Traffic Theory Solutions - 3 July 2013

Problem 1 -Three cards are distributed in positions 0, 1, 2. At each step, the pair of positions 0 and 1 is selected with probability 1/3 and the cards exchange positions. Otherwise, the pair 1 and 2 is selected and the cards exchanged. If an Ace is placed in position 0, what is the distribution of its position after two steps? and after an infinite number of steps? What if, at each step, all pairs are selected with the same probability and the cards are exchanged?

Solution The position of the Ace is a homogeneous, irreducible Markov chain whose transition matrix is

$$P = \begin{vmatrix} 2/3 & 1/3 & 0 \\ 1/3 & 0 & 2/3 \\ 0 & 2/3 & 1/3 \end{vmatrix}$$

We have

$$\mathbf{\Pi}(0) = [1, 0, 0]$$

$$\mathbf{\Pi}(2) = \mathbf{\Pi}(0)\mathbf{P}^2 = [5/9, 2/9, 2/9]$$

Asymptotically we must solve the balance equations to get

$$\mathbf{\Pi} = [1/3, 1/3, 1/3]$$

The latter can be also inferred by the fact that the matrix has columns that sum to one.

When all pairs are selected with the same probability the matrix becomes

$$P = \begin{vmatrix} 1/3 & 1/3 & 1/3 \\ 1/3 & 1/3 & 1/3 \\ 1/3 & 1/3 & 1/3 \end{vmatrix}$$

From this we see that at each step we have distribution again equal to $[1/3, 1/3, 1/3]$.

Problem 2 -An internet session generates a traffic flows that is composed of busy periods whose length is exponential distributed with rate μ. Busy periods are separated by idle periods whose length is exponential distributed with rate λ. Find

a) the traffic on the channel, the frequency of activation instants (when the busy period starts), and the average number of activation instants in a period of length τ;

b) assuming stationarity, the probability that the channel is active (busy) at a given time t, and, conditional to this, the probability that the same busy period is still active at $t + \tau$;

c) if the channel is busy at time t the probability that it is still busy (maybe with a different busy period) at time $t + \tau$.

d) the traffic distribution when n equal and independent sources are considered.

Solution

a) The state of the channel is a binary chain with parameters λ and μ. By the definitions we have

$$S = \frac{\lambda}{\lambda + \mu}$$

$$\lambda_a = \frac{1}{m_X + m_Y} = \frac{\lambda\mu}{\lambda+\mu}$$

$$\lambda_a\tau = \frac{\lambda\mu}{\lambda+\mu}\tau$$

b)

$$P(Z=1) = S = \frac{\lambda}{\lambda+\mu}$$

$$P(Z(d)=1, t<d\le t+\tau | Z(t)=1) = P(X\ge\tau) = e^{-\mu\tau},$$

where the latter comes from the fact that the duration of the busy period, X is negative exponential, memoryless.

c)

$$P(Z(t+\tau)=1|Z(t)=1) = \pi_1(t+\tau) = \frac{\lambda}{\lambda+\mu}[1-e^{-(\lambda+\mu)\tau}] + e^{-(\lambda+\mu)\tau}$$

This comes from the transient solution explained in Example 1.77 of the class notes.

d) Since the component traffics are equal and independent the traffic is binomially distributed.

$$P(Z=k) = \binom{n}{k}\left(\frac{\lambda}{\lambda+\mu}\right)^k\left(\frac{\mu}{\lambda+\mu}\right)^{n-k} \qquad (0\le k\le n)$$

Problem 3 - Jobs are served by two processors. When only one job is present both processors serve this one job, thus doubling the processing power. When two job are present, immediately each one gets one processor. When more the two jobs are present, only two jobs at a time are served, one job per processor, and the others wait in the queue. Assuming that arrivals are Poisson at rate λ, and their service time is negative exponential with rate μ when served by a single processor, doubled with two processors, find

a) the asymptotic distribution of the jobs in the system;
b) the average waiting time in the queue and in the service respectively and compare with the same figures of an M/M/2 with parameters λ, μ;
c) the asymptotic distribution when both processors are given to the same job only when the job is alone in the system when starting service. In addition neither of the two processors assigned in this way are released before the service ends (the occupation process is not markovian but can be made markovian by....).

Solution

a)

When there is only one job in the system, it is served by two servers and the service rate is, therefore, 2μ. When there are two or more jobs, each one is served by a one server, each one with service rate μ, and, therefore, the state finishing rate is again 2μ. This show that the state diagram is exactly the one of an M/M/1 with descending rate 2μ. The distribution is then

$$\pi_i = (1-\rho)\rho^i, \qquad \rho = \lambda/(2\mu), \qquad i\ge 0.$$

The service time can not be evaluated directly, since the same job can be served at any instant by different number of servers. We use the Little's result by evaluating the average number in the service as

$$E[N_s] = \pi_1 + 2\sum_{i=2}^{\infty} \pi_i = (1-\rho)\left(\rho + 2\sum_{i=2}^{\infty} \rho^i =\right) = (1-\rho)\left(\rho + 2\left(\frac{1}{1-\rho} - 1 - \rho\right)\right) =$$

$$= (1-\rho)\rho + 2 - 2(1-\rho^2) = \rho + \rho^2$$

Notice that for $\rho = 0$ the average is zero, while for $\rho = 1$ the average is two, as it must be. The average service time is then,

$$E[T_s] = \frac{1}{2\mu}(1+\rho),$$

always comprised between $1/(2\mu)$ e $1/\mu$.
The average time in the queue is the difference between the time in the system and the time in the service, where the former is equal to the one in M/M/1, since the average number is the same. Therefore we have

$$E[W] = \frac{1}{2\mu(1-\rho)} - \frac{1}{2\mu}(1+\rho) = \frac{\rho^2}{2\mu(1-\rho)}.$$

In an M/M/2 the average service time is $1/\mu$, while the distribution can be derived directly as shown in Example 3.41. We have

$$\pi_i = \pi_0 2\rho^i \qquad i \geq 1, \qquad \rho = \lambda/(2\mu).$$

$$\pi_0 = \frac{1-\rho}{1+\rho}$$

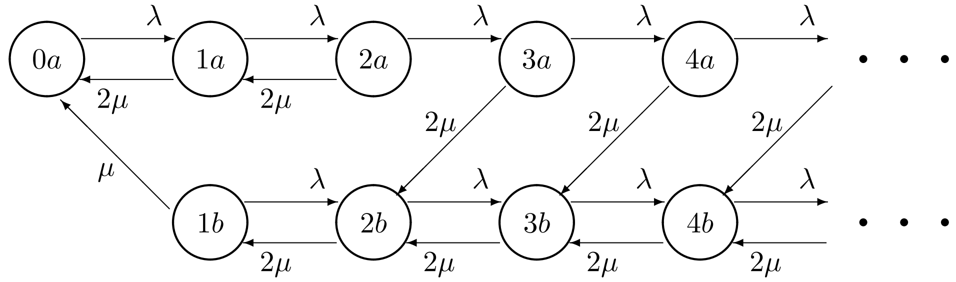
The former shows that the average number in the system is twice that of M/M/1, provided that the correct π_0 is used. Hence

$$E[N] = \frac{\rho}{1-\rho}\frac{2}{1+\rho} = \frac{2\rho}{1-\rho^2}$$

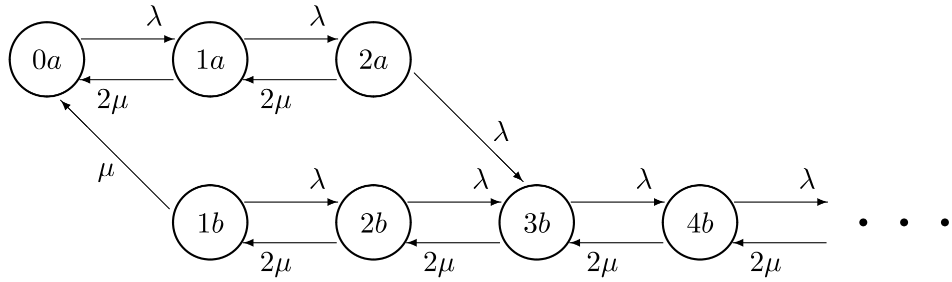
$$E[V] = \frac{1}{\mu(1-\rho^2)}$$

$$E[W] = \frac{\rho^2}{\mu(1-\rho^2)}$$

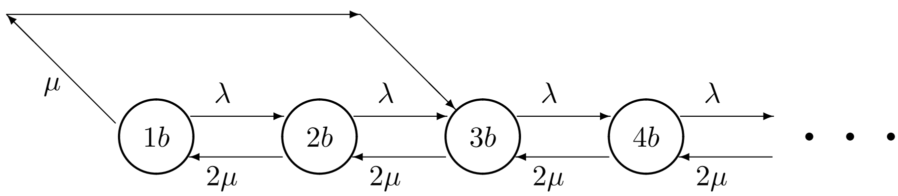
c) the state diagram is



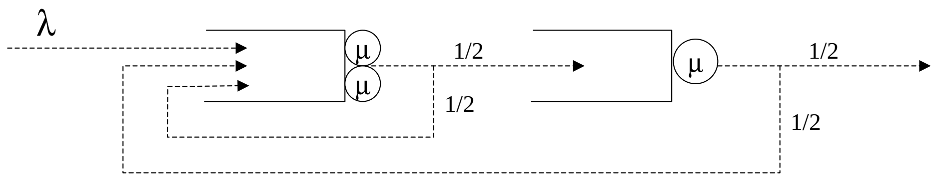
where states a are those where the two servers are dedicated to a single job, and states b otherwise. However, states ia with $i \geq 3$ have the same input and output rates as states ib; therefore we need not distinguish a and b for such states. This yields the following diagram



The solution of such a chain can be simplified by solving first the flow of the following chain, which comes from the horizontal cut of the chain above



Problem 4 -The open network in the figure below is composed by two systems, 1 nd 2, the first with two servers and the latter with one server, all with service rate μ. The input flow is Poisson, with the routing probabilities shown in the figure. Find



a) the global (between birth and death) average time spent by a user in the system with one server.
b) the average number of users in the network.
c) the same as in a) and b) assuming that the input flow is dropped, so that the network becomes closed, and that there are three users in the network.

Solution
a) We easily find $\lambda_1 = 4\lambda$ and $\lambda_2 = 2\lambda$. The average number in system 2 is $\rho_2/(1-\rho_2)$ with $\rho_2 = 2\lambda/(\mu)$, and the time requested is attained dividing by $\lambda_0 = \lambda$. Therefore

$$E[D_2] = \frac{2}{\mu(1-\rho_2)}, \qquad \rho_2 = 2\lambda/\mu.$$

The average number in the network is provided by the sum of the average in the two systems. The average number in system 1 has been evaluate in $3b$:

$$E[N_1] = \frac{2\rho_1}{1-\rho_1^2}, \qquad \rho_1 = 4\lambda/(2\mu).$$

$$E[N] = E[N_1] + E[N_2] = \frac{2\rho_1}{1-\rho_1^2} + \frac{\rho_2}{1-\rho_2}.$$