Traffic Theory - 1 february 2018

Problem 1 - A Markov Chain $X(n)$ presents the following one-step transition matrix at all times n.

$$\mathbf{P}(n) = \begin{vmatrix} 0.4 & 0.2 & 0.4 \\ 0.5 & 0.5 & 0 \\ 0 & 0.4 & 0.6 \end{vmatrix}.$$

The distribution of the chain at time 0 is $\mathbf{\Pi}(0) = [0.25 \;\; 0.25 \;\;\; 0.5]$. Find

a) the second order distribution $\mathbf{\Pi}(n, n+1)$ (all terms) at times $n = 1$ and $n = \infty$;

b) $P(X(n) = 0|X(n+1) = 2)$ at times $n = 1$ and $n = \infty$;

c) the asymptotic distribution of process $Y(n)$, that is derived in the following way. At all times $3k$, for all $k = 0, 1, 2, \ldots$, an independent experiment is performed and the chain is forced to state i, $i = 0, 1, 2$, with the same probabilities as $\mathbf{\Pi}(0)$, while at other times $\mathbf{P}(n)$ is used as usual. Is $Y(n)$ a Markov Chain?

d) Write the intensity matrix $\mathbf{Q}$ of a continuous-time Markov Chain having the same asymptotic distribution as $X(n)$.

Solution a)

$$p_{jk}(1, 2) = \pi_j(1)p_{jk},$$

with

$$\mathbf{\Pi}(1) = \mathbf{\Pi}(0)\mathbf{P} = [0.225 \;\; 0.375 \;\; 0.400].$$

Therefore we get

$$\mathbf{\Pi}(1, 2) = \begin{vmatrix} 0.09 & 0.045 & 0.09 \\ 0.1875 & 0.1875 & 0 \\ 0 & 0.16 & 0.24 \end{vmatrix}.$$

Similarly we have

$$p_{jk}(n, n+1) = \pi_j(n)p_{jk},$$

where $\pi_j(n)$ is the element of the asymptotic distribution, which is $(n = \infty)$

$$\mathbf{\Pi}(n) = [0.3125 \;\; 0.3750 \;\; 0.3125].$$

Therefore we get

$$\mathbf{\Pi}(n, n+1) = \begin{vmatrix} 0.1250 & 0.0625 & 0.1250 \\ 0.1875 & 0.1875 & 0 \\ 0 & 0.1250 & 0.1875 \end{vmatrix}.$$

b) We have

$$P(X(n) = 0|X(n+1) = 2) = P(X(n+1) = 2|X(n) = 0)\frac{P(X(n) = 0)}{P(X(n+1) = 2)} =$$

$$= p_{02}\,\frac{\pi_0(n)}{\pi_2(n+1)}.$$

Then, at time $n = 1$ we need

$$\mathbf{\Pi}(1) = [0.225 \;\; 0.375 \;\; 0.400],$$

$$\mathbf{\Pi}(2) = [0.2775 \;\; 0.3925 \;\; 0.3300],$$

and we have

$$P(X(n) = 0|X(n+1) = 2) = 0.4 \; \frac{0.225}{0.33} = 0.2727.$$

At time $n = \infty$ we need thye asymptotic distribution already evaluated above, and we have

$$P(X(n) = 0|X(n+1) = 2) = 0.4 \; \frac{0.3125}{0.3125} = 0.4.$$

c) $Y(n)$ is a non-homogenous MC, whose distribution is

$$\mathbf{\Pi}(0) = [0.25 \;\; 0.25 \;\;\; 0.5],$$

$$\mathbf{\Pi}(1) = [0.225 \;\; 0.375 \;\; 0.400],$$

$$\mathbf{\Pi}(2) = [0.2775 \;\; 0.3925 \;\; 0.3300],$$

$$\mathbf{\Pi}(3) = [0.25 \;\; 0.25 \;\;\; 0.5],$$

$$\mathbf{\Pi}(4) = [0.225 \;\; 0.375 \;\; 0.400],$$

and repeats periodically as shown above. The asymptotic (generalized) distribution is

$$\mathbf{\Pi} = \frac{\mathbf{\Pi}(0) + \mathbf{\Pi}(1) + \mathbf{\Pi}(0)}{2} = [0.2508 \;\; 0.3392 \;\; 0.41].$$

d) For continuous-time MC, the transition probability in time Δt is

$$p_{jk}(t, t + \Delta t) = q_{jk}\Delta t, \qquad j \neq k.$$

Since balance equations with p_{jk} are the same as with q_{jk}, the required q_{jk} must be proportional to p_{jk}, for $j \neq k$ and $q_{jj} = -\sum_{j \neq k} q_{jk}$. Therefore, with any finite k we have

$$\mathbf{Q} = \begin{vmatrix} -k \times 0.6 & k \times 0.2 & k \times 0.4 \\ k \times 0.5 & -k \times 0.5 & 0 \\ 0 & k \times 0.4 & -k \times 0.6 \end{vmatrix}.$$

Problem 2 - A channel alternates Idle and Busy periods. Idle periods have negative exponential pdf with rate λ, while Busy periods have a constant duration T. A user activates at time t, in stationary conditions, and transmits for a constant period T. Find

a) the probability that the transmission completely fits in an Idle period;
b) if the user senses the channel Busy, the pdf of the waiting time till the channel is sensed idle;
c) if the user transmits in case b) the pdf of the length of the new Busy period that includes the new transmission.

Solution a) The probability that the user starts the transmission within the Idle period is

$$P_I = \frac{1/T}{\lambda + 1/T}$$

The remaining Idle period length is again a negative exponential variable Z with rate λ. Therefore the required probability is

$$\frac{1/T}{\lambda + 1/T} Pr(Z > T) = \frac{1/T}{\lambda + 1/T} e^{-\lambda T}.$$

b) Since the user senses the channel Busy in a time instant which is uniform within period T, the required pdf is uniform:

$$f_X(x) = \frac{1}{T}, \qquad 0 \leq x \leq T.$$

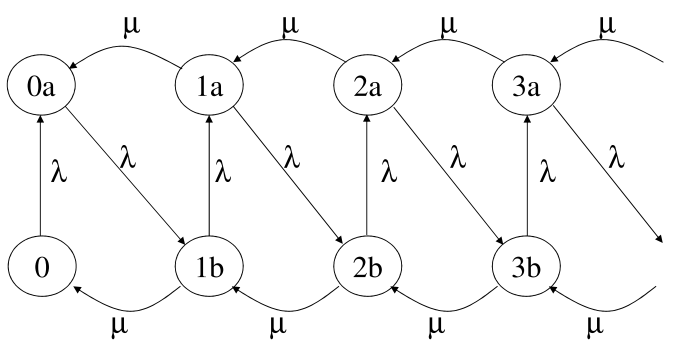
c) The new busy period is the summation of RV Y, distance of the sensing instant from the beginning of the Busy period (uniform variable as above) and the transmission time T, Therefore, the required pdf is a uniform, translated variable V:

$$f_V(x) = \frac{1}{T}, \qquad T \leq x \leq 2T.$$

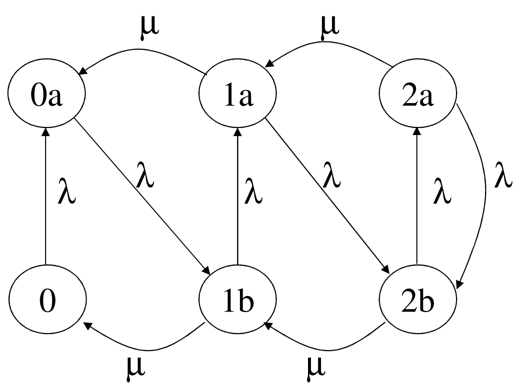
Problem 3 - In an M/M/1 system (λ, μ), only arrivals with even order number are admitted in the system.

a) Draw the Markov chain that describes the number of users in the system;
b) find the asymptotic distribution when arrivals that see two users in the systems are blocked and cleared;
c) compare the average time spent in the system above with the classic $M/M/1$ with arrival rate $\lambda/2$, again with at most two users in the system.

Solution a) We must distinguish even (a) and odd (b) arrivals in order to know what to do. Hence. this information must be included in the state information, and the state diagram is as shown below:



b) the state diagram is changed as shown below:



c) The average time in the system can be evaluated as

$$E[V] = \frac{2}{\lambda}E[N] = \frac{2}{\lambda}(\pi_1 + 2\pi_2),$$

where $\pi_i = \pi_{ia} + \pi_{ib}$. In the M/M/1 case, we have

$$\pi_i = \frac{1-\rho}{1-\rho^3}\rho^i, \qquad i = 0, 1, 2, \qquad \rho = \frac{\lambda}{2\mu}.$$

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Problem 4 - A closed network is composed of two queues A and B with markovian single servers at rate μ. Users leaving queue A enter queue B, and those leaving queue B enters queue A with probability p, otherwise enter again queue B. Assuming there are M users in the network, find

a) the joint and the marginals distribution of the queues;
b) the average time spent at each of the queues.
c) Assuming now that all users enter the network from outside at queue A, enters B and return to A as above, leaving the network after the third departure from A, find the birth rate λ so that the average number in the network is M (Notice that this point is completely independent from the answers in a) and b)).

Solution a) According to the general solution, the joint distribution is

$$\pi(n_1, n_2) = B\left(\frac{\nu_A}{\mu}\right)^{n_1}\left(\frac{\nu_B}{\mu}\right)^{n_2}, \qquad n_1 + n_2 = M,$$

with

$$\nu_A = \frac{p}{1+p}, \qquad \nu_A = \frac{1}{1+p}.$$

For the marginals we have

$$\pi_A(n_1) = B\left(\frac{\nu_A}{\mu}\right)^{n_1}\left(\frac{\nu_B}{\mu}\right)^{M-n_1} = \frac{B}{(\mu(1+p))^M}\,p^{n_1} = \frac{1-p}{1-p^{M+1}}p^{n_1},$$

since

$$1 = \frac{B}{(\mu(1+p))^M}\sum_{n_1=1}^{M} p^{n_1} = \frac{B}{(\mu(1+p))^M}\frac{1-p^{M+1}}{1-p}.$$

$$\pi_B(n_2) = B\left(\frac{\nu_A}{\mu}\right)^{M-n_2}\left(\frac{\nu_B}{\mu}\right)^{n_2} = \frac{B}{(\mu(1+p))^M}\, p^{M-n_2} = \frac{1-p}{1-p^{M+1}} p^{M-n_2}.$$

Since only one variable is required, the distribution can also be easily derived from the simple state diagram of the system, which is a Birth and Death process.

b) We need flow rates, that can be derived as

$$\lambda_A = \sum_{n_1=1}^{M} \mu\pi_A(n_1) = \mu(1-\pi_A(0)),$$

$$\lambda_B = \sum_{n_2=1}^{M} \mu\pi_B(n_2) = \mu(1-\pi_B(0)),$$

and

$$E[N_A] = \sum_{n_1=1}^{M} n_1\pi_A(n_1) = \frac{p-(M+1-Mp)p^{M+1}}{(1-p)(1-p^{M+1})},$$

$$E[N_B] = M - E[N_1],$$

which yields

$$E[V_A] = \frac{1}{\mu}\,\frac{p-(M+1-Mp)p^{M+1}}{(1-p)(p-p^{M+1})}.$$

c) Here we have an open network of M/M/1 queues, so that

$$E[N_A] = \frac{\rho_A}{1-\rho_A}, \qquad E[N_B] = \frac{\rho_B}{1-\rho_B}$$

with

$$\lambda_A = 3\lambda, \qquad \lambda_B = \frac{2\lambda}{p}.$$

Then, we must have

$$M = \frac{\rho_A}{1-\rho_A} + \frac{\rho_B}{1-\rho_B} = \frac{3\lambda}{\mu-3\lambda} + \frac{2\lambda}{p\mu-2\lambda}$$