Traffic Theory Solutions - 28 February 2013

Problem 1 -A Markov Chain presents the following one-step transition matrix at all times n.

$$\begin{vmatrix} 0.4 & 0.6 & 0 \\ 0.4 & 0 & 0.6 \\ 0 & 0.4 & 0.6 \end{vmatrix}$$

It starts at time 0 in state 0 and at all time instants $n = 3k$, $k = 1, 2, 3, \ldots$, is forced to assume value 0. Find

a) the distribution at times 1, 2 e 3.
b) the asymptotic distribution ($n \to \infty$) at times $3n$, $3n + 1$ e $3n + 2$, $n = 0, 1, 2, \ldots$;
c) the asymptotic distribution (at time n, $n \to \infty$) (i.e., in a RIP).

Solution
a) With the usual procedure we find,

$$\mathbf{\Pi}(1) = \mathbf{\Pi}(0)\mathbf{P} = [0.4, 0.6, 0]$$

$$\mathbf{\Pi}(2) = \mathbf{\Pi}(1)\mathbf{P} = [0.4, 0.24, 0.36]$$

$$\mathbf{\Pi}(3) = \mathbf{\Pi}(0) = [1, 0, 0,]$$

b)

$$\mathbf{\Pi}(3n) = \mathbf{\Pi}(0)$$

$$\mathbf{\Pi}(3n + 1) = \mathbf{\Pi}(1)$$

$$\mathbf{\Pi}(3n + 2) = \mathbf{\Pi}(2)$$

c)

$$\mathbf{\Pi} = (1/3)(\mathbf{\Pi}(3n) + \mathbf{\Pi}(3n + 1) + \mathbf{\Pi}(3n + 2)) = [0.6, 0.28, 0.12]$$

Problem 2 - A continuous-time markovian random walk with parameters λ and μ presents two reflecting barriers at states $-N$ and N. Find

a) the distribution π_i, $-N \leq i \leq N$.
b) the distribution ν_i, $-N \leq i \leq N$, of the correspondent transition chain;
c) the average return time to N, ie, the time between two consecutive entrances into N. (Hint: this can be related to the average cycle time....)

Solution
a) With $\rho = \lambda/\mu$:

$$\pi_i = \frac{1 - \rho}{1 - \rho^{2N+1}} \rho^{i+N}, \qquad -N \leq i \leq N.$$

b) The transition chain has the following transition probabilities:

$$p_{-N,-N+1} = 1, \qquad p_{i,i+1} = \frac{\lambda}{\lambda+\mu}, \qquad p_{i,i-1} = \frac{\mu}{\lambda+\mu}, \qquad p_{N,N-1} = 1$$

$$\pi_i = \pi_0 \frac{\lambda+\mu}{\mu}\left(\frac{\lambda}{\mu}\right)^{i-1+N}, \qquad -N+1 \le i \le N-1.$$

$$\pi_N = \pi_0 \left(\frac{\lambda}{\mu}\right)^{2N-1}.$$

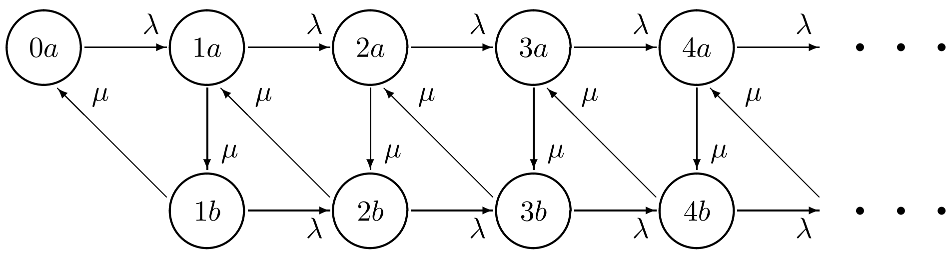
c) Assuming the entrance in state N as regeneration instants, the average return time coincides with the cycle period, while the average time in N in a cycle is $1/\mu$. Therefore, by the fundamental Theorem we have

$$\pi_{N,N} = \frac{1/\mu}{m_{N,N}}, \qquad \rightarrow \qquad m_{N,N} = \frac{1}{\mu \pi_{N,N}}$$

Problem 3 - A queue has only one server that, when entered by a user, provides service in two cascaded phases, a and b, both of them lasting for independent periods of time, each of them with negative exponential pdf with rate μ. When the server ends the first phase a it immediately begins the second phase b and, when b finishes, the user leaves (this means that the global service time $a+b$ is the sum of two exponential variables, and is not markovian), and a new user, if any in the queue, can enter the server again. Assuming Poisson arrivals, the number in the system (server + queue) n is not markovian. However, we can suitably "enlarge" the state variable n to a new one that takes into account the Phase of the service and that is markovian.

a) draw the markovian state diagram with the new state variable;
b) assuming that a maximum of two users can be in the system (one in the server and one in the queue), find the distribution π_i $i = 0, 1, 2$ of the users in the queue.

Solution
a)



Problem 4 - An open network of 4 markovian queues, 1,2,3,4, serves two Poisson flows of users, the first at rate λ, enters the network at node 1, while the second at rate 2λ, enters the network at node 2. The two flows have different routing matrices among nodes 0,1,2,3,4, as given below

$$\begin{vmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0.5 & 0 & 0.5 \\ 0.5 & 0 & 0 & 0.5 & 0 \\ 0 & 0.5 & 0.5 & 0 & 0 \\ 0.5 & 0.5 & 0 & 0 & 0 \end{vmatrix} \qquad \begin{vmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0.5 & 0 & 0 & 0.5 \\ 1 & 0 & 0 & 0 & 0 \\ 0.5 & 0 & 0 & 0.5 & 0 \end{vmatrix}$$

Queue 2 has an *infinite* number of markovian servers, each of rate μ, while queues 1,3 and 4, have a single server at rates 2μ. Find

a) the bottleneck queue, ie, the one with highest loading factor ρ;
b) the average network crossing delay;
c) the average network crossing delay of users of the first flow;

Solution
a) We have two routing matrices. One way is to solve for the flows of both separately, and sum the flows. We have

$$\lambda_1^{(1)} = (12/7)\lambda, \qquad \lambda_2^{(1)} = (8/7)\lambda, \qquad \lambda_3^{(1)} = (4/7)\lambda, \qquad \lambda_4^{(1)} = (6/7)\lambda,$$

$$\lambda_1^{(2)} = \lambda, \qquad \lambda_2^{(2)} = 2\lambda, \qquad \lambda_3^{(2)} = (3/2)\lambda, \qquad \lambda_4^{(2)} = \lambda.$$

$$\rho_1 = \frac{19}{14}\frac{\lambda}{\mu}, \qquad \rho_3 = \frac{29}{28}\frac{\lambda}{\mu}, \qquad \rho_4 = \frac{13}{14}\frac{\lambda}{\mu}.$$

The bottleneck queue is queue 1 since it presents the highest loading factor. Node 2 cannot be a bottleneck since it has an infinite number of servers.
b)

$$D = \frac{19}{21}\frac{1}{2\mu(1-\rho_1)} + \frac{22}{21}\frac{1}{\mu} + \frac{29}{42}\frac{1}{2\mu(1-\rho_3)} + \frac{13}{21}\frac{1}{2\mu(1-\rho_4)}$$

c)

$$D = \frac{12}{7}\frac{1}{2\mu(1-\rho_1)} + \frac{8}{7}\frac{1}{\mu} + \frac{4}{7}\frac{1}{2\mu(1-\rho_3)} + \frac{6}{7}\frac{1}{2\mu(1-\rho_4)}$$