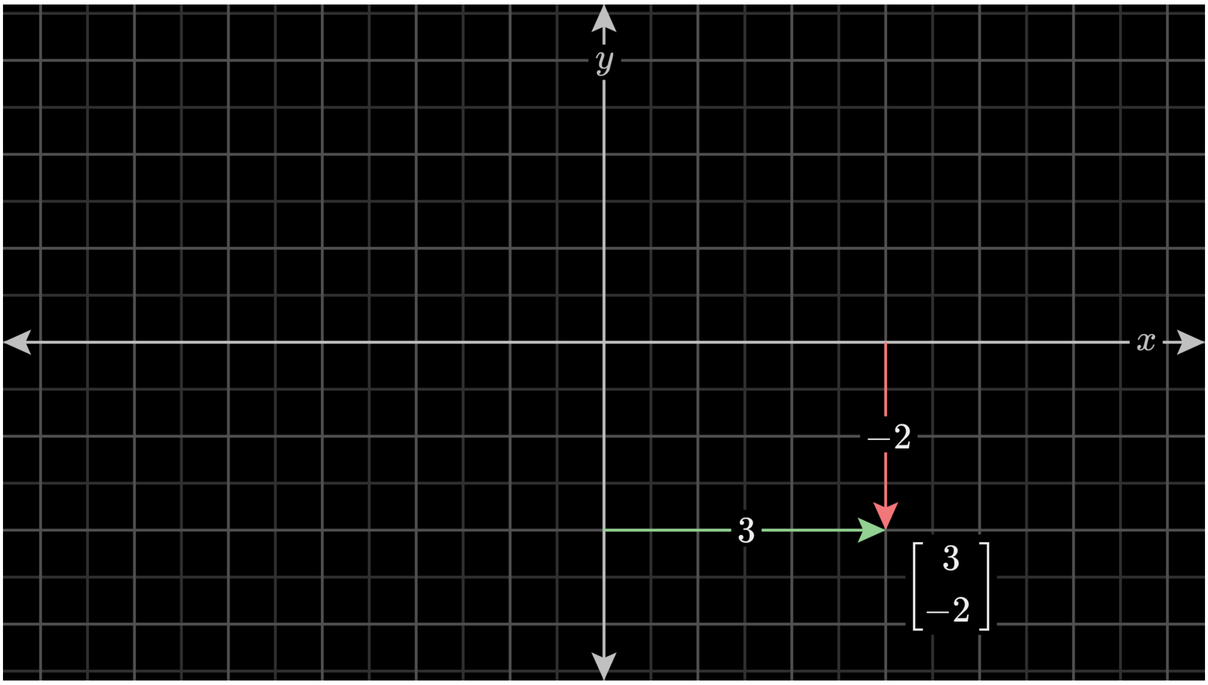
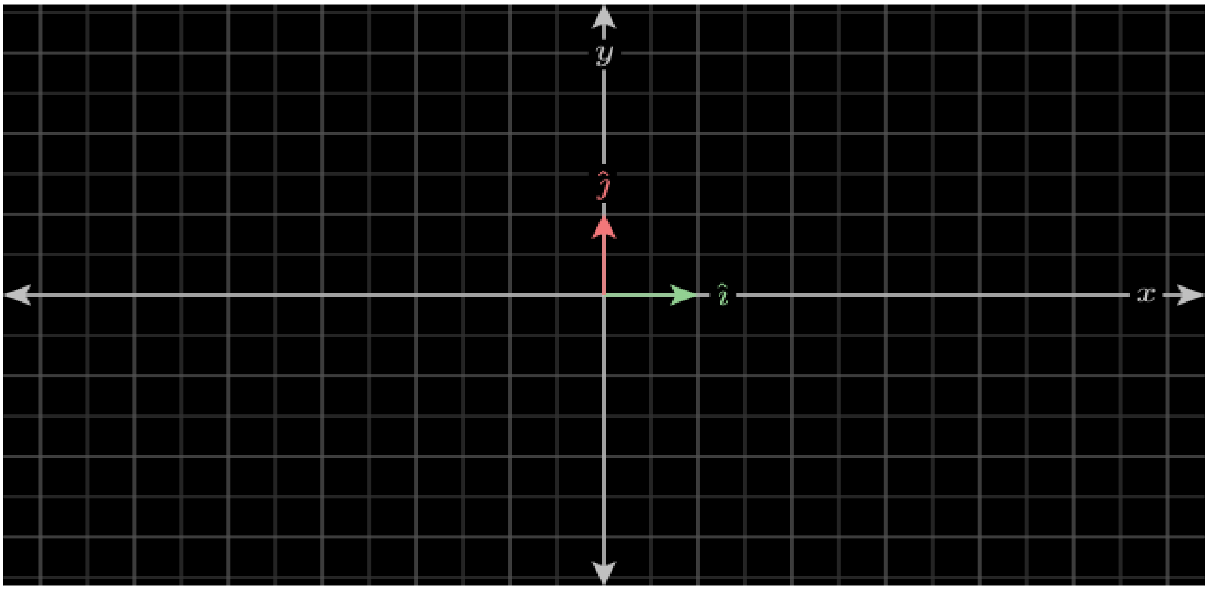
there's another interesting way to think about vector coordinates, which is central to linear algebra. When we have a pair of numbers meant to describe a vector, like (3,−2), each coordinate is a scalar, meaning how each one stretches or squishes vectors.



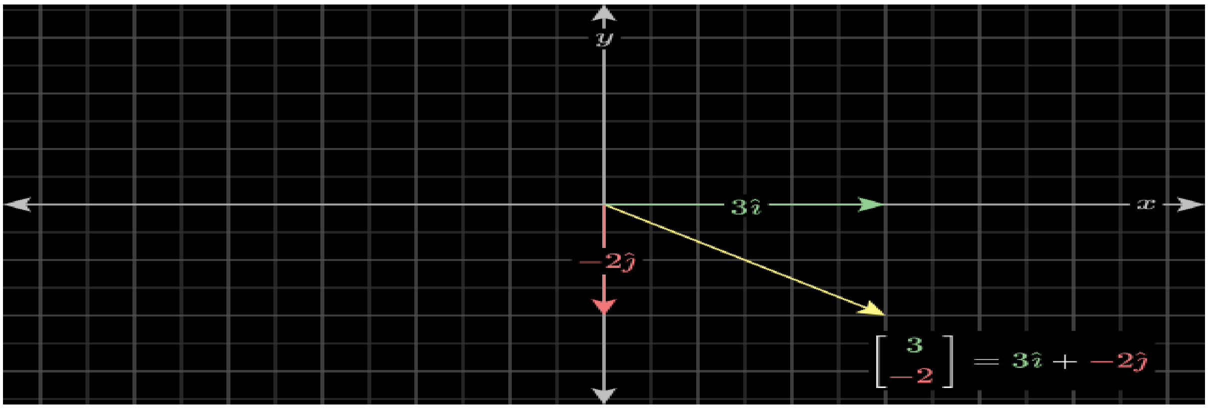
n the xy-coordinate system, there are two special vectors. The one pointing to the right with length 1, commonly called "i hat" or "the unit vector in the x-direction". The other one is pointing straight up with length 1, commonly called "j hat" or "the unit vector in the y-direction".



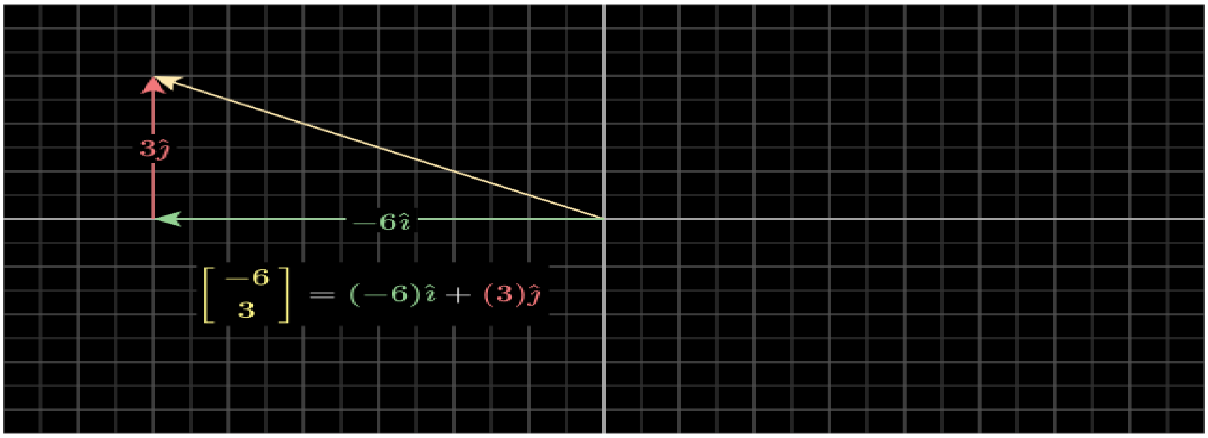
The **x-coordinate** can be seen as a scalar that **scales** the basis vector i, stretching it by a certain factor. For instance, if the x-coordinate is 3, it means i is stretched by a factor of 3.

Similarly, the **y-coordinate** scales the basis vector *j*, but with the potential to **flip** it depending on the sign. If the y-coordinate is -2, this means *j* is flipped (because of the negative sign) and then stretched by a factor of 2.

In this sense, the vector described by the coordinates is the sum of two scaled vectors: one scaled version of *i* and one scaled and possibly flipped version of *j*. The overall vector represents a combination of horizontal and vertical displacements, reflecting the contribution of both the x and y components.

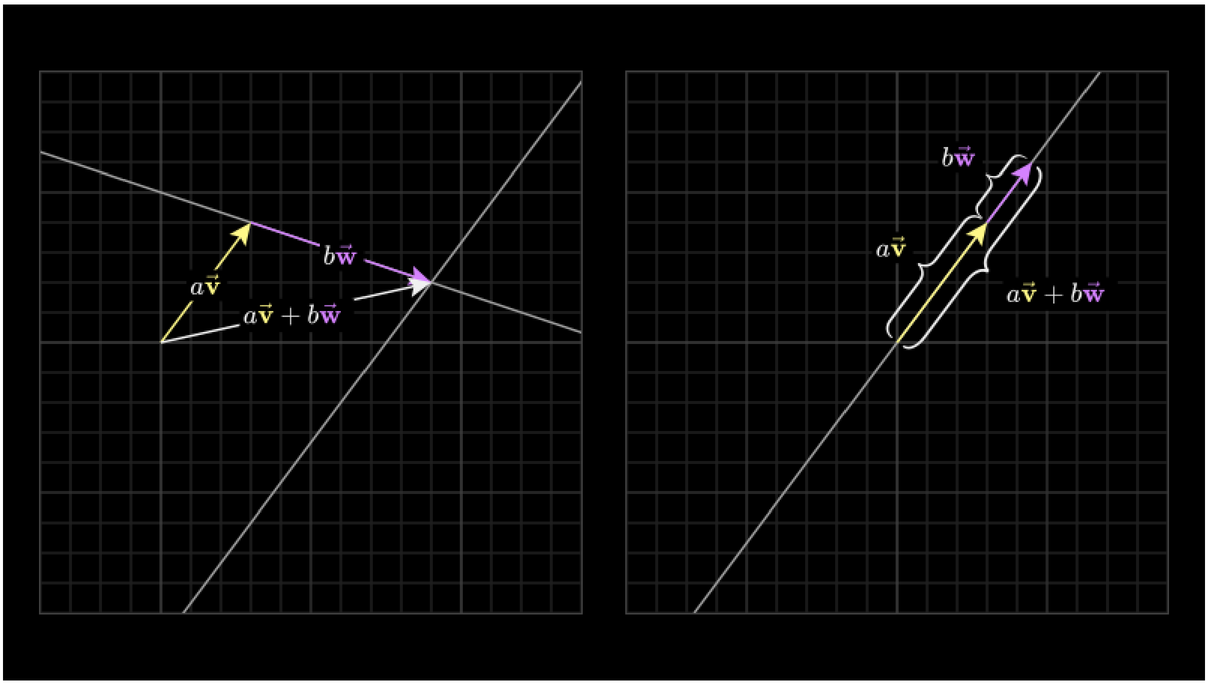


This idea of adding together two scaled vectors is a surprisingly important concept. Those two vectors i and j have a special name: Together they are called the "basis" of the coordinate system. What this means is that when you think about coordinates as scalars, the basis vectors are what those scalars actually scale.



Span

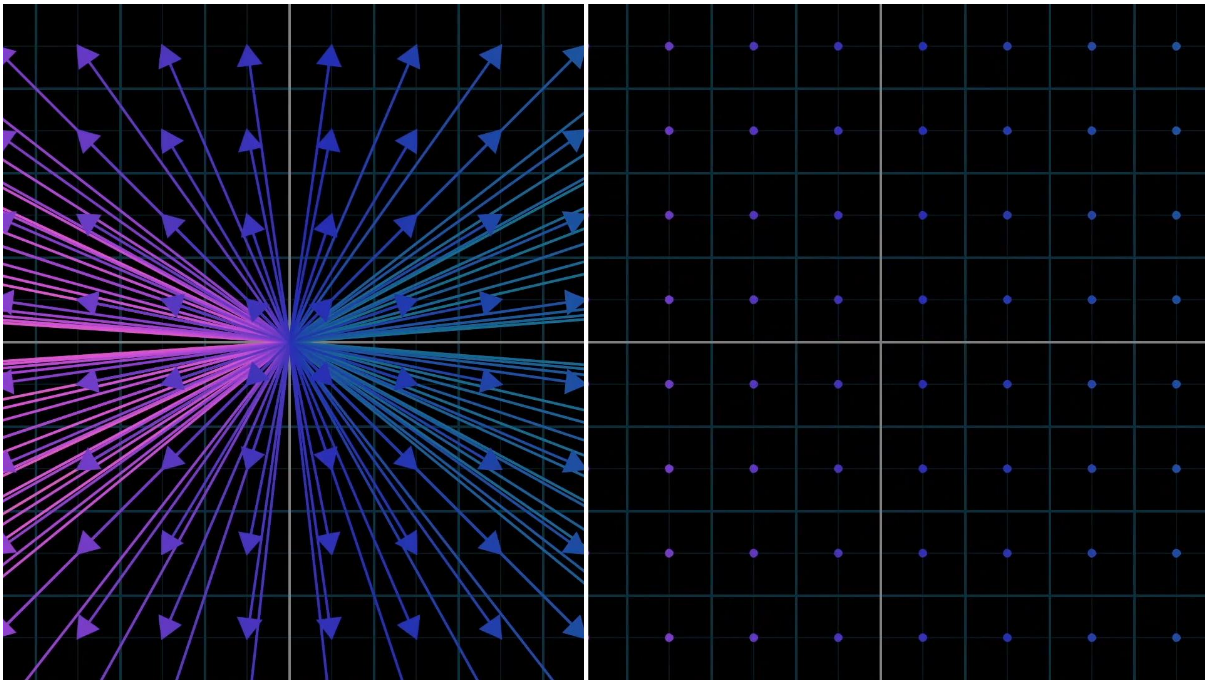
The set of all possible vectors you can reach with linear combinations of a given pair of vectors is called the "span" of those two vectors. Restating what we just saw in this lingo, the span of most pairs of 2D vectors is all vectors in 2D space, but when they line up, their span is all vectors whose tip sit on a certain line.



based on how we said linear algebra revolves around vector addition and scalar multiplication? The span of two vectors is basically a way of asking what are all the possible vectors we can reach using these two by only using those fundamental operations of vector addition and scalar multiplication.

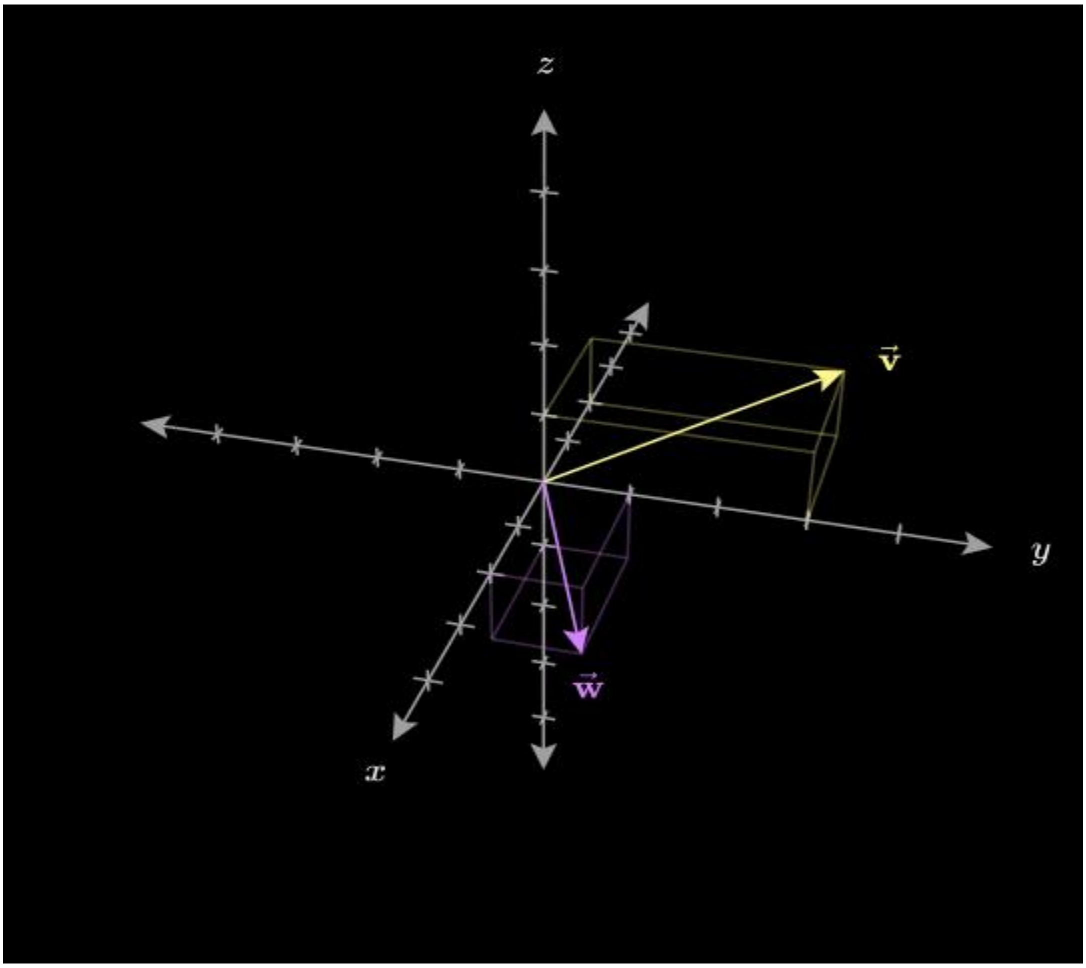
Vectors vs Points

To consider all possible two-dimensional vectors, each can be visualized as the point where its tip is located, starting from the origin. By focusing on the positions of their tips rather than the arrows themselves, the entire collection of vectors can be conceptualized as an infinite flat plane. This plane represents the entirety of two-dimensional space, encompassing all possible vectors.



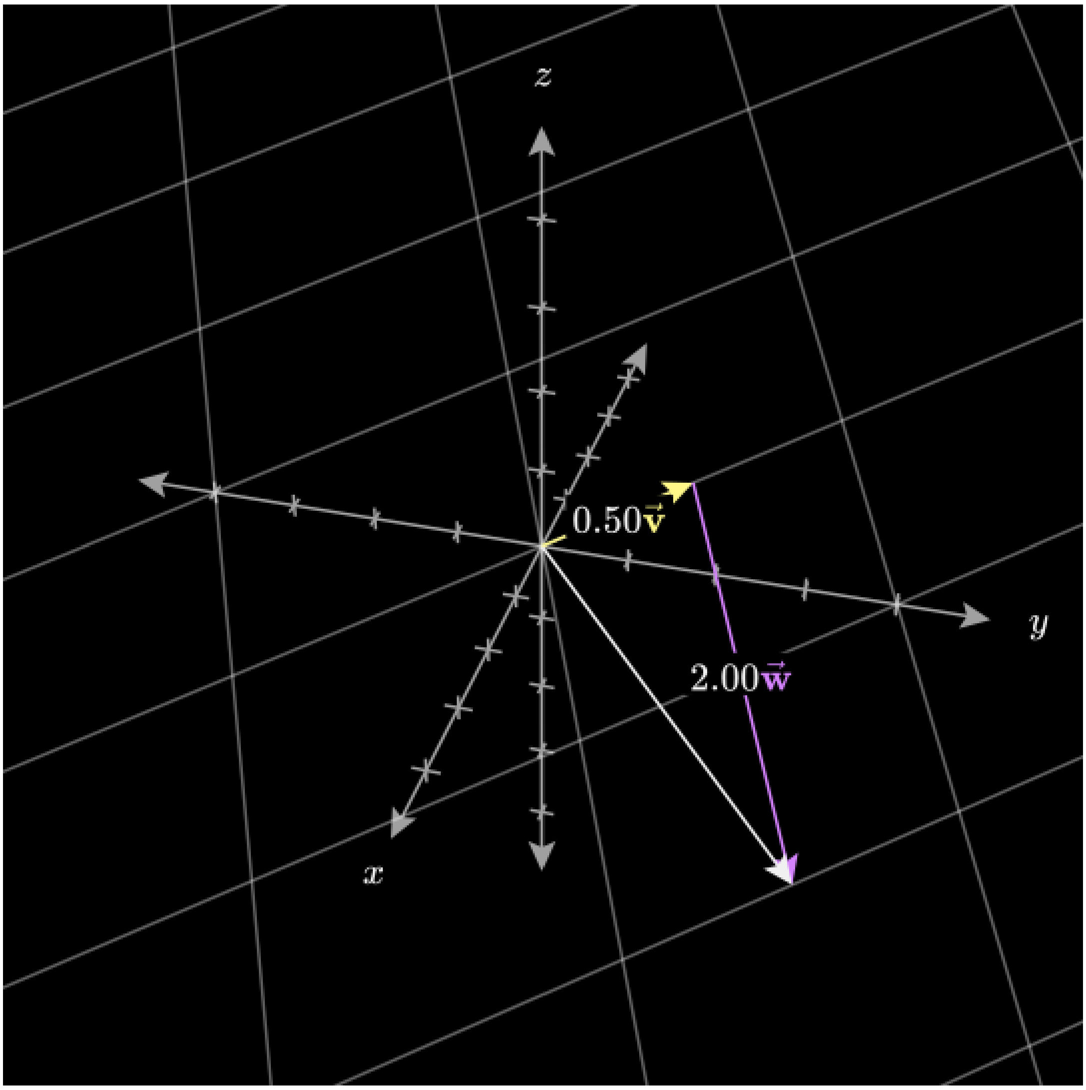
Span in 3D

The idea of span gets more interesting when thinking about vectors in three-dimensional space. For example, taking two vectors in three-dimensional space that are not pointing in the same direction, what does it mean to take their span?



Well, their span is the collection of all possible linear combinations of those two vectors, meaning all possible vectors got by scaling each of the two you start with in some way, then adding them together.

Visualizing two scalars defining a linear combination is like turning two knobs, each controlling the scaling of two basis vectors. Adjusting these knobs scales the vectors and adds them together, resulting in a new vector. The tip of this vector moves, tracing out a flat sheet that passes through the origin in three-dimensional space. This sheet represents all possible linear combinations of the two basis vectors.



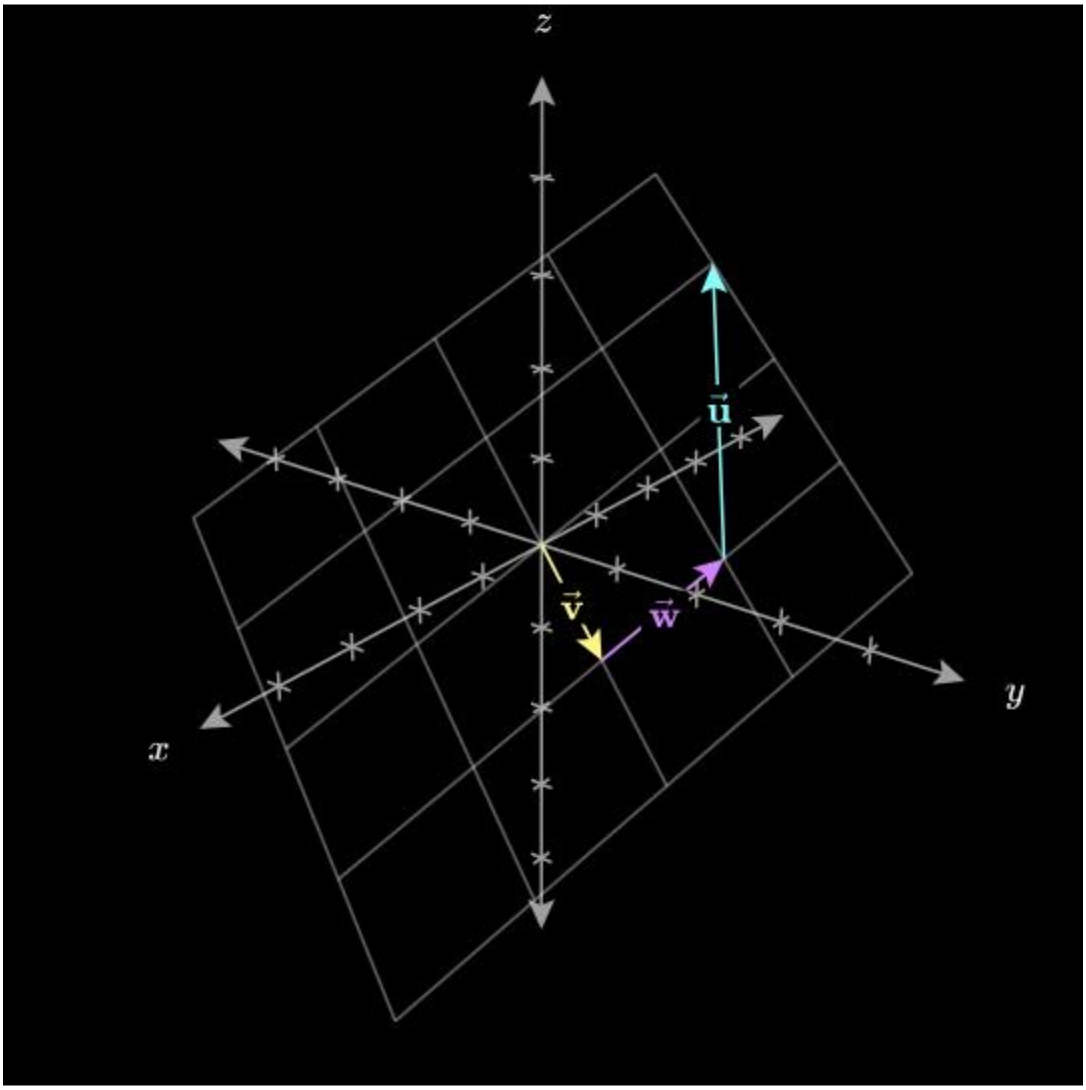
This flat sheet is the span of the two vectors. Or more precisely, the set of all possible vectors whose tips sit on this flat sheet is the span of your two vectors. Isn't that a beautiful mental image?

What happens if you add on a third vector, and consider the span of all three vectors? A linear combination of three vectors is defined pretty much the same way as for two: Choose three scalars, use them to scale each of your vectors, then add them all together. And again, the span of these vectors is the set of all possible linear combinations.

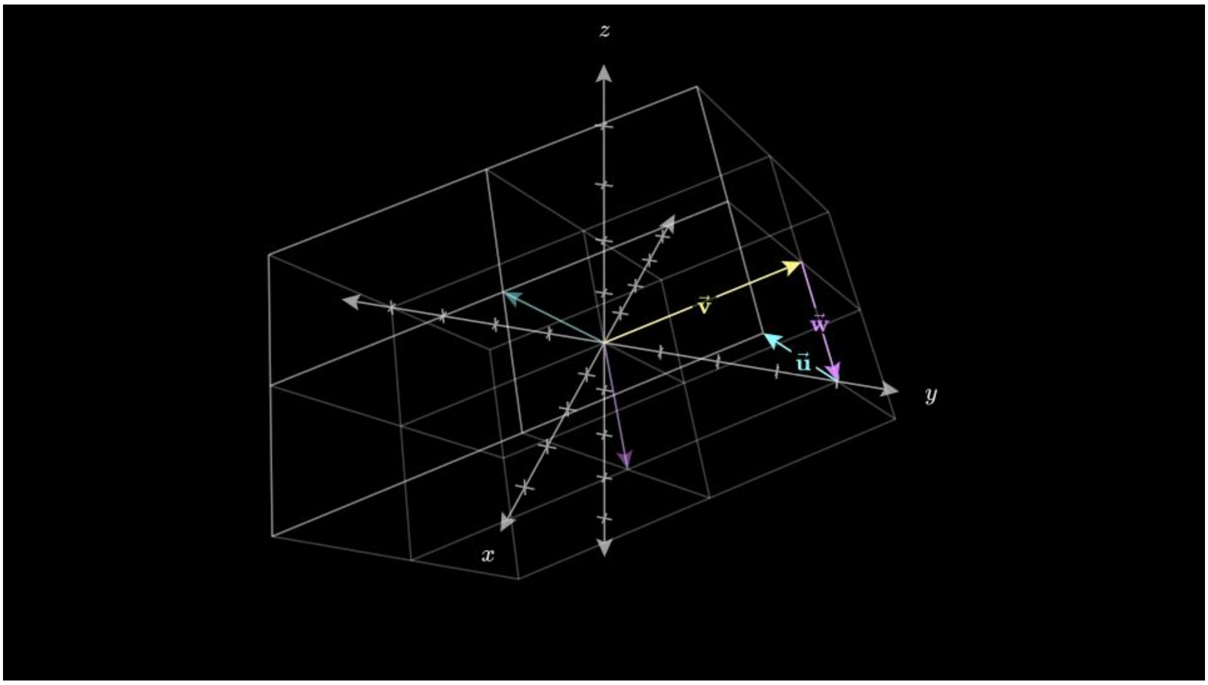
Two things can happen when we add a third vector:

1. If thethird vector happens to be sitting on the span of the first two, then the span doesn't change, it’s sort of trapped on that same flat sheet. In other words,

adding a scaled version of the third vector to linear combinations of the first two doesn't give access to any new vectors.



Choosing a third vector randomly will almost always result in a vector that does not lie in the span of the first two. Since this third vector points in a direction independent of the first two, it introduces a new dimension of movement. Scaling this third vector allows it to sweep through space in a way that shifts the plane formed by the first two vectors, effectively covering all of three-dimensional space. Together, these three vectors form a basis for the entire space, unlocking access to every possible three-dimensional vector through their linear combinations



In the case where the third vector was sitting on the span of the first two, or the case where two vectors happen to line up, we want some terminology to describe the fact that at least one of these vectors is redundant, not adding anything to the span. Whenever this happens, where there is multiple vectors, and is possible to remove one without reducing their span, the relevant terminology is to say they are “linearly dependent”.

Another way of phrasing this would be to say that one of the vectors can be expressed as a linear combination of the others. That is, it's already in the span of the other two.

On the other hand, if each vector really does add another dimension to the span, they are said to be “linearly independent”.