Mathematical Modeling

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Contents

Abstract

This paper presents a comprehensive review of various models of population growth, with a particular emphasis on the Malthus-Condorcet-Mill model developed by Joel E. Cohen. The Malthus-Condorcet-Mill model seeks to address an often overlooked aspect of human choice that cannot be adequately captured by the simplistic ecological notion of carrying capacity. Moreover, this paper delves into the significance of the Allee effect in population dynamics and endeavors to integrate this effect into Cohen's model. By combining these two concepts, the aim is to deepen our understanding of population dynamics and offer valuable insights into the implications of the Allee effect on population growth. Through this comprehensive examination and integration of the Malthus-Condorcet-Mill model and the Allee effect, this paper strives to contribute to the advancement of our knowledge in the field of population dynamics and provide valuable perspectives on the complex dynamics that influence population growth.

1 Introduction

Joel E. Cohen [1], a prominent researcher in population dynamics, has made significant contributions to understanding the role of human carrying capacity in population models. Through his exploration of various existing models, Cohen recognized the impact of technology and human decisions on the human carrying capacity. However, he also realized that this capacity cannot be infinitely expanded due to inherent limitations.

To address this issue and synthesize the works of influential thinkers such as Malthus, Condorcet, and Mill, Cohen developed the Malthus-Condorcet-Mill model. This model seeks to provide a comprehensive framework for understanding population growth dynamics by considering factors beyond pure technological advancement and human decisions.

Despite the advancements made by Cohen's model, it fails to incorporate an essential component: positive density dependence, commonly referred to as the Allee effect. The Allee effect highlights the positive relationship between population density and individual fitness or growth rates, which plays a crucial role in shaping population dynamics [2].

This paper aims to address the gap in Cohen's model by integrating the Allee effect. By doing so, we enhance our understanding of population growth and gain insights into the implications of positive density dependence on population dynamics. Through this comprehensive exploration, we aim to bridge the theoretical and empirical gaps and contribute to a more holistic understanding of population dynamics.

2 Malthus-Condorcet-Mill model by Joel E. Cohen

2.1 Logistic equation

One of the simple models of population growth is the exponential model, which posits that the human population grows exponentially without the presence of any limiting factors.

$$\frac{dP}{dt} = rP(1 - \frac{P}{K}) \tag{1}$$

Where $P(t)$ is the population at time t, K is the carrying capacity of the environment. This model assumes that population growth follows a constant rate of multiplication, resulting in an ever-accelerating expansion of the population over time. However, it is important to note that this model oversimplifies the complex dynamics of real-world populations, as it does not consider the availability of resources, environmental constraints, or other factors that ultimately influence the carrying capacity and sustainability of human populations.

2.2 Malthusian model

Thomas Robert Malthus (1766-1834) was one of the first scientists to contribute to the concept of the human carrying capacity. In 1978, he described the relationship between human carrying capacity and human population. He described that the human population depends how fast it outstrips the resources available to it.

A Malthusian catastrophe in Figure 1 refers to a hypothetical event or situation in which population growth outpaces the availability of resources, leading to dire consequences. Thomas Malthus argued that population tends to increase at a faster rate than the production of food and resources necessary to sustain it.

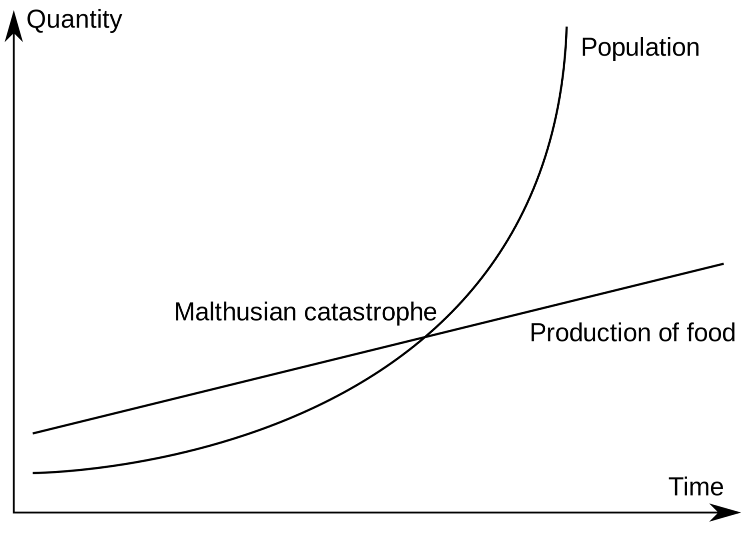

Figure 1: Malthusian Catastrophe [4]

According to Malthus, population growth occurs exponentially, while resource production increases only arithmetically. He believed that this imbalance would eventually result in a point where the population exceeds the carrying capacity of the environment, leading to a catastrophic scenario characterized by famine, disease, poverty, and other forms of suffering. Malthus suggested that these conditions would act as natural checks on population growth, helping to restore a balance between population and resources [1].

Thus, Malthus opposes the optimism of the Marquis de Condorcet (1743-1794). Condorcet's ideas were well-aligned with the main principles of Enlightenment, a belief in the possibility of human perfection. Condorcet believed that the population can overcome whatever obstacles it will encounter in its way to growth. One thing that Marquis missed in his analysis was the fact of how much population itself can contribute to its carrying capacity.

Joel E. Cohen, in "Population Growth and Earth's Human Carrying Capacity" [1] describes an idealized mathematical model that involves human population and human carrying capacity.

$$\frac{dP(t)}{dt} = rP(t)[K(t) - P(t)] \tag{2}$$

The equation is the same as the logistic equation, except that carrying capacity is defined as a function of t, instead of being a constant in the logistic equation developed by Pierre-François Verhulst. Here, the rate of change of carrying capacity is also dependent on the population itself. Because additional people are able to create new medicine, invent technologies, and thus increase the human carrying capacity. At the same time additional people can start wars, create and spread viruses, thus decreasing the human carrying capacity.

2.3 Condorcet

Then by supposing that the rate of change of the human carrying capacity is directly proportional to the rate of the change of the population itself, we get the following equation, named after Condorcet:

$$\frac{dK(t)}{dt} = c\frac{dP(t)}{dt} \tag{3}$$

Here, c is called the Condorcet parameter and it can take negative and positive values and be zero.

- When $c = 1$: population and carrying capacity grow at the same rate. Every extra person increases the carrying capacity by exactly how much they consume. Population grows exponentially.
- When $c > 1$: carrying capacity grows at a faster rate than the population. Every extra person increases the carrying capacity by exactly how much they consume + some extra amount. Population grows faster than the exponential growth.
    - If $c = 0$: $K' = K(t)$.
    - If $0 < c < 1$: every extra person is only able to increase the carrying capacity by less than the amount one person consumes. $K' > K(0)$.

- When $c < 1$: carrying capacity grows at a slower rate than the population. The overall effect of the $K(t)$ to the $P(t)$ is almost like having a constant carrying capacity K'.

  - If $c < 0$, every extra person decreases the carrying capacity. $K' < K(0)$

This Malthus-Condorcet model, combines the following population models:

1. Exponential growth model of Euler in the 18th.
2. Logistic growth model of Verhulst in the 19th.
3. Doomsday (faster-than-exponential) growth model of von Foerster et al. in the 20th. [1]

2.4 Miller's parameter

In the aforementioned conjectures, the parameter c determines an individual's contribution to the population's carrying capacity. This contribution relies on the available resources to that person. As resources are shared among individuals, the constant c should be replaced by a variable $c(t)$ that decreases with population growth. This variable adaptation reflects the diminishing availability of resources per capita as the population expands.

Let us suppose that where,

$$\frac{dK(t)}{dt} = \frac{L}{P(t)} \frac{dP(t)}{dt} \tag{4}$$

where L is a positive constant, called the Miller parameter. Replacing our constant c by $L/P(t)$, gives the Condorcet-Mill equation. Note that here we are assuming that the resources are shared equally among individuals.

If $c(0) = L/P(0) > 1$, then population initially grows faster than exponential growth. When $L = P(t)$ population will have a brief exponential growth. When $P(t) > L$, the population grows sigmoidally [1]. The model is illustrated in Figure 2, where $K(t)$ is shown by a dashed line, $P(t)$ is shown by a solid line, and the estimated actual human population is represented as solid rectangles.

[1] Doomsday growth model is a model that assumes the conditions of a paradise, where there are no environmental hazards and an unlimited food supply, among other factors. In this model, the rate of change of the population size, denoted as $\frac{dN}{dt}$, is described by the equation $\frac{dN}{dt} = a_0 N$. Here, N represents the number of elements in the population, and a_0 is called the productivity of the individual element [5].

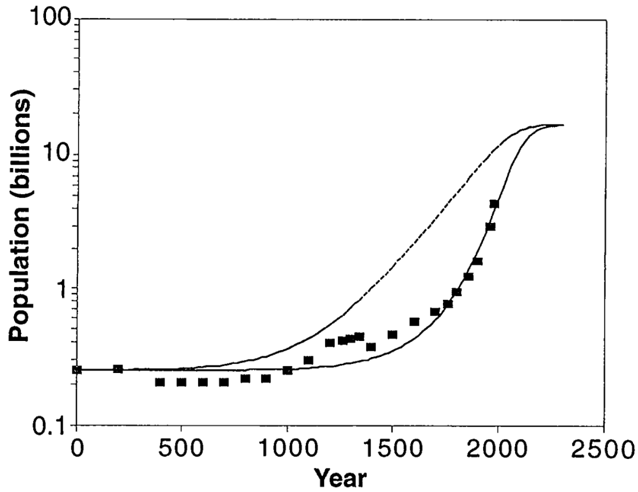

Figure 2: Numerical Illustrations of Malthus-Condorcet-Mill model.

2.5 Remarks

There are a few noteworthy remarks to be made about this model:

Firstly, it is essential to acknowledge that this model does not encompass the direct relationship between human carrying capacity and the human population. It lacks an understanding of the cultural, economic, and environmental resources possessed by each individual, making it difficult to quantify the extent to which each person can increase the carrying capacity.

Secondly, it is important to highlight that the model cannot accurately predict the future. Historical records reveal instances of faster-than-exponential population growth, as well as advancements in technology and medicine that have improved human life. These factors suggest that population growth could either continue to expand or follow a sigmoidal pattern, reaching a certain size and then stabilizing.

Thirdly, it is worth noting the optimistic perspective that humans are capable of infinitely expanding their carrying capacity. According to this viewpoint, each individual would need to increase the carrying capacity by at least the amount they consume, if not more. However, it is important to approach this notion with caution, as the feasibility and sustainability of such infinite expansion warrant careful consideration [1].

3 Malthus-Condorcet-Mill-Allee

3.1 Allee effect

The Allee effect is a fascinating phenomenon that elucidates the decline in population size and the subsequent decrease in growth rate, often leading to zero or even negative values. It can be best described as a form of inverse density dependence. This intriguing concept primarily arises from three key factors.

Firstly, the loss of genetic diversity and the occurrence of inbreeding play significant roles in the Allee effect.

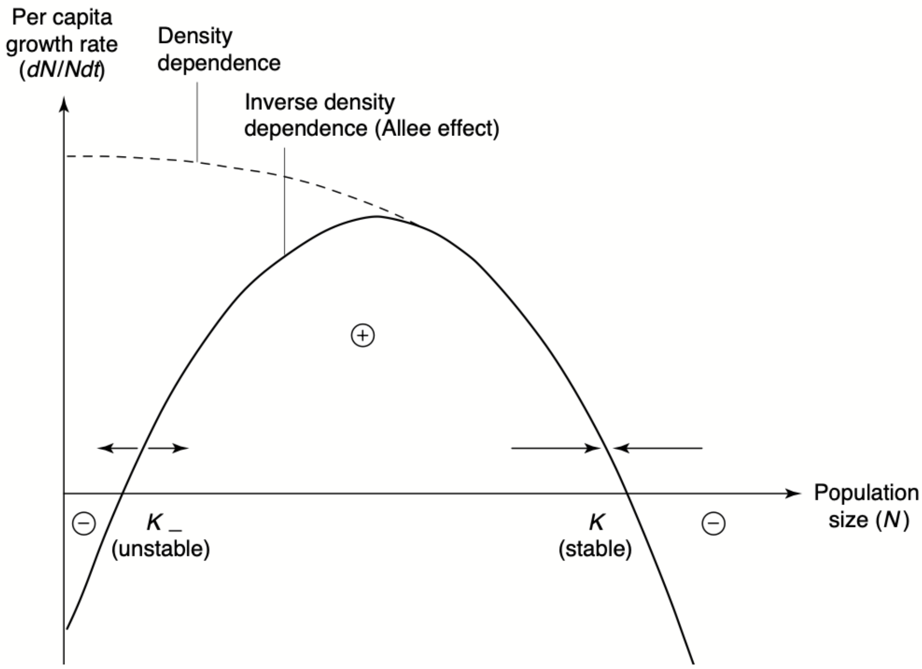

Figure 3: Illustration of the Allee effect, from a very simple mathematical model of population dynamics [3].

As populations become smaller, there is a heightened risk of reduced genetic variation due to limited breeding opportunities. Inbreeding, in turn, can lead to various detrimental effects, such as reduced fitness and increased vulnerability to diseases and environmental stressors.

Secondly, sex-ratio fluctuations contribute to the Allee effect. In small populations, chance events can cause significant imbalances in the proportion of males to females, disrupting the mating dynamics necessary for reproductive success. Such fluctuations can further impede successful breeding and ultimately hinder population growth.

Thirdly, the Allee effect highlights the impact of diminished cooperative strategies crucial for survival in small populations. Many species rely on cooperative behaviors, such as group defense against predators, efficient foraging, or collaborative parental care, to thrive. However, when population numbers dwindle, the effectiveness of these cooperative strategies can diminish, leading to reduced reproductive success and overall population decline.

The Allee effect has many implications in population dynamics. Franck Courchamp and et. al. states that in lower densities parasitism and kleptoparasitism can also be affected. In prey-predator systems, not only the prey population, but predator systems may also be affected [2].

A simple logistic equation could be rewritten as follows, including the Allee effect.

$$\frac{dP(t)}{dt} = rP(t)(1-\frac{P(t)}{K(t)})(\frac{P(t)}{K_a(t)}-1) \tag{5}$$

In the equation above the $(\frac{P}{K_a}-1)$ presents the Allee effect. The equation 5 has three equilibrium points: $P=0$,

$P = K$, and $P = K_a$. The equilibrium point $N = K_a$ introduces the Allee effect, where populations with sizes less than K_a experience a negative growth rate. Among these equilibrium points, $P = 0$ and $P = K_a$ are classified as stable fixed points, while $P = K_a$ is an unstable fixed point.

A significant difference between this model and the logistic model is that $P = 0$ is a stable fixed point in this case, whereas it is an unstable fixed point in the logistic model. Therefore, for population sizes less than K_a, the population tends to decline towards zero, which serves as an attractive stable fixed point representing extinction. Conversely, for population sizes greater than K_a, the population strives towards the stable fixed point K, which represents the carrying capacity. Hence, the value K_a, introduced by the Allee effect, can be referred to as a "threshold for extinction".

3.2 Modeling K_a

As Joel E. Cohen emphasizes in his work, carrying capacity is not a constant but a variable influenced by the population itself. Similarly, considering the threshold for extinction as a variable can be useful in understanding the dynamics of population survival. In line with this concept, the Condorcet equation suggests that additional people can contribute to the creation of new medicine and technology, thereby decreasing the extinction threshold. This implies that even with a small population density, the availability of advancements can enhance survival prospects. However, it is essential to note that an excessively high rate of population growth can exert pressure on infrastructure, potentially leading to strain and negative consequences. Destructive impacts on the environment, resource scarcity, and challenges in maintaining essential services may arise from an unbalanced growth rate.

So by making similiar assumptions, as Cohen did to the carrying capacity, we can say that the growth rate of the extinction threshold is proportional to the growth rate of population itself.

$$\frac{dK_a}{dt} = c_a \frac{dP}{dt} \tag{6}$$

If we do the same analysis to this model:

- When $c_a = 1$: Population and the extinction threshold grows at the same rate. Every extra person increases the extinction threshold by one.
- When $c_a > 1$: The extinction threshold grows faster than the population. Every extra person increases the extinction threshold by more than 1. The extinction becomes more possible with a greater population size.
- When $c_a < 1$: The population grows faster than the extinction threshold. Every extra person decreases the extinction threshold by more than 1. The extinction becomes less possible with a greater population size.

As previously, c is the contribution of the individual to the extinction threshold. So $c_a = \frac{L_a}{P(t)}$.

Now we have

$$\frac{dP}{dt} = rP(t)(K(t)-P(t))(P(t)-K_a(t)), \tag{7}$$

$$\frac{dK(t)}{dt} = \frac{L}{P(t)}\frac{dP(t)}{dt} \tag{8}$$

$$\frac{dK_a(t)}{dt} = \frac{L_a}{P(t)}\frac{dP(t)}{dt} \tag{9}$$

After substituting equation 7 into equations 8, 9:

$$\frac{dK(t)}{dt} = Lr(K(t)-P(t))(P(t)-K_a(t)) \tag{10}$$

$$\frac{dK_a(t)}{dt} = L_a r(K(t)-P(t))(P(t)-K_a(t)) \tag{11}$$

From equations 10, 11 we get that the throughput growth rate and the extinction threshold in this model are proportional:

$$\frac{1}{L}\frac{dK}{dt} = \frac{1}{L_a}\frac{dK_a}{dt} \tag{12}$$

This observation suggests that if we have information about the growth rate of the carrying capacity, we can also determine a constant multiple of the growth rate of the extinction threshold. If we let $\frac{L}{L_a} = \alpha$, then $\frac{dK}{dt} = \alpha\frac{dK_a}{dt}$.

3.3 Illustrations

3.3.1 Population Growth Rate Graph with Allee effect.

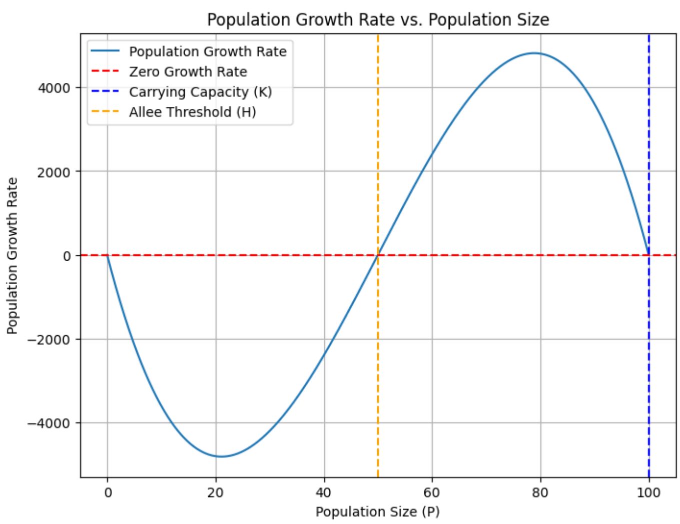

Figure 4

3.3.2 Population Size Graph with Allee Effect

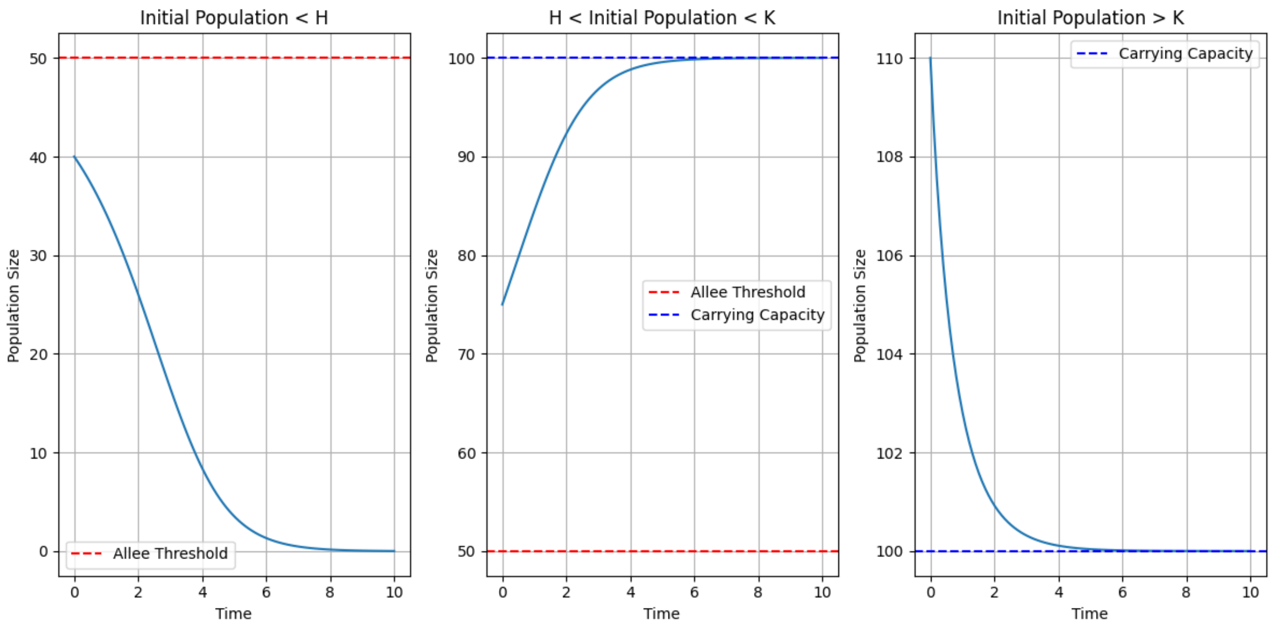

Figure 5

3.3.3 Population Size and Carrying Capacity Graph with Allee Effect

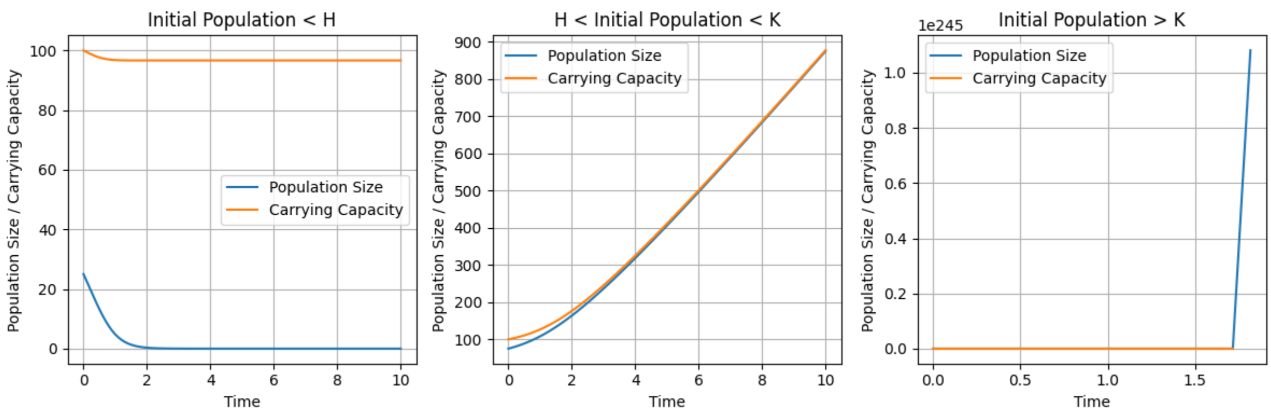

Figure 6

4 Conclusion

This paper provides a comprehensive overview of several population growth models, with a particular focus on understanding the concept of the Earth's carrying capacity. Throughout this article, I have drawn inspiration from such outstanding scientists as Malthus, Condorcet and Mill, who have made invaluable contributions to understanding how the dynamics of the human population affects throughput.

Cohen highlights how these authors emphasized the relationship between population size and the Earth's ability to support it. Their findings emphasize the importance of taking into account not only the population itself, but also the complex dynamics resulting from the interaction between individuals and their environment.

Expanding our study of population dynamics, we delved into the concept of the Allee effect. This phenomenon creates an additional level of complexity due to the inclusion of the concept of the Alley threshold. The Allee threshold represents a critical population below which the growth rate decreases or even leads to a reduction in the population. Understanding the relationship between Alley threshold and throughput gives us a deeper understanding of the dynamics affecting population growth.

Despite the fact that our model successfully combines elements of Malthus, Condorcet, Mill and taking into account the Allee effect, it is extremely important to recognize its limitations. One notable limitation is the lack of individual contribution to the carrying capacity and the extinction threshold. While overlooking the diverse impacts that each individual can have on the environment and sustainability, our findings may not fully reflect the complexity and nuances of population dynamics.

Further research is needed to address these limitations. Future research should aim to refine our understanding by taking into account individual contributions to carrying capacity and examining the implications for extinction risk. By addressing these knowledge gaps, we can improve the accuracy and completeness of our models and use them as a basis for more effective population management strategies.

In conclusion, the integration of the ideas of Malthus, Condorcet, Mill and the study of the Allee effect give us a broader view of population growth and its complex relationship with capacity. However, while recognizing the need for ongoing research, we stress the importance of addressing gaps in our understanding to improve our ability to effectively assess and manage population dynamics. Thanks to these efforts, we can develop more reliable models and strategies to ensure the sustainable coexistence of people and our environment.

References

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