STAT 525

Chapter 10
Diagnostics in Multiple Regression

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Model Adequacy for a Predictor

Review: Partial Coefficient of Determination

Consider model

$$Y_i = \beta_0 + \beta_1 X_{i1} + \beta_2 X_{i2} + \cdots + \beta_{p-1} X_{i,p-1} + \varepsilon_i$$

- Define two variables:
  - residuals of predicting Y as function of $X_2, \cdots, X_{p-1}$

$$e_i(Y|X_2, \cdots, X_{p-1}) = Y_i - \hat{Y}_i(X_2, \cdots, X_{p-1})$$

  - residuals of predicting X_1 as function of $X_2, \cdots, X_{p-1}$

$$e_i(X_1|X_2, \cdots, X_{p-1}) = X_{i1} - \hat{X}_{i1}(X_2, \cdots, X_{p-1})$$

- $R^2_{Y1|2:(p-1)}$ equals to R^2 for regressing $e_i(Y|X_2, \cdots, X_{p-1})$ on $e_i(X_1|X_2, \cdots, X_{p-1})$
  - Measures *additional* information in X_1 helping predict Y
  - Equals to squared partial correlation $r^2_{Y1|2:(p-1)}$
- Similarly consider each X_j, $j = 2, \cdots, p-1$

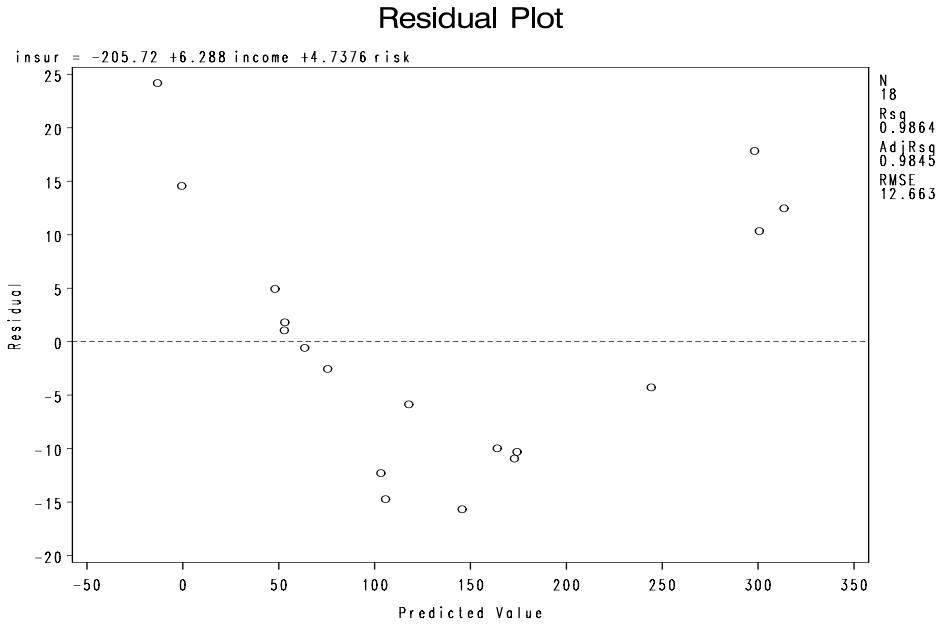
Partial Regression Plots

- For X_j, plot $e_i(Y|all\ X_k\ with\ k \neq j)$ vs $e_i(X_j|all\ X_k\ with\ k \neq j)$
- Also called added variable plots or adjusted variable plots
- Shows strength of the marginal relationship given other variables in the model model
  - Provides visual display of relationship
  - Allows check of "adjusted" relationship
- Can detect:
  - Nonlinear relationship
  - Heterogeneous variance
  - Unusual observations
- NOTE: don't confuse Partial Regression Plots with *Partial Residual Plots*
  - Partial Residual Plots: $e_i + \widehat{\beta}_j X_{ij}$ vs. X_{ij}, where $\widehat{\beta}$ is based on the full model.
  - Partial Regression Plots is a more meaningful way to show the relationship between Y and X_j (why?)

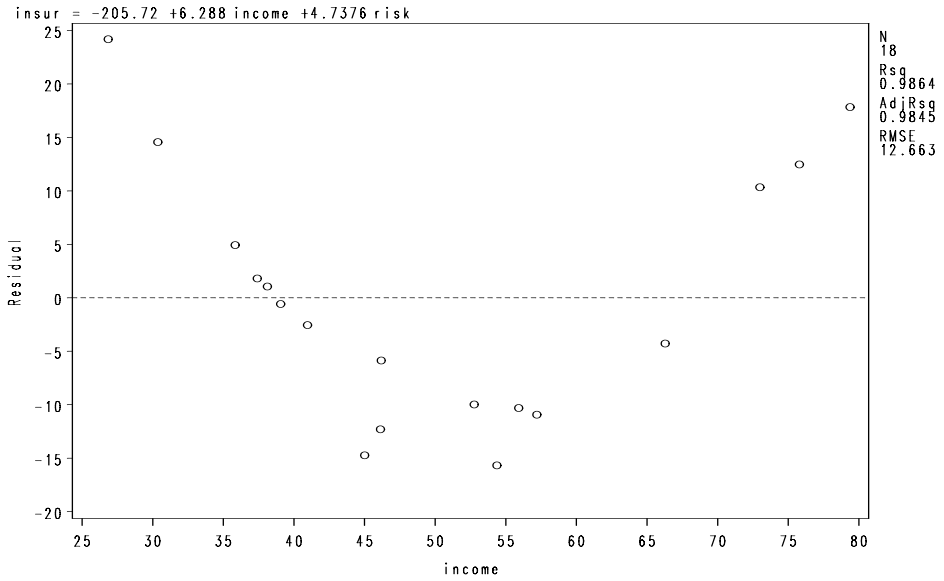
Example on Page 386

- Surveyed 18 managers age 30-39. Interested in relating the amount of life insurance carried to risk aversion and salary.
- Y is amount of life insurance carried
- Two predictor variables
  - X_1 average annual income during past two years
  - X_2 risk aversion score (higher $\rightarrow$ more averse)

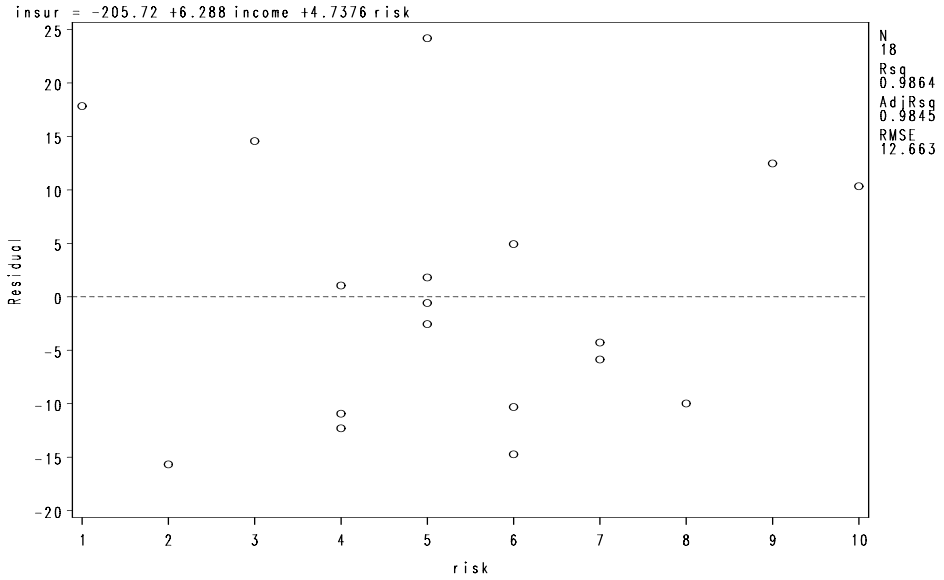
```
data a1;
    infile 'D:\nobackup\tmp\CH10TA01.TXT';
    input income risk insur;

proc reg data=a1;
    model insur = income risk;
    output out=out1 r=resid p=pred;

proc sort data=out1; by pred; run;
symbol v=circle i=none c=black;
proc gplot data=out1;
    plot resid*pred resid*income resid*risk/vref=0;
run;
```



Residual Plot



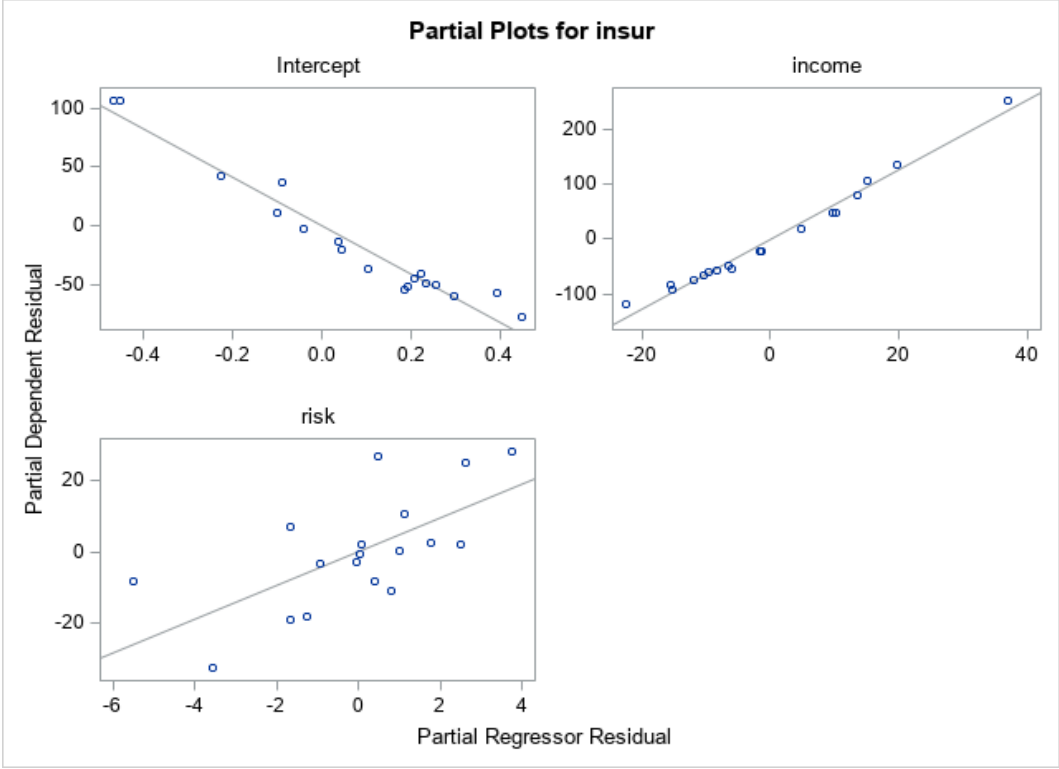
Residual Plot



Partial Regression Plots from PROC REG

```
proc reg data=a1;
    model insur = income risk/partial;
run;
```

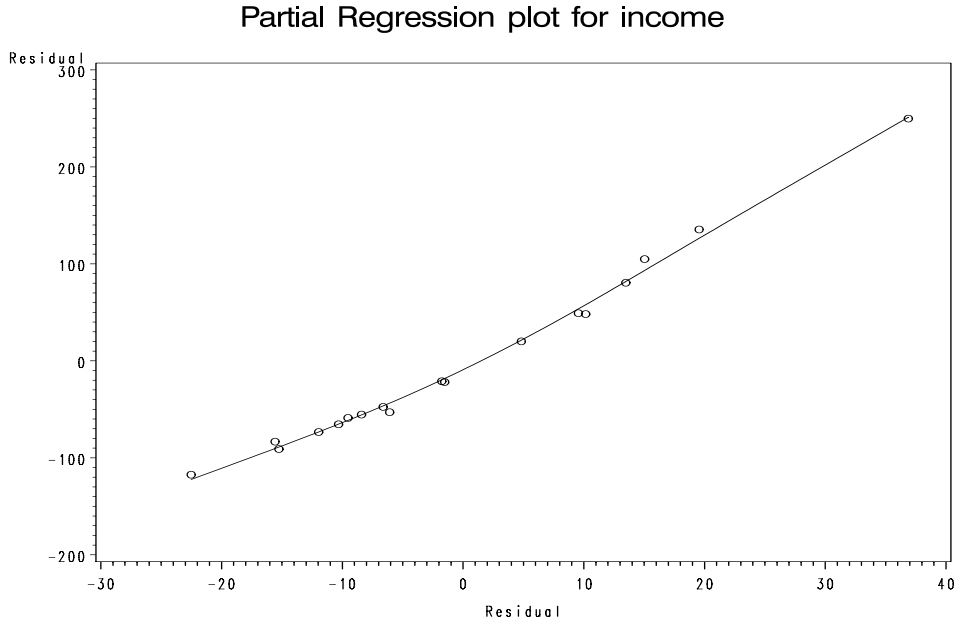
- Latest version of SAS provides partial regression plots for each predictor including intercept. Old version of SAS gives line printer plots only.



Partial Regression Plot of X_1 (Income)

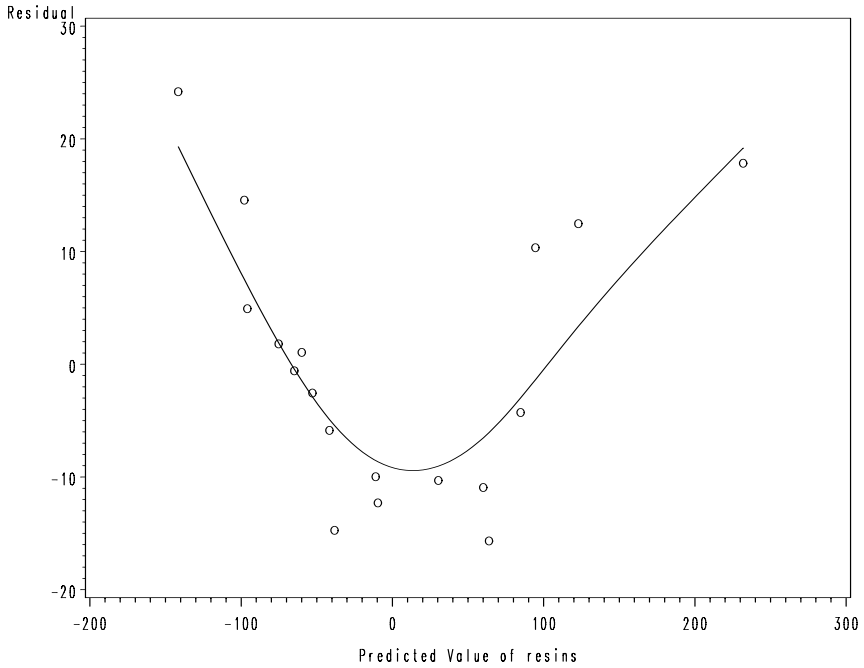
Generate partial regression plots manually.

```
proc reg data=a1;
    model insur income = risk;
    output out=a2 r=resins resinc;

proc sort data=a2; by resinc;

symbol v=circle i=sm70 c=black;
proc gplot data=a2;
    plot resins*resinc;
run;
```



e(insurance|risk) vs. e(income|risk)

```
/* --- Is a linear relationship appropriate? ---*/
proc reg data=a2;
    model resins = resinc;
    output out=new1 r=res p=pred;
run;

symbol v=circle i=sm70 c=black;
proc gplot data=new1;
    plot res*pred;
run; quit;
```



residuals of `model resins = resinc`

Output of `model resins = resinc`

Parameter Estimates

Variable	Label	DF	Parameter Estimate	Standard Error	t Value	Pr > |t|
Intercept	Intercept	1	1.10593E-14	2.88985	0.00	1.0000
resinc	Residual	1	6.28803	0.19767	31.81	<.0001

Output of `model insur=income risk`

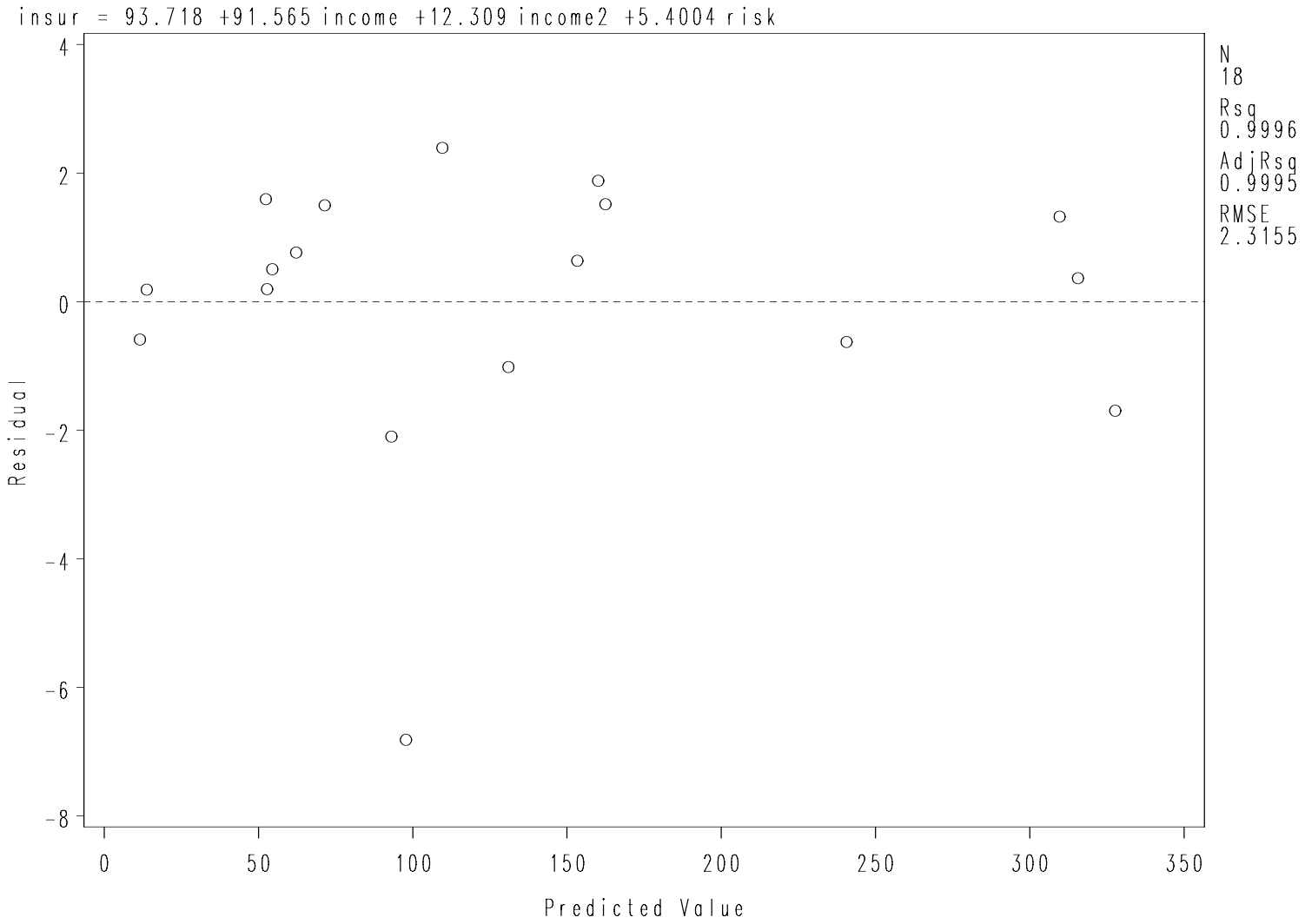
Parameter Estimates

Variable	DF	Parameter Estimate	Standard Error	t Value	Pr > |t|
Intercept	1	-205.71866	11.39268	-18.06	<.0001
income	1	6.28803	0.20415	30.80	<.0001
risk	1	4.73760	1.37808	3.44	0.0037

- Note that the parameter estimate for the slope is the same as the parameter estimate for `risk` in the full model

```
data a2; set a1;
    sincome = income;

proc standard data=a2 out=a3 mean=0 std=1;
    var sincome;

data a3; set a3;
    sincome2=sincome*sincome;

/* --- Include quadratic term of income in the model ---*/
proc reg data=a3;
    model insur = sincome sincome2 risk;
    output out=out2 r=resid p=pred;
run;

proc sort data=out2; by pred; run;
symbol v=circle i=none c=black;
proc gplot data=out2;
    plot resid*pred/vref=0;
run; quit;
```

Residual Plot



Output

Analysis of Variance

Source	DF	Sum of Squares	Mean Square	F Value	Pr > F
Model	3	176249	58750	10958.0	<.0001
Error	14	75.05895	5.36135		
Corrected Total	17	176324			

Root MSE	2.31546	R-Square	0.9996
Dependent Mean	134.44444	Adj R-Sq	0.9995
Coeff Var	1.72224		

Parameter Estimates

Variable	DF	Parameter Estimate	Standard Error	t Value	Pr > |t|
Intercept	1	93.71759	1.63501	57.32	<.0001
sincome	1	91.56523	0.65352	140.11	<.0001
sincome2	1	12.30855	0.59042	20.85	<.0001
risk	1	5.40039	0.25399	21.26	<.0001

Residuals for Diagnostics

- $\mathbf{e} = \mathbf{Y} - \hat{\mathbf{Y}} = (\mathbf{I} - \mathbf{H})\mathbf{Y}$
  - $\mathbf{I} - \mathbf{H}$ symmetric and idempotent
- Expected value $E(\mathbf{e}) = \mathbf{0}$
- Covariance matrix

$$\begin{aligned} \boldsymbol{\sigma}^2(\mathbf{e}) &= \sigma^2(\mathbf{I} - \mathbf{H})(\mathbf{I} - \mathbf{H})' \\ &= \sigma^2(\mathbf{I} - \mathbf{H}) \end{aligned}$$

  - $\text{Var}(e_i) = \sigma^2 \cdot (1 - h_{ii})$ where $h_{ii} = \mathbf{X}_i'(\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}_i$
  - $Cov(e_i, e_j) = \sigma^2 \cdot (0 - h_{ij}) = -\sigma^2 h_{ij}$
- Estimated variance and covariance
  - $\widehat{Var}(e_i) = MSE \cdot (1 - h_{ii})$
  - $\widehat{Cov}(e_i, e_j) = -MSE \cdot h_{ij}$

Residuals

- Ordinary residual

$$e_i = Y_i - \widehat{Y}_i \quad \rightarrow \quad \mathbf{e} \sim \text{MVN}(\mathbf{0}, (\mathbf{I} - \mathbf{H})\sigma^2)$$

  - residuals do not have the same variance, but depend on $\mathbf{X}_i$

- Semi-studentized residual

$$r_i = \frac{e_i}{\sqrt{\text{MSE}}}$$

  - denominator is not an estimate of SD of e_i

- (Internally) Studentized Residual

$$r_i = \frac{e_i}{\sqrt{\text{MSE}(1 - h_{ii})}}$$

  - denominator is the estimate of SD of e_i

- "Studentized" residual doesn't follow the student t distribution (but a τ distribution)

- Outlier may not have a outstanding studentized residual

Deleted Residual

- Deleted residual (a refinement of residual)

$$d_i = Y_i - \hat{Y}_{i(i)} = \frac{e_i}{1 - h_{ii}}$$

  - $(\mathbf{X}_i, Y_i)$ was not used to fit the model
  - can calculate d_i in a single model fit

- Standard deviation of deleted residuals

$$\begin{aligned} s^2\{d_i\} &= MSE_{(i)} \cdot (1 + \mathbf{X}_i'(\mathbf{X}_{(i)}'\mathbf{X}_{(i)})^{-1}\mathbf{X}_i) \\ &= \frac{MSE_{(i)}}{1 - h_{ii}} \end{aligned}$$

- Studentized deleted residual (externally studentized residual)

$$\begin{aligned} t_i &= \frac{d_i}{s\{d_i\}} = \frac{e_i}{1 - h_{ii}} \cdot \sqrt{\frac{1 - h_{ii}}{MSE_{(i)}}} \\ &= \frac{e_i}{\sqrt{MSE_{(i)}(1 - h_{ii})}} \end{aligned}$$

Studentized Deleted Residuals

- If there is only one outlier, its studentized deleted residual will be outstanding
- Useful for identifying outlying Y observation
  - Test $H_{i0} : E[Y_i] = X_i\beta$ vs $H_{ia} : E[Y_i] \neq X_i\beta$
- If there are no outlying observations,

$$t_i \sim t_{n-1-p}$$

  - can compare t_i to this reference distribution
  - adjust for n tests using Bonferroni
  - an outlier has $|t_i| > t_{1-\alpha/(2n)}(n-1-p)$
  - t_i are not independent

Example on Page 386 (Continued)

```
/* INFLUENCE: to report studentized deleted residuals */
proc reg data=a3;
    model insur = sincome sincome2 risk/r influence;
run; quit;
```

`r`: requests an analysis of the residuals, includes mean predicted values, residual values and their respective standard errors, the studentized residual, and Cooks D statistic

`influence`: requests influence statistics, includes deleted residual values and other statistics to be discussed later.

Output

Newer version of SAS can provide graphics for studentized residual plots

Output Statistics

Obs	Dependent Variable	Predicted Value	Std Error Mean Predict	Residual	Std Error Residual	Student Residual
1	91.0000	97.8164	0.7181	-6.8164	2.201	-3.097
2	162.0000	160.1201	0.9577	1.8799	2.108	0.892
3	11.0000	11.5901	1.5574	-0.5901	1.713	-0.344
4	240.0000	240.6278	0.8580	-0.6278	2.151	-0.292
5	73.0000	71.5019	0.6656	1.4981	2.218	0.675
6	311.0000	309.6777	1.4363	1.3223	1.816	0.728
7	316.0000	315.6359	2.0100	0.3641	1.150	0.317
8	154.0000	153.3645	0.9829	0.6355	2.096	0.303
9	164.0000	162.4847	0.8211	1.5153	2.165	0.700
10	54.0000	52.4068	0.7346	1.5932	2.196	0.726
11	53.0000	52.8060	0.8340	0.1940	2.160	0.0898
12	326.0000	327.6975	1.4378	-1.6975	1.815	-0.935
13	55.0000	54.4957	0.7142	0.5043	2.203	0.229
14	130.0000	131.0179	1.2720	-1.0179	1.935	-0.526
15	112.0000	109.6080	0.8185	2.3920	2.166	1.104
16	91.0000	93.0992	0.8093	-2.0992	2.169	-0.968
17	14.0000	13.8135	1.2042	0.1865	1.978	0.0943
18	63.0000	62.2363	0.6776	0.7637	2.214	0.345

Output Statistics

Obs	-2-1 0 1 2	Cook's D	RStudent	Hat Diag H	Cov Ratio	DFFITS
1	|******| |	0.255	-5.3155	0.0962	0.0147	-1.7339
2	| |* |	0.041	0.8848	0.1711	1.2842	0.4020
3	| | |	0.025	-0.3333	0.4524	2.3742	-0.3029
4	| | |	0.003	-0.2822	0.1373	1.5215	-0.1126
5	| |* |	0.010	0.6618	0.0826	1.2842	0.1986
6	| |* |	0.083	0.7153	0.3848	1.8735	0.5656
7	| | |	0.077	0.3063	0.7535	5.3027	0.5356
8	| | |	0.005	0.2931	0.1802	1.5981	0.1374
9	| |* |	0.018	0.6866	0.1258	1.3342	0.2604
10	| |* |	0.015	0.7127	0.1006	1.2830	0.2384
11	| | |	0.000	0.0866	0.1297	1.5420	0.0334
12	| *| |	0.137	-0.9308	0.3856	1.6912	-0.7373
13	| | |	0.001	0.2210	0.0951	1.4643	0.0717
14	| *| |	0.030	-0.5120	0.3018	1.7786	-0.3366
15	| |** |	0.044	1.1138	0.1249	1.0675	0.4209
16	| *| |	0.033	-0.9653	0.1222	1.1616	-0.3601
17	| | |	0.001	0.0909	0.2705	1.8390	0.0553
18	| | |	0.003	0.3338	0.0856	1.4216	0.1022

With 18 observations and 3 predictors, the df for the studentized deleted residuals are 13. The P-value associated with Obs #1 is 0.00014. Using Bonferroni, we'd compare this to 0.1/(2*18) = 0.00278. Conclusion: observation does appear to be unusual.

Identifying Outlying X: Hat Matrix Diagonals

- Diagonals $0 \leq h_{ii} \leq 1$ and sum to p
- Also known as the leverage of ith case
- Is a measure of distance between the X value and the mean of the X values for all n cases $(\overline{X}_1, \overline{X}_2, ..., \overline{X}_{p-1})$
- Since $\hat{\mathbf{Y}} = \mathbf{HY}$

$$\hat{Y}_i = h_{i1}Y_1 + h_{i2}Y_2 + \ldots + h_{in}Y_n$$

- Thus h_{ii} is a measure of how much Y_i is contributing to the prediction of $\hat{Y}_i$

Hat Matrix Diagonals

- Residual

$$\begin{aligned} \mathbf{e} &= (\mathbf{I}-\mathbf{H})\mathbf{Y} \\ Var(\mathbf{e}) &= (\mathbf{I}-\mathbf{H})\sigma^2 \\ Var(e_i) &= (1-h_{ii})\sigma^2 \end{aligned}$$

- Large h_{ii} means small residual variance
  - $\hat{Y}_i$ will be close to Y_i (i.e., model is forced to fit this observation closely)

- Observations with large h_{ii} considered influential

  - large h_{ii} if it is more than double of the average value, i.e., $h_{ii} > 2p/n$

- Can compute $\mathbf{X}'_{new}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}_{new}$ to check for hidden extrapolation

Example on Page 386 (Continued)

```
options nocenter ls=75;

proc reg data=a3;
    model insur = sincome sincome2 risk/r influence;
    id sincome risk;
run; quit;
```

Output

Obs	sincome	risk	Hat Diag H	Cov Ratio	DFFITS
1	-0.323145222	6	0.0962	0.0147	-1.7339
2	0.4607431878	4	0.1711	1.2842	0.4020
3	-1.490427964	5	0.4524	2.3742	-0.3029
4	1.0448345518	7	0.1373	1.5215	-0.1126
5	-0.583241377	5	0.0826	1.2842	0.1986
6	1.4759281779	10	0.3848	1.8735	0.5656
7	1.8863221101	1	0.7535	5.3027	0.5356
8	0.175447406	8	0.1802	1.5981	0.1374
9	0.377944412	6	0.1258	1.3342	0.2604
10	-0.765938675	4	0.1006	1.2830	0.2384
11	-0.912636506	6	0.1297	1.5420	0.0334
12	1.6559255166	9	0.3856	1.6912	-0.7373
13	-0.811837997	5	0.0951	1.4643	0.0717
14	0.2789458757	2	0.3018	1.7786	-0.3366
15	-0.24754634	7	0.1249	1.0675	0.4209
16	-0.251146287	4	0.1222	1.1616	-0.3601
17	-1.264531303	3	0.2705	1.8390	0.0553
18	-0.705639567	5	0.0856	1.4216	0.1022

The critical value in this case would be if a diagonal value was greater than $2(4)/18 = 0.44$. It does appear that there are some outlying X observations (Obs #3 and #7). For Obs #7, the largest income and lowest risk. For Obs #3, the smallest income.

Identifying Influential Cases

DFFITS

- Difference between the fitted values $\hat{Y}_i$ and the predicted values $\hat{Y}_{i(i)}$
- Measures influence of case i on $\hat{Y}_i$

$$\text{DFFITS}_i = \frac{\hat{Y}_i - \hat{Y}_{i(i)}}{\sqrt{\text{MSE}_{(i)} h_{ii}}} = t_i \sqrt{\frac{h_{ii}}{1 - h_{ii}}}$$

  - The denominator is the estimated standard deviation of $\hat{Y}_i$, obtained using the $MSE_{(i)}$
  - t_i is the studentized deleted residual

- Adjusts studentized deleted residual by function of h_{ii}
- Concern if absolute value greater than 1 for small data sets, or greater than $2\sqrt{p/n}$ for large data sets

Output

Obs	sincome	risk	Hat Diag H	Cov Ratio	DFFITS
1	-0.323145222	6	0.0962	0.0147	-1.7339
2	0.4607431878	4	0.1711	1.2842	0.4020
3	-1.490427964	5	0.4524	2.3742	-0.3029
4	1.0448345518	7	0.1373	1.5215	-0.1126
5	-0.583241377	5	0.0826	1.2842	0.1986
6	1.4759281779	10	0.3848	1.8735	0.5656
7	1.8863221101	1	0.7535	5.3027	0.5356
8	0.175447406	8	0.1802	1.5981	0.1374
9	0.377944412	6	0.1258	1.3342	0.2604
10	-0.765938675	4	0.1006	1.2830	0.2384
11	-0.912636506	6	0.1297	1.5420	0.0334
12	1.6559255166	9	0.3856	1.6912	-0.7373
13	-0.811837997	5	0.0951	1.4643	0.0717
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17	-1.264531303	3	0.2705	1.8390	0.0553
18	-0.705639567	5	0.0856	1.4216	0.1022

This is a small data set, so we'll be concerned about values greater than 1 in scale. In this case, Obs #1 has strong influence. Recall this observation had a very large studentized deleted residual. None of the others are a concern.

Cook's Distance

- Measures influence of a case on all $\widehat{Y}_i$'s
- Standardized version of sum of squared differences between fitted values with and without case i

$$D_i = \frac{\sum_{j=1}^{n} (\widehat{Y}_j - \widehat{Y}_{j(i)})^2}{p \cdot \text{MSE}}$$

  - can be obtained in a single fit
- Compare with $F(p, n-p)$
- Concern if D_i is above the 50%-tile of $F(p, n-p)$

Output

Output Statistics

Obs	sincome	risk	-2-1 0 1 2	Cook's D	RStudent
1	-0.323145222	6	|******|	0.255	-5.3155
2	0.4607431878	4	|*|	0.041	0.8848
3	-1.490427964	5	| |	0.025	-0.3333
4	1.0448345518	7	| |	0.003	-0.2822
5	-0.583241377	5	|*|	0.010	0.6618
6	1.4759281779	10	|*|	0.083	0.7153
7	1.8863221101	1	| |	0.077	0.3063
8	0.175447406	8	| |	0.005	0.2931
9	0.377944412	6	|*|	0.018	0.6866
10	-0.765938675	4	|*|	0.015	0.7127
11	-0.912636506	6	| |	0.000	0.0866
12	1.6559255166	9	*| |	0.137	-0.9308
13	-0.811837997	5	| |	0.001	0.2210
14	0.2789458757	2	*| |	0.030	-0.5120
15	-0.24754634	7	|**	0.044	1.1138
16	-0.251146287	4	*| |	0.033	-0.9653
17	-1.264531303	3	| |	0.001	0.0909
18	-0.705639567	5	| |	0.003	0.3338

With 18 observations and 3 predictors, the df for the F are 4 and 14. The 30, 40, and 50%-tiles are 0.553, 0.707, and 0.881 respectively. None of the observations appear to have an undue amount of influence.

DFBETAS

- Measures influence of case i on each of the regression coefficients
- Standardized version of the difference between regression coefficient computed with and without case i

$$\text{DFBETAS}_{k(i)} = \frac{b_k - b_{k(i)}}{\sqrt{\text{MSE}_{(i)} c_{kk}}}$$

  where c_{kk} is the k-th diagonal element of $(\mathbf{X}'\mathbf{X})^{-1}$

  - The denominator is the SE of β_k, obtained using $MSE_{(i)}$ to estimate σ^2

- Concern if greater than 1 for small data sets or greater than $2/\sqrt{n}$ for large data sets

Output

	Output Statistics					
			DFBETAS			
Obs	sincome	risk	Intercept	sincome	sincome2	risk
1	-0.323145222	6	-0.4440	0.0662	0.9168	-0.3686
2	0.4607431878	4	0.3372	0.2513	-0.2579	-0.2064
3	-1.490427964	5	0.0874	0.2513	-0.2312	-0.0525
4	1.0448345518	7	-0.0067	-0.0692	0.0230	-0.0299
5	-0.583241377	5	0.0831	-0.0566	-0.0580	-0.0108
6	1.4759281779	10	-0.3129	0.1183	0.1704	0.3901
7	1.8863221101	1	0.2554	0.2235	0.2233	-0.3381
8	0.175447406	8	-0.0162	0.0245	-0.0712	0.0788
9	0.377944412	6	0.1121	0.1333	-0.1799	0.0084
10	-0.765938675	4	0.1267	-0.0988	-0.0084	-0.0773
11	-0.912636506	6	-0.0064	-0.0244	0.0091	0.0126
12	1.6559255166	9	0.3453	-0.1728	-0.3486	-0.3821
13	-0.811837997	5	0.0137	-0.0427	0.0063	0.0030
14	0.2789458757	2	-0.3279	-0.1746	0.1861	0.2583
15	-0.24754634	7	-0.0046	-0.0195	-0.2036	0.2003
16	-0.251146287	4	-0.2937	-0.0774	0.2177	0.1654
17	-1.264531303	3	0.0101	-0.0383	0.0317	-0.0150
18	-0.705639567	5	0.0310	-0.0471	-0.0097	-0.0003

Nothing looks real troubling here except for Obs #1 and its influence on the quadratic coefficient. Since this had such a large residual, we will remove it and refit the model.

Analysis of Variance

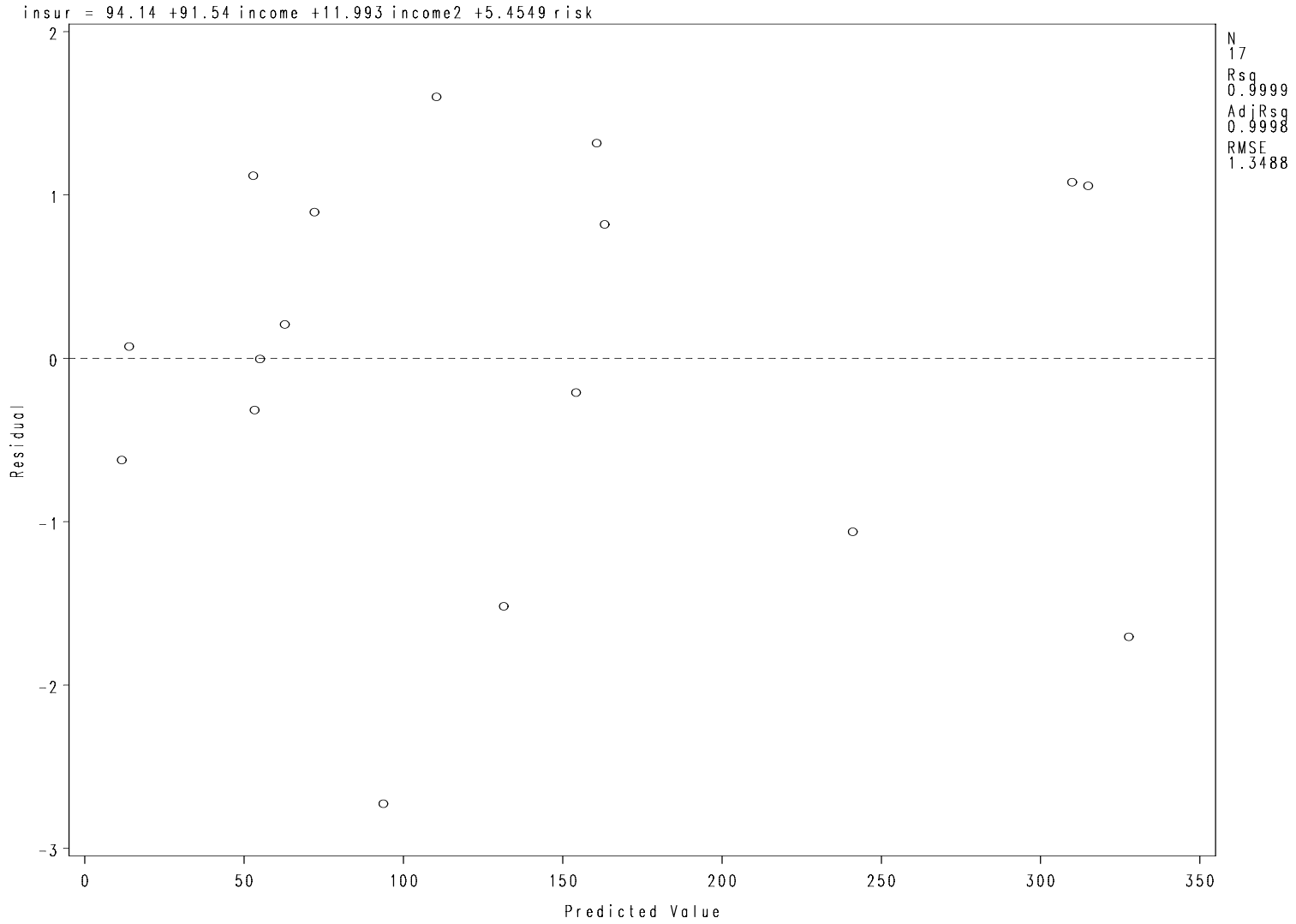
Source	DF	Sum of Squares	Mean Square	F Value	Pr > F
Model	3	174302	58101	31934.2	<.0001
Error	13	23.65205	1.81939		
Corrected Total	16	174326			

Root MSE	1.34885	R-Square	0.9999
Dependent Mean	137.00000	Adj R-Sq	0.9998
Coeff Var	0.98456		

Parameter Estimates

Variable	DF	Parameter Estimate	Standard Error	t Value	Pr > |t|
Intercept	1	94.14049	0.95577	98.50	<.0001
sincome	1	91.54004	0.38073	240.43	<.0001
sincome2	1	11.99324	0.34902	34.36	<.0001
risk	1	5.45493	0.14831	36.78	<.0001

Residual Plot

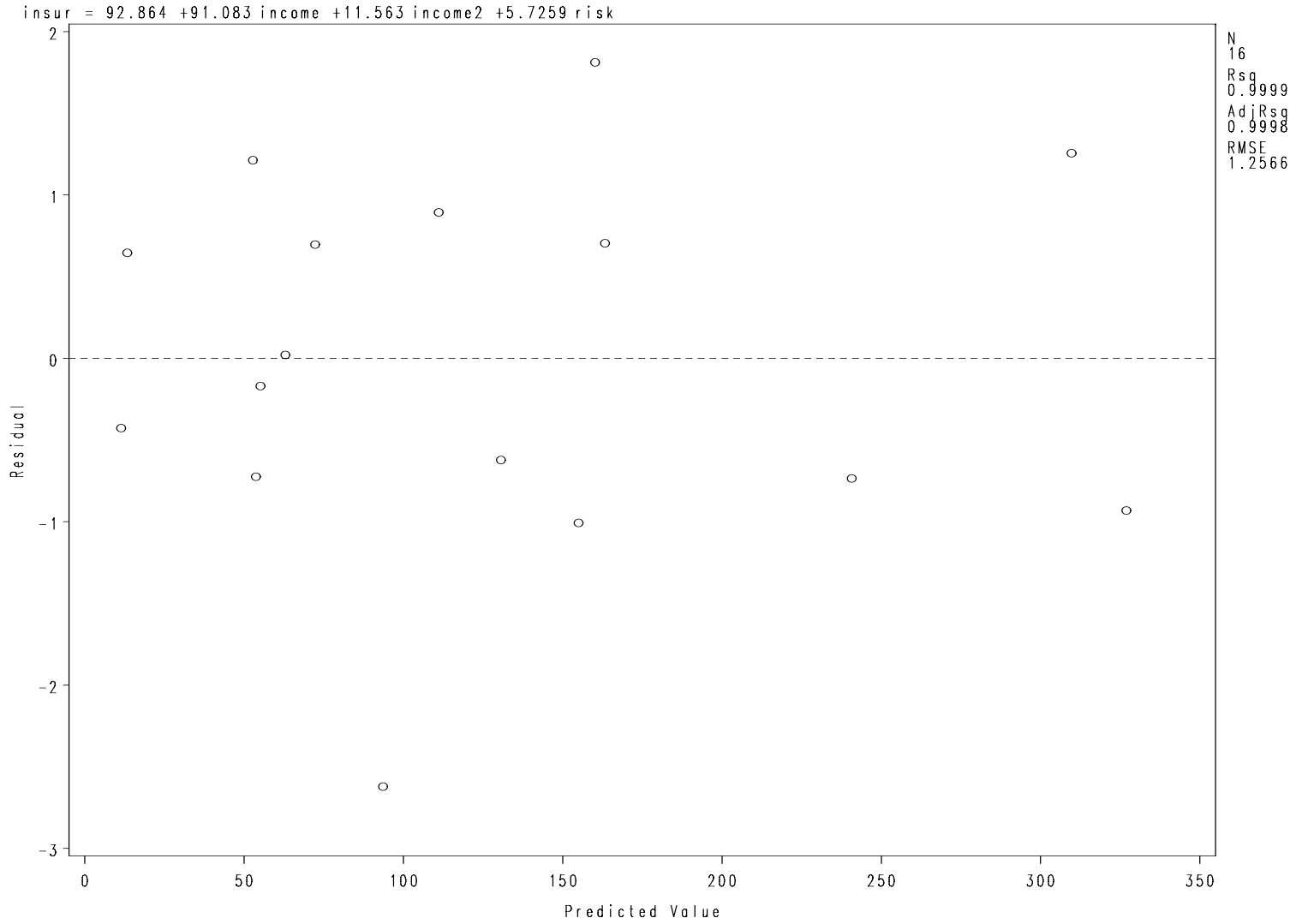


Output

			DFBETAS			
Obs	income	risk	Intercept	income	income2	risk
1	0.4607431878	4	0.4210	0.3079	-0.3285	-0.2467
2	-1.490427964	5	0.1587	0.4590	-0.4154	-0.0960
3	1.0448345518	7	-0.0246	-0.2058	0.0768	-0.0927
4	-0.583241377	5	0.0906	-0.0592	-0.0692	-0.0069
5	1.4759281779	10	-0.4422	0.1685	0.2325	0.5595
6	1.8863221101	1	1.4336	1.2882	1.3223	-1.9612
7	0.175447406	8	0.0074	-0.0138	0.0439	-0.0465
8	0.377944412	6	0.1100	0.1238	-0.1770	0.0124
9	-0.765938675	4	0.1591	-0.1213	-0.0204	-0.0899
10	-0.912636506	6	0.0163	0.0690	-0.0221	-0.0367
11	1.6559255166	9	0.6402	-0.3214	-0.6388	-0.7095
12	-0.811837997	5	-0.0002	0.0006	-0.0000	-0.0001
13	0.2789458757	2	-0.9070	-0.4778	0.5234	0.6995
14	-0.24754634	7	0.0076	-0.0251	-0.2646	0.2479
15	-0.251146287	4	-0.8138	-0.2068	0.6230	0.4303
16	-1.264531303	3	0.0068	-0.0254	0.0205	-0.0098
17	-0.705639567	5	0.0155	-0.0221	-0.0066	0.0007

Now Obs #6 is influential. This was Obs #7 before we discarded the first observation. It would be worth investigating how much the model changes with and without this observation.

Residual Plot



Conclusive comments

- Outlier vs. Infuential observation: former ones have unusual X or/and Y values; latter ones have larger impact on the prediction or/and estimation.
- Treatment of unusual observations
  - To decide whether we shall keep/discard them
  - If keep, (i) diagnose the appropriateness of the modeling or (ii) reduce the weights of these observations

Multicollinearity Diagnostics: VIF

- Use **V**ariance **I**nflation **F**actor (VIF) for quantitative diagnostic of multicollinearity
- VIF_k is the the kth diagonal element of r_{XX}^{-1} (inverse of sample correlation matrix)

$$\text{VIF}_k = (r_{XX}^{-1})_{kk} = \frac{1}{1 - R_k^2}$$

  - where R_k^2 is the coefficient of multiple determination of X_k regressed versus all other $p - 2$ variables.

- In standardized regression

$$\begin{aligned} Var(\mathbf{b}^*) &= (\sigma^*)^2 \ \mathbf{r}_{X'X}^{-1} \\ Var(\mathbf{b}_k^*) &= (\sigma^*)^2 (r_{XX}^{-1})_{kk} = (\sigma^*)^2 \ \text{VIF}_k \end{aligned}$$

  - the larger VIF_k, the larger the variance of the estimated coefficient

- VIF of 10 or more suggests strong multicollinearity
- Also compare mean VIF to 1

$$\overline{\text{VIF}} = \frac{\sum \text{VIF}_k}{p-1}$$

  - $\overline{\text{VIF}}$ considerably larger than one indicates serious multicollinearity problems

- Tolerance(TOL) = 1/VIF

```
/* TOL: tolerance = 1/VIF*/
proc reg data=a3;
    model insur = sincome sincome2 risk/tol;
    id sincome risk;
run; quit;
```

Output

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sincome2	1	11.99324	0.34902	34.36	<.0001	0.79731
risk	1	5.45493	0.14831	36.78	<.0001	0.92021

Chapter Review

- Partial Regression Plots to Check Model Adequacy for a Predictor Variable
- Identifying Outlying Y Observations
  - Use studentized deleted residuals
- Identifying Outlying X's
- Identifying Influential Cases
  - DFFITS, DFBETAS
  - Cook's Distance
- Multicollinearity Diagnostics Using Variance Inflation Factor