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Department of Civil, Geo and Enviromental Engineering
Technische Universität München

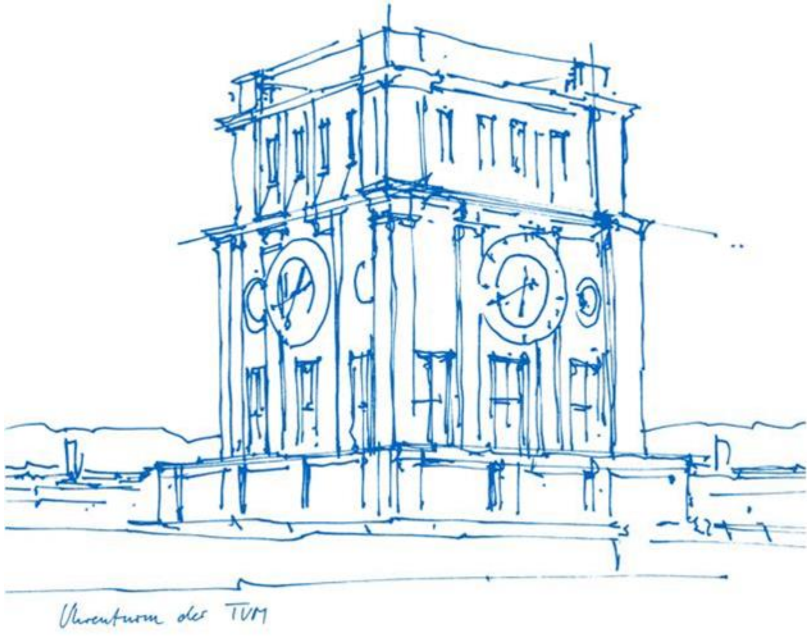
Application of weak Dirichlet boundary conditions for the variational multiscale stabilized finite element discretization of the two-dimensional transient Navier-Stokes equations

Technische Universität München

Ingenieurfakultät Bau Geo Umwelt

Lehrstuhl für Statik

München, 24. Juli 2019



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Overview and Motivation

Stabilized FEM for fluids:

- Simulations of incompressible Navier-Stokes equation(has nonlinear and unsymmetrical terms) with classical Galerkin method may suffer from spurious oscillations arising from:
- 1. Advective – diffusive character of the equation
- 2. Mixed formulation character of equations (pressure and velocity)

- To deal with these shortcomings stabilization is done

$$\boxed{\varepsilon P + div\, \mathbf{u} = 0} \quad \boxed{\lambda = 1/\varepsilon}$$

$$\boxed{P = -\frac{1}{\varepsilon} div\, \mathbf{u}}$$

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u}\,.\,\nabla)\,\mathbf{u} = -\nabla p + \frac{1}{\mathrm{Re}}\nabla^2 \mathbf{u} + f \quad in \quad \Omega \qquad (1)$$

$$div\, \mathbf{u} = 0 \quad in \quad \Omega \qquad (2)$$

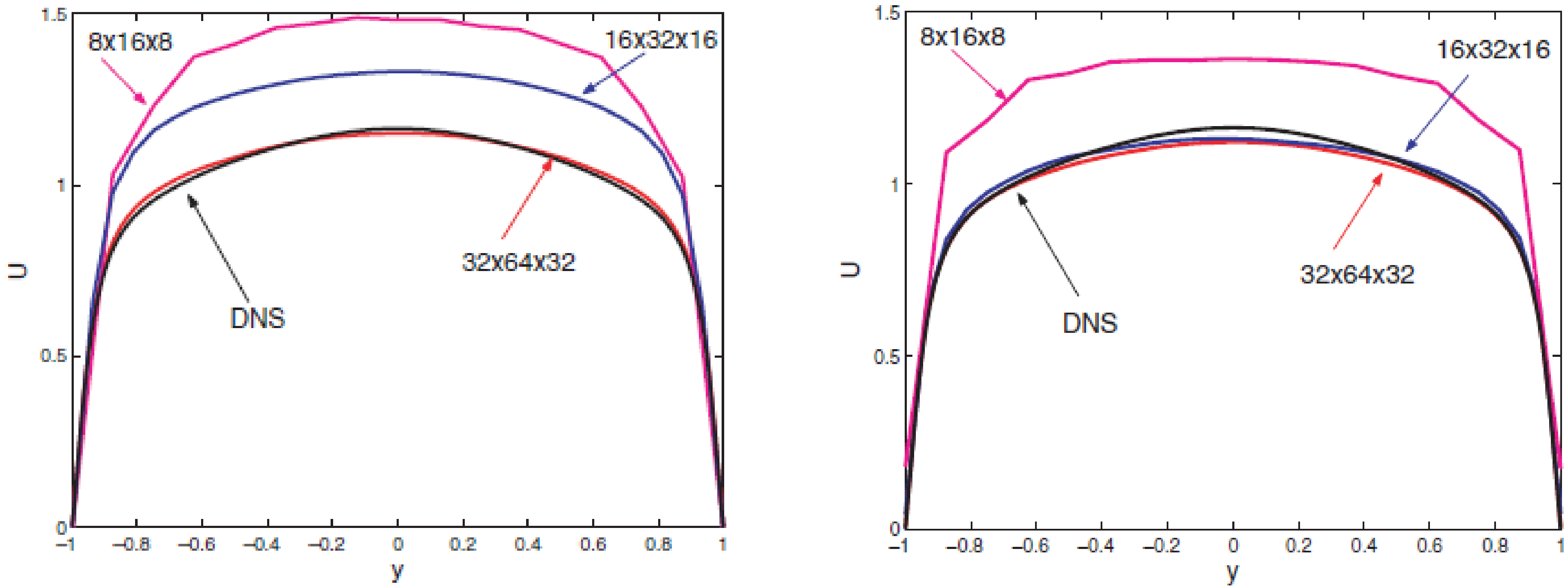
How exactly?

- There are different methods of stabilization, the one which we look at is: Variational Multiscale stabilization.
- This is spatial stabilization for both velocity and pressure fields
- Stabilized methods added at almost no cost a term that improved the accuracy of the Galerkin method
- This smoothens out the oscillations by addition of numerical diffusivity without affecting the accuracy of the solution

Boundary conditions in fluid problems

- Major problem :
  - coupling of pressure and velocity fields
  - implementation of pressure boundary conditions.
- Accuracy of boundary conditions largely determines the flow characteristics.
- Dirichlet boundary condition (e.g. no-slip, fixing pressure at outlet)
- Neumann conditions

Overview and Motivation



(a) Strong (b) Weak

Turbulent channel flow. Convergence of the mean flow. Comparison between weak and strong imposition of the no-slip boundary condition.

Y. Bazilevs, T.J.R. Hughes / Computers & Fluids 36 (2007) 12–26

Problems with strong boundary conditions

- It has been repeatedly noted over the years that strongly enforced outflow boundary conditions give rise to spurious oscillations even for methods with otherwise good stability properties. So maybe strong satisfaction is not such a good idea.
- Applying the constraints strongly tends to pose as a problem
- Strong velocity might put impossible pressure requirements
- Fixing pressure loses the ability to make it incompressible
- It is intuitive to keep the velocity and pressure unknowns and solving for them by applying them in a weakly constrained way.
- Although there are shortcomings which we will discuss in the successive parts of the presentation

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Weak Constraints Techniques

Concept of Weak Boundary Conditions:

- No exact fullfilmet of BC‘s at prescribed DOF‘s
- Terms added to the variational equations
- Allow for calculating accurate fluxes

- → Values of D.O.F dictated by a constraint condition :
- Penalty Method
- Lagrange Multipliers

$$\boxed{[\mathbf{C}]\{\mathbf{D}\} - \{\mathbf{Q}\} = \{\mathbf{0}\}}$$

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Weak Constraints Techniques

Penalty Method Formulation:

- Add penalty function for violating constraint.
- Doesn‘t increase number of unkonws.
- Alter typology of system matrix.
- Penalty factor choice.

$$[\mathbf{C}]\{\mathbf{D}\} - \{\mathbf{Q}\} = \{\mathbf{0}\}$$

$$\{\mathbf{t}\} = [\mathbf{C}]\{\mathbf{D}\} - \{\mathbf{Q}\}$$

$$\Pi_p = \frac{1}{2}\{\mathbf{D}\}^T[\mathbf{K}]\{\mathbf{D}\} - \{\mathbf{D}\}^T\{\mathbf{R}\} + \frac{1}{2}\{\mathbf{t}\}^T\lceil\boldsymbol{\alpha}\rfloor\{\mathbf{t}\}$$

$$\rho\left(\frac{\partial \mathbf{u}}{\partial t},\omega\right) - \mu(\nabla\mathbf{u},\nabla\omega) - (\mathbf{p},\nabla\omega) + \frac{\alpha}{2}(\chi,\omega)_\Gamma = \rho\langle\mathbf{f},\omega\rangle + \langle\omega,t\rangle_{\Gamma_N}$$

$$(q,\nabla\mathbf{u}) = 0$$

$$\left([\mathbf{K}] + [\mathbf{C}]^T\lceil\boldsymbol{\alpha}\rfloor[\mathbf{C}]\right)\{\mathbf{D}\} = \{\mathbf{R}\} + [\mathbf{C}]^T\lceil\boldsymbol{\alpha}\rfloor\{\mathbf{Q}\}$$

Weak Constraints Techniques

Lagrange Multipliers Formulation:

- Introduce undetermined multipliers
- Increase number of unkonws.
- No change to originial system matrix.
- Saddle point problem

$$\Pi_p = \frac{1}{2}\{\mathbf{D}\}^T[\mathbf{K}]\{\mathbf{D}\} - \{\mathbf{D}\}^T\{\mathbf{R}\} + \{\boldsymbol{\lambda}\}^T\left([\mathbf{C}]\{\mathbf{D}\} - \{\mathbf{Q}\}\right)$$

$$\begin{bmatrix} \mathbf{K} & \mathbf{C}^T \\ \mathbf{C} & \mathbf{0} \end{bmatrix} \begin{Bmatrix} \mathbf{D} \\ \boldsymbol{\lambda} \end{Bmatrix} = \begin{Bmatrix} \mathbf{R} \\ \mathbf{Q} \end{Bmatrix}$$

$$\rho\left(\frac{\partial \mathbf{u}}{\partial t}, \omega\right) - \mu\left(\nabla \mathbf{u}, \nabla \omega\right) - \left(p, \nabla \omega\right) + \langle \lambda, \psi \rangle_{0,\Gamma_C} + \langle \chi, \mu \rangle_{0,\Gamma_C} = L(v)$$

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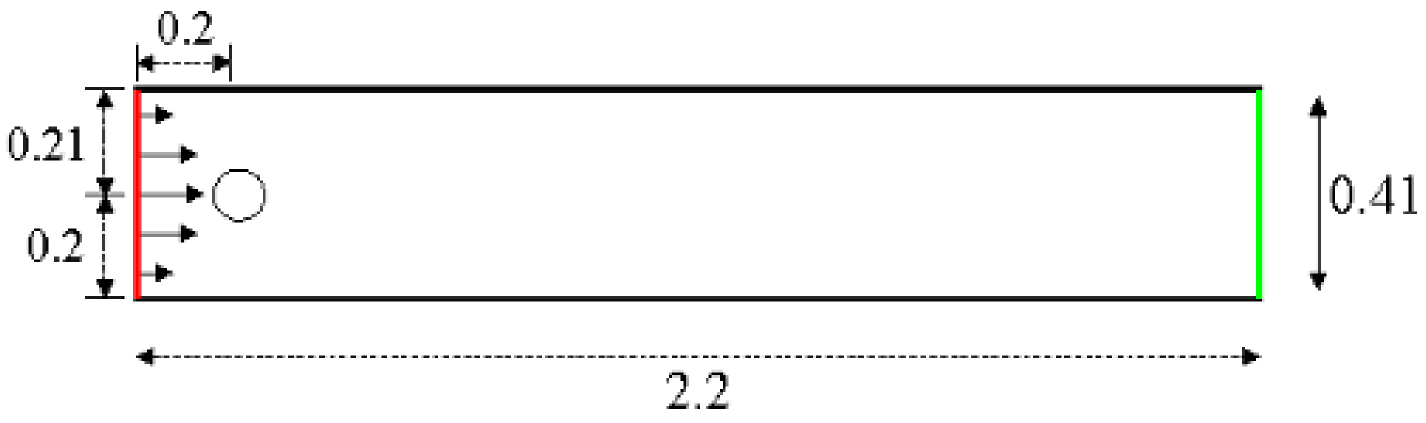
Results and Discussion

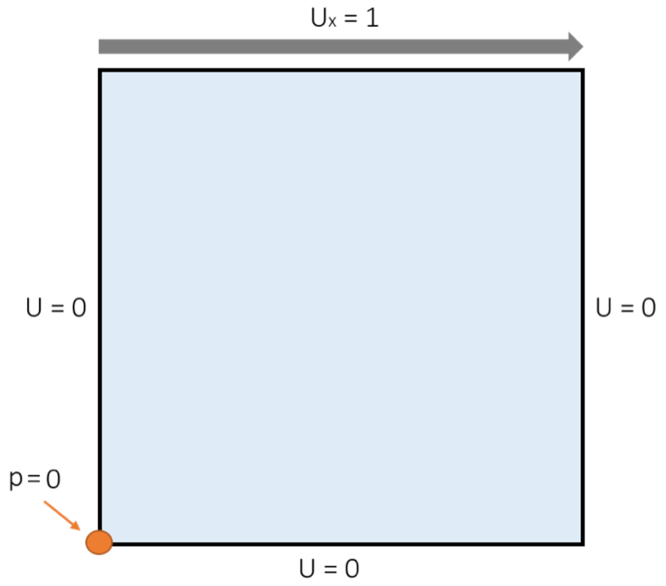
❑ Flow around a cylinder

Parameter	Value
U	1.5 m/s
r	0.05 m
ρ	1 kg/m³
v	0.001 m²/s



❑ Driven Cavity

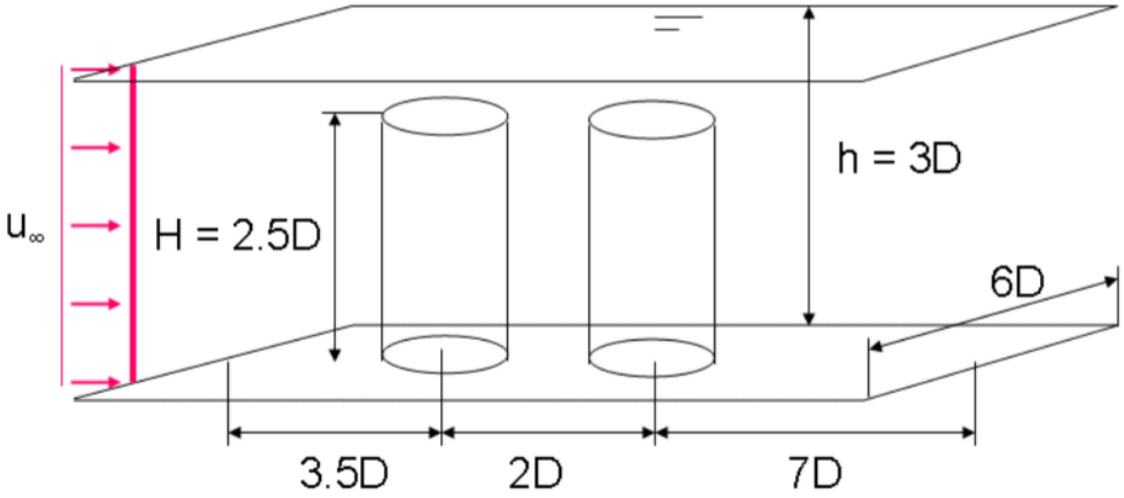
Parameter	Value
U	1.0 m/s
L	0.1 m
ρ	1000 kg/m³
Re	100000



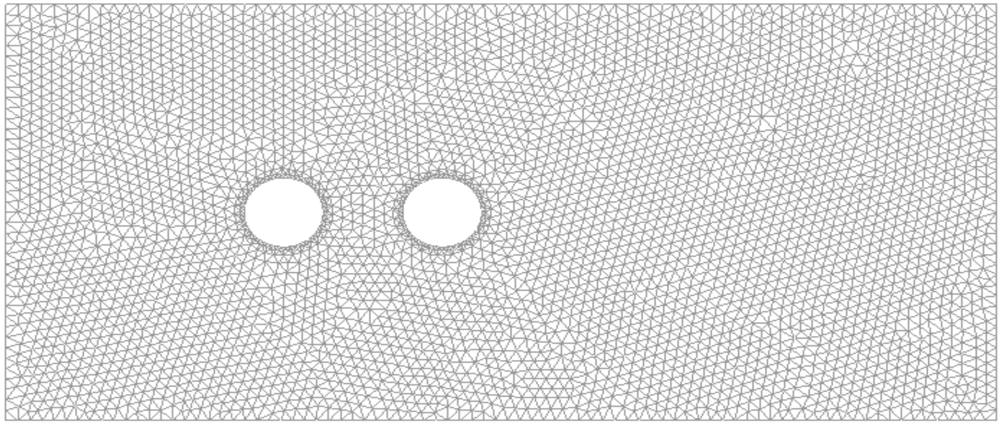
Results and Discussion

❑ Flow around 2 cylinders

Parameter	Value
D	20 mm
u	0.0125
Re	1500
Fluid	Water



Parameter	Value
Number of nodes	4036
Element type	triangle



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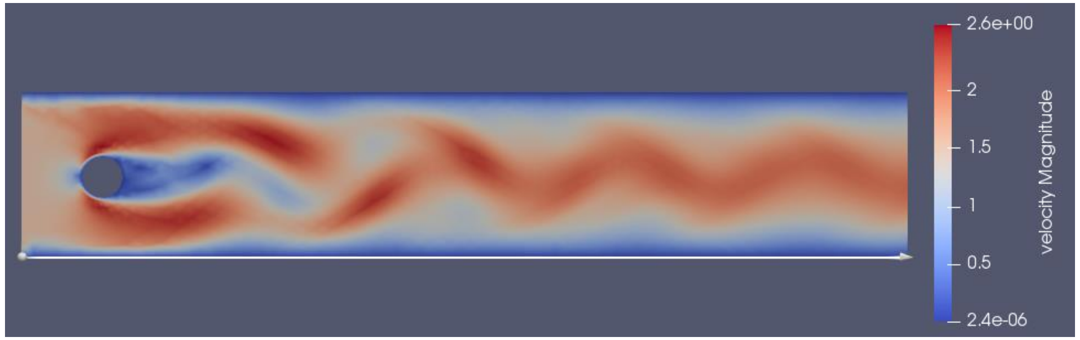
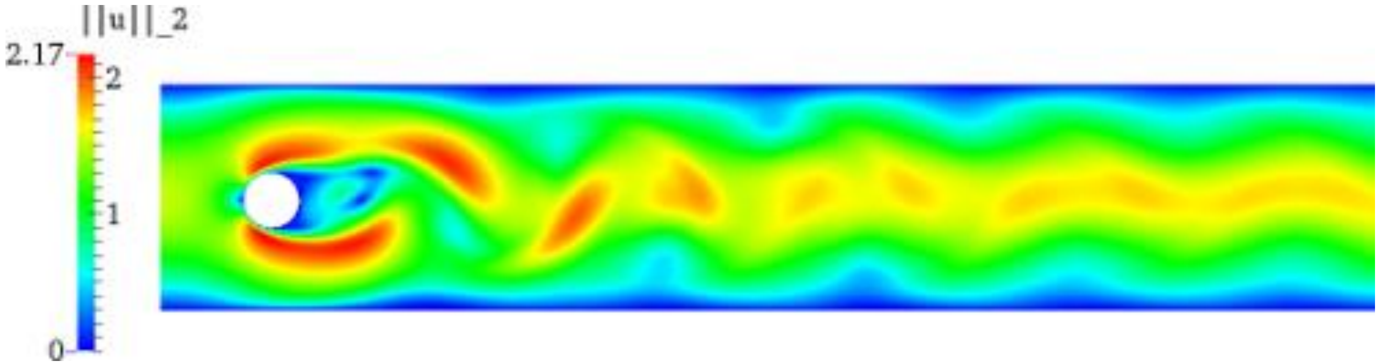
Results and Discussion

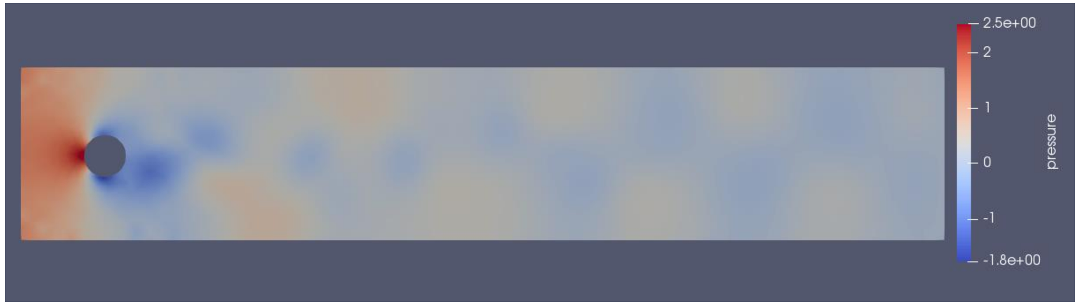
❑ Flow around a cylinder

ParaView Results

Benchmark case Results







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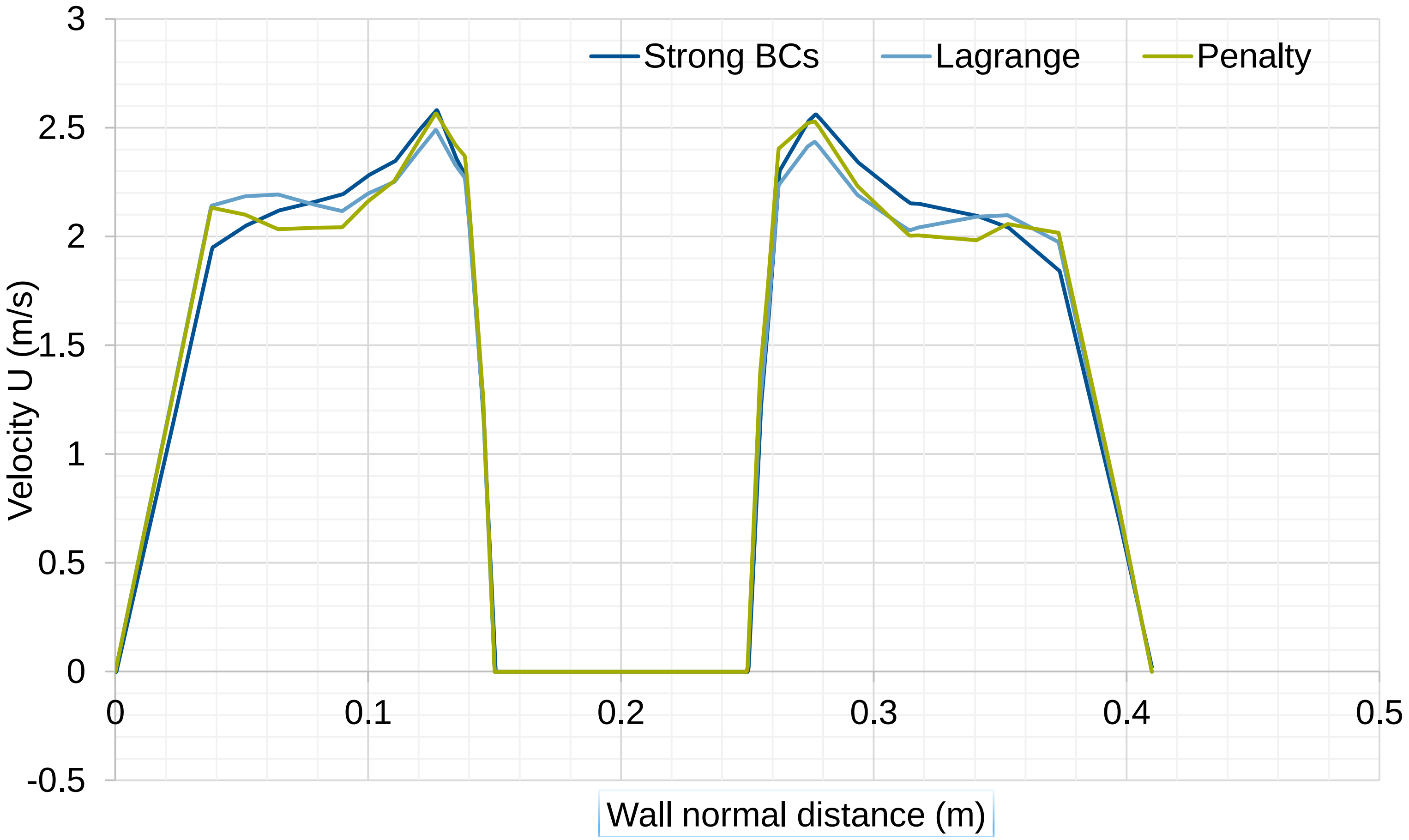
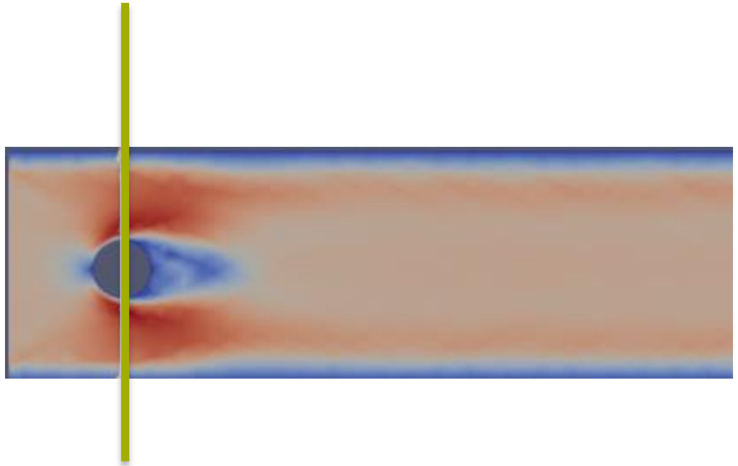
Results and Discussion

❑ Flow around a cylinder





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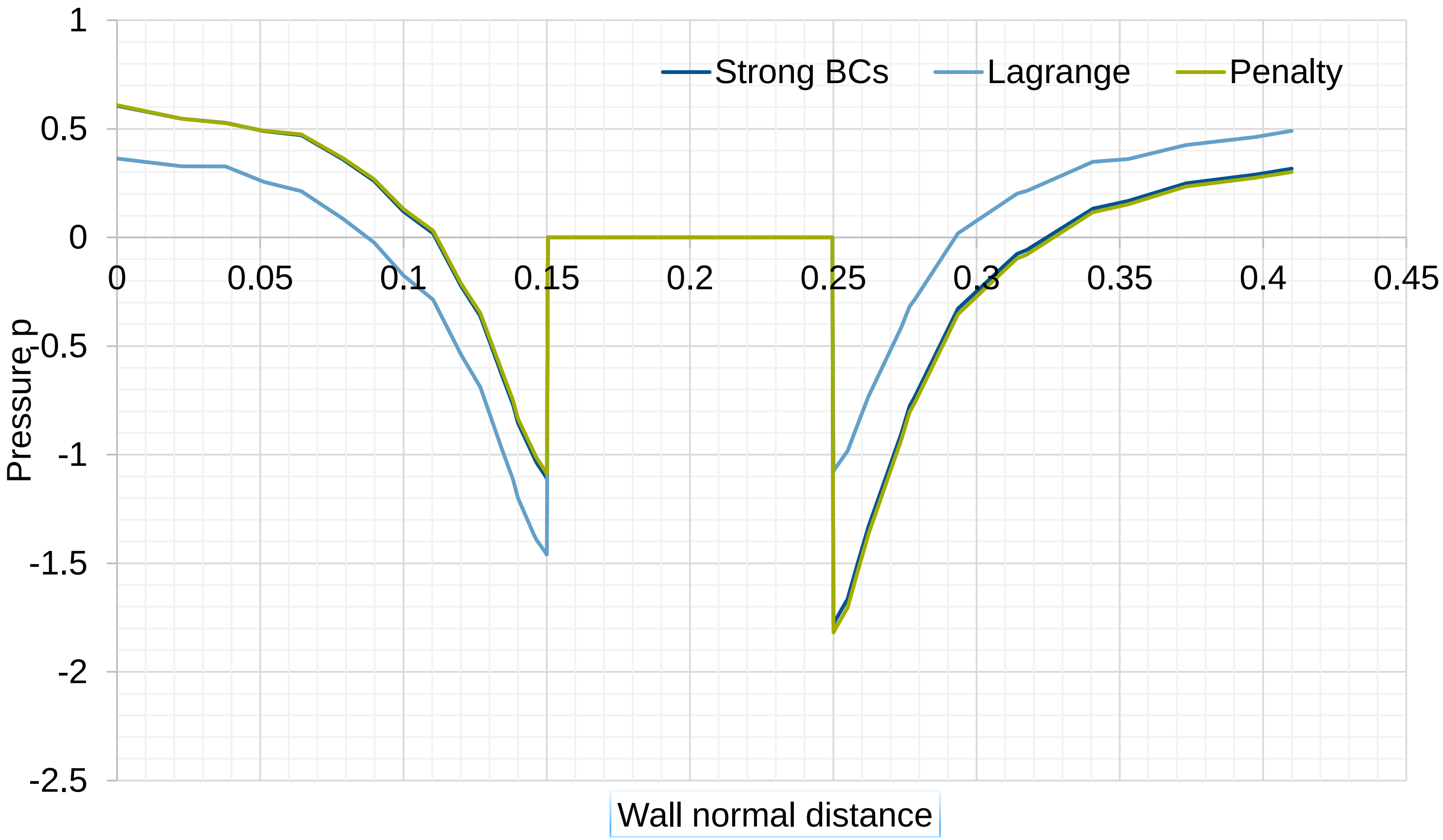
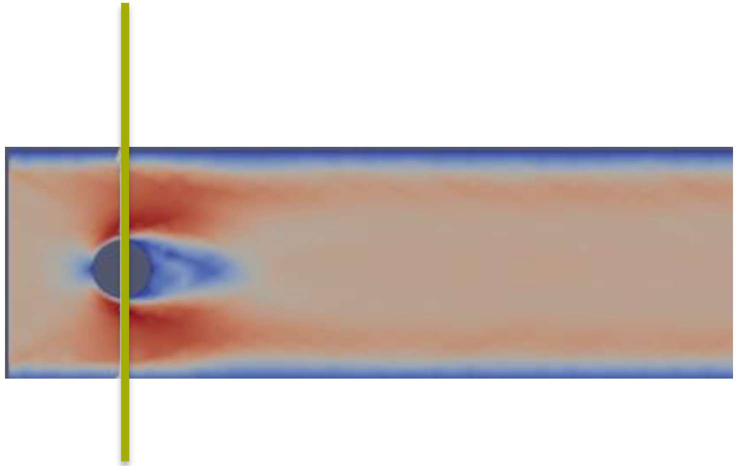
Results and Discussion

❑ Flow around a cylinder





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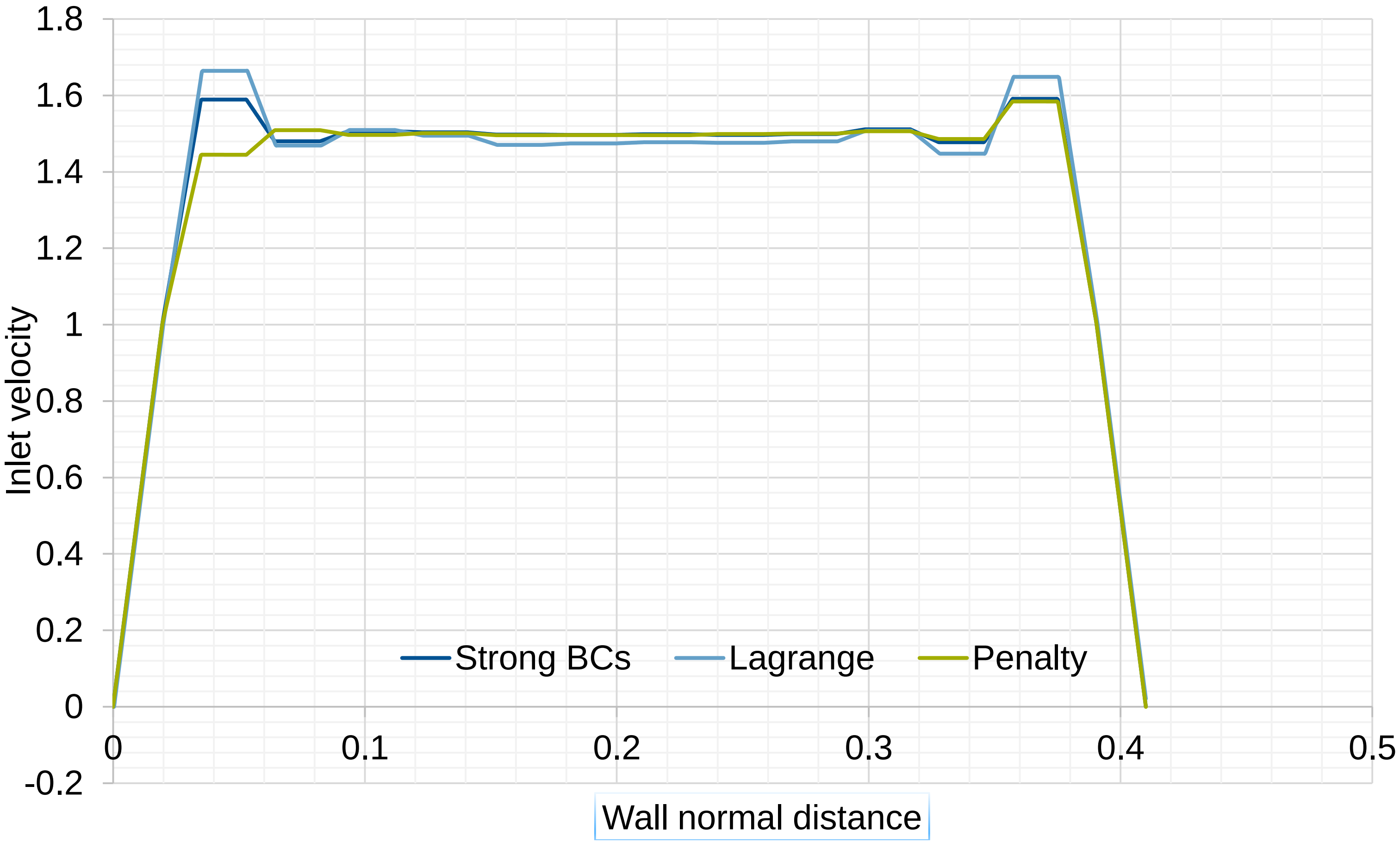
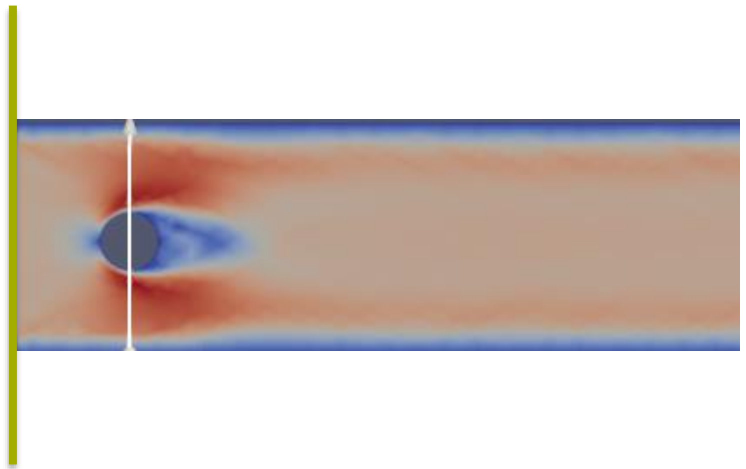
Results and Discussion

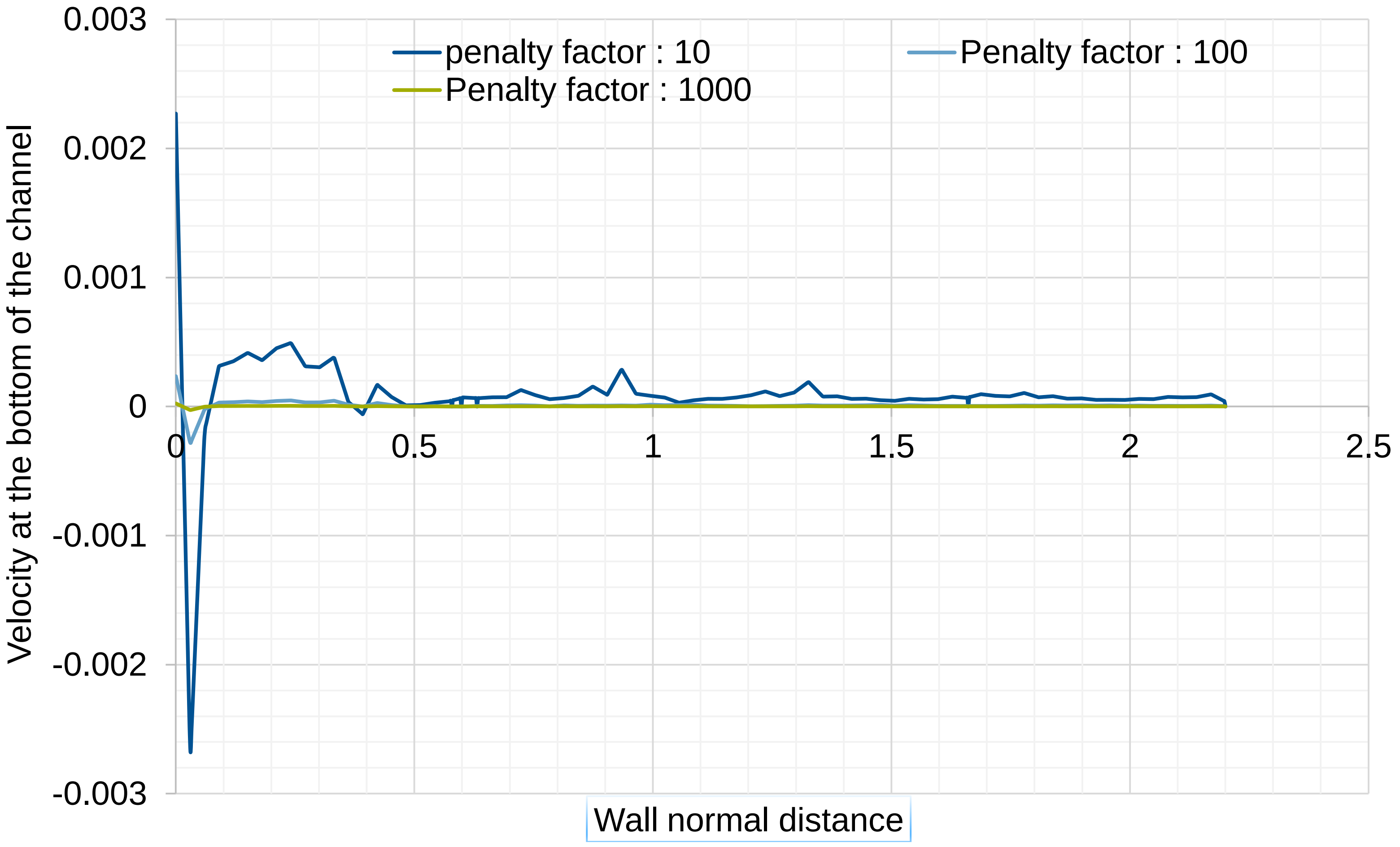
❑ Flow around a cylinder

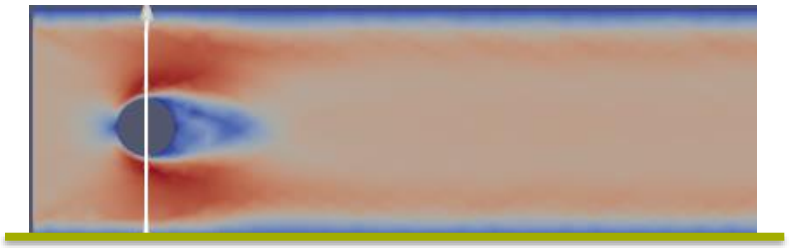




Results and Discussion

❑ Flow around a cylinder





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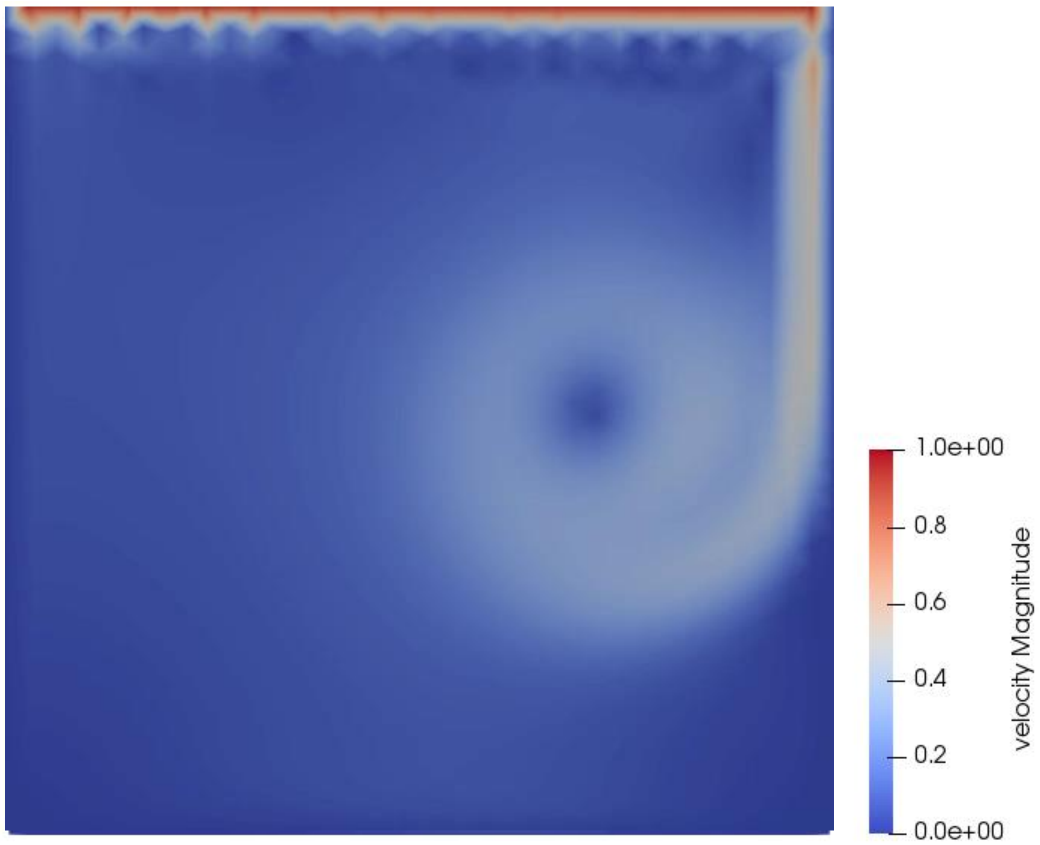
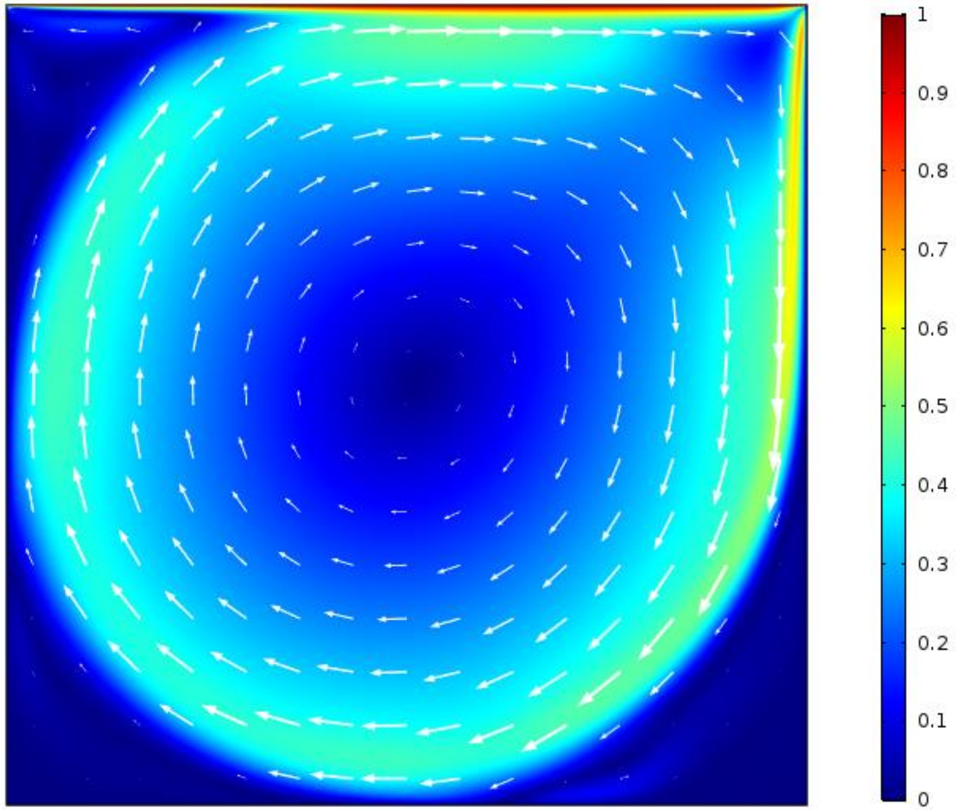
Results and Discussion

❑ Driven Cavity

ParaView Results



Benchmark case Results



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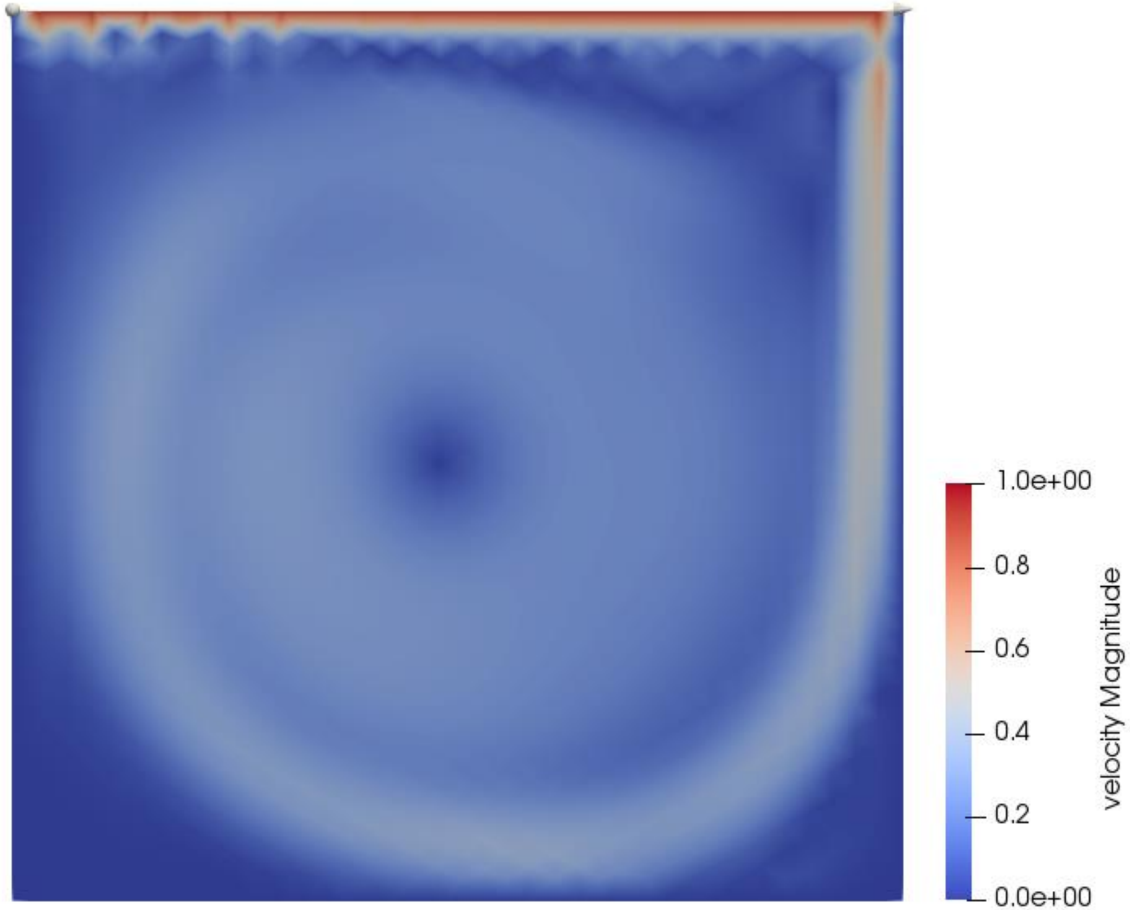
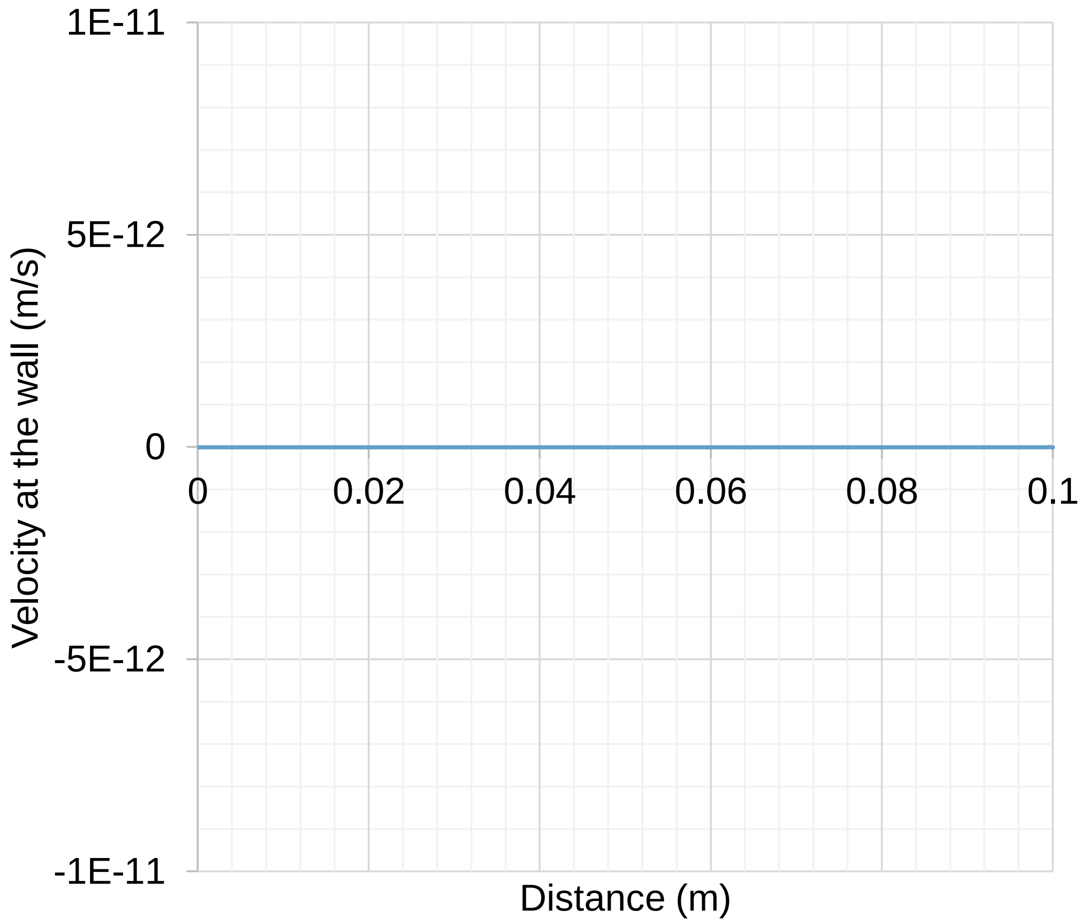
Results and Discussion

❑ Driven Cavity

➢ Lagrange Boundary Condition

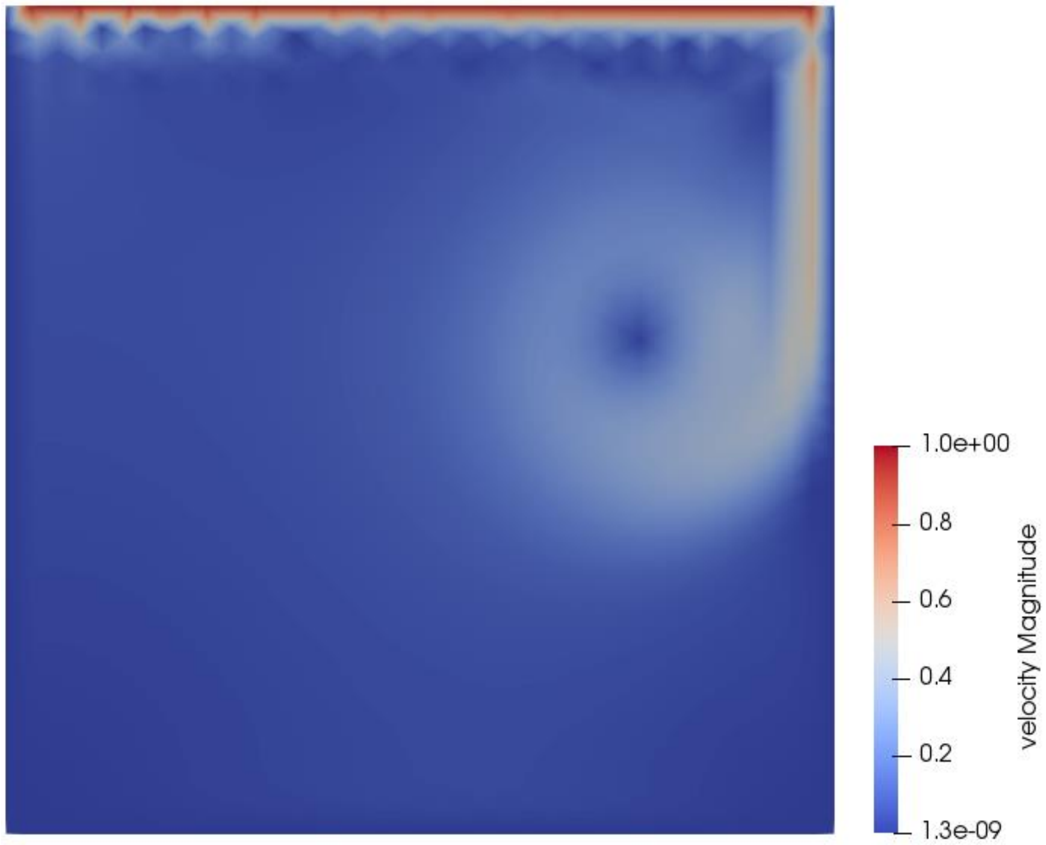


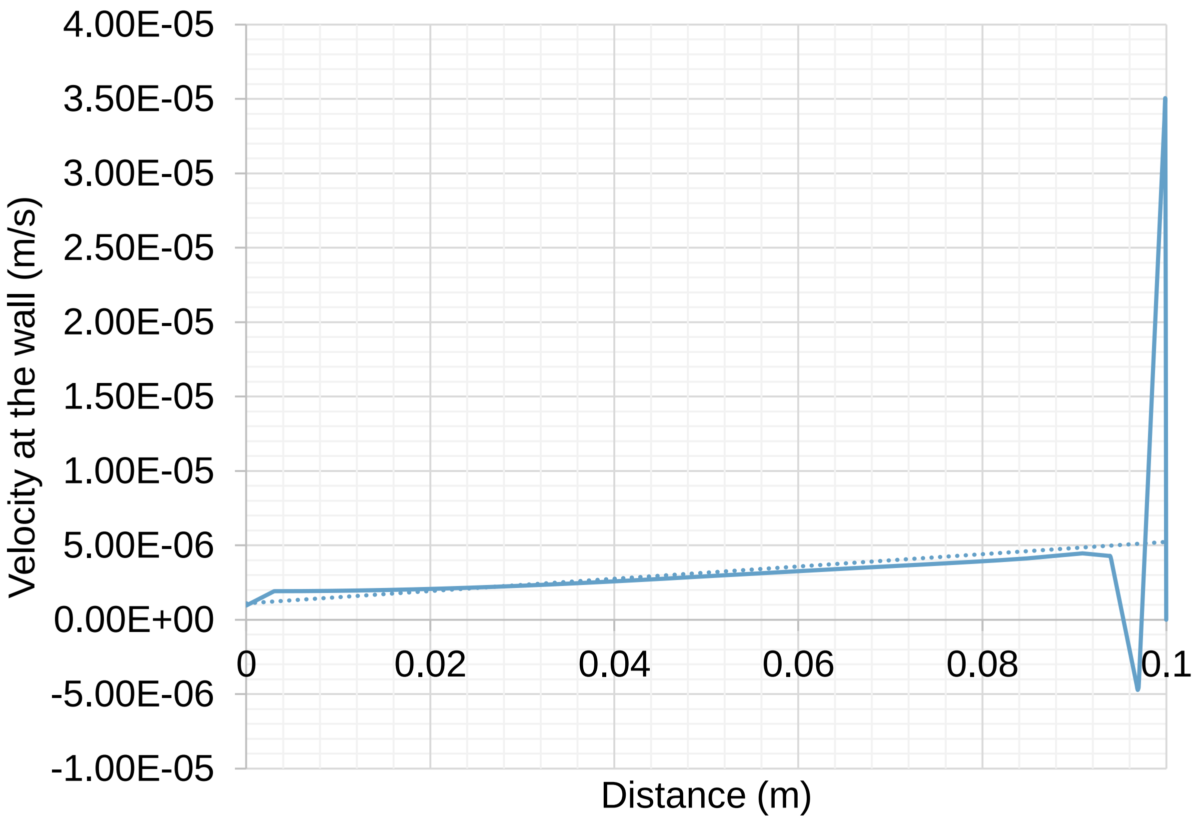


Results and Discussion

❑ Driven Cavity

➢ Penalty Boundary Condition (*λ=10)*

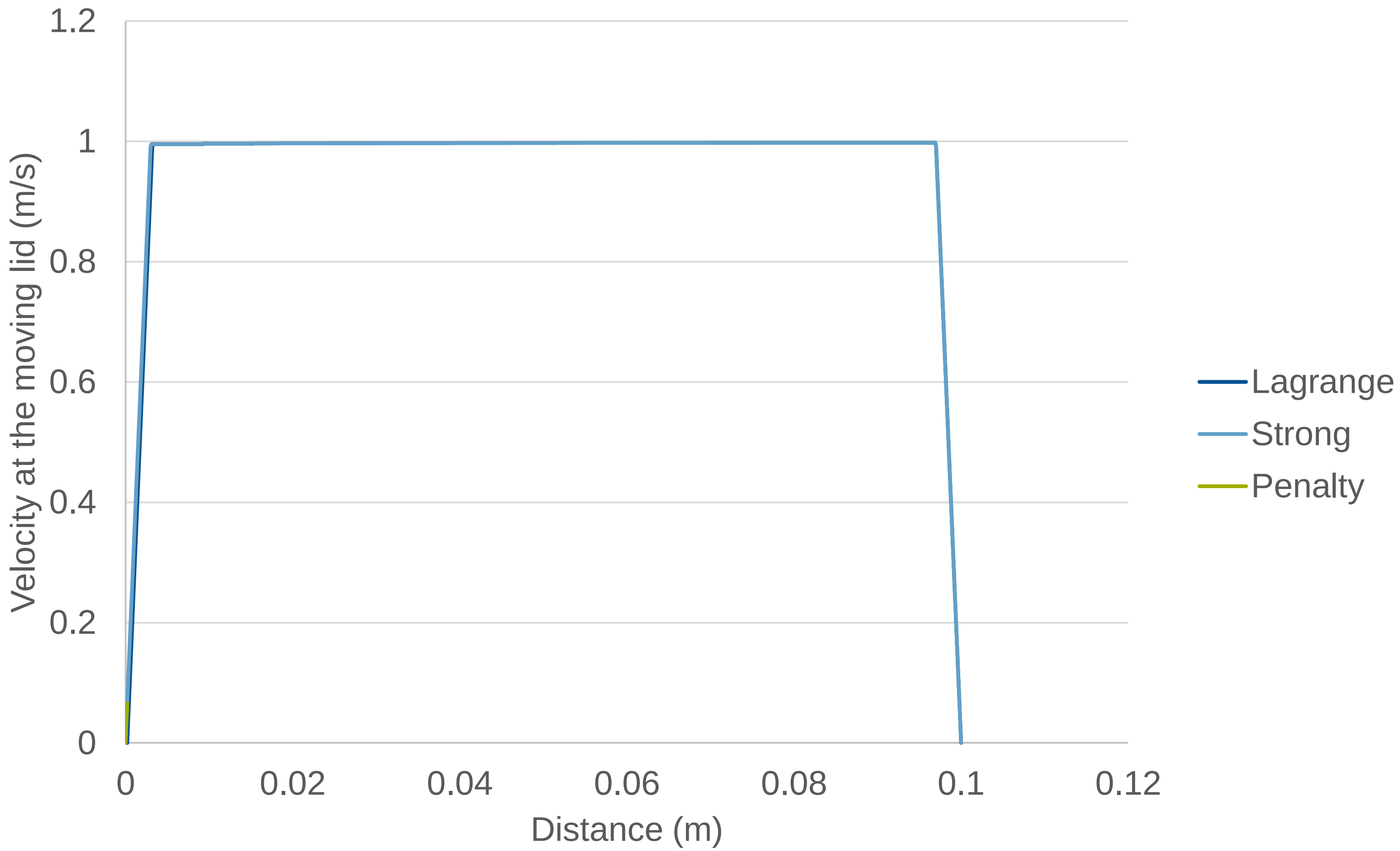




Results and Discussion

❑ Driven Cavity

➢ Velocity at the moving lid



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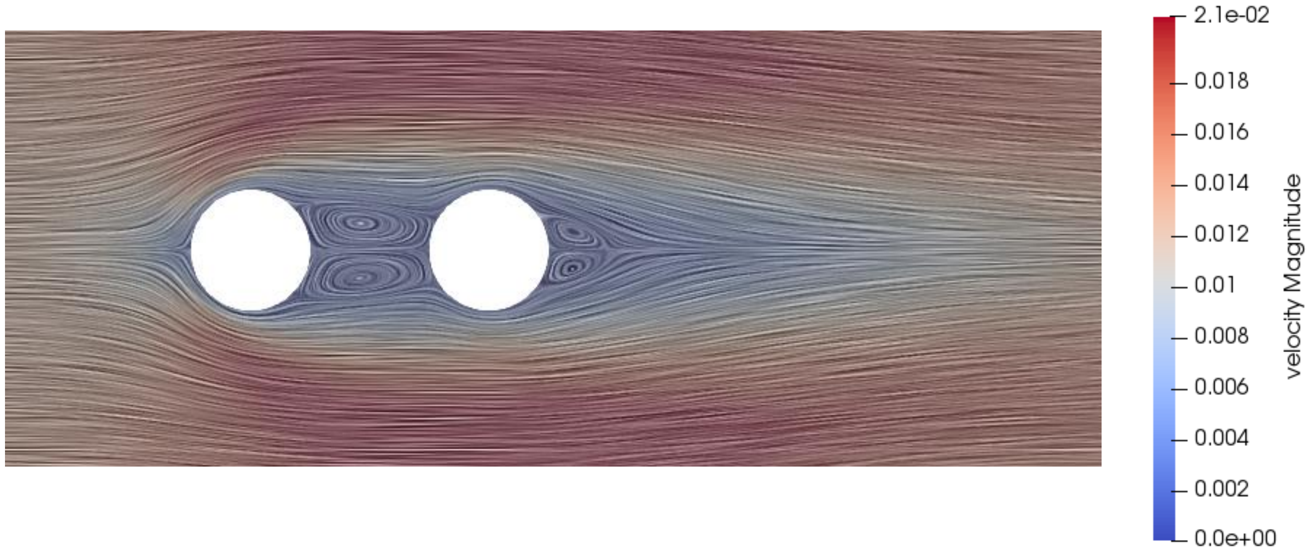
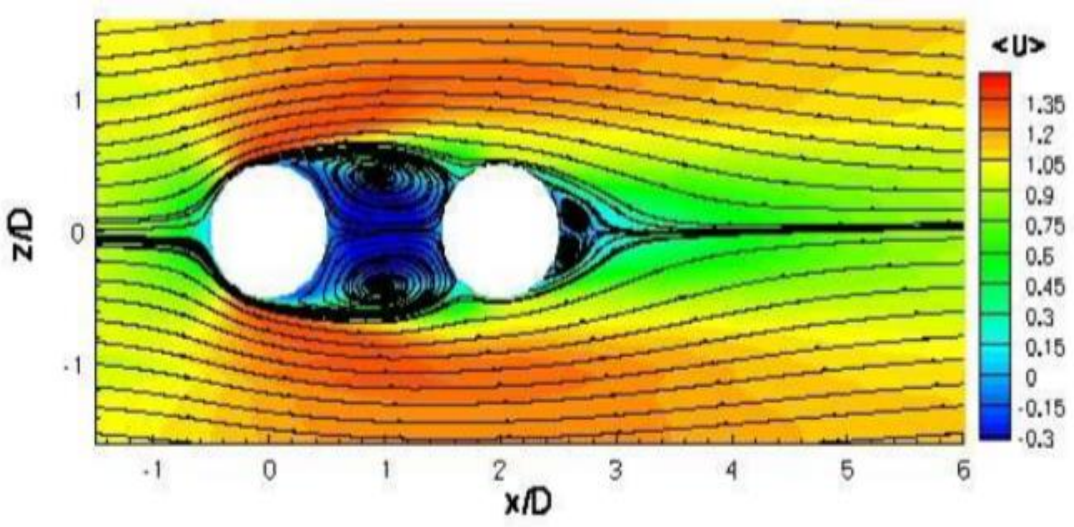
Results and Discussion

- **Flow around 2 cylinders**

ParaView Results



Benchmark case Results



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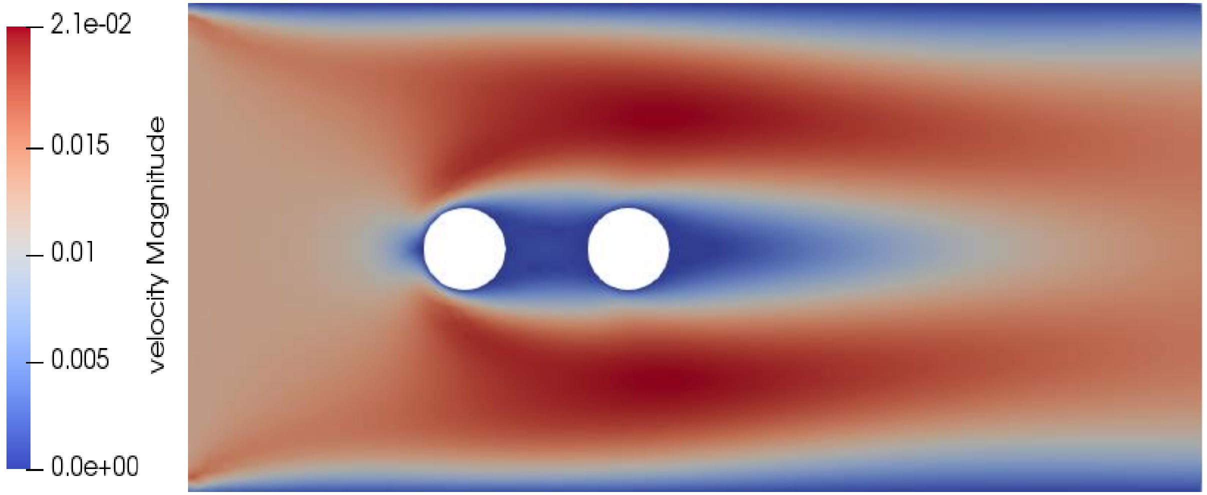
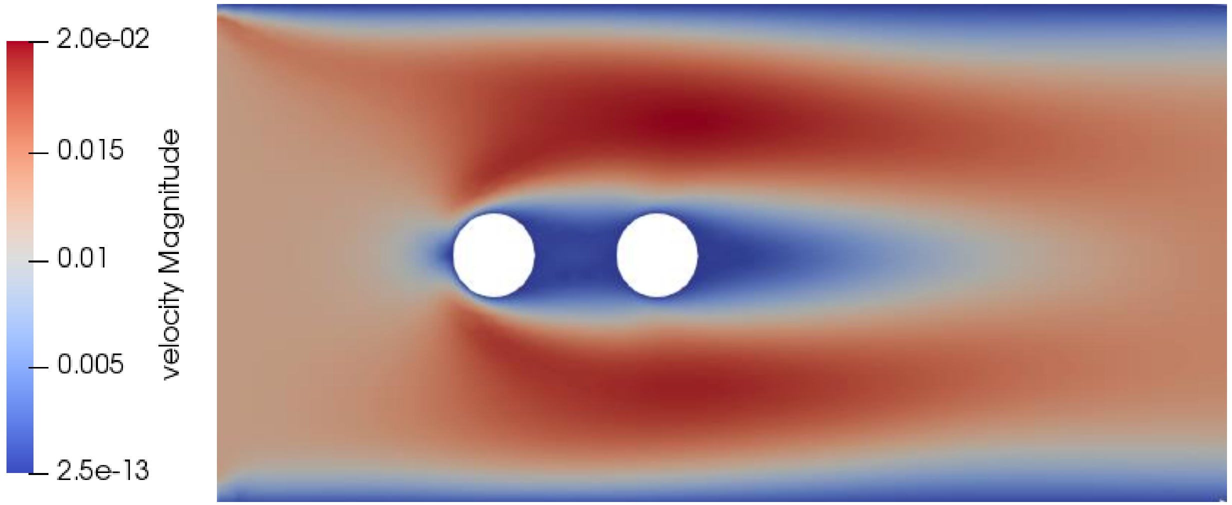
Results and Discussion

- **Flow around 2 cylinders**
  - *Weak Dirichlet Boundary Condition*



Lagrange Multiplier

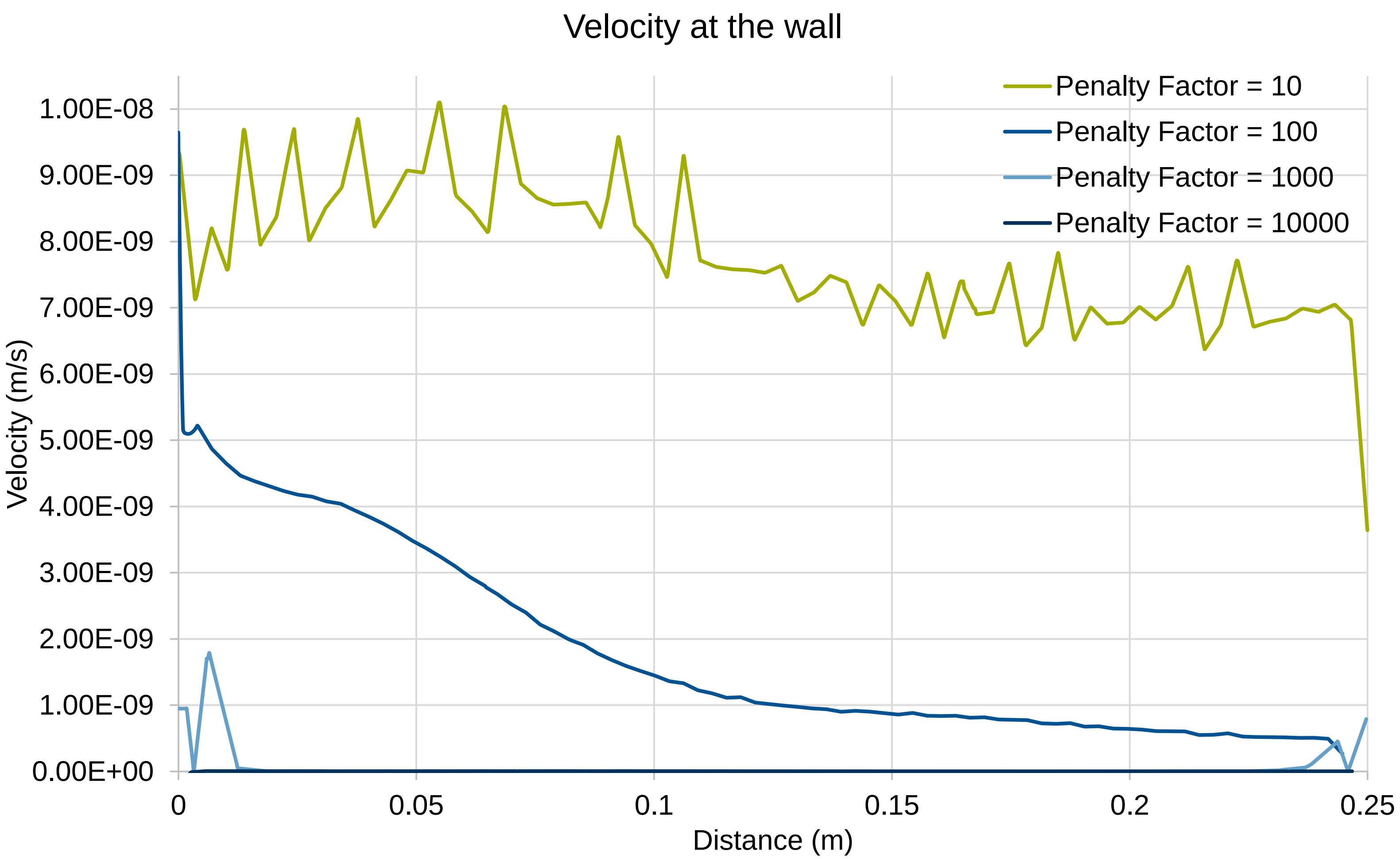


Penalty (λ=10000)

Results and Discussion

❑ Flow around 2 cylinders

➢ *Penalty Dirichlet boundary condition (λ=10,100,1000,10000)*



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Thank you for listening

References

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