Traffic Theory - 1 december 2016

Problem 1 - A homogeneous Markov Chain $X(n)$ presents the following one-step transition matrix:

$$\begin{vmatrix} 0.6 & 0.4 & 0 \\ 0 & 0.5 & 0.5 \\ 0.5 & 0 & 0.5 \end{vmatrix}$$

Given the initial distribution [0.5 0.25 0.25], find

a) $\mathbf{P}(0,2)$ and the asymptotic transition matrix $\lim_{n\to\infty} \mathbf{P}(0,n) = \mathbf{P}^*$;

From the above Markov chain we define a binary chain $Y(n)$ with states a and b, where a is the union of states 0 and 1, and b coincides with state 2. Again given the same initial distribution [0.5 0.25 0.25], for this new chain find

b) the asymptotic distribution $\mathbf{\Pi}'$ of $Y(n)$;

c) the transition probabilities $P(Y(1) = b|Y(0) = a)$ and $P(Y(1) = a|Y(0) = b)$ (notice: you must refer to chain $X(n)$; start with the definition of conditional....);

d) the transition probabilities $P(Y(2) = b|Y(1) = a, Y(0) = a)$ and $P(Y(2) = b|Y(1) = a)$;

e) the transition probabilities $\lim_{n\to\infty} P(Y(n+1) = b|Y(n) = a)$.

Solution

a)

$$\mathbf{P}(0,2) = \begin{vmatrix} 0.36 & 0.44 & 0.20 \\ 0.25 & 0.25 & 0.5 \\ 0.55 & 0.2 & 0.25 \end{vmatrix} \qquad \mathbf{P}^* = \begin{vmatrix} 5/13 & 4/13 & 4/13 \\ 5/13 & 4/13 & 4/13 \\ 5/13 & 4/13 & 4/13 \end{vmatrix}$$

b)

$$\mathbf{\Pi}' = [\pi_0 + \pi_1 \quad \pi_2] = [9/13 \quad 4/13].$$

c)

$$P(Y(1) = b|Y(0) = a) = \frac{P(Y(1) = b; Y(0) = a)}{P(Y(0) = a)} =$$

$$= \frac{P(\{X(1) = 2; X(0) = 0\} + \{X(1) = 2; X(0) = 1\})}{P(X(0) = 0) + P(\{X(0) = 1)} =$$

$$= \frac{P(X(1) = 2; X(0) = 0) + P(X(1) = 2; X(0) = 1)}{0.75} =$$

$$= \frac{P(X(1) = 2|X(0) = 0)P(X(0) = 0) + P(X(1) = 2|X(0) = 1)P(X(0) = 1)}{0.75} =$$

$$= \frac{P(X(1) = 2|X(0) = 1)P(X(0) = 1)}{0.75} = \frac{0.125}{0.75} = \frac{1}{6}$$

and

$$P(Y(1) = a|Y(0) = b) = P(\{X(1) = 0\} + \{X(1) = 1\}|X(0) = 2)$$

$$= P(X(1) = 0|X(0) = 2) + P(X(1) = 1|X(0) = 2) = 0.5$$

Problem 2 -At a bus stop busses arrive according to a Poisson process of rate μ. Passengers arrive at the bus stop according to a Poisson process with rate λ.

a) Find the distribution of the number of passengers when the bus arrives;

b) as in a) when busses arrive according to a renewal process with distance pdf equal to an Erlang-2 ($\mu^2 x e^{-\mu x}$).

Solution

a) This is exactly the question solved in Problems 1.1 and 1.2.

b) Following the argument in a) we have

$$p_k = \int_0^\infty \frac{(\lambda x)^k}{k!} e^{-\lambda x} \mu^2 x e^{-\mu x} dx$$

The integral is solved by parts, (is the integral of a Gamma function) or noting that it can be written as

$$p_k = (k+1)\left(\frac{\mu}{\lambda+\mu}\right)^2 \left(\frac{\lambda}{\lambda+\mu}\right)^k \int_0^\infty \frac{((\lambda+\mu)x)^{k+1}}{(k+1)!}(\lambda+\mu)e^{-(\lambda+\mu)x}dx = (k+1)\frac{\lambda^k \mu^2}{(\lambda+\mu)^{k+2}}$$

since the function within the integral is a pdf, namely the Erlang k +2, whose area is 1.

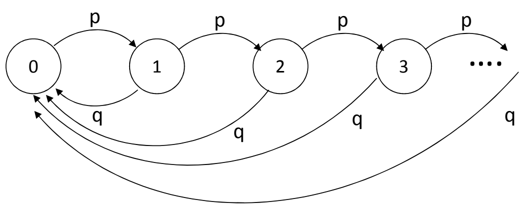
Problem 3- At each step one user arrives with probability p (Bernoulli arrivals) and enters a room. The room empties completely at each time with probability q.

a) Draw the Markov chain that describes the number of users in the room;

b) find the asymptotic distribution, the conditions for its existence, and the average number in the room.

Solution

a) The state diagram is shown in the figure



b) Taking the balance across state $i+1$ provides

$$\pi_{i+1}(p+q) = \pi_i p, \qquad \text{or} \qquad \pi_{i+1} = \pi_i \frac{p}{p+q},$$

which shows that the solution is the geometric distribution

$$\pi_i = \frac{q}{p+q}\left(\frac{p}{p+q}\right)^i,$$

which always exists for $q > 0$, and whose average is p/q.

Problem 4 - A random walk on the plane is given by the discrete-time process $(X(n), Y(n))$, representing the coordinates, positive or null, of the walk. The only possible movements are as follows

- $(i, j) \to (i+1, j)$ occurs with probability p, if $i, j \geq 0$;
- $(i, j) \to (i-1, j+1)$ occurs with probability r if $i \geq 1$, zero otherwise;
- $(i, j) \to (i, j-1)$ occurs with probability q if $j \geq 1$, zero otherwise.

a) Check whether the solution is of the type

$$\pi_{ij} = \pi_{00}\left(\frac{p}{r}\right)^i\left(\frac{p}{q}\right)^j.$$

and find π_{00}.

b) Check whether the solution is still as in a) when the walk has a boundary at $i = 3$ and the transitions that cross the boundary are dropped;

c) check whether the solution is still as in a) when the walk has a boundary at $i + j = 3$ and the transitions that cross the boundary are dropped.

Solution
a) Drawing the state diagram, and taking the balance of flows at state (i, j), $i, j > 0$, we get

$$\pi_{ij}(p+q+r) = \pi_{i,j+1}q + \pi_{i+1,j-1}r + \pi_{i-1,j}p.$$

Replancing the suggested π_{ij} in both sides we verify that the LHS equals the RHS. The same procedure applied to border states shows again that the LHS equals the RHS. Hence the flows are in equilibrium with the suggested solution which, then, is the solution. We also have

$$\pi_{00} = (1 - p/r)(1 - p/q).$$

b) Proceeding as above at border nodes $(3, j)$, the suggested solution does not balance the flows.

c) Proceeding as above at border nodes $(i, 3-i)$, the suggested solution does balance the flows.