu + \alpha_i + \beta_j + (\alpha\beta)_{ij} + \varepsilon_{ijt}, i = \overset{2\text{ levels}}{1, 2}, j = \overset{2\text{ levels}}{1, 2}, t = 1, \ldots, r_{ij}.$$

This is called *the two-way complete model* or *two-way ANOVA model*. It is just a rewriting of model ($???$). Note

- $\alpha_i, \beta_j, (\alpha\beta)_{ij}$ are not unique since there are multiple ways to decompose μ_{ij} as in ($???$). One way is

$$\alpha_i = \overset{AB}{\bar{\mu}_{i\cdot}} - \bar{\mu}_{\cdot\cdot}, \quad \beta_j = \overset{AB}{\bar{\mu}_{\cdot j}} - \bar{\mu}_{\cdot\cdot},$$
$$(\alpha\beta)_{\underset{AB}{ij}} = \mu_{ij} + \bar{\mu}_{\cdot\cdot} - \bar{\mu}_{i\cdot} - \bar{\mu}_{\cdot j}.$$

- $(\alpha\beta)_{ij}$ are called the interaction effects. When there is no interaction, $(\alpha\beta)_{ij} = 0$ for all i and j.
- The two-way ANOVA model facilitates the tests about the interaction effects.

The Meaning of Interaction

An example: Factor A with 3 levels and Factor B with 2 levels. Consider two cases for the 6 treatment means as shown below

A : 3 levels B : 2 levels

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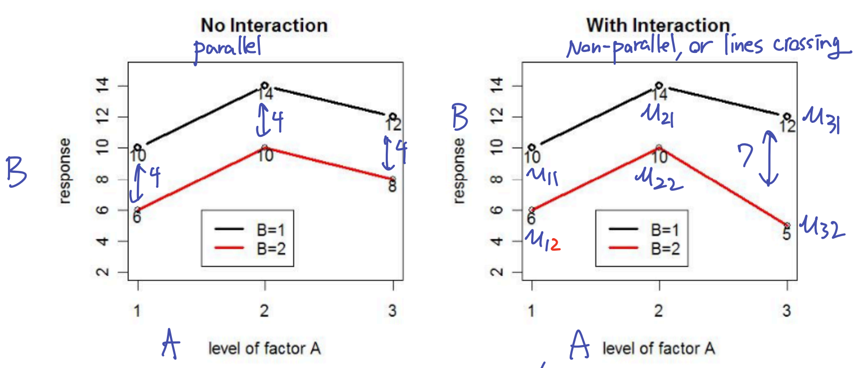

The left-side indicates no interaction effects, i.e., $(\alpha\beta)_{ij} = 0$ for all i, j. Then

$$\mu_{ij} = \mu + \alpha_i + \beta_j.$$

Hence the differences between the two levels of B, say, 1 and 2, are the same regardless the level of A. Specifically,

$$\mu^{AB}_{i2} - \mu^{AB}_{i1} = \beta_2 - \beta_1 = 4, \text{ for all } i.$$

However, the right-side shows that the differences between the two levels of Factor B depends on the levels of A. Specifically

$$\mu_{12} - \mu_{11} = 4, \mu_{22} - \mu_{21} = 4, \mu_{32} - \mu_{31} = 7.$$

We will say there are interaction effects in the right-side case, and no interaction effects in the left-side case.

In reality, the means μ_{ij} are unknown and we shall plot the sample means instead. This results in an interaction plot.

Two-Way Main Effects Model

When no interaction effects exist, we have a two-way main effects model

$$Y_{ijt} = \mu + \alpha_i + \beta_j + \varepsilon_{ijt}, i = 1, 2, j = 1, 2, t = 1, \ldots, r_{ij}.$$

Contrast in the Two-Way Complete Model

As in the one-way model, a contrast is of the form $\sum_i \sum_j c_{ij}\mu_{ij}$ where $\sum_i \sum_j c_{ij} = 0$. Linear combination of means where coefficients sum to 0.

Interaction Contrasts

If the coefficients satisfy the additional conditions $\sum_i c_{ij} = 0$ for each j and $\sum_j c_{ij} = 0$ for each i,

Interaction Contrast ① + ②

contrast only ① Requirement 1: sum of coefficients = 0

② Requirement 2: sum of rows/columns = 0

$$\sum_i \sum_j c_{ij}\mu_{ij} = \sum_i \sum_j c_{ij}(\alpha\beta)_{ij}$$

	B1	B2	
A1	u_{11} $\frac{1}{2}$	u_{12} $-\frac{1}{2}$	0
A2	u_{21} $\frac{1}{2}$	u_{22} $-\frac{1}{2}$	0
A3	u_{31} -1	u_{32} $+1$	0
	0	0	

is called an interaction contrast.

Example of interaction contract: Yes :

$$(\overset{+1}{\mu_{12}} - \overset{-1}{\mu_{11}}) - (\overset{-1}{\mu_{22}} - \overset{+1}{\mu_{21}}). = 0$$

$\frac{1}{2}(u_{11}+u_{21}) - \frac{1}{2}(u_{12}+u_{22}) - u_{31} + u_{32}$

$\cancel{\frac{1}{2}(1+1) - \frac{1}{2}(1+1) - 1 + 1} = 0$

Q1: Is $\frac{\mu_{11}+\mu_{21}}{2} - \frac{\mu_{12}+\mu_{22}}{2} - (\mu_{31} - \mu_{32})$ a contrast? Yes Is it an interaction contrast? Yes

Q2: Is $\frac{\mu_{11}+\mu_{21}+\mu_{31}}{2} - \frac{\mu_{12}+\mu_{22}+\mu_{32}}{2}$ a contrast? Yes Is it an interaction contrast? NO

$\cancel{\frac{1}{2}(1+1+1) - \frac{1}{2}(1+1+1)} = 0$

	B1	B2	
A1	u_{11} $\frac{1}{2}$	u_{12} $-\frac{1}{2}$	0 ✓
A2	u_{21} $\frac{1}{2}$	u_{22} $-\frac{1}{2}$	0 ✓
A3	u_{31} $\frac{1}{2}$	u_{32} $-\frac{1}{2}$	0 ✓
	× 1.5	× -1.5	

Main Effects Contrasts

How to measure the difference between two levels of Factor B in the presence of interaction effects? A reasonable choice is

A : 3 levels B : 2 levels

Factor B : 2

B=2 constant

$$(\mu_{12} + \mu_{22} + \mu_{32})/3 - (\mu_{11} + \mu_{21} + \mu_{31})/3 = \bar{\mu}_{\cdot 2} - \bar{\mu}_{\cdot 1} = (\beta_2 + (\alpha\beta)_{\cdot 2}) - (\beta_1 + (\alpha\beta)_{\cdot 1}).$$

A varies (B) level 2 − (B) level 1

Factor A : 3

The average of the 3 levels of A. when B=2

In general, a contrast in the main effects of Factor B takes the form how many levels of A

$$\sum_j k_j \bar{\mu}_{.j} = \sum_j k_j \left(\beta_j + \overline{(\alpha\beta)}_{.j}\right),$$

A varies
B constant

where $\Sigma k_j = 0$ and $\overline{(\alpha\beta)}_{.j} = \frac{1}{a}\Sigma_i (\alpha\beta)_{ij}$.

Note the contrast here is NOT $\sum_j k_j \beta_j$.

A contrast in the main effects if Factor A takes the form

how many levels of B

$$\sum_i c_i \bar{\mu}_{i.} = \sum_i c_i \left(\alpha_i + \overline{(\alpha\beta)}_{i.}\right)$$

A constant
B varies

where $\sum_i c_i = 0$ and $\overline{(\alpha\beta)}_{i.} = \frac{1}{b}\sum_{j=1}^{b} (\alpha\beta)_{ij}$.

Note the main effect contrast here is NOT $\sum_j k_j \beta_j$ or $\sum_i c_i \alpha_i$.

The Essence of Contrast

Any contrast is a linear combination of the treatment means with the contrast coefficients adding up to be 0. The interaction contrasts and main effects contrasts are particular contrasts with additional requirements on the coefficients.

Contrast for Two Particular Treatments

Unlike in the one-way model, the contrast for two treatment means, say, $\mu_{22} - \mu_{11}$, involves both the main effects and the interaction effects.

$$\underset{AB}{\mu_{22}} - \underset{AB}{\mu_{11}} = (\cancel{\mu} + \overset{\text{main}}{\alpha_2} + \overset{\text{main}}{\beta_2} + \overset{\text{interaction}}{(\alpha\beta)_{22}}) - (\cancel{\mu} + \alpha_1 + \beta_1 + (\alpha\beta)_{11}) = (\alpha_2 - \alpha_1) + (\beta_2 - \beta_1) + (\alpha\beta_{22} - \alpha\beta_{11})$$

The first two terms are main effects contrasts but the third term is not an interaction contrast!

Analysis of the Two-Way Complete Model

Least Squares Estimators

The model parameters are σ^2 and $\mu_{ij}, i = 1, 2, \ldots, a, j = 1, 2, \ldots, b$. These are the same parameters as in the one-way model. Therefore, the least squares estimators are the sample means:

estimator for $\hat{\mu}_{ij}$

$$\hat{\mu}_{ij} = \bar{Y}_{ij.}$$

$\bar{Y}_{ij}$ has a normal distribution with mean μ_{ij} and variance σ^2/r_{ij}, where r_{ij} denotes the sample size of treatment ij.

estimator for σ^2

Estimator for σ^2 is the same as in the one-way model: MSE.

The confidence upper limit is given by

$$\boxed{\sigma^2 \le \frac{ssE\text{stimate}}{\chi^2_{n-ab,1-\alpha}}}$$

Contrasts and Inferences for Contrasts

A 100(1- α)% confidence interval for a single contrast is of the form

Control for SE for simultaneous CI.

$$\boxed{\text{estimate} \pm t_{n-ab,\alpha/2}\ (\text{standard error ofestimate}).}$$

Multiple Comparisons for the Complete Model

Tukey's method for all pairwise comparisons and Dunnett's method for treatment-versus-control are applicable to main effects contrasts or interaction contrasts. However, these methods do not provide a simultaneous error rate for both main effects and interaction contrasts are considered. One can apply Bonferroni method to achieve a simultaneous error rate. For example, one intends to have 95% simultaneous for all pairwise comparisons of the main effects of Factor A, and all pairwise comparisons for all the interaction interaction effects. The Bonferroni method implies the simultaneous error rate will be 95% if one obtains the 97.5% simultaneous confidence intervals for the all pairwise comparisons of Factor A and 97.55% simultaneous confidence intervals for all pairwise comparisons of the interaction contrasts. Then all the intervals together have a simultaneous error rate 95%.

For multiple contrasts, the 100(1- $\alpha)\%$ simultaneous confidence intervals are of the form $\text{estimate} \pm w \times (\text{std error of estimate})$, where w varies with each method.

For example, for comparing main effects of factor A, use the w values provided in the following table:

Method	Bonferroni	Scheffe	Tukey	Dunnett
w	$t_{n-ab,\alpha/(2m)}$	$\sqrt{(a-1)F_{a-1,n-ab,\alpha}}$	$\frac{q_{a,n-ab,\alpha}}{\sqrt{2}}$	needs the multivariate t-distribution
Table	A.4, p802 preplanned contrasts, but not after analysis.	F value in A.6,804 Applied to any contrasts	q value in A.8, p814 All pairwise comparison	A.10, 818 Treatment v.s. control

The w values can be given similarly for the multiple contrasts concerning the interaction contracts or the main effects contracts for Factor B.

w values for interaction contrasts:

Method	Bonferroni	Scheffe	Tukey	Dunnett
w	$t_{n-ab,\alpha/(2m)}$	$\sqrt{(b-1)F_{b-1,n-ab,\alpha}}$	$\frac{q_{b,n-ab,\alpha}}{\sqrt{2}}$	needs the multivariate t-distribution
Table	A.4, p802	F value in A.6,804	q value in A.8, p814	A.10, 818

w values for interaction contrasts:

Method	Bonferroni	Scheffe	Tukey	Dunnett
w	$t_{n-ab,\alpha/(2m)}$	$\sqrt{(ab-1)F_{ab-1,n-ab,\alpha}}$	$\frac{q_{ab,n-ab,\alpha}}{\sqrt{2}}$	needs the multivariate t-distribution
Table	A.4, p802	F value in A.6,804	q value in A.8, p814	A.10, 818

```
data react;
input Order Trtmt A  B    y;
lines;
  1     6   2  3  0.256
  2     6   2  3  0.281
  3     2   1  2  0.167
  4     6   2  3  0.258
  5     2   1  2  0.182
  6     5   2  2  0.283
  7     4   2  1  0.257
  8     5   2  2  0.235
  9     1   1  1  0.204
 10     1   1  1  0.170
 11     5   2  2  0.260
 12     2   1  2  0.187
 13     3   1  3  0.202
 14     4   2  1  0.279
 15     4   2  1  0.269
 16     3   1  3  0.198
 17     3   1  3  0.236
 18     1   1  1  0.181
;
run;
PROC GLM data=react;
    CLASS A B;
    MODEL Y = A B A*B;
    LSMEANS A/PDIFF CL ALPHA=0.01;
    LSMEANS B/PDIFF = ALL CL ADJUST = TUKEY ALPHA = 0.01;
    LSMEANS A*B/pdiff=ALL ajust=Tukey;
run;

proc glm data=react;
class Trtmt;
model Y=trtmt;
lsmeans trtmt/pdiff=all adjust=Tukey;
run;
```

Note the order of factors in the `CLASS` statement determines how the treatments are coded. The order of factors in the `MODEL` statement makes no difference. Run the following code and compare with the `LSMEANS` output from the previous results.

```
PROC GLM data=react;
```

```
    CLASS B A;
    MODEL Y = A B A*B;
    LSMEANS A/PDIFF CL ALPHA=0.01;
    LSMEANS B/PDIFF = ALL CL ADJUST = TUKEY ALPHA = 0.01;
    LSMEANS A*B/pdiff=ALL ajust=Tukey;
run;
```

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