Traffic Theory Solutions - 8 February 2013

Problem 1 - A discrete-time MC has the following transition matrix over states $0, 1, 2, 3, 4$:

$$\mathbf{P} = \begin{vmatrix} 0.2 & 0.8 & 0 & 0 & 0 \\ 0.4 & 0.6 & 0 & 0 & 0 \\ 0 & 0.5 & 0 & 0.5 & 0 \\ 0 & 0 & 0 & 0.8 & 0.2 \\ 0 & 0 & 0 & 0.3 & 0.7 \end{vmatrix}$$

Find

a) the distribution vector at time $n = 2$ in the two cases when it starts at time $n = 0$ in states $X = 0$ and $X = 4$ respectively;
b) the asymptotic distributions in the two cases in a)
c) verify whether the vector $[1/6; 2/6; 0; 3/10; 2/10]$ represents an asymptotic (stationary) distribution of the chain;

Solution
a) With the usual procedure we find, starting from 0 and 4 respectively:

$$\mathbf{\Pi}(2) = \mathbf{\Pi}(0)\mathbf{P}^2 = [0.36, 0.64, 0, 0, 0]$$

$$\mathbf{\Pi}(2) = \mathbf{\Pi}(4)\mathbf{P}^2 = [0, 0, 0, 0.45, 0.55]$$

b) By observing the state space of the chain we see that subsets $0, 1$ and $3, 4$ are irreducible. This means that once you start in each of them you can not leave. In other words, if you start in the former (state 0) you have a binary chain with asymptotic distribution $[1/3, 2/3]$, while if you start in the latter (state 4) you have a binary chain with asymptotic distribution $[3/5, 2/5]$. This leads to the following answer: , starting from 0 and 4 respectively we have:

$$\mathbf{\Pi} = [1/3, 2/3, 0, 0, 0]$$

$$\mathbf{\Pi} = [0, 0, 0, 3/5, 2/5]$$

c) We must verify whether the suggested distribution satisfies the balance equations

$$[1/6; 2/6; 0; 3/10; 2/10] = [1/6; 2/6; 0; 3/10; 2/10]\mathbf{P}.$$

The answer is Yes. In fact, the suggested distribution coincides with the asymptotic distribution when starting at time 0 in state 2.

Problem 2 -Arrival on the time axis occur at constant distance T.

a) find the pdf of the asymptotic waiting time to the next arrival (remember that asymptotic distributions are those seen by a RIP)
b) the same as a) when interarrival times alternate between constants T_1 and T_2.
c) the same as a) when interarrival times are drawn equal to T_1 with probability α and equal to T_2 with probability $1 - \alpha$.
d) in the three cases a), b) and c), find the probability that an arrival occurs in the next Δt.

Solution

a) The asymptotic distributions and pdf are those seen by a Random Inspection Point, which is uniform. This means that, since the interarrival is constant, the pdf of the waiting time is uniform in $[0, T]$:

$$f_Y(y) = 1/T, \qquad 0 \le y \le T.$$

b) With the same argument and the total probability theorem, we have

$$f_Y(y) = \frac{T_1}{T_1 + T_2} r_1(t) + \frac{T_2}{T_1 + T_2} r_2(t),$$

where

$$r_1(t) = \frac{1}{T_1}, \qquad 0 \le y \le T_1,$$

$$r_2(t) = \frac{1}{T_2}, \qquad 0 \le y \le T_2.$$

c) We can apply the formula of the Renewal Paradox. The pdf of the interarrival is

$$f_X(x) = \alpha\delta(x - T_1) + (1 - \alpha)\delta(x - T_2)$$

and the waiting time pdf is

$$f_Y(y) = \frac{1 - F_X(y)}{m_X} = \frac{\alpha T_1 r_1(t) + (1 - \alpha)T_2 r_2(t)}{\alpha T_1 + (1 - \alpha)T_2}$$

Otherwise, we can proceed as in b) recognizing that the probability that the RIP lies in interval 1 is

$$p_1 = \frac{\alpha T_1}{\alpha T_1 + (1 - \alpha)T_2}$$

Problem 3 - An M/M/1 queue works with this modification: whenever the the system empties and the last served user leaves, the server immediately closes the system (no user can enter the queue) and takes a vacation period distributed like a negative exponential with rate β. After the vacation, the system opens again, users enter and are served. The users arrived during the vacation period are dropped and no longer considered. Determine

a) the stationary distribution of the number of users in the system, together with the maximum arrival rate λ allowed for stability;

b) the average rate of served users and their average time in the system;

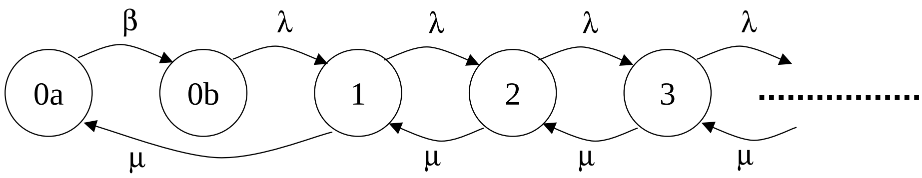
c) as in a) when the server's vacation period is a constant equal to T;

d) draw the markovian state diagram in the case when the server, when in vacation, leaves the system open, and users that arrive during this period stay in the queue waiting for the server return.

Solution

a)

The MC is shown in the figure



By the vertical cuts before states 1, 2,..., it turns out

$$\pi_i = \pi_{0b} \left(\frac{\lambda}{\mu}\right)^i, \qquad i \geq 0b, \qquad \lambda < \mu,$$

and by the balance at node $0b$:

$$\pi_{0b} = \pi_{0a}\frac{\beta}{\lambda}.$$

By imposing the normalization condition we have

$$\pi_{0b} = \frac{1 - \lambda/\mu}{1 + (\lambda/\beta)(1 - \lambda/\mu)}$$

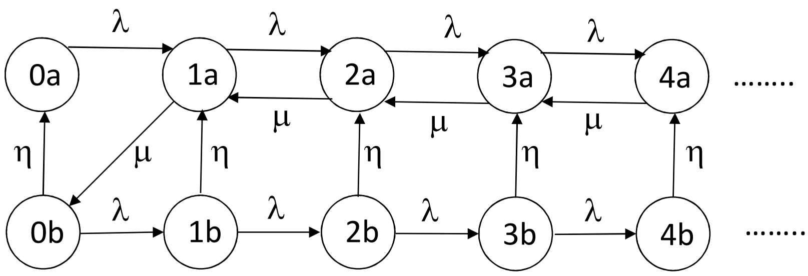
$$\pi_{0a} = \frac{(\lambda/\beta)(1 - \lambda/\mu)}{1 + (\lambda/\beta)(1 - \lambda/\mu)}.$$

b) The served rate is $\lambda(1 - \pi_{0a})$. As for the waiting time, note that this does not change with respect to the plain M/M/1. Which means

$$E[V] = \frac{1}{\mu}\frac{1}{1 - \lambda/\mu}$$

c) The

d)



Problem 4 - A closed network of 4 markovian queues, 1,2,3,4, has $M = 100$ users and the following routing matrix

$$\begin{vmatrix} 0 & 0.5 & 0 & 0.5 \\ 0 & 0.5 & 0.5 & 0 \\ 0.5 & 0.5 & 0 & 0 \\ 0.5 & 0 & 0 & 0.5 \end{vmatrix}$$

All the queues have *infinite* servers whose rates are equal to the index of the node. Find

a) the flow rates at each queue;
b) the average delay spent in the subnetwork composed of nodes $2, 3, 4$, by a user between the exit and the return to node 1;

(Note that with an infinite number of servers there is no queue and the time in the node is equal to...)

Solution a) The average time $E[V_i]$ spent at each queue equals the reverse of the service rate (no queue), and, therefore, is equal to $1, 1/2, 1/3, 1/4$, respectively. The flow rates can be evaluated by

$$\lambda_i = \frac{\nu_i M}{\sum_{k=1}^{J} \nu_k E[V_k]}$$

where ν_i is the distribution of the routing chain. In our case we get

$$\nu_1 = 1/5, \qquad \nu_2 = 2/5, \qquad \nu_3 = 1/5, \qquad \nu_3 = 1/5,$$

$$\lambda_1 = 1200/31, \qquad \lambda_2 = 2400/31, \qquad \lambda_3 = 1200/31, \qquad \lambda_3 = 1200/31,$$

b) It is exactly provided as in open networks, where instead of node 0 here we have node 1.

$$D = \sum_{i \neq 1} \frac{\lambda_i}{\lambda_i} E[V_i] = 19/12$$

Problem 5 - Let consider an M/M/m/m system, m even, ie Poisson arrivals at rate λ, m markovian servers at rate μ, no queue (for example, each server is a transmitter of a given speed).
a) Write the distribution of the number in the system.
b) Assume now that each arrival seizes two servers at a time and, when finished, with the same rate μ as before, releases both servers. Again, write the distribution of the number in the system.
c) Assume now that at rate λ_A users of the first type arrive, while at rate λ_B users of the second type (that size two servers) arrive,
c_1) draw the state diagram for the joint number of users $(n_A,\ n_B)$ in the system;
c_2) verify whether the solution is the product form of the distributions in A and B.

Solution
a) It is the case known as Blocking System and the distribution is

$$\pi_i = \frac{A^i/i!}{\sum_{k=0}^{m} A^k/k!} \qquad i = 0, 1, \ldots, m.$$

with $A = \lambda/\mu$.
b) If each user takes two servers, the distribution is the same as before with $m/2$ available resources:

$$\pi_j = \frac{A^j/j!}{\sum_{k=0}^{m/2} A^k/k!} \qquad j = 0, 1, \ldots, m/2.$$

c) If we had $m = \infty$ we would have two non interfering systems (statistically independent), each one with the Poisson distribution, and the joint distribution would be the product. To see if the product form also holds with finite m we should verify if this satisfies the balance equation at a generic node:

$$\pi_{i,j}(\lambda + j\mu + i\mu) = \pi_{i,j+1}(j+1)\mu + \pi_{i+1,j}(i+1)\mu + \pi_{i,j-1}\lambda(1-\alpha) + \pi_{i-1,j}\lambda\alpha$$

which is indeed satisfied by the product of the distributions above. We should also check the balance for border states, and these also satisfy the product form.