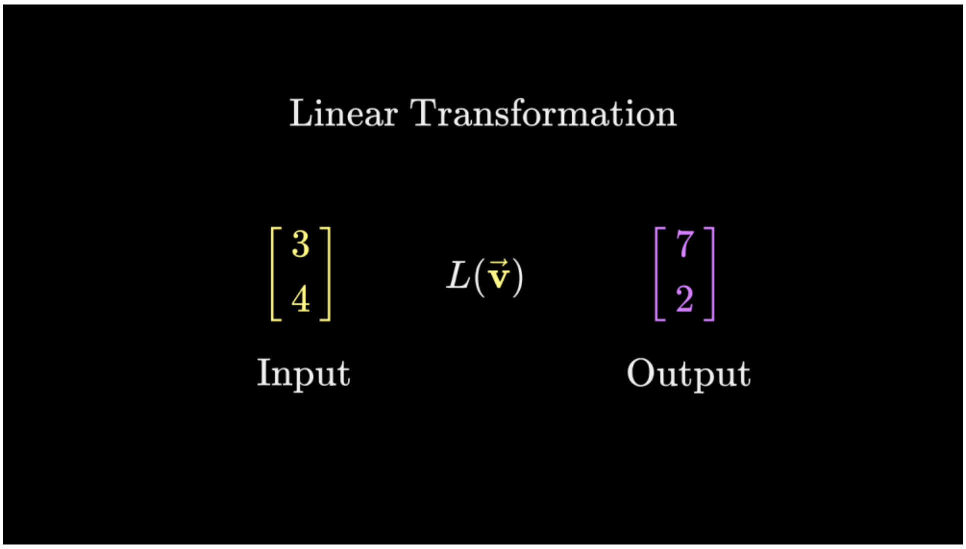
Chapter 3: Linear transformations and matrices

If there was one topic that makes all of the others in linear algebra start to click, it might be this one. We'll be learning about the idea of a linear transformation and its relation to matrices. For this chapter, the focus will simply be on what these linear transformations look like in the case of two dimensions, and how they relate to the idea of matrix-vector multiplication. In particular, matrix multiplication that doesn't rely on memorization.

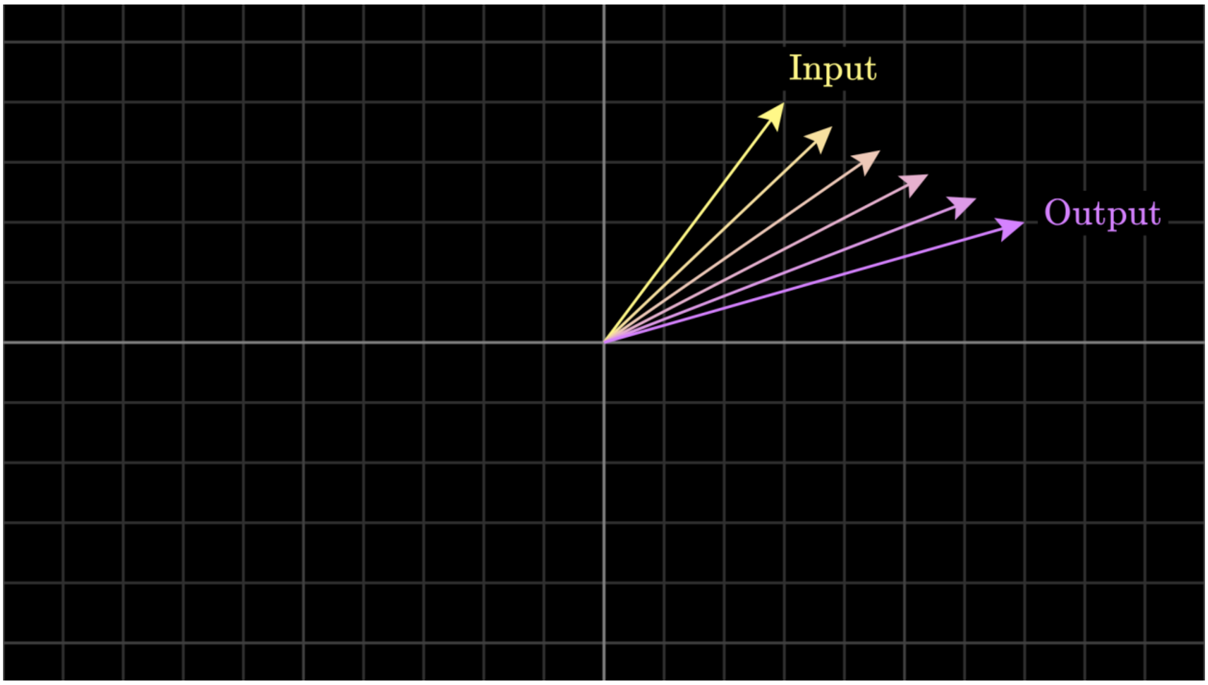
Transformations Are Functions

To start, let's parse this term: "Linear transformation". *Transformation* is essentially a fancy word for function; it's something that takes in inputs, and spits out some output for each one. Specifically, in the context of linear algebra, we think about transformations that take in some vector and spit out another vector.



So why use the word "transformation" instead of "function" if they mean the same thing? It's to be suggestive of a certain way to visualize this input-output relation. Rather than trying to use something like a graph, which really only works in the case of functions that take in one or two numbers and output a number, a great way to understand functions of vectors is to use *movement*.

If a transformation takes some input vector to some output vector, we imagine that input vector *moving* to the output vector.



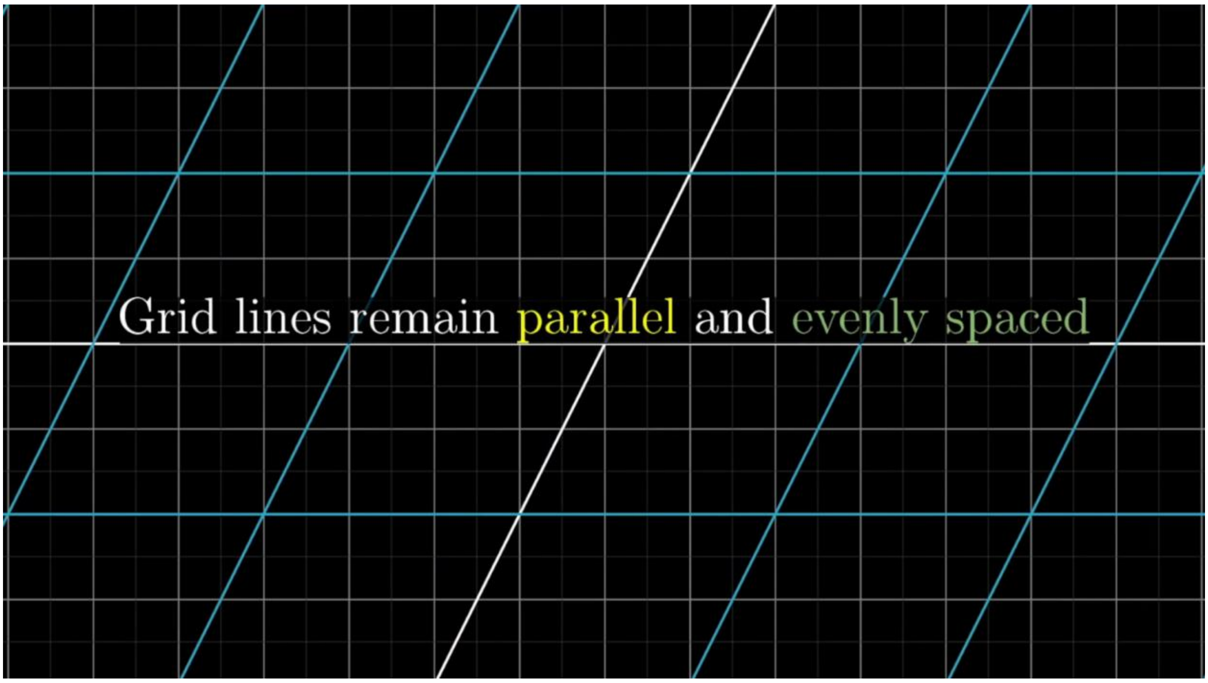
It gets very crowded to think about all vectors all at once, each as an arrow. So let's think of each vector not as an arrow, but as a single point: the point where its tip sits. That way, to think about a transformation taking every possible input vector to its corresponding output vector, we watch every point in space move to some other point.

In the case of transformations in two dimensions, to get a better feel for the shape of a transformation, we can do this with all the points on an infinite grid. It can also be helpful to keep a static copy of the grid in the background, just to help keep track of where everything ends up relative to where it starts.

What makes a transformation "linear"?

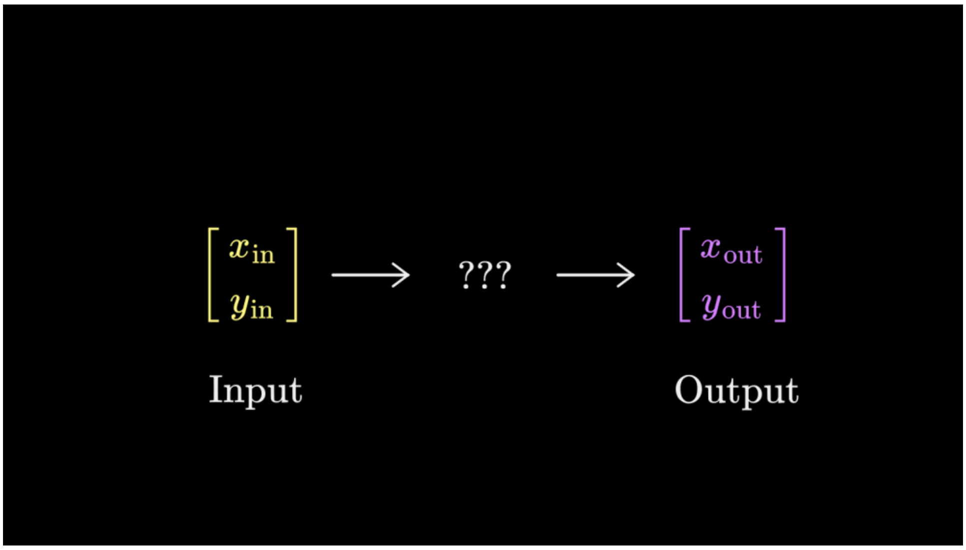
As you imagine, though, arbitrary transformations can look pretty complicated, but luckily linear algebra limits itself to a special type of transformation that's easier to understand called *Linear* transformations. Visually speaking, a transformation is "linear" if it has two properties: all lines must remain lines, without getting curved, and the origin must remain fixed in place.

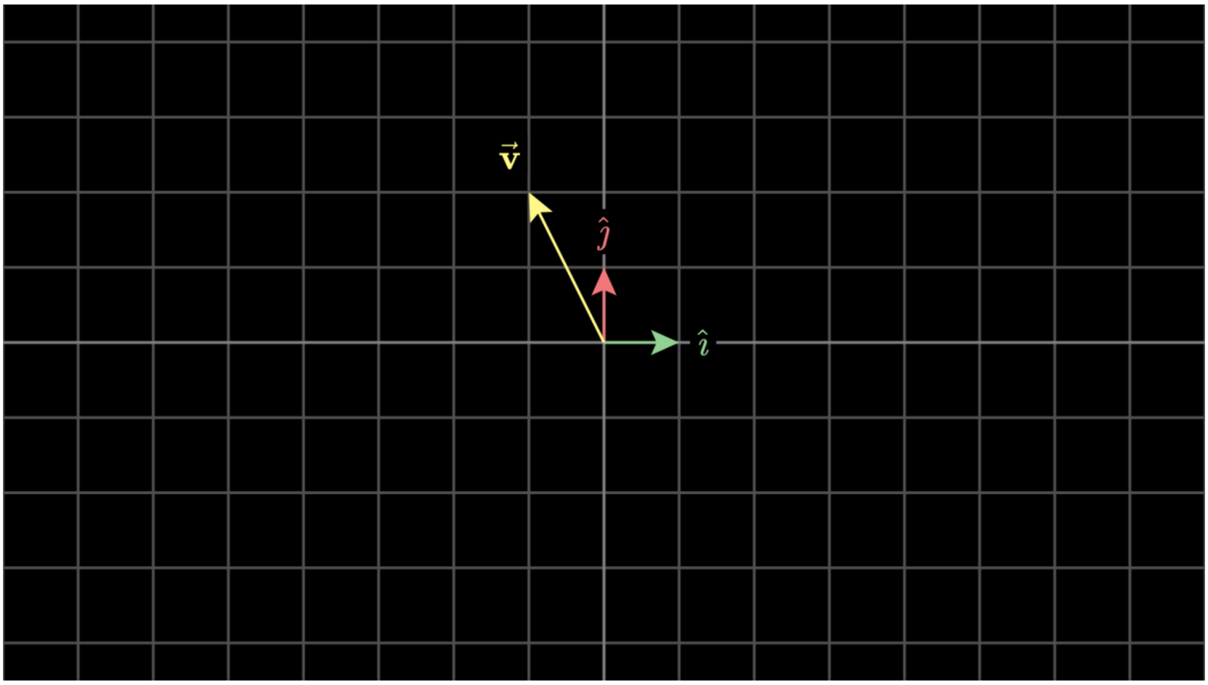
In general, linear transformations is keeping grid lines parallel and evenly spaced, although they might change the angles between perpendicular grid lines. Some linear transformations are simple to think about, like rotations about the origin.



Matrices

the coordinates of where that vector lands.

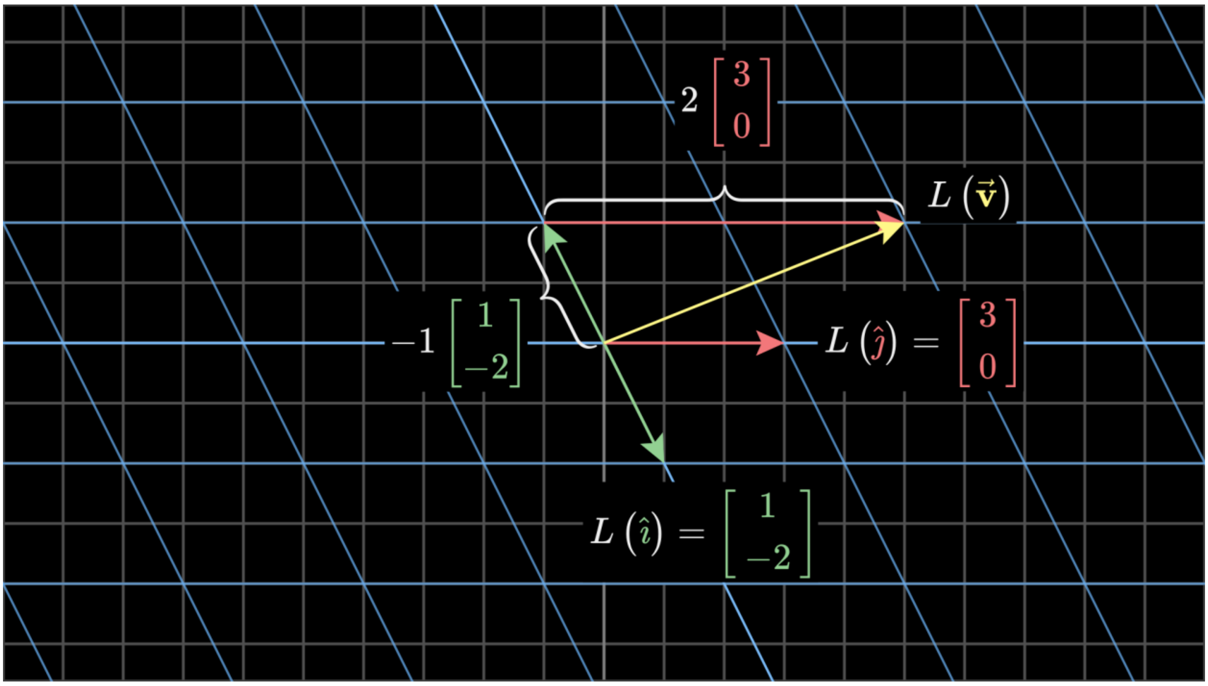




$\vec{\mathbf{v}}$
$\hat{\jmath}$
$\hat{\imath}$

If we play some transformation and follow where all three of these vectors go, the property that grid lines remain parallel and evenly spaced has a really important consequence: the place where v $\vec{v}$ lands will be (-1)(−1) times the vector where $\imath$ ^⃞^ landed, plus $2$2 times the vector where $\jmath$ ^⃞^ landed.

In other words, it started off as a certain linear combination of $\imath$ ^⃞^ and $\jmath$ ^⃞^ , and it ended up at that same linear combination of where those two vectors landed.



Now, given that we're actually showing you the full transformation, we could have just looked to see that v $\vec{v}$ has coordinates $[5\ 2]$[52], but the cool part here is that this gives us a technique to deduce where the vector lands without needing to watch the transformation.

Writing the vector with more general coordinates, xx and yy: It will land on xx times the vector where $\imath$ lands, $[1\ {-2}]$[1−2], plus yy time the vector where $\jmath$ lands, $[3\ 0]$[30].

$$L(\vec{v})=x[1\ {-2}]+y[\ 3\ 0]$$
L(v)=x[1−2]+y[30]

Carrying out that sum, we see that it lands on $[1x+3y\ {-2x}+0y]$[1x+3y−2x+0y]. Given any vector you can tell where it lands using this formula.

$$L(\vec{v})=[1x+3y\ {-2x}+0y\ ]$$
L(v)=[1x+3y−2x+0y]

What all of this is saying is that the two-dimensional linear transformation is completely described by just four numbers: The two coordinates for where $\imath$ ^⃞^ lands, and the two coordinates for where $\jmath$ lands.

It's common to package these four numbers into a 2x2 grid of numbers, called a “2x2 matrix”, where you can interpret the columns as the two special vectors where $\imath$ and $\jmath$ land.

In a 2x2 matrix describing a linear transformation, and a specific vector, and we want to know where the linear transformation takes that vector, you take the coordinates of that vector, multiply them by the corresponding column of the matrix, then add together what you get. This corresponds with the idea of adding scaled versions of our new basis vectors.