Traffic Theory - 9 September 2015

Problem 1 -A Markov Chain presents the following one-step transition matrix at all times n.

$$\begin{vmatrix} 0.4 & 0.6 & 0 \\ 0.4 & 0 & 0.6 \\ 0 & 0.4 & 0.6 \end{vmatrix}$$

a) Find the asymptotic (stationary) distribution.

b) Knowing that at time n_0 the distribution of the chain is the stationary one, find the distribution at time $n_0 + 2$.

c) Knowing that at time n_0 the chain is in state 0, find the distribution at time $n_0 + 2$.

Solution
a) $[4/19,\ 6/19,\ 9/19]$

b) If in n_0 is stationary, then it is stationary in any other $n + k$, $k > 0$. Therefore the distribution is as in a).

c) $[0.4,\ 0.24,\ 0.36]$.

Problem 2 - Arrivals occur with distances that are independent negative exponential RVs, but with rates that periodically change at each arrival according to the periodical sequence, λ, 2λ and 3λ. Arrivals occurring at the cited rates are referred to as A,B, and C types. Find, asymptotically,

a) the rate of arrival,
b) chosen an arrival, the pdf of the distance to the next arrival;
c) taken a Random Inspection Point, the probabilities that the next arrival is of type A, B, and C;
d) taken a Random Inspection Point, the pdf of the waiting time to the next arrival

Solution
a)

$$\eta = \frac{3}{m_A + m_B + m_C} = \frac{3}{1/\lambda + 1/(2\lambda) + 1/(3\lambda)} = \frac{18}{11}\lambda.$$

b)

$$f_X(x) = \frac{1}{3}\lambda e^{-\lambda x} + \frac{1}{3}2\lambda e^{-2\lambda x} + \frac{1}{3}3\lambda e^{-3\lambda x}.$$

c) The probabilities that the random point lies in an interarrival of type A,B and C are respectively

$$p_A = \frac{1/\lambda}{1/\lambda + 1/(2\lambda) + 1/(3\lambda)} = \frac{6}{11}, \qquad p_B = \frac{3}{11}, \qquad p_C = \frac{2}{11}.$$

d) By point c) we have

$$f_Y(x) = \frac{6}{11}\lambda e^{-\lambda x} + \frac{3}{11}2\lambda e^{-2\lambda x} + \frac{2}{11}3\lambda e^{-3\lambda x} = \frac{6}{11}\left(\lambda e^{-\lambda x} + \lambda e^{-2\lambda x} + \lambda e^{-3\lambda x}\right).$$

Problem 3 - Referring to an M/M/2 system,

a) write the expression for the average time spent in the system;

Now, the two servers are devoted to a single user in the following way. The served user receives in parallel two different and independent services X_1 and X_2, both RV negative exponentially distributed at rate μ. The user leaves the service (meaning that the next in queue enter the service room) when either of the parallel services finishes (i.e., the user leaves when the shorter service ends).

b) Draw the state diagram of the MC number of users in the system and write its asymptotic distribution;

c) Draw the state diagram of the MC used to represent the different case in which the user leaves service when the longer of the parallel services finishes (i.e., both parallel service must be finished).

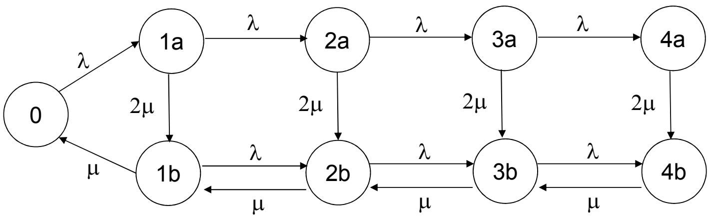
Solution

a) Besides general theory, this very example is shown in the class notes.

$$E[W] = \frac{1}{\mu}\frac{\rho^2}{(1-\rho)^2}.$$

b) The rate of service is practically the same, 2μ, as we have in an M/M/2 system. The difference is that here the rate is still 2μ also when only one user is present. Therefore, the state diagram, and distribution, are the same as with an M/M/1 with service rate equal to 2μ.

c) The state diagram is as shown below, where states a represent the case where the two servers are runn ger service is running.



Problem 4 A network of four nodes $1, 2, 3, 4$, is such that node 1 has an infinite number of servers, node 2 has two servers and the others have one server each. All servers have a negative exponential service time at rate μ. A Poisson flow A with rate λ enters the network from node 0 and is routed according the matrix given below.

$$\begin{vmatrix} 0 & 0.5 & 0 & 0.5 & 0 \\ 0 & 0 & 0.5 & 0 & 0.5 \\ 0.5 & 0 & 0 & 0.5 & 0 \\ 0 & 0.5 & 0.5 & 0 & 0 \\ 0.5 & 0.5 & 0 & 0 & 0 \end{vmatrix}$$

An additional Poisson flow B at rate $\lambda/2$ enters the network reaching nodes of the path $1, 2, 3, 4$ and then $2, 3, 4$, after which leaves the network. Find

a) the flow rate at each node;
b) the average network crossing delay of users of flow B;
c) the average network crossing delay of users that enter the network at node 1.

Solution

a) We have two routing mechanisms. We must solve for the flows of both separately, and sum the flows. We have

$$\lambda_1^{(A)} = (10/7)\lambda, \qquad \lambda_2^{(A)} = (9/7)\lambda, \qquad \lambda_3^{(A)} = (8/7)\lambda, \qquad \lambda_4^{(A)} = (5/7)\lambda,$$

$$\lambda_1^{(B)} = \lambda/2, \qquad \lambda_2^{(B)} = \lambda, \qquad \lambda_3^{(B)} = \lambda, \qquad \lambda_4^{(B)} = \lambda.$$

The total flow at the nodes is the summation:

$$\lambda_1 = (27/14)\lambda, \qquad \lambda_2 = (16/7)\lambda, \qquad \lambda_3 = (15/7)\lambda, \qquad \lambda_4 = (12/7)\lambda.$$

b) The average time spent at the nodes is respectively $E[V_1]$, equal to the service time since we have no queue, $E[V_2]$ is the time in the queue of 3a) plus the service time, and the remaining are those we have in M/M/1 systems. Hence the required crossing time is

$$D^{(B)} = \sum_{i=1}^{4} \frac{\lambda_i^{(B)}}{\lambda^{(B)}} E[V_i] = \frac{\lambda_1^{(B)}}{\lambda^{(B)}} \frac{1}{\mu} + \frac{\lambda_2^{(B)}}{\lambda^{(B)}} \left(\frac{1}{\mu} \frac{\rho^2}{(1-\rho)^2} + \frac{1}{\mu} \right) + \frac{\lambda_3^{(B)}}{\lambda^{(B)}} \frac{1}{\mu(1-\rho_3)} + \frac{\lambda_4^{(B)}}{\lambda^{(B)}} \frac{1}{\mu(1-\rho_4)}$$

with

$$\lambda^{(B)} = \sum \lambda_i^{(B)}, \qquad \rho_i = \lambda_i/\mu.$$

c) We must derive the flows caused by users that enter the network at node 1. This can be done by canceling other entering flows. We still have flows A and B, and proceed as above.