Traffic Theory - 8 february 2017

Problem 1 - A random walk $X(n)$ with parameters $p = 0.25$ and $q = 0.5$, and reflecting barriers in 0 and 3, is used to build the binary process $Y(n)$, such that we have $Y(n) = 0$ whenever $X(n) = 0$ or $X(n) = 1$, and $Y(n) = 1$ otherwise. Find

a) the asymptotic distribution $\Pi(n)$ of $Y(n)$;

b) the asymptotic probabilities $P(Y(n+1) = 1|Y(n) = 0)$ and $P(Y(n+1) = 1|Y(n) = 1)$;

c) the asymptotic probability $P(Y(n+2) = 1|Y(n+1) = 0, Y(n) = 0)$.

Solution

a) the asymptotic distribution of the walk is derived from the general solution

$$\pi_i = \frac{1 - p/q}{1 - (p/q)^4}(p/q)^i, \qquad 0 \le i \le 4,$$

and provides

$$\mathbf{\Pi} = [\frac{8}{15} \;\; \frac{4}{15} \;\; \frac{2}{15} \;\; \frac{1}{15}].$$

The distribution of $Y(n)$ is then

$$\mathbf{\Pi} = [\frac{12}{15} \;\; \frac{3}{15}].$$

b) Formally we have

$$P(Y(n+1) = 1|Y(n) = 0) = \frac{P(Y(n+1) = 1, Y(n) = 0)}{P(Y(n) = 0)} =$$

$$= \frac{P(\{X(n+1) = 2\} + \{X(n+1) = 3\}, \{X(n) = 1\} + \{X(n) = 0\})}{P(Y(n) = 0)} =$$

$$= \frac{P(X(n+1) = 2, X(n) = 1)}{P(Y(n) = 0)} = \frac{4}{15}\frac{1}{4}\frac{15}{12} = \frac{1}{12}$$

The last row can be written immediately, because $\{X(n+1) = 2, X(n) = 1\}$ represents the only way we can have $\{Y(n+1) = 1, Y(n) = 0\}$.

$$P(Y(n+1) = 1|Y(n) = 1) = \frac{P(Y(n+1) = 1, Y(n) = 1)}{P(Y(n) = 1)} =$$

$$= \frac{P(\{X(n+1) = 2\} + \{X(n+1) = 3\}, \{X(n) = 2\} + \{X(n) = 3\})}{P(Y(n) = 1)} =$$

$$= \frac{P(\{X(n{+}1){=}2, X(n){=}2\}{+}\{X(n{+}1){=}3, X(n){=}2\}{+}\{X(n{+}1){=}2, X(n){=}3\}{+}\{X(n{+}1){=}3, X(n){=}3\})}{P(Y(n){=}1)} =$$

$$\frac{15}{3}\left(\frac{2}{15}\frac{1}{4} + \frac{2}{15}\frac{1}{4} + \frac{1}{15}\frac{1}{2} + \frac{1}{15}\frac{1}{4}\right) = \frac{7}{12}.$$

Problem 2 - A continuous-time Markov Chain $X(t)$ presents the following transition-rate matrix:

$$\begin{vmatrix} -10 & 4 & 6 \\ 5 & -10 & 5 \\ 3 & 2 & -5 \end{vmatrix}.$$

Find

a) the asymptotic distribution Π, and the average sojourn time in state 0;

b) given the transition chain $Y(n)$ of $X(t)$, find its transition matrix and its asymptotic distribution;

c) find the probability that, starting in state 0, with the second transition the chain reaches state zero.

Solution
a)

$$[\frac{5}{19}\ \ \frac{4}{19}\ \ \frac{10}{19}]$$

The average sojourn time in state 0 is 1/10.

b)

$$[\frac{5}{14}\ \ \frac{4}{14}\ \ \frac{4}{14}]$$

c) 0.56.

Problem 3- The M/M/1 system (λ, μ) is modified in the following way. At the arrival of a new user the service rate is set to 2μ, while at the end of each service at rate 2μ the service rate is changed to μ and all subsequent users are served with this rate until a new user arrives, at which time the service rate is set, again to 2μ, and the procedure repeats.

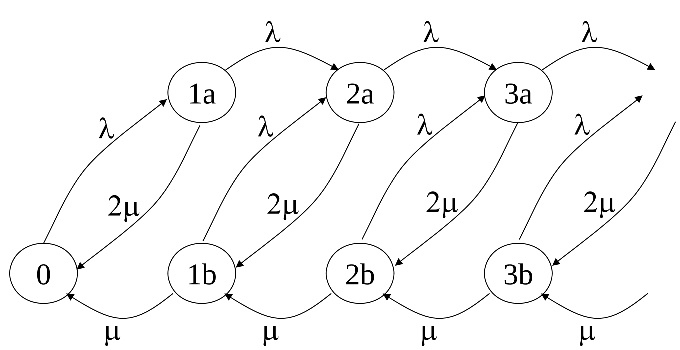
a) Draw the Markov chain that describes the number of users in the system ;

b) find the asymptotic distribution, the conditions for its existence, and the average number in the sistem;

c) find the the average number in the service (traffic) and the average service time.

Solution
a) The state diagram is shown in the figure

Solution a)



b) From a diagonal cut we have

$$\pi_{ia}\lambda = \pi_{ib}\mu, \qquad i \geq 1,$$

and using this into the balance equation to a nodes we have

$$\pi_{i+1,a}(\lambda + 2\mu) = \pi_{i,a}\lambda + \pi_{i,b}\lambda = \pi_{i,a}\lambda + \pi_{i,a}\frac{\lambda^2}{\mu}, \qquad i \geq 1,$$

which provides

$$\pi_{i+1,a} = \pi_{ia}\frac{\lambda(\lambda+\mu)}{\mu(\lambda+2\mu)}, \qquad i \geq 1.$$

$$\pi_{1,a} = \pi_0\frac{\lambda}{(\lambda+2\mu)}.$$

Denoted

$$\eta = \frac{\lambda(\lambda+\mu)}{\mu(\lambda+2\mu)},$$

we can write

$$\pi_{ia} = \pi_0\frac{\mu}{\lambda+\mu}\eta^i \quad i \geq 1,$$

$$\pi_{ib} = \pi_0\frac{\lambda}{\lambda+\mu}\eta^i \quad i \geq 1,$$

and also

$$\pi_i = \pi_{ia} + \pi_{ib} = \pi_0\eta^i \quad i \geq 0.$$

and finally

$$\pi_0 = (1-\eta),$$

where the convergence of the solution exists for $\eta < 1$ or $\lambda/\mu < \sqrt{2}$.

The average number in the system is

$$E[N] = \sum_{i=1}^{\infty} i\,\pi_i = \frac{\eta}{1-\eta}.$$

c) The average number in the service, coincides with the probability the server is busy, $1-\pi_0 = \eta$, and the average time in service is

$$(1-\pi_0)/\lambda = \frac{\eta}{\lambda} = \frac{1}{\mu}\frac{(\lambda+\mu)}{(\lambda+2\mu)},$$

shorter than $1/\mu$.

Problem 4 - Users, arriving according to Poisson with rate λ, require two independent services, each of them with a time exponentially distributed with rate μ. The service can be performed using three different systems. In system A the user enters two cascaded queues, each of them providing one of the two required services (network of queues). In System B the user enters only one queue and, once in the server, gets the two services in sequence (average $2/\mu$). It leaves the server only at the end of the two services. In order to get the same server load factor in the two cases, in system B the queues are doubled and each user chooses its queue at random with probability 0.5, gets the two services and leaves. System C is similar to system B but, after ending the first service, the users exits the service and reaches the end of the queue. After ending the second service, it leaves (network of queues). Find the whole average time the users spend in the system in the three cases (In system B the whole time spent in the server is not exponential, therefore the variation coefficient c is needed).

Solution

a) The two cascaded queues represent a Jackson Network, and the average time is

$$E[D] = \frac{2}{\mu(1-\rho)}, \qquad \rho = \lambda/\mu.$$

b) The system is an $M/E_2/1$, i.e., the service time is an Erlang-2. Hence we need the Pollaczeck-Khinchine formula

$$E[V] = \frac{\rho}{1-\rho}E[Z] + m_x = \frac{\rho}{1-\rho}\frac{m_x}{2}(1+c^2) + m_x,$$

where, in our case, $c^2 = 1/2$, $\rho = \lambda/\mu$, and $m_x = 2/\mu$. Then

$$E[V] = \frac{\rho}{1-\rho}\frac{1}{\mu}\frac{3}{2} + \frac{2}{\mu} = \frac{4-\rho}{2\mu(1-\rho)}$$

a smaller value than in a).

c) The network is not Jackson's, however we have seen that the solution is exactly the same when the flow rates are the same. The load of the server is still the same, $\rho = \lambda/\mu$, and the time for two passages is again as in a).