Analysis of Incomplete Block Designs

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Least Squares Estimation

The model for an incomplete block design is

$$\begin{gathered} Y_{hi} = \mu + \theta_h + \tau_i + \epsilon_{hi}, \\ \epsilon_{hi} \sim N\left(0, \sigma^2\right), \\ h = 1, \ldots, b; i = 1, \ldots, v; \quad (h, i) \text{ in the design}, \\ \epsilon_{hi}\text{'s are mutually independent} \end{gathered}$$

The least squares estimates for θ_h and τ_i minimize

$$\sum_{(h,i) \text{ in the design}} (Y_{hi} - \mu - \theta_h - \tau_i)^2 = \sum_h \sum_i n_{hi} (Y_{hi} - \mu - \theta_h - \tau_i)^2$$

subject to $\sum \tau_i = \sum \theta_h = 0$, where $n_{hi} = 1$ if (h, i) is in the design and $n_{hi} = 0$ otherwise.

One major difference is that the sum is not over all pairs (h, i) but only over those that appeared in the design. The least squares estimators for the treatment parameters in the model for an incomplete block design must include an adjustment for blocks. This means that the least squares estimator for the pairwise comparison $\tau_p - \tau_i$ is not the unadjusted estimator $\bar{Y}_{.p} - \bar{Y}_{.i}$ as it would be for a randomized complete block design.

The least squares estimates are a solution to

$$r(k-1)\hat{\tau}_i - \sum_{p \neq i} \lambda_{pi} \hat{\tau}_p = kQ_i, \quad \text{for } i = 1, \ldots, v \sum_i \hat{\tau}_i = 0, \sum_h \hat{\theta}_h = 0.$$

where λ_{pi} is the number of blocks containing both treatments p and i, and

$$Q_i = T_i - \frac{1}{k} \sum_{h=1}^{b} n_{hi} B_h, \quad T_i = \sum_{h=1}^{b} n_{hi} y_{hi}, \quad B_h = \sum_{i=1}^{v} n_{hi} y_{hi}.$$

Confidence intervals for a contrast $\sum_i c_i \tau_i$ are of the form

$$\text{estimate} \pm w(\text{standard error}).$$

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For a single $100(1-\alpha)\%$ confidence interval, $w = t_{\alpha/2, df}$ where df is the number of degrees of freedom for error.

For simultaneous confidence intervals, the Bonferroni and Scheffé methods can be used for all incomplete block designs. Tukey's and Dunnett's methods can be used for balanced incomplete block designs.

Analysis of Variance

The following is the ANOVA table for the incomplete block design.

Source of variation	Degrees of freedom	Sum of squares	Mean square	Ratio
Blocks (adj)	$b-1$	$ss\theta_{\text{adj}}$	$\text{ms}\theta_{\text{adj}}$	—
Blocks (unadj)	$b-1$	$ss\theta$	—	—

Treatments (adj)	$v-1$	ssT_{adj}	msT_{adj}	$\frac{msT_{\text{adj}}}{msE}$
Error	$bk-b-v+1$	ssE	msE	
Total	$bk-1$	$sstot$		

where

$$
\begin{array}{ll}
ss\theta = \sum_{h=1}^{b} B_h^2/k - G^2/(bk) & ssE = sstot - ss\theta - ssT_{\text{adj}} \\
ssT_{\text{adj}} = \sum_{i=1}^{v} Q_i \hat{\tau}_i & \text{sstot} = \sum_{h=1}^{b}\sum_{i=1}^{v} n_{hi} y_{hi}^2 - G^2/(bk) \\
Q_i = T_i - \sum_{h=1}^{b} n_{hi} B_h/k. & ss\theta_{\text{adj}} = sstot - ssE - \left(\sum_{i=1}^{v} T_i^2/r - G^2/(bk)\right)
\end{array}
$$

Analysis of Balanced Incomplete Block Designs

For the balanced incomplete block design the estimates of treatment effects have a simple form:

$$
\hat{\tau}_i = \frac{k}{\lambda v} Q_i, \quad \text{for } i = 1, \ldots, v,
$$

where λ is the number of times that every pair of treatments occurs together in a block, and Q_i is the adjusted treatment total. Thus, the sum of squares for treatments adjusted for blocks in becomes

$$
ssT_{\text{adj}} = \sum_{i=1}^{v} \frac{k}{\lambda v} Q_i^2,
$$

and the least squares estimator of contrast $\sum c_i \tau_i$ is

$$
\sum_{i=1}^{v} c_i \hat{\tau}_i = \frac{k}{\lambda v} \sum_{i=1}^{v} c_i Q_i.
$$

It can be shown that the corresponding variance can be calculated as

$$
\operatorname{Var}\left(\sum_{i=1}^{v} c_i \hat{\tau}_i\right) = \sum_{i=1}^{v} c_i^2 \left(\frac{k}{\lambda v}\right) \sigma^2.
$$

The simultaneous confidence intervals for a set of contrasts $\sum c_i \tau_i$ are given by

$$
\frac{k}{\lambda v} \sum_{i=1}^{v} c_i Q_i \pm w \sqrt{\sum_{i=1}^{v} c_i^2 \left(\frac{k}{\lambda v}\right) msE}
$$

where the critical coefficients for the four methods are, respectively,

$$
\begin{gathered}
w_B = t_{bk-b-v+1, \alpha/2m}; w_S = \sqrt{(v-1) F_{v-1, bk-b-v+1, \alpha}}; \\
w_T = q_{v, bk-b-v+1, \alpha}/\sqrt{2}; w_{D2} = |t|_{v-1, bk-b-v+1, \alpha}^{(0.5)}
\end{gathered}
$$

SAS Code

SAS code is the same as for the randomized complete block designs. See Sec. 11.8 for examples.

Sample Size

Given the number of treatments v and the block size k, how many blocks b are required to achieve confidence intervals of a specified length or a hypothesis test of specified power? Since for most purposes the balanced incomplete block design is the best incomplete block design when it is available, we start by calculating b and the treatment replication $r = bk/v$ for this design. Then if a balanced incomplete block design cannot be found with b and r close to the calculated values, a group divisible, cyclic, or other incomplete block design can be considered. Since balanced incomplete block designs are the most efficient, other incomplete block designs would generally require b and r to be a little larger.

Suppose we use Tukey's method for all pairwise comparisons. Then

minimum significant difference

$$msd = \underbrace{(q_{v,df,\alpha}/\sqrt{2})}_{\text{Tukey's}} \times \sqrt{2(mse)\frac{k}{\lambda v}}$$

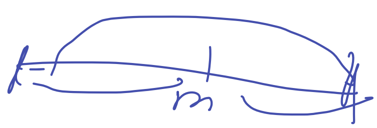
where $\lambda = r(k-1)/(v-1)$, $df = vr - b - v + 1$ and $b = vr/k$.

Consider the case when $v = 5, k = 3$ and mse is bounded by 2. How many replicates r per treatment are sufficient for the 95% simultaneous confidence intervals of all pairwise comparisons to have a width less than 3?

```
data size;
input r @@;
alpha=0.05;
mse=2;
v=5;
k=3;
b=v*r/k;
df=v*r-b-v+1;
lambda=r*(k-1)/(v-1);
q=probmc('range',.,1-alpha,df,v);
msd=(q/sqrt(2))*sqrt(2*mse*k/(lambda*v)); Tukey's
lines;
14 15 16 17 18 19
;
proc print;
var r b lambda msd;
run;
```

mean ± msd

2msd



2×msd=width

Obs	r	b	lambda	msd
1	14	23.3333	7.0	1.66753
2	15	25.0000	7.5	1.60593
3	16	26.6667	8.0	1.55072
4	17	28.3333	8.5	1.50086
5	18	30.0000	9.0	1.45554
6	19	31.6667	9.5	1.41410

We see that we need r to be at least 18 for the width to be less than 3. When $r = 18$, we have $b = 30$ and $\lambda = 9$. Both are integers so that a balanced incomplete block design may exist.

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Activity Details

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