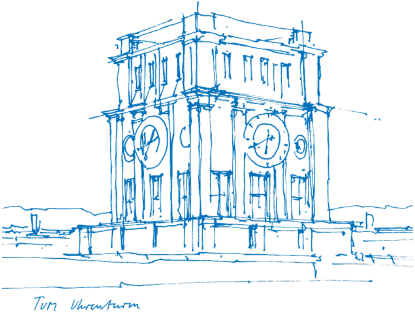
Mathematical Basics 2

SS 2017

Jörg Encke, Korbinian Steger

Prof. Werner Hemmert
Technical University of Munich
Bio-Inspired Information Processing



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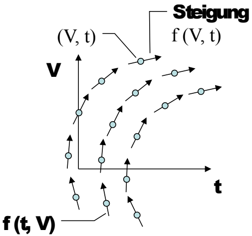
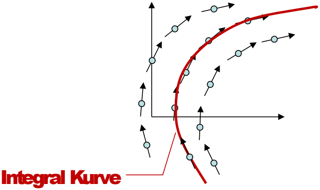
analytically $\Rightarrow$ **geometrically**



$$\frac{dV}{dt} = f(V, t) \quad \Rightarrow \quad \text{directional field}$$

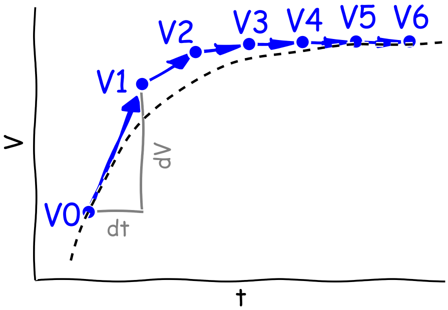


V_1 (solution) $\Rightarrow$ integral curve

Some Definitions

$$t_{n+1} = t_n + \Delta t$$
$$V_n = V(t_n)$$
$$V_{n+1} = V(t_{n+1})$$

Use the slope $f(V_n, t_n)$ to compute the position of the next point V_{n+1}.

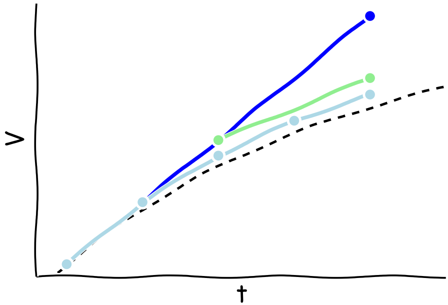


$$\frac{dV}{dt} = f(V, t)$$

$$\frac{V_{n+1} - V_n}{\Delta t} = f(V, t)$$

$$V_{n+1} = V_n + f(V_n, t_n) \cdot \Delta t$$

The error due to the numeric approximation depends on the incremental step size. The correlation of those variables is given by the **order of consistency**.



Local Error

The **local error** is the error at a given time t

Derivation using the taylor series:

$$V(t+\Delta t) = V(t) + \Delta t V`(t) + \underbrace{\frac{1}{2}\Delta t^2 V``(t) + O(\Delta t^3)}_{\text{Error}}$$

$\Rightarrow$ For small Δt the **local error** is proportional to Δt^2

Global Error

The global error is the sum of all local errors ($\propto \Delta t^2$) until a given time t

The number of steps for reaching the time t is given by:

$$\frac{t - t_0}{\Delta t}$$

Therefore the global error is approximately proportional to Δt

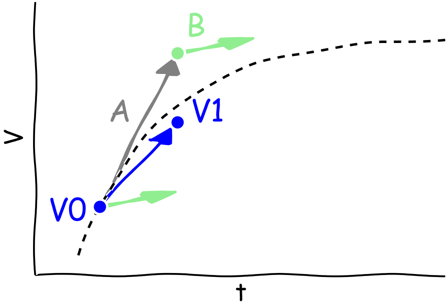
The explicit euler approximation is a 1st order method:
$\Rightarrow$ **Half step size half error**

Is there something better than explicit euler method?

The Runge-Kutta methods are a family of **single step methods**.

- Heun method (2. Order)
- Runge-Kutta (4. Order)

Use the average slope of $f(V_n, t_n)$ and **one** more support point to compute the position of the next point V_{n+1}.



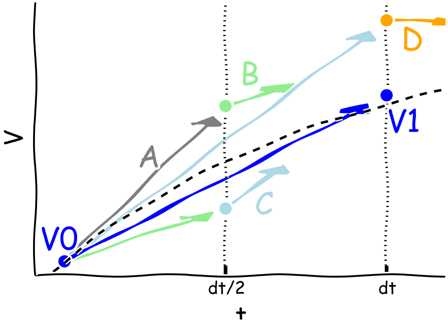
$$A = f(V_n, t_n)$$
$$\tilde{V} = V_n + A \cdot \Delta t$$
$$B = f(\tilde{V}, t_{n+1})$$

$$V_{n+1} = V_n + \frac{A + B}{2} \cdot \Delta t$$

Method of 2nd order $\Rightarrow$ **Half step size, 1/4 error.**

Runge Kutta Method (4th order)

Use the average slope of $f(V_n, t_n)$ and **three** more support points to compute the position of the next point V_{n+1}.



$$A = f(V_n, t_n)$$

$$\tilde{V_A} = V_n + A \cdot \frac{\Delta t}{2}$$

$$B = f(\tilde{V_A}, t_{n+0.5})$$

$$\tilde{V_B} = V_n + B \cdot \frac{\Delta t}{2}$$

$$C = f(\tilde{V_B}, t_{n+0.5})$$

$$\tilde{V_C} = V_n + C \cdot \Delta t$$

$$D = f(\tilde{V_C}, t_{n+1})$$

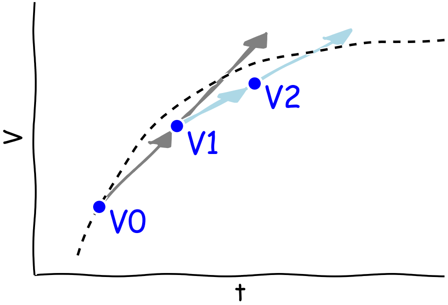
Method of 4th order $\Rightarrow$ **Half step size, 1/16 error.**

$$V_{n+1} = V_n + \frac{A + 2B + 2C + D}{6} \cdot \Delta t$$

Is there more?

- Implicit methods
- Exponential methods

Use the slope at the next point $f(V_{n+1}, t_)$ to compute the position of the next point V_{n+1}.



$$\frac{dV}{dt} = f(V, t_n)$$
$$\frac{V_{n+1} - V_n}{\Delta t} = f(V, t)$$

$$V_{n+1} = V_n + f(V_{n+1}, t_{n+1}) \cdot \Delta t$$

1st order method $\Rightarrow$ **Half step size, half error**

For calculation the following equation must be solved:

$$0 = V_{n+1} - V_n + f(V_{n+1}, t_{n+1}) \cdot \Delta t$$

Depending on the characteristics of $f(V_{n+1}, t_{n+1})$ this can be done either analytically or numerically.

 # Blackboard Notes #

Advantages ??

Exponential methods can be used with equations looking like

$$\frac{dV}{dt} = A(t)V(t) + B(V,t)$$

In the case of the 1st order exponential euler method such an equation can be solved by:

$$V_{n+1} = V_n e^{A(t_n)\Delta t} + \frac{B(t_n)}{A(t_n)}(e^{A(t_n)\Delta t} - 1)$$

Advantages ??

 # Demo #

Using Dalquist's Test Equation

$$\frac{dV}{dt} = \lambda V$$

the following applies for the exact solution $V(t) = e^{\lambda t}$:

$$\lim_{t \to \infty} |V(t)| = \begin{cases} 0 & Re\{\lambda\} < 0 \\ \infty & Re\{\lambda\} > 0 \end{cases}$$

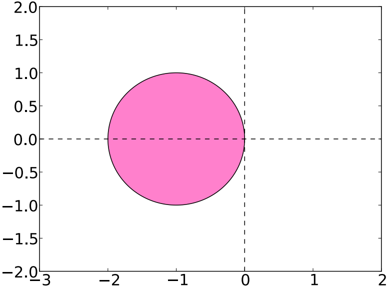
For $Re\{\lambda\} < 0$ as well as for the numerical solution applies:

$$|V_n| \to 0 \text{ für } n \to \infty$$

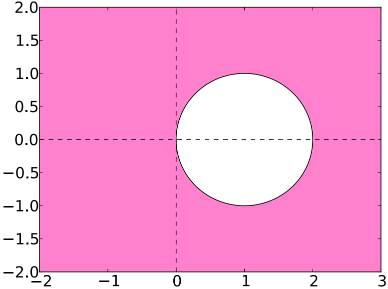
Calculation

 # See notes on blackboard #

Area of Stability



Expliziter Euler

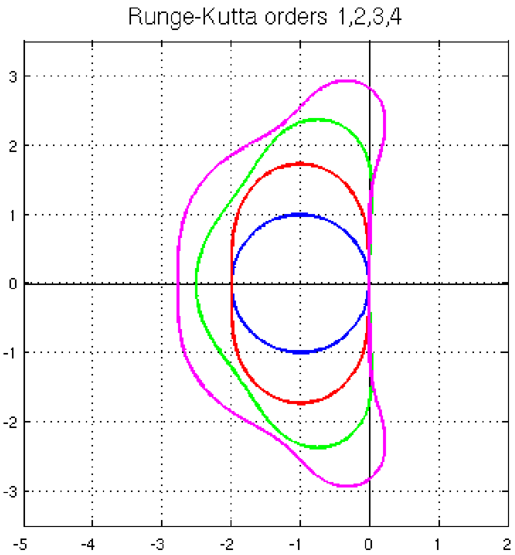


Impliziter Euler

- Small area of stability
- Not suitable for stiff equations

- A-stable
- Not suitable for unstable equations

Area of Stability



Questions?