Traffic Theory - 10 January 2013 - Solutions

Problem 1 - The binary MC, X_n, ($x_n = 0, 1$), presents the following one step transition matrix

$$\mathbf{P}(n, n+1) = \begin{vmatrix} 1/4 & 3/4 \\ 1/2 & 1/2 \end{vmatrix}, \qquad \forall n.$$

If the chain starts in $X = 0$ at time $n = 0$, find

a) the first order distribution $\pi_j(n)$ (all terms) at times $n = 2$ and $n = \infty$;

b) the second order distribution $\pi_{jk}(n-1, n)$ (all terms) at times $n = 2$ and $n = \infty$

c) Find the vector of the initial conditions $[\pi_0(0), \pi_1(0)]$, that makes $\pi_j(n)$ stationary at all $n > 0$.

d) Find the transition matrices $\mathbf{P}(0, 2)$ and $\mathbf{P}(0, n)$ with $n \to \infty$ (suggestion: note that $p_{jk}(0, \infty) =$ ...).

Solution a) We have to evaluate

$$\mathbf{\Pi}(2) = \mathbf{\Pi}(0) \times \mathbf{P}^2 = [7/16; 9/16].$$

The second answer is clearly the asymptotic distribution:

$$\lim_{n\to\infty} \mathbf{\Pi}(n) = [2/5; 3/5].$$

b) Each element of the distribution can be evaluated as

$$\pi_{ij}(1, 2) = \pi_i(1)p_{ij}.$$

The result is

$$\mathbf{\Pi}(1, 2) = \begin{vmatrix} 1/16 & 3/16 \\ 6/16 & 6/16 \end{vmatrix}.$$

Again, each element of the distribution can be evaluated as

$$\lim_{n\to\infty} \pi_{i,j}(n-1, n) = \lim_{n\to\infty} \pi_i(n-1)p_{ij} = (\lim_{n\to\infty} \pi_i(n-1))p_{ij}.$$

The result is

$$\lim_{n\to\infty} \mathbf{\Pi}(n-1, n) = \begin{vmatrix} 2/20 & 6/20 \\ 6/20 & 6/20 \end{vmatrix}.$$

c) We must start with the stationary (asymptotic) distribution

$$\mathbf{\Pi}(0) = \mathbf{\Pi}(n) = [2/5; 3/5],$$

since this assure us that $\mathbf{\Pi}(1) = \mathbf{\Pi}(0)\mathbf{P} = \mathbf{\Pi}(0)$.

d)

$$\mathbf{P}(0,2) = \mathbf{P}^2 = \begin{vmatrix} 7/16 & 9/16 \\ 6/16 & 10/16 \end{vmatrix}.$$

For the second part, we must note that

$$\lim_{n\to\infty} p_{i,j}(0,n) = \pi_j,$$

since the chain is irreducible and the initial condition (state i) does not affect the asymptotic distribution. So we have

$$\lim_{n\to\infty} \mathbf{P}(0,n) = \begin{vmatrix} 2/5 & 3/5 \\ 2/5 & 3/5 \end{vmatrix}.$$

Problem 2 -A transmission channel alternates busy (transmission) and idle periods. Busy periods are independent, negative exponential RVs with rate μ, while idle periods are independent, negative exponential RVs with rate λ. A user wants to transmit on this channel during idle periods, aborting its transmission if the channel becomes busy. Find

a) the probability that the channel is idle at time t, asymptotically.

b) Assuming the channel is sensed idle at time t and that the user starts a transmission of duration T in such instant, find the probability that such transmission can be terminated without abortion.

c) repeat point b) when the transmission duration is a negative exponential RV with average T.

Solution a)

$$\frac{1/\lambda}{1/\lambda + 1/\mu} = \frac{\mu}{\lambda + \mu}$$

b) The sought probability is the probability that the idle lastst for more than T:

$$P(X > T) = e^{-\lambda t}$$

c)

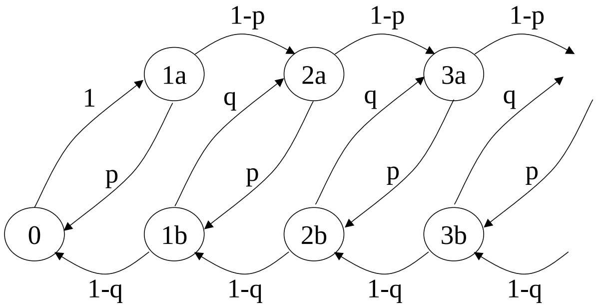
$$P(X > Y) = \frac{1}{\lambda T + 1}$$

Problem 3 - A random walk process $N(n)$ is such that it moves one step at a time upward. At each step the direction of the step is changed with probability p. Similarly, when it moves in the opposite direction it reverses the direction with probability q. Upon entering the origin, the direction is immediately reversed. Process $N(n)$ is not markovian, but we can derive its distribution defining a new process $Y(n)$ that is markovian.

a) draw the state diagram of process $Y(n)$;

b) find the asymptotic distribution of $N(n)$ together with the conditions on the parameters such that the asymptotic distribution exists.

Solution a)



b) From a diagonal cut we have

$$\pi_{ia}(1-p) = \pi_{ib}(1-q), \qquad i \geq 1,$$

and fom balance to b nodes

$$\pi_{ib} = \pi_{i+1,b}(1-q) + \pi_{i+1,a}p, \qquad i \geq 0.$$

By substituting we get

$$\pi_{i+1,b} = \pi_{ib}\frac{1-p}{1-q}, \qquad i \geq 0,$$

which yields

$$\pi_{ib} = \pi_0 \left(\frac{1-p}{1-q}\right)^i \qquad i \geq 0,$$

$$\pi_{ia} = \pi_0 \left(\frac{1-p}{1-q}\right)^{i-1} \qquad i \geq 1,$$

$$\pi_0 = \frac{p-q}{2(1-q)},$$

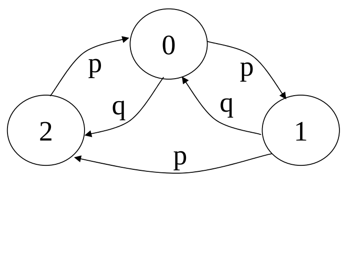
where the convergence of the solution exists for $p > q$. The distribution of the original chain is then

$$\pi_i = \pi_{ia} + \pi_{ib} \quad i \geq 1$$

Problem 4 -The discrete-time Markov chain shown in the figure is changed into a continuos-time chain making the sojourn times in states 0,1,2 all constant and equal respectively to 1,1,2 . Assuming $p = q = 0.5$, find

a) the asymptotic distribution of the discrete-time chain;

b) the asymptotic distribution of the continuous-time chain;



Solution

a) The continuous-time chain is semimarkovian, and its distribution is related to discrete time ν_i by

$$\pi_i = \frac{\nu_i E[Z_i]}{\sum\limits_k \nu_k E[Z_k]}$$

where Z_i represent the sojourn time in state i. Otherwise, we can solve the balance equations with transition rates

$$q_{jk} = (1/E[Z_j])p_{jk}$$

where p_{jk} are the transition probabilities of the discrete-time chain. we have

$$\nu_0 = 2/6, \qquad \nu_1 = 1/6, \qquad \nu_2 = 3/6.$$

$$\pi_0 = 2/9, \qquad \pi_1 = 1/9, \qquad \pi_2 = 6/9.$$