Traffic Theory - 11 february 2015

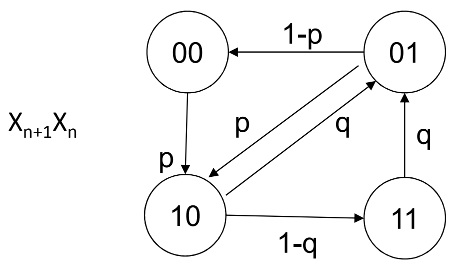
Problem 1 - Given a two-state (01) discrete-time Markov Chain $X(n),\ n = 0, 1, \ldots$ with transition probabilities p and q, derive chain $Y(n),\ n = 1, 2, \ldots$ such that $Y(n+1) = X(n) \oplus X(n+1)$, with $\oplus$ meaning binary summation. Find

a) the first order distribution $\mathbf{\Pi}(n)$ of $Y(n)$ at time $n = 2$ starting at time $n = 0$ in with $X(0) = 0$ and $Y(0) = 0$.

b) the asymptotic distribution $\mathbf{\Pi}$ of $Y(n)$.

c) the asymptotic second order distribution $\mathbf{\Pi}(n, n+1)$, i.e., $n = \infty$.

Solution - The state of chain $Y(n+1)$ (not Markov) depends on both states $X(n+1)$ and $X(n)$. Therefore we must represent as state of Y the joint states of both the latter, as shown in the following diagram.



a) Looking at the diagram we can derive the transition matrix $\mathbf{P}$ and find $\mathbf{\Pi}(2)$. Alternatively, starting from state 00 after one step we have $\pi_{00}(1) = 1 - p$, $\pi_{10}(1) = p$, and after two steps we have

$$\pi_{00}(2) = (1-p)^2, \qquad \pi_{10}(2) = p(1-p), \qquad \pi_{01}(2) = pq, \qquad \pi_{11}(2) = p(1-q),$$

and remembering the relationship $Y(n+1) = X(n) \oplus X(n+1)$ we finally have for Y

$$\pi'_0(2) = (1-p)^2 + p(1-q), \qquad \pi'_1(2) = p(1-p) + pq.$$

b) Again we must find the asymptotic distribution of chain X_{n+1}, X_n, from the diagram above. By the balance equations we have

$$\pi_{00} = \frac{(1-p)q}{p+q}, \qquad \pi_{10} = \pi_{01} = \frac{pq}{p+q}, \qquad \pi_{11}(2) = \frac{(1-q)p}{p+q},$$

and, hence,

$$\pi'_0 = \frac{(1-p)q + (1-q)p}{p+q}, \qquad \pi'_1 = \frac{2pq}{p+q}.$$

c) Again looking at the diagram we have

$$\pi'_{00} = \pi_{00,00} + \pi_{11,11} = \frac{(1-p)q}{p+q}(1-p) + \frac{(1-q)p}{p+q}(1-q)$$

$$\pi'_{01} = \pi_{00,10} + \pi_{11,01} = \frac{(1-p)q}{p+q}p + \frac{(1-q)p}{p+q}q$$

$$\pi'_{10} = \pi_{10,11} + \pi_{01,00} = \frac{pq}{p+q}(1-q) + \frac{pq}{p+q}(1-p)$$

$$\pi'_{11} = \pi_{10,01} + \pi_{01,10} = \frac{pq}{p+q}q + \frac{pq}{p+q}p$$

Problem 2 - Type A arrivals occur each time a continuous-time binary (01) MC, with transition rates λ and μ, enters state 0. Type B arrivals occur each time the same MC enters state 1. Find

a) the arrival frequency for both types and for the merge of the two (type C);

b) the pdf of the distance of two consecutive type C arrivals;

c) the pdf of the distance of two consecutive type A arrivals;

d) the asymptotic probability that the next arrival is of type A

(Hint: do not forget the relationship between arrivals and the MC).

Solution- The binary chain alternates states 0 and 1 after times negative exponentially distributed with rates μ and λ respectively. Therefore, arrivals of type A and B alternates with the same rules.

a)

$$\lambda_A = \frac{1}{1/\lambda + 1/\mu} = \frac{\lambda\mu}{\lambda+\mu}, \qquad \lambda_B = \lambda_A, \qquad \lambda_C = 2\lambda_A$$

b) We can have distance AB and distance BA with equal probability. Hence, by the Total Probability Theorem

$$f_X(x) = (1/2)\lambda e^{-\lambda x} + (1/2)\mu e^{-\mu x}.$$

c) Is the convolution of the two exponentials.

d) Is the probability that the RIP lies in the interval BA rather than in the interval AB, which is

$$P = \frac{1/\mu}{1/\lambda + 1/\mu} = \frac{\lambda}{\lambda+\mu}.$$

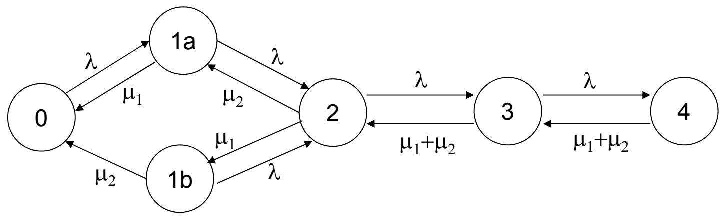
Problem 3 - Let consider an $M/M/2$ where servers have different service speeds, respectively equal to μ_1 and μ_2, $\mu_1 > \mu_2$, and Poisson arrivals have frequency λ. When both servers are available, users always choose the fastest server, otherwise they use the one that is free upon arrival if no queue exists, or the one that becomes free first when at the head of the queue. All servers can not be released before the end of the service.

a) Draw the markovian state diagram;

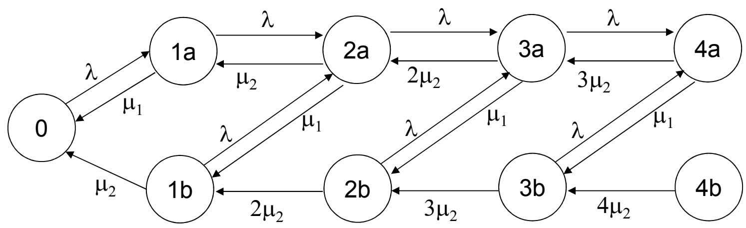
b) when $\mu_1 = 2\mu$, find the asymptotic occupancy distribution, the load of each server and the average time a user spends in service;

c) Draw the markovian state diagram when an infinite number of servers of rate μ_2 is added and operate with the rules above;

Solution - Problems a) and b) refer to a very small variation of Example 3.63 in class notes, and



c) When we have $i \geq 2$ users in the system the overall service rate can be either $\mu_1 + (i-1)\mu_2$ if the fastest server is taken, or $i\mu_2$ if the fastest server is idle. So, all states must be split and transitions



Problem 4 A closed network of two nodes is such that users have a sojourn time equal to T and $2T$ in the two nodes. Users go from node 1 to node 2 and out of node 2 return to node 2 exactly one time and then go to node 1, and the procedure repeats. If users are M in number:

a) find the average number of users at each node.

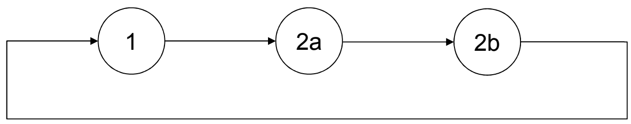
In the network with the users above other users arrive from outside the network at node 1 according to a Poisson flow of rate λ. These users are routed exactly as the users above, except that they leave the network at each node with probability p.

b) As in a).

c) If nodes are replaced by M/M/1 systems with the same service rate μ, draw the markovian state diagram of the occupancy of nodes in case a).

d) Verify whether a product form of the Birth-and-Death type balance the diagram in c.

Solution - a) the memory in returning to node 2 can be taken into account as shown below



We may use Little's result and evaluate

$$\lambda_i = \frac{\nu_i M}{\sum_{k=1}^{J} \nu_k E[V_k]}, \qquad E[N_i] = M \frac{\nu_i E[V_i]}{\sum_{k=1}^{J} \nu_k E[V_k]}.$$

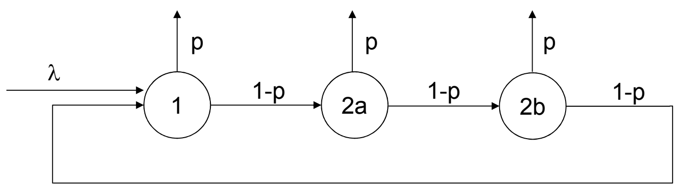
We have $\nu_i = 1/3$, and get

$$E[N_1] = M/5, \qquad E[N_{2a}] = E[N_{2b}] = 2M/5,$$

$$E[N_1] = M/5, \qquad E[N_2] = 4M/5.$$

A shortcut is observing that the time spent in node 2 is 4 times the time spent in 1, which provides 4 times users in node 2 than in node 1.

b) the added users are routed as shown below



with flow balance we get

$$\lambda_1 = \frac{\lambda}{1-(1-p)^3}, \qquad \lambda_{2a} = \frac{\lambda(1-p)}{1-(1-p)^3}, \qquad \lambda_{2b} = \frac{\lambda(1-p)^2}{1-(1-p)^3}$$

and finally

$$E[N_1] = \frac{\lambda T}{1-(1-p)^3}, \qquad E[N_2] = \frac{\lambda(1-p)2T}{1-(1-p)^3}(2-p).$$

Then these numbers must be added to those in a).

c) and d) can be answered as shown in section 5.2.5 in class notes.