Experiments with Two Crossed Treatment Factors

ANOVA for Two-Way Complete Models

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ANOVA for the Two-Way Complete Model

There are three standard hypotheses that are usually examined when the two-way complete model is used. The first hypothesis is that the interaction between treatment factors A and B is negligible; that is,

Meaning:

Ho: no interaction $$H_0^{AB} : (\alpha\beta)_{ij} = 0, \text{ for all } i, j. \quad \mu_{ij} = \mu + \alpha_i + \beta_j + \cancel{(\alpha\beta)_{ij}} \tag{1}$$

Note the textbook states the same hypothesis in an alternative but equivalent way:

(B) alternative=q (A) alternative=s A B

$$H_0^{AB} : \{[\underbrace{(\alpha\beta)_{ij} - (\alpha\beta)_{iq}}_{0}] - [\underbrace{(\alpha\beta)_{sj} + (\alpha\beta)_{sq}}_{0}] = 0 \text{ for all } i \neq s, j \neq q\}. \tag{2}$$

One can verify (2) can be expressed as

$$H_0^{AB} : \{(\mu_{ij} - \mu_{iq}) - (\mu_{sj} + \mu_{sq}) = 0 \text{ for all } i \neq s, j \neq q\}. \quad ✓ \tag{3}$$

The interpretation of (3) is clear: the pairwise difference for any two levels of a factor does not depend on the level of another factor. That is what no-interaction means.

The other two standard hypotheses are the main-effect hypotheses:

Split A and B ✓

$$(H_0^A : \bar{\mu}_{1\cdot} = \ldots = \bar{\mu}_{a\cdot})(H_0^B : \bar{\mu}_{\cdot 1} = \ldots = \bar{\mu}_{\cdot b})$$

where $\bar{\mu}_{i\cdot} = (1/b)\sum_{j=1}^{b} \mu_{ij}$ and $\bar{\mu}_{\cdot j} = (1/a)\sum_{i=1}^{a} \mu_{ij}$.

Again, note the textbook expresses these two hypothesis in a different but equivalent way by introducing more notations.

Testing Interactions

interaction present.

We test H_0^{AB} in (1) against the alternative hypothesis H_A^{AB} : {the interaction is not negligible}. The idea is to compare the sum of squares for error ssE under the two-way complete model with the sum of squares for error ssE_0^{AB} under the reduced model obtained when H_0^{AB} is true. The difference

✓ Essentially just the main effect model.

Reject Ho: { If ssAB ↑↑↑, means there is ↑↑↑ difference between reduced and full.

Reduced - Full

$$\boxed{ssAB = ssE_0^{AB} - ssE}$$

is called the sum of squares for the interaction AB, and the test rejects H_0^{AB} in favor of H_A^{AB} if $ssAB$ is large relative to ssE. Under H_0^{AB}, $ssAB/\sigma^2$ has an F distribution with degrees of freedom equal to the number of parameters reduced.

Note H_0^{AB} has $(a-1)(b-1)$ constraints and each constraint reduces the number of parameter by 1. Hence the degree of freedom of the F distribution is $(a-1)(b-1)$. Hence

$$\boxed{F = \frac{ssAB/(a-1)(b-1)}{ssE/(n-ab)} \sim F_{\underbrace{(a-1)(b-1)}_{df1}, \underbrace{n-ab}_{df2}}}$$

We reject H_0^{AB} at significance level α if the test statistic $F > F_{(a-1)(b-1), n-ab, \alpha}$ or the p-value $< \alpha$.

df1 df2 α

Note here I use $ssAB$ to denote a random variable (when talking about distribution) as well as a particular value obtained from the experimental data. The book distinguishes these two by using different notations $SS(AB)$ and $ssAB$.

Let us see how $ssAB$ is calculated when the sample sizes all equal to r.

Write $\mu_{ij} = \mu + \alpha_i + \beta_j + (\alpha\beta)_{ij}$ where

$$\alpha_i = \bar{\mu}_{i.} - \bar{\mu}_{..},\ \beta_j = \bar{\mu}_{.j} - \bar{\mu}_{..},$$

$$(\alpha\beta)_{ij} = \mu_{ij} + \bar{\mu}_{..} - \bar{\mu}_{i.} - \bar{\mu}_{.j}.$$

Under H_0^{AB}, $(\alpha\beta)_{ij} = 0$. We therefore have

move $\bar{u}_{..}$, $-\bar{u}_{i.}$, $-\bar{u}_{.j}$ to the other side of the equation

$$\mu_{ij} = \bar{\mu}_{i.} + \bar{\mu}_{.j} - \bar{\mu}_{..}.$$

whose least squares estimator when the sample sizes are equal is

estimate using:

$$\hat{\mu}_{ij} = \bar{Y}_{i..} + \bar{Y}_{.j.} - \bar{Y}_{...}.$$

LSE
least Squares Estimate

Then

Reduced

$$\begin{aligned} ssE_0^{AB} &= \sum_i \sum_j \sum_t (y_{ijt} - \bar{y}_{i..} - \bar{y}_{j.} + \bar{y}_{....})^2 \\ &= \sum_i \sum_j \sum_t (y_{ijt} - \bar{y}_{ij.})^2 + r \sum_i \sum_j (\bar{y}_{ij.} - \bar{y}_{i.} - \bar{y}_{j.} + \bar{y}_{...})^2 \\ &= ssE + r \sum_i \sum_j (\bar{y}_{ij.} - \bar{y}_{i.} - \bar{y}_{j.} + \bar{y}_{...})^2. \end{aligned}$$

Therefore,

$$\begin{aligned} ssAB = ssE_0^{AB} - ssE \\ &= r \sum_i \sum_j (\bar{y}_{ij.} - \bar{y}_{i.} - \bar{y}_{.j.} + \bar{y}_{...})^2 \\ &= r \sum_i \sum_j \bar{y}_{ij.}^2 - br \sum_i \bar{y}_{i.}^2 - ar \sum_j \bar{y}_{.j.}^2 + abr\bar{y}_{...}^2 \end{aligned}$$

This formula holds only when the sample sizes are all equal to r. When sample sizes are unequal, the formula becomes more complex but can be expressed using matrix and vector notations.

Testing Main Effects

Consider testing $H_0^A : \bar{\mu}_{1.} = \ldots = \bar{\mu}_{a.}$. This reduced model reduces the number of parameters by $(a-1)$. To fined the SSE under the reduced model, first note the least squares estimator of μ_{ij} is

$$\bar{y}_{ij.} - \bar{y}_{i..} + \bar{y}_{....}$$

Hence

$$\begin{aligned} ssE_0^A &= \sum_i \sum_j \sum_t (y_{ijt} - \bar{y}_{ij.} + \bar{y}_{i..} - \bar{y}_{....})^2 \\ &= \sum_{i=1}^a \sum_{j=1}^b \sum_{t=1}^r (y_{ijt} - \bar{y}_{ij.})^2 + br \sum_{i=1}^a (\bar{y}_{i..} - \bar{y}_{...})^2 \\ &= ssE + br \sum_{i=1}^a (\bar{y}_{i..} - \bar{y}_{...})^2 \end{aligned}$$

and

$$\boxed{ssA = ssE_0^A - ssE} = br \sum_{i=1}^a (\bar{y}_{i..} - \bar{y}_{...})^2$$

H^A

Reject H_0^A if

$$\boxed{\frac{msA}{msE} > F_{a-1,n-ab,\alpha},}$$

where $\boxed{msA = ssA/(a-1) \text{ and } msE = ssE/(n-ab)}$.

(V) Interaction

Testing the main effects of Factor B is similar. The following is called the ANOVA table for the two way complete model:

Source of Variation	Degrees of Freedom	Sum of Squares	Mean Squares $\frac{SS}{df}$	Ratio F	P value
A	$a-1$	ssA	ssA/(a-1)	msA/msError	
B	$b-1$	ssB	msB/(b-1)	msB/msE	
AB	$(a-1)(b-1)$	ssAB	msAB/(a-1)(b-1)	msAB/msE	
Error Complete	n-ab	ssE	ssE/(n-ab)		
Total	n-1	ssTotal			

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