Analysis of the Two-Way Main Effects Model

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Least Squares Estimation

The two-way main-effects model is

(handwritten: no interaction, $\alpha\beta_{ij} = 0$)

(handwritten: model parameters: $\alpha-1$, $\beta-1$, sum parameters $(\alpha-1)+(\beta-1)+1 = \alpha+\beta-1$)

$$Y_{ijt} = \mu + \alpha_i + \beta_j + \epsilon_{ijt},$$

(handwritten: Contrains)

$$\sum_{i=1}^{a} \alpha_i = 0, \quad \sum_{j=1}^{b} \beta_j = 0,$$

$$\epsilon_{ijt} \sim N\left(0, \sigma^2\right),$$

ϵ_{ijt}'s are mutually independent, $t = 1, \ldots, r_{ij};\quad i = 1, \ldots, a;\quad j = 1, \ldots, b$.

Note there is no equivalent one-way model to the two-way main effects model, unlike the two-way complete model. This is a simplified model than the two-way complete model. There are $1 + (a-1) + (b-1) = a + b - 1$ number of freely-changing parameters. The number of degrees of freedom for error is therefore $n - a - b + 1$. *(handwritten: dof)*

I also note that the least squares estimator for μ_{ij} is no longer the sample mean $\bar{Y}_{ij\cdot}$. This is because μ_{ij} are now constrained. The precise formula for these estimators can be given but more complex. They can be derived and expressed through matrix and vector notations. We will skip that part in this class.

In case of equal sample size, we have

$$\hat{\mu} = \bar{Y}_{\ldots}, \hat{\alpha}_i = \bar{Y}_{i\cdot\cdot} - \bar{Y}_{\ldots}, \hat{\beta}_j = \bar{Y}_{\cdot j\cdot} - \bar{Y}_{\ldots}.$$

(handwritten: grand mean)

Hence the least squares estimator for μ_{ij} is $\hat{\mu} + \hat{\alpha}_i + \hat{\beta}_j = \bar{Y}_{i\cdot\cdot} + \bar{Y}_{\cdot j\cdot} - \bar{Y}_{\ldots}$. Note that the three terms are not independent. Therefore the variance of $\hat{\mu}_{ij}$ is NOT the sum of variances of three additive terms.

Derivation of LSE When Sample Sizes are Equal

We estimate μ, α_i, β_j by minimizing

$$\sum_{i=1}^{a} \sum_{j=1}^{b} \sum_{t=1}^{r} \left(y_{ijt} - (\mu + \alpha_i + \beta_j)\right)^2$$

subject to $\sum_{i=1}^{a} \alpha_i = 0, \sum_{j=1}^{b} \beta_j = 0,$.

Taking derivatives with respect to μ, α_i and β_j, we get

$$y_{\ldots} - abr\hat{\mu} - br\sum_i \hat{\alpha}_i - ar\sum_j \hat{\beta}_j = 0,$$

$$y_{i\cdot\cdot} - br\hat{\mu} - br\hat{\alpha}_i - r\sum_j \hat{\beta}_j = 0, \quad i = 1, \ldots, a,$$

$$y_{\cdot j\cdot} - ar\hat{\mu} - r\sum_i \hat{\alpha}_i - ar\hat{\beta}_j = 0, \quad j = 1, \ldots, b.$$

The solution:

$$\hat{\mu} = \bar{y}_{\ldots},$$
$$\hat{\alpha}_i = \bar{y}_{i..} - \bar{y}_{\ldots}, \quad i = 1, \ldots, a,$$
$$\hat{\beta}_j = \bar{y}_{.j.} - \bar{y}_{\ldots}, \quad j = 1, \ldots, b$$

Main Affects Contrasts for Factor A

Recall a contrast for the main effects of A is $\sum_{i=1}^a c_i \bar{\mu}_{i\cdot}$ with $\sum_{i=1}^a c_i = 0$. Therefore

$$\sum_{i=1}^a c_i \bar{\mu}_{i\cdot} = \sum_{i=1}^a c_i \alpha_i, \text{ with } \sum_i c_i = 0.$$

The contrast is estimated by $\sum_{i=1}^a c_i \hat{\alpha}_i = \sum_i c_i (\bar{Y}_{i..} + \bar{Y}_{.j.} - \bar{Y}_{...}) = \sum_i c_i \bar{Y}_{i...}$

It is variance is

$$\text{Var}\left(\sum_i c_i \bar{Y}_{i..}\right) = \frac{\sigma^2}{br} \sum_i c_i^2$$

For example, $\alpha_2 - \alpha_1$ has least squares estimator and associated variance

$$\hat{\alpha}_2 - \hat{\alpha}_1 = \bar{Y}_{2..} - \bar{Y}_{1..} \quad \text{with} \quad \text{Var}\left(\bar{Y}_{2..} - \bar{Y}_{1..}\right) = \frac{2\sigma^2}{br}.$$

Main Effects Contrasts of Factor B

For Factor B, a main-effect contrast $\sum k_j \beta_j$ with $\sum_j k_j = 0$ has least squares estimator and associated variance

$$\sum_j k_j \hat{\beta}_j = \sum_j k_j \bar{Y}_{.j.} \quad \text{and} \quad \text{Var}\left(\sum_j k_j \bar{Y}_{.j.}\right) = \frac{\sigma^2}{ar} \sum_j k_j^2$$

Estimation of σ^2 in the Main-Effects Model

First note

$$SSE = \sum_i \sum_j \sum_t \left(Y_{ijt} - \bar{Y}_{i..} - \bar{Y}_{.j.} + \bar{Y}_{...}\right)^2.$$

We can show

$$E[SSE] = (n - a - b + 1)\sigma^2 \text{ where } n = abr.$$

Hence the unbiased estimator for σ^2 is

$$MSE = SSE/(n - a - b + 1)$$

It can be shown that SSE/σ^2 has a chi-squared distribution with $(n - a - b + 1)$ degrees of freedom. Given a dataset, the calculated SSE is denoted by ssE. An upper $100(1 - \alpha)\%$ confidence bound for σ^2 is therefore given by

$$\sigma^2 \leq \frac{\text{ssE}}{\chi^2_{n-a-b+1, 1-\alpha}}$$

Confidence Interval for a Single Contrast

The $100(1 - \alpha)\%$ confidence interval of a single contrast is of the form

$$\text{estimate } \pm t_{n-a-b+1,\alpha/2} \times (\text{ standard error}).$$

For example, the 95% confidence interval for the main effects contrast $\alpha_2 - \alpha_1$ is

$$(\bar{y}_{2..} - \bar{y}_{1..}) \pm t_{n-a-b+1,\alpha/2} \times \sqrt{msE \times 2/(br)}.$$

Note the number of degrees of freedom of t-distribution is always the degrees of freedom of MSE.

Simultaneous Confidence Intervals

We may obtain simultaneous confidence intervals for some main effects contrasts for Factor A, or for the main effects contrasts for Factor B. These intervals are given similar to those in the one-way model or two-way complete model and are given in the same form

$$\text{estimate } \pm w \times (\text{ standard error}).$$

where w depends on the particular method to be used for constructing the simultaneous confidence intervals. Given below are the w values for different methods for the main-effects contrasts for Factor A:

$$w_B = t_{n-a-b+1,\alpha/2m};$$
$$w_S = \sqrt{(a-1)F_{a-1,n-a-b+1,\alpha}}$$
$$w_T = q_{a,n-a-b+1,\alpha}/\sqrt{2};$$
$$w_{D2} = |t|^{(0.5)}_{a-1,n-a-b+1,\alpha}$$

The w values for simultaneous confidence intervals for main-effects contrasts for Factor B are given similarly. These values are given similarly to those for the one-way model or the two-way complete model except the degrees of freedom for the error differ.

Unequal Variances

When the variances are unequal, i.e., $Y_{ijt} \sim N(0, \sigma_{ij}^2)$ with σ_{ij}^2 being unequal, we may use a variance stabilizing transformation or use Satterthwaite's approximation to the degrees of freedom. The provide the option `DDFM=Satterthwaite` in the model statement in SAS.

Analysis of Variance

For the two-way main effects model, we may test two hypotheses

$$H_0^A : \alpha_1 = \alpha_2 = \cdots = \alpha_a$$

and

$$H_0^B : \beta_1 = \beta_2 = \cdots = \beta_b.$$

Often, these are the hypothesis we test first. If the null hypothesis is rejected, we may use contrasts for further analysis to find out more about the differences that exist among the main effects.

The F-test is similar to those in the two-way complete model by comparing the difference between the SSE under H_0 and the SSE under the full model (which is now the two-way main effects model).

The exact formula can be easily given when the sample sizes are equal. For example, under H_0, the model is reduced to a one-way model with a treatments each ofwhich has a sample size br. Then

$$ssE_0^B = \sum_{i=1}^{a} \sum_{j=1}^{b} \sum_{t=1}^{r} (y_{ijt} - \bar{y}_{i..})^2.$$

$$ssB = \sum_{i=1}^{\breve{w}} \sum_{j=1}^{\breve{v}} \sum_{t=1}^{'} (y_{ijt} - \bar{y}_{i..})^2 - \sum_{i=1}^{\breve{w}} \sum_{j=1}^{\breve{v}} \sum_{t=1}^{'} ((y_{ijt} - \bar{y}_{i..}) - (\bar{y}_{.j.} - \bar{y}_{.....}))^2$$

$$= ar \sum (\bar{y}_{.j.} - \bar{y}_{...})^2$$

Therefore, when H_0^B is true,

$$\boxed{\frac{SSB/(b-1)\sigma^2}{SSE/(n-a-b+1)\sigma^2} = \frac{MSB}{MSE} \sim F_{b-1,n-a-b+1},}$$

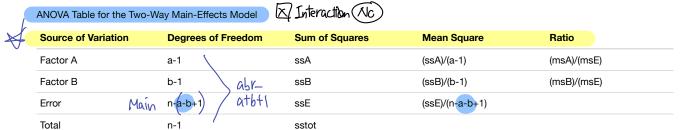and the decision rule for testing H_0^B against H_A^B is

$$\text{reject } H_0^B \text{ if } \frac{msB}{msE} > F_{b-1,n-a-b+1,\alpha}.$$

Similarly,

$$ssA = br \sum (\bar{y}_{i..} - \bar{y}_{...})^2.$$

We reject

$$H_0^{\textcircled{A}} \text{ if } \frac{ms\textcircled{A}}{msE} > F_{\textcircled{a}-1,n-a-b+1,\alpha}.$$

ANOVA Table for the Two-Way Main-Effects Model   ☒ Interaction Ⓝⓒ

Source of Variation	Degrees of Freedom	Sum of Squares	Mean Square	Ratio
Factor A	a-1	ssA	(ssA)/(a-1)	(msA)/(msE)
Factor B	b-1	ssB	(ssB)/(b-1)	(msB)/(msE)
Error	n-a-b+1	ssE	(ssE)/(n-a-b+1)	
Total	n-1	sstot		

abr — Main ... a+b+1

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Activity Details

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🟦 Side-by-Side Comparison: Two-Way Main Effects vs Two-Way Complete Model

Feature	Two-Way Main Effects Model	Two-Way Complete Model
Model form	$Y_{ijt} = \mu + \alpha_i + \beta_j + \varepsilon_{ijt}$	$Y_{ijt} = \mu + \alpha_i + \beta_j + (\alpha\beta)_{ij} + \varepsilon_{ijt}$
Interaction term	❌ *Not included*	✅ *Included*
When to use	When prior evidence suggests *no interaction*	When testing for or expecting interaction
Hypotheses	$H_0^A : \alpha_i = 0; H_0^B : \beta_j = 0$	Same + $H_0^{AB} : (\alpha\beta)_{ij} = 0$
Error degrees of freedom	$ab \cdot r - a - b + 1$	$ab(r-1)$
Interpretability	Easier if model is correct (more power for A, B)	More flexible, but more complex if interaction present
ANOVA table	A, B, Error, Total	A, B, A×B, Error, Total
Common in...	Surveys, observational studies	Full-factorial experiments, lab trials
Visual check	Parallel lines in interaction plot	Use interaction plot + test A×B effect