Row-Column Designs

When you have 2 blocking factors – and one treatment applied exactly once in each row & column.

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Introduction

In a block design, the blocks could be the combinations of levels of two blocking factors. When the two blocking factors do not interact, the experiment can be designed with only one observation per combination of the levels of the blocking factors.

I will use a hypothetical example to demonstrate the idea. Suppose there are 9 blocks which represent the combinations of levels of two blocking factors, denotes by K (with three levels) and L (with three levels). Previously, we denote the block effects by θ_i, and the model for the block design with no block by treatment interaction is

$$Y_{hi} = \mu + \theta_h + \tau_i + \epsilon_{hi}.$$

The block effects have 8 degrees of freedom and the treatment effects have $(v-1)$ degrees of freedom. Assume $v = 3$. Then the total sample size has to be 12 or more in order for the error terms has a positive degree of freedom.

If the two blocking factors do not interact, then the block effects have $2+2$ degrees of freedom. Then it is possible to design an experiment with $n = 9$ observations if $v = 3$. This leads to a row-column design.

Row-Column Designs

Design Plans and Randomization

In a row-column design, there are two blocking factors whose levels are called row blocks and columns blocks. The following table shows an experimental plan with $b = 7$ row blocks and $c = 4$ column blocks, $v = 7$ treatments.

only 4 blocks for 7 treatments (Incomplete Block Design)

(4) Column blocks

		I	II	III	IV
(7) Row blocks (Randomized Complete Block Design)	I	1	3	6	7
	II	2	4	7	1
	III	3	5	1	2
	IV	4	6	2	3
	V	5	7	3	4
	VI	6	1	4	5
	VII	7	2	5	6

# of level of second blocking factor

v = 7 treatments (1~7)

# of level of first blocking factor

Youden Design

Given an experimental plan, random assignments are taken in a few steps as follows:

i. The row block labels in the design are randomly assigned to the levels of the first blocking factor.
ii. The column block labels in the design are randomly assigned to the levels of the second blocking factor.
iii. The treatment labels in the design are randomly assigned to the levels of the treatment factor.

Total of (3) Random Assignments

Since there is only one experimental unit in each cell, there is no need for random assignment of experimental units to treatment labels within a cell.

Estimability

It is difficult to directly verify if all treatment contrasts are all estimable for a given row-column design. One easy and practical way is enter a set of hypothetical data for the design into a computer package (such as SAS) and see whether the required contrasts are estimable.

The row-column designs to be introduced next guarantee all treatment contrasts are estimable.

Latin Square Designs

A $v \times v$ Latin square is an arrangement of v Latin letters into a $v \times v$ array (a table with v rows and v columns) in such a way that each letter occurs once in each row and once in each column. For example, the following 3×3 array is a 3×3 Latin square:

same # of v of rows, columns, treatments

	I	II	III
I	A	B	C
II	B	C	A
III	C	A	B

A Latin square design has v treatment labels, and v^2 experimental units arranged in v row blocks and v column blocks.

Latin square designs are often used in experiments involving subjects, especially where the subjects are allocated a sequence of treatments over time and where the time effect is thought to have a major effect on the response. For example, in an experiment to compare the effects of v drugs, the rows of the Latin square might correspond to v subjects to whom the drugs are given, and the columns might correspond to v time periods, with each subject receiving one drug during each time period. An experiment of this type, in which each subject receives a sequence of treatments, is called a *crossover experiment*.

Several (say s) Latin squares may be stacked over to yield an s-replicate Latin square.

Youden Designs

when you have b rows and v columns but only v treatments (incomplete in rows)

A $v \times c$ Youden square is an arrangement of v Latin letters into a $v \times c$ array (with $c < v$) in such a way that each letter occurs once in each column and at most once in each row. In addition, every pair of treatments occurs in the same number of rows, so the columns form complete blocks and the rows form a balanced incomplete block design.

Replication of Youden Squares

We can obtain $r = cs$ observations per treatment by stacking s Youden squares one above another (giving $b = vs$ row blocks and $c(< v)$ column blocks). This results in large column-block sizes but may allow for estimation of some column $\times$ treatment interaction contrasts after adjusting for row block effects. The row blocks still form a balanced incomplete block design, and the column blocks still form a complete block design.

Analysis of Row-Column Designs

Model for a Row-Column Design

For a row-column design with b rows and c columns, the row-column-treatment model is

column block

row block

# of treatment

$$\begin{gathered} Y_{hqi} = \mu + \theta_h + \phi_q + \tau_i + \epsilon_{hqi}, \\ \epsilon_{hqi} \sim N\left(0, \sigma^2\right), \\ \epsilon_{hqi}\text{'s are mutually independent}, \\ h = 1, \ldots, b; q = 1, \ldots, c; i = 1, \ldots, v; (h, q, i) \text{ in the design}, \end{gathered} \tag{1}$$

where θ_h, ϕ_q, and τ_i denote the effects on the response of row block h, column block q, and treatment i, respectively.

The ANOVA table for this model is given below.

Source of variation	Degrees of freedom	Sum of squares	Mean square	Ratio
Rows (unadj)	$b-1$	$ss\theta$	–	–
Columns (unadj)	$c-1$	$ss\phi$	–	–
Treatments(adj)	$v-1$	ssT_{adj}	msT_{adj}	msT_{adj}/msE
Error	$bc-b-c-v+2$	ssE	msE	
Total	$bc-1$	$sstot$		

row + column

n = b × c

→ rows SS + columns SS + treatments SS + error SS

Formulae

$$Q_i = T_i - \frac{1}{c}\Sigma_{h=1}^{b} n_{h.i} B_h - \frac{1}{b}\sum_{q=1}^{c} n_{.qi} C_q + \frac{r}{bc} G$$
$$sstot = \Sigma_{h=1}^{b}\Sigma_{q=1}^{c}\Sigma_{i=1}^{v} n_{hqi} y_{hqi}^2 - \frac{1}{bc} G^2$$
$$ssE = sstot - ss\theta - ss\phi - ssT_{\text{adj}}$$
$$T_i = \Sigma_{h=1}^{b}\Sigma_{q=1}^{c} n_{hqi} y_{hqi}$$
$$C_q = \Sigma_{h=1}^{b}\Sigma_{i=1}^{v} n_{hqi} y_{hqi}$$
$$ssT_{\text{adj}} = \Sigma_{i=1}^{v} Q_i \hat{\tau}_i$$
$$ss\theta = \frac{1}{c}\Sigma_{h=1}^{b} B_h^2 - \frac{1}{bc} G^2$$
$$ss\phi = \frac{1}{b}\Sigma_{q=1}^{c} C_q^2 - \frac{1}{bc} G^2$$
$$B_h = \Sigma_{q=1}^{c}\Sigma_{i=1}^{v} n_{hqi} y_{hqi}$$
$$G = \Sigma_{h=1}^{b}\Sigma_{q=1}^{c}\Sigma_{i=1}^{v} n_{hqi} y_{hqi}$$

Figure 1: Analysis of variance table for a row-column design

The least squares estimates and the various sums of squares are derived similarly to incomplete block designs we introduced before.

For simultaneous confidence intervals, the methods of Bonferroni and Scheffé are applicable to all row-column design. Tukey and Dunnett methods for pairwise comparisons can be used for Latin square designs and Youden designs.

Analysis of Latin Square Designs

Consider an s-replicate $v \times v$ Latin square design. Let Y_{hqi} denote the observation in the hth row, qth column and for the ith treatment. The model for the design is (1).

The treatment effect are estimated by

$$\hat{\tau}_i = \bar{Y}_{\cdot\cdot i} - \bar{Y}_{\cdots}. \tag{2}$$

For any treatment contrast $\sum d_i \tau_i$, its estimate is

$$\sum d_i \hat{\tau}_i = \sum d_i \bar{Y}_{\cdot\cdot i}.$$

with variance

with variance

$$\text{Var}\left(\sum d_i\hat{\tau}_i\right) = \sum d_i^2\left(\frac{\sigma^2}{vs}\right)$$

The multiple-comparison methods of Bonferroni, Scheffé, Tukey, and Dunnett can be used for s replicate Latin square designs. Formulae for confidence intervals for treatment contrasts $\Sigma d_i\tau_i$ are of the form

$$\left(\frac{1}{vs}\Sigma_i d_i T_i \pm w\sqrt{\text{msE}\left(\frac{\Sigma d_i^2}{vs}\right)}\right),$$

Total Sum for Treatment i: T_i

$\frac{T_i}{vs} = \bar{Y}_{\cdots i}$

where the appropriate critical coefficients w for the four methods are,

$$w_B = t_{(vs-2)(v-1),\alpha/2m}; \quad w_S = \sqrt{(v-1)F_{v-1,(vs-2)(v-1),\alpha}};$$
$$w_T = q_{v,(vs-2)(v-1),\alpha}/\sqrt{2}; \quad w_{D2} = |t|^{(0.5)}_{v-1,(vs-2)(v-1),\alpha}.$$

If we know the upper bound of msE, we can choose an appropriate s in order to have a desired width for the confidence intervals.

Examples and SAS Code

The Exercise Bicycle Experiment is described in Section 12.2.4 on page 404. There are three treatment factors each with 2 levels, which yield 8 treatments. There are 8 row blocks representing day of exercise, and 3 column blocks representing 3 experimenters. The design and data are shown in Table 12.6 on page 404.

```
* Data by observtion, as in Table 12.10, p417;
DATA BIKE;
  INPUT DAY SUBJECT PULSE DURAT SPEED PEDAL;
  LINES;
        1     1      45    1      1      2
        1     2      25    2      1      2
        1     3      18    2      2      1
        2     1      27    2      2      2
        2     2      20    2      1      1
        2     3      32    2      1      2
        3     1      40    2      1      2
        3     2      23    2      2      2
        3     3      28    1      1      2
        4     1      17    2      2      1
        4     2      32    1      1      2
        4     3      24    1      2      2
        5     1      30    2      1      1
        5     2      36    1      1      1
        5     3      20    2      2      2
        6     1      29    1      2      2
        6     2      13    2      2      1
        6     3      20    1      2      1
        7     1      34    1      1      1
        7     2      18    1      2      1
        7     3      25    2      1      1
        8     1      21    1      2      1
        8     2      22    1      2      2
        8     3      34    1      1      1
;
run;
proc print;
run;

proc glm data=bike;
class day subject durat speed pedal;
model pulse=day subject durat|speed|pedal;
run;
```

In the above analysis, we included all main effects and all possible interaction terms of the three treatment factors. Therefore it shall be equivalent to the following analysis when the treatment combinations are used:

```
DATA BIKE;
  INPUT DAY SUBJECT PULSE DURAT SPEED PEDAL;
  trtmt=100*durat+10*speed+pedal;
  LINES;
        1     1      45    1      1      2
        1     2      25    2      1      2
        1     3      18    2      2      1
        2     1      27    2      2      2
        2     2      20    2      1      1
        2     3      32    2      1      2
        3     1      40    2      1      2
```

```
        3     2     23    2     2     2
        3     3     28    1     1     2
        4     1     17    2     2     1
        4     2     32    1     1     2
        4     3     24    1     2     2
        5     1     30    2     1     1
        5     2     36    1     1     1
        5     3     20    2     2     2
        6     1     29    1     2     2
        6     2     13    2     2     1
        6     3     20    1     2     1
        7     1     34    1     1     1
        7     2     18    1     2     1
        7     3     25    2     1     1
        8     1     21    1     2     1
        8     2     22    1     2     2
        8     3     34    1     1     1
;
run;
proc print;
run;

proc glm data=bike;
class day subject trtmt;
model pulse=day subject trtmt;
run;
```

The two models have the same msE. The p-value for treatment is 0.0238, which indicate a significant difference among the treatments. However, it does not tell use anything about the interactions of the three factors, not does it reveal anything about the main effects of each treatment factor. That is why it is preferable to use the factorial model.