Traffic Theory - 4 March 2016

Problem 1 - A discrete-time random walk is modified in this way. Upon entering a state $i > 0$ the chain remains in the state for three time units, then leaves and with probability p and $1-p$ goes to state $i+1$ and $i-1$ respectively. Leaving state 0, again after three time units, the chain reaches state 1 with probability 1.

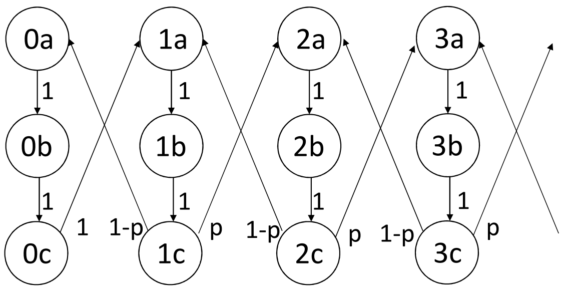
a) Draw the state diagram and find the asymptotic distribution of the chain and the stability conditions on p;

b) the same as the above if the sojourn time is changed to a negative exponential variable with rate μ;

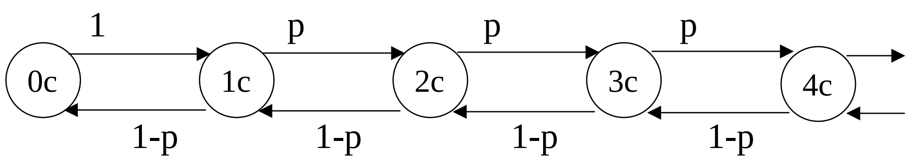
c) the same as point a) if sojourn time in state 0 is changed to one unit time.

Solution

a) The state diagram is as follows



The balance at node b shows $\pi_{ia} = \pi_{ib} = \pi_{ic}$ (The problem is similar to P.1.28 of the class notes, and the solution is the same). In this way the equations reduce to those of the chain in the figure below



whose solution is well known:

$$\pi_{ic} = \pi_{0c}\frac{p^{i-1}}{(1-p)^i}, \qquad i \geq 1,$$

and exists for $1-p > p$, or $p < 0.5$. Then we have

$$\pi_i = \pi_{ia} + \pi_{ib} + \pi_{ic} = \pi_0\frac{p^{i-1}}{(1-p)^i}, \qquad i \geq 1$$

$$\pi_0 = \frac{1-2p}{2-2p}.$$

b) The asymptotic distribution of a Markov chain only depends on the average of the the sojourn time, and does not change if this average is multiplied by a constant, which is our case, from average 3 to average 1μ.

c) The first diagram above is changed dropping states $0b$ and $0c$, all the remaining being the same. The second figure is still valid if we replace π_{0c} with π_{0a}. We have

$$\pi_{ic} = \pi_{0a}\frac{p^{i-1}}{(1-p)^i}, \qquad i \geq 1,$$

$$\pi_i = \pi_{ia} + \pi_{ib} + \pi_{ic} = 3\pi_{0a}\frac{p^{i-1}}{(1-p)^i} = 3\pi_0\frac{p^{i-1}}{(1-p)^i}, \qquad i \geq 1,$$

$$\pi_0 = \frac{1-2p}{4-2p}.$$

Problem 2 - Three sources activate in a periodical sequence, alternating Idle periods and Busy periods in the sequence $I_1, B_1, I_2, B_2, I_3, B_3$, always repeating the sequence. Idle and busy periods are negative exponential RVs of rate λ_1, λ_2 and λ_3, and μ_1, μ_2 and μ_3 respectively. Assuming stationarity:

a) find the frequency at which busy periods start (service requests); find the average traffic;
b) taken a Random Inspection Point (RIP), and assuming it lies in an idle period, find the pdf of the waiting time to service (the end of the idle period);
c) if we merge n of such independent sources, find the traffic distribution.

Solution
a) The frequency at which busy periods start is

$$\nu = \frac{3}{1/\lambda_1 + 1/\mu_1 + 1/\lambda_2 + 1/\mu_2 + 1/\lambda_3 + 1/\mu_3}.$$

As only one source is active at one time, the average traffic coincides with the stationary probability that a source is active, which is (Random Inspection Point)

$$S = \frac{1/\mu_1 + 1/\mu_2 + 1/\mu_3}{1/\lambda_1 + 1/\mu_1 + 1/\lambda_2 + 1/\mu_2 + 1/\lambda_3 + 1/\mu_3}.$$

b) Due to the memoryless property of the negative exponential, if the RIP is in period i the pdf is negative exponential with rate λ_i. By the Total Probability Theorem we have

$$f_X(x) = \sum_{i=1}^{3} \frac{1/\lambda_i}{1/\lambda_1 + 1/\lambda_2 + 1/\lambda_3}\lambda_i e^{-\lambda_i x}$$

c) Because sources are independent the traffic distribution is Binomial with parameters n and S.

Problem 3 -An access point shares the bandwidth in complete fairness among users. If bandwidth requests occur according to a Poisson flow of rate λ, and if service time is negative exponential with rate μ when a user is using the entire bandwidth, find

a) the probability that at a stationary time instant only one user is using the entire bandwidth;
b) the probability that, upon arrival, a user can use the entire bandwidth;

c) the probability that a user is using the entire bandwidth from the beginning to the end of service (hint: he must find the system empty upon arrival, and no one is allowed to arrive during service)

d) the average time in the system, i.e, from the beginning to the end of service.

Solution

It can be easily seen that, as proven in class notes, that the occupancy distribution of a complete-sharing system is exactly the one we get for an M/M/1. Therefore,

a)

$$\pi_1 = (1-\rho)\rho, \qquad \rho = \lambda/\mu.$$

b)

$$\pi_0 = 1-\rho.$$

c) This question has the same answer as question 3 (or c) in Problem 3.16 of class notes

$$p = \frac{\mu}{\lambda+\mu}.$$

d) This time is exactly the average time spent in the M/M/1 System:

$$E[V] = \frac{1}{\mu}\frac{1}{1-\rho}$$

Problem 4 - An open network of 3 markovian, single-server queues with service rate μ has the following routing matrix

$$\begin{vmatrix} 0 & 0.4 & 0 & 0.6 \\ 0 & 0.6 & 0.4 & 0 \\ 0.5 & 0.25 & 0 & 0.25 \\ 0 & 0.5 & 0.5 & 0 \end{vmatrix}$$

where 0 is the source node and 2λ the source rate of the Poisson input flow. A second Poisson flow of rate $\lambda/2$ enters the network at node 2, then goes round to nodes 3, 1 and back to 2. This for exactly two rounds and finally it leaves the network. Find

a) The bottleneck queue, i.e., the one that first becomes unstable when λ increases, and the corresponding maximum λ;

b) the average network crossing delay, and the average network crossing delay suffered by the second flow.

c) the average of the total time spent by a user of the second flow in node 3.

Solution

a)We must distinguish the rates of the two different flows. For the first and the second we have respectively

$$\begin{cases} \lambda_1^{(1)} = \dfrac{29}{4}\lambda \\[2ex] \lambda_2^{(1)} = 4\lambda \\[2ex] \lambda_3^{(1)} = \dfrac{11}{5}\lambda \end{cases}$$

$$\begin{cases} \lambda_1^{(2)} = \lambda \\[2ex] \lambda_2^{(2)} = \dfrac{3}{2}\lambda \\[2ex] \lambda_3^{(2)} = \lambda \end{cases}$$

and the global rates

$$\begin{cases} \lambda_1 = \dfrac{33}{4}\lambda \\[2ex] \lambda_2 = \dfrac{11}{2}\lambda \\[2ex] \lambda_3 = \dfrac{16}{5}\lambda \end{cases}$$

Since the servers have the same service rate, the maximum load factor is the one with the higher flow rate, i.e., queue 1. The corresponding maximum λ allowed for stability is then $\lambda < (4/33)\mu$.

The average crossing delay is

$$E[D] = \sum \frac{\lambda_i}{\lambda_{in}} E[V_i] = \sum \frac{\lambda_i}{(5/2)\lambda} \; \frac{1}{\mu - \lambda_i}$$

and the average network crossing delay suffered by the second flow

$$E[D_2] = \sum \frac{\lambda_i^{(2)}}{\lambda_{in}^{(2)}} E[V_i] = \sum \frac{\lambda_i^{(2)}}{(1/2)\lambda} \; \frac{1}{\mu - \lambda_i}$$

By the generalized Little's result we have

$$E[T_2] = \frac{\lambda_3^{(2)}}{\lambda_{in}^{(2)}} E[V_3] = \frac{\lambda_3^{(2)}}{(1/2)\lambda} \; \frac{1}{\mu - \lambda_3} = \frac{10}{5\mu - 16\lambda_3}$$