Traffic Theory - 4 december 2014

Problem 1 - A Markov Chain presents the following one-step transition matrix at all times n.

$$\mathbf{P} = \begin{vmatrix} 0 & 0.6 & 0.4 \\ 0.4 & 0.6 & 0 \\ 0.6 & 0 & 0.4 \end{vmatrix}$$

If the chain starts in $X = 0$ at time $n = 0$, find

a) the first order distribution $\mathbf{\Pi}(n)$, and the transition matrix $\mathbf{P}(0, n)$ at time $n = 2$;

b) the first order distribution $\mathbf{\Pi}(n)$, and the transition matrix $\mathbf{P}(0, n)$ at time $n = \infty$;

c) the second order distribution $\mathbf{\Pi}(n - 1, n)$ (all terms) at times $n = 2$ and $n = \infty$;

d) the vector of the initial distribution $\mathbf{\Pi}(0)$ that makes the chain stationary at all $n \geq 0$.

Solution

a)

$$\mathbf{P}(0,2) = \begin{vmatrix} 0.48 & 0.36 & 0.16 \\ 0.24 & 0.6 & 0.16 \\ 0.24 & 0.36 & 0.4 \end{vmatrix} \qquad \mathbf{\Pi}(n) = [0.48 \;\; 0.36 \;\; 0.16]$$

b)

$$\mathbf{P}(0,\infty) = \begin{vmatrix} 6/19 & 9/19 & 4/19 \\ 6/19 & 9/19 & 4/19 \\ 6/19 & 9/19 & 4/19 \end{vmatrix} \qquad \mathbf{\Pi}(\infty) = [6/19 \;\; 9/19 \;\; 4/19]$$

c) the elements of $\mathbf{\Pi}(n - 1, n)$ at time $n = 2$

$$\mathbf{\Pi}(1,2) = \begin{vmatrix} 0 & 0 & 0 \\ 0.24 & 0.36 & 0 \\ 0.24 & 0 & 0.16 \end{vmatrix} \qquad \mathbf{\Pi}(\infty,\infty+1) = \begin{vmatrix} 0 & 36/190 & 24/190 \\ 36/190 & 54/190 & 0 \\ 24/190 & 0 & 16/190 \end{vmatrix}$$

d)

$$\mathbf{\Pi}(0) = [6/19 \;\; 9/19 \;\; 4/19]$$

Problem 2 - A continuous time random walk $X(t)$, from 0 to infinity, is such that each time the walk returns to state 0 the upward transition rate of the entire walk is selected between the two values λ and 2λ in a uniform way, while the downward rate is always μ. Find

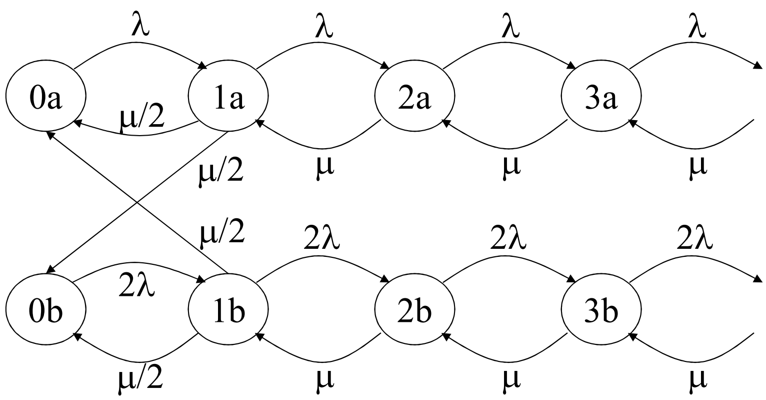
a) the asymptotic distribution of $X(t)$ and the conditions on λ under which it exists;

b) the the asymptotic distribution of chain $Y(t)$, which is equal to $X(t)$ but now the selection between λ and 2λ is deterministically alternated;

c) referring to an asymptotic time t in case a), write the pdf of the time to the next upward transition in the two cases when you know, and when you do not know, which upward rate is used (the experiment we refer is the one that selects only samples that present next an upward transition).

Solution

a) The state diagram is shown below.



We easily recognize that

$$\pi_{ia} = \pi_{0a} \left(\frac{\lambda}{\mu}\right)^i, \quad i \geq 0, \qquad\qquad \pi_{ib} = \pi_{0b} \left(\frac{2\lambda}{\mu}\right)^i, \quad i \geq 0,$$

which both exist for $2\lambda < \mu$. Then we have, for example

$$\pi_{0a}\lambda = (\pi_{1a} + \pi_{1b})(\mu/2)$$

$$\pi_{0a}\lambda = \left(\pi_{0a}\frac{\lambda}{\mu} + \pi_{0b}\frac{2\lambda}{\mu}\right)(\mu/2)$$

$$\pi_{0a}\lambda = \pi_{0a}\lambda/2 + \pi_{0b}\lambda$$

which provides

$$\pi_{0a} = 2\pi_{0b}$$

Taking the summation over all states we have

$$2\pi_{0b}\frac{1}{1-\lambda/\mu} + \pi_{0b}\frac{1}{1-2\lambda/\mu} = 1$$

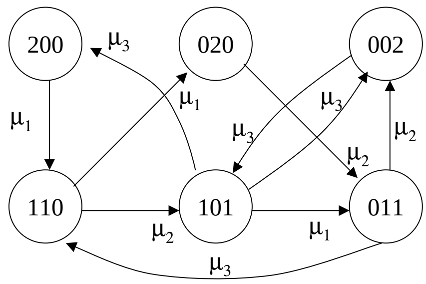
which provides π_{0b}.

b) The state diagram is as the old one but the rate $1a \to 0b$ is now μ while the rate $1a \to 0a$ is zero. Similarly on the other side. The substitution proves that the solution is the same.

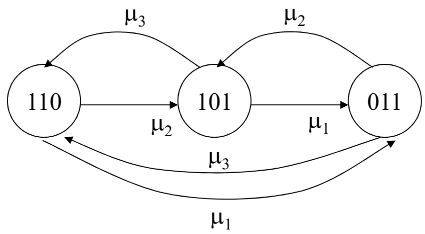
c) If we know the upward rate, the time to the next upward transition is negative exponential with known rate, either λ or 2λ. Otherwise we have

$$\pi_a \lambda e^{-\lambda x} + \pi_b \lambda e^{-2\lambda x}.$$

Problem 3 -Determine the asymptotic distribution of the continuous-time MC whose state diagram is shown in the figure below.



Solution
By reducing the flows that have only one path we get the following state diagram



We have

$$\pi_{011}(\mu_2 + \mu_3) = \pi_{101}\mu_1 + \pi_{110}\mu_1$$

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and solving with respect π_{110}:

$$\pi_{011}(\mu_2 + \mu_3) = \pi_{011}\frac{\mu_1\mu_2}{\mu_1 + \mu_3} + \pi_{110}\frac{\mu_1\mu_2}{\mu_1 + \mu_3} + \pi_{110}\mu_1$$

$$\pi_{011}(\mu_1\mu_3 + \mu_2\mu_3 + \mu_3^2) = \pi_{110}(\mu_1\mu_2 + \mu_1^2 + \mu_1\mu_3)$$

$$\pi_{011} = \pi_{110}\frac{\mu_1}{\mu_3} \qquad \pi_{101} = \pi_{110}\frac{\mu_2}{\mu_3}$$

and finally, from the original diagram:

$$\pi_{200} = \pi_{110}\frac{\mu_2}{\mu_1} \qquad \pi_{020} = \pi_{110} \qquad \pi_{002} = \pi_{110}\frac{\mu_2}{\mu_3}\frac{\mu_1 + \mu_3}{\mu_3}$$

Finally, the summation provides π_{110}.

Problem 4 A renewal arrival flow A, with interarrival time X drawn from an Erlang-2 pdf of parameter λ, is merged with a Poisson arrival flow B of parameter μ. Find

a) the distribution of the number of type B arrivals that lie within two consecutive type A arrivals;
b) the pdf of the distance of two consecutive type A arrivals knowing that between them there is exactly one type B arrival;

c) the probability that an arrival taken from the merged flow is of type A;

d) the probability that the arrival next to a type A arrival is still of type A (observe that in between we must have 0....);

e) the probability that the arrival next to t, asymptotically, is of type A (assume $\mu = \lambda$).

Solution

a) By the Total Probability Theorem

$$P(N_B = k) = \int_0^\infty \frac{(\mu x)^k}{k!} e^{-\mu x} \lambda^2 x e^{-\lambda x} dx = \frac{(k+1)\lambda^2 \mu^k}{(\lambda+\mu)^{k+2}} \int_0^\infty \frac{((\lambda+\mu)x)^{k+1}}{(k+1)!} (\lambda+\mu) e^{-(\lambda+\mu)x} dx.$$

The function under the integral sign is a pdf and the integral equals one, yielding

$$P(N_B = k) = (k+1) \left(\frac{\lambda}{\mu}\right)^2 \left(\frac{\mu}{(\lambda+\mu)}\right)^{k+2}.$$

b) Using Bayes' Theorem:

$$f_X(x/N_B(x) = 1) = \frac{P(N_B(X) = 1/X = x) f_X(x)}{P(N_B(X) = 1)} = \frac{\mu x e^{-\mu x} \lambda^2 x e^{-\lambda x}}{2\left(\frac{\lambda}{\mu}\right)^2 \left(\frac{\mu}{(\lambda+\mu)}\right)^3} =$$

$$= \frac{(\lambda+\mu)^3}{2} x^2 e^{-(\lambda+\mu)x},$$

Erlang-3 with parameter $\lambda + \mu$.

c) The rate of the merged flow is $\lambda/2 + \mu$. The probability that an arrival taken from the merged flow is of type A is

$$P_A = \frac{\lambda/2}{\lambda/2 + \mu} = \frac{\lambda}{\lambda + 2\mu}.$$

d) Coincides with (question a)):

$$P(N_B = 0) = \left(\frac{\lambda}{\mu}\right)^2 \left(\frac{\mu}{(\lambda+\mu)}\right)^2 = \left(\frac{\lambda}{(\lambda+\mu)}\right)^2.$$

Note that $\lambda/(\lambda+\mu)$ is the probability that ....