Chapter 12

Fourier Series

12.1 Motivation

Many problems in physics involve vibrations and oscillations. Often the oscillatory motion is simple (e.g. weights on springs, pendulums, harmonic waves etc.) and can be represented as single sine or cosine functions. However, in many cases, (electromagnetism, heat conduction, quantum theory, etc.) the waveforms are not simple and, unlike sines and cosines, can be difficult to treat analytically.

Fourier methods give us a set of powerful tools for representing any periodic function as a sum of sines and cosines. The *Fourier series* of a function, $f(x)$, with period $2L$ is,

$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} \left(a_n \cos\frac{n\pi x}{L} + b_n \sin\frac{n\pi x}{L} \right). \tag{12.1}$$

To see how this works, let us consider a sound wave. The vibration of a tuning fork produces a sound wave of a given frequency. If we plot the pressure as a function of distance, x, or time, t, it looks like a single sine wave, or a 'pure tone' (Fig. 12.1: left). When a note is played on a flute, we get a more complex sound (Fig. 12.1: centre). The note that we get is made up from the sum of many pure tones: the fundamental and different harmonics with frequencies 2,3,4,... times the frequency of the fundamental (Fig. 12.1: right). This is the Fourier series.

Fourier methods are used very heavily in signal and data analysis. By Fourier analysing a signal - essentially by expanding it in the form of Eq. (12.1) - we can immediately tell which harmonics are the important ones. For ex-

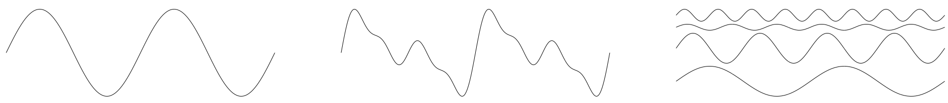


Figure 12.1: Left: a pure sine wave, $\sin(\omega t)$. Centre: example waveform, $f(t)$, from a flute. Right: the note from the flute is made up of the sum of the fundamental sine wave and a series of harmonics. In this example, $f(t) = \sin(\omega t) + \sin(2\omega t) + 0.2\sin(3\omega t) + 0.4\sin(4\omega t)$. This is the Fourier series.

ample, in the note from the flute (Fig. 12.1), the harmonic at frequency 2ω has relatively large amplitude, while the harmonic at 3ω is small. If, for example, a poorly designed speaker filtered out the harmonic at 2ω it would greatly change the character of the sound, while filtering out the harmonic at 3ω would have a much less discernible effect.

Fourier methods are also commonly used in mathematical physics. In this chapter we will focus on using them to solve differential equations, and the wave equation in particular. We will examine Fourier half range series and Fourier full range series, study some applications of Fourier series, then finish by introducing Fourier transforms and the convolution theorem.

Using these notes

This set of notes contains the minimum amount of information you will need to complete the course. You should also study other sources. In particular, use maths textbooks (for example, *Mathematical methods in the Physical Sciences*, Mary L. Boas) as a source of problems and worked examples to supplement the few that are contained within these notes.

Prerequisites

To complete the chapter you will need to be proficient at integrating by parts, be able to manipulate trigonometric functions, and understand the symmetry of even and odd functions. There is a short revision section on the symmetry of functions in the appendix to these notes.

You should also be familiar with the contents of Chapter 11, partial differential equations.

12.2 Fourier half range sine series

In chapter 11 we calculated the separable solutions for a wave on a string that is fixed at both ends, at $x = 0$ and at $x = L$. In general, the displacement of such a string is,

$$y(x,t) = \sum_{n=1}^{\infty} \sin\frac{n\pi x}{L}\left(a_n \sin\frac{n\pi ct}{L} + b_n \cos\frac{n\pi ct}{L}\right), \tag{12.2}$$

where each of the a_n and b_n for $n = 1, 2, 3\ldots$ is an arbitrary constant that we can set once we know the boundary conditions for any given problem. The range 0 to L is called the *half range* because it is half the maximum wavelength or spatial period.

Consider the case when the string is initially at rest and has initial displacement $y(x, 0) = f(x)$. Then, by substituting $t = 0$ into Eq. (12.2) we find

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{L}. \tag{12.3}$$

Given any function, $f(x)$, can we find the coefficients, b_n, such that Eq. (12.3) is satisfied? Remarkably, yes! This is Fourier's theorem.

Equation (12.3) is the Fourier half range sine series of a function, $f(x)$. This is a very powerful result. It tells us that, within the range 0 to L, we can write *any* (physically reasonable) function as a sum of sine waves.

Fourier sine series coefficients

The Fourier half-range sine series coefficients, b_n, are given by,

$$b_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}dx. \tag{12.4}$$

Derivation of the Fourier sine series coefficients

The formula for the b_n can be derived directly from the Fourier series representation (Eq. (12.3)). First, multiply both sides of Eq. (12.3) by $\sin(m\pi x/L)$ then integrate from 0 to L. This gives,

$$\int_0^L f(x)\sin\frac{m\pi x}{L}dx = \sum_{n=1}^{\infty} b_n \int_0^L \sin\frac{n\pi x}{L}\sin\frac{m\pi x}{L}dx.$$

The integral on the right hand side is a standard integral (Eq. (A.2)) with result $(L/2)\delta_{nm}$, where δ_{nm} is a Kronecker delta defined in Eq. (A.1). Essentially, when the two sine waves in the integral on the right have a differing wavelength they interfere destructively and cancel to zero. We only get a non-zero result for the integral when the wavelengths are the same and $n = m$.

Substituting in the result from Eq. (A.2), we have,

$$\int_0^L f(x)\sin\frac{m\pi x}{L}dx = \sum_{n=1}^{\infty} b_n \frac{L}{2}\delta_{nm} = b_m\frac{L}{2}.$$

Finally, rearranging this equation and replacing the symbol m with n we find Eq. (12.4), the formula for the Fourier series sine coefficients.

Using the Fourier series results

We can now use the results from Eq. (12.3) and Eq.(12.4) to find the Fourier half range sine series for any function, $f(x)$.

Example 12.1. Calculate the Fourier series representation of the function, $f(x) = 1$ in $0 \le x < L$.

We wish to represent the function, $f(x)$, as a Fourier sine series,

$$f(x) = 1 = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{L}.$$

To do this, we simply need to calculate the appropriate Fourier coefficients, b_n, using Eq. (12.4),

$$\begin{aligned} b_n &= \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}dx = \frac{2}{L}\int_0^L \sin\frac{n\pi x}{L}dx, \\ &= -\frac{2}{L}\frac{L}{n\pi}\left[\cos\frac{n\pi x}{L}\right]_0^L, \\ &= -\frac{2}{L}\frac{L}{n\pi}\left(\cos(n\pi) - 1\right). \end{aligned}$$

We can simplify this equation using the result that $\cos(n\pi) = (-1)^n$. Then,

$$\begin{aligned} b_n &= \frac{2}{n\pi}\left(1 - (-1)^n\right), \\ &= \begin{cases} 0 & \text{if } n \text{ is even} \\ \frac{4}{n\pi} & \text{if } n \text{ is odd.} \end{cases} \end{aligned} \tag{12.5}$$

So, the Fourier sine series of $f(x) = 1$ for $0 \leq x < L$ is,

$$1 = \sum_{n,\text{odd}} \frac{4}{n\pi} \sin \frac{n\pi x}{L}, \tag{12.6}$$

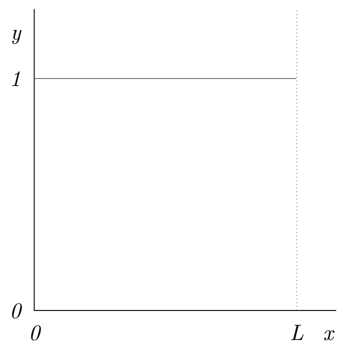
where the notation 'n, odd', simply means to take only the odd integer terms in the summation. We could also write this explicitly by defining a new integer counter, $m = 0, 1, 2, \ldots \infty$, and setting $n = 2m + 1$ so that n is always odd,

$$1 = \sum_{m=0}^{\infty} \frac{4}{(2m+1)\pi} \sin \frac{(2m+1)\pi x}{L}.$$

Figure 12.2 illustrates how the representation of $f(x) = 1$ is built up by adding together sine waves from the series. Writing out the first few terms in Eq. (12.6) explicitly, we have

$$1 = \frac{4}{\pi} \left(\sin \frac{\pi x}{L} + \frac{1}{3} \sin \frac{3\pi x}{L} + \frac{1}{5} \sin \frac{5\pi x}{L} + \ldots \right).$$

As we add each successive sine term from the infinite series we get closer and closer to an exact representation of the function (Fig. 12.2: right).



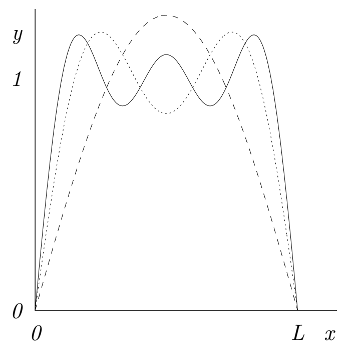


Figure 12.2: Left: the function, $f(x) = 1$ in $0 \leq x < L$. Right: the first three partial sums of its Fourier sine series, dashed line: $f_1 = 4\sin(\pi x/L)/\pi$, dotted line: $f_2 = f_1 + 4\sin(3\pi x/L)/(3\pi)$, solid line: $f_3 = f_2 + 4\sin(5\pi x/L)/(5\pi)$.

Exercise 12.1. If $f(x)$ is given by $f(x) = x$ in $0 \leq x < L$, show that the Fourier sine series coefficients are given by $b_n = 2L(-1)^{n+1}/(n\pi)$. Hence show that the Fourier sine series for $f(x)$ is,

$$x = \frac{2L}{\pi} \left(\sin \frac{\pi x}{L} - \frac{1}{2} \sin \frac{2\pi x}{L} + \frac{1}{3} \sin \frac{3\pi x}{L} - \ldots \right). \tag{12.7}$$

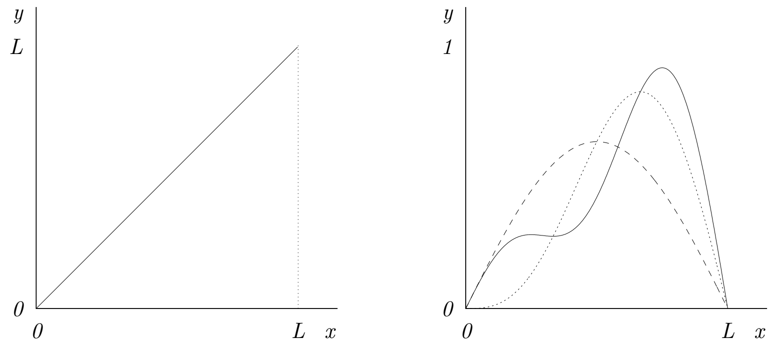


Figure 12.3: Left: the function, $f(x) = x$ in $0 \leq x < L$. Right: the first three partial sums of its Fourier sine series, dashed line: $f_1 = 2L\sin(\pi x/L)/\pi$, dotted line: $f_2 = f_1 - 2L\sin(2\pi x/L)/(2\pi)$, solid line: $f_3 = f_2 + 2L\sin(3\pi x/L)/(3\pi)$.

Exercise 12.2. A function, $g(x)$ is defined by,

$$g(x) = \begin{cases} x/L & \text{if } 0 \leq x < L/2 \\ 1 - x/L & \text{if } L/2 \leq x < L. \end{cases}$$

By expanding $g(x)$ as a Fourier sine series show that,

$$g(x) = \sum_{n,\text{odd}} \frac{4(-1)^{\frac{n-1}{2}}}{n^2\pi^2} \sin\frac{n\pi x}{L}.$$

Hint: The integral for b_n can be split into the sum of two parts, for example $\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$ where $a < c < b$.

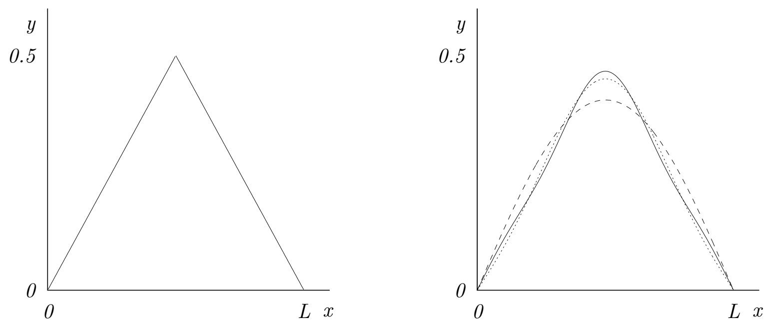


Figure 12.4: Left: the function, $g(x)$ from Ex. 2. Right: the first three partial sums of its Fourier sine series, dashed line: $f_1 = 4\sin(\pi x/L)/\pi^2$, dotted line $f_2 = f_1 - 4\sin(3\pi x/L)/(9\pi^2)$, solid line $f_3 = f_2 + 4\sin(5\pi x/L)/(25\pi^2)$.

12.2.1 Application to differential equations

The wave equation

The separable solutions to the wave equation for a string fixed at $x = 0$ and $x = L$ are,

$$y(x,t) = \sum_{n=1}^{\infty} \sin\frac{n\pi x}{L}\left(A_n \sin\frac{n\pi ct}{L} + B_n \cos\frac{n\pi ct}{L}\right). \tag{12.8}$$

Armed with results (12.3) and (12.4) we can now find the coefficients A_n and B_n given a set of initial conditions.

Let us examine the general case when the string is given an initial displacement, $y(x,0) = p(x)$ and an initial velocity, $y_t(x,0) = q(x)$. By substituting $t = 0$ into Eq. (12.8) we immediately find that

$$y(x,0) = p(x) = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{L}.$$

This looks like the Fourier sine series of $p(x)$ with Fourier series coefficients, B_n. So, to find the coefficients, B_n, we simply need to apply the formula in Eq. (12.4),

$$B_n = \frac{2}{L}\int_0^L p(x)\sin\frac{n\pi x}{L}dx. \tag{12.9}$$

We can follow a similar process to find the A_n. First, find the transverse velocity of the string at $t = 0$,

$$y_t(x,t) = \frac{\partial y}{\partial t} = \sum_{n=1}^{\infty} \sin\frac{n\pi x}{L}\left(A_n\frac{n\pi c}{L}\cos\frac{n\pi ct}{L} - B_n\frac{n\pi c}{L}\sin\frac{n\pi ct}{L}\right).$$

So, at $t = 0$,

$$y_t(x,0) = q(x) = \sum_{n=1}^{\infty}\left(\frac{A_n n\pi c}{L}\right)\sin\frac{n\pi x}{L}.$$

Again, this looks like the Fourier sine series of $q(x)$ with Fourier series coefficients $A_n n\pi c/L$. So, again, to find $A_n n\pi c/L$ we simply need to apply the formula in Eq. (12.4),

$$\begin{aligned}\left(\frac{A_n n\pi c}{L}\right) &= \frac{2}{L}\int_0^L q(x)\sin\frac{n\pi x}{L}dx,\\ \Rightarrow A_n &= \frac{2}{n\pi c}\int_0^L q(x)\sin\frac{n\pi x}{L}dx. \qquad (12.10)\end{aligned}$$

Example 12.2. A string fixed at $x = 0$ and at $x = L$ is given constant initial velocity, $y_t(x, 0) = v$, and zero initial displacement, $y(x, 0) = 0$. Find $y(x, t)$.

In general,

$$y(x,t) = \sum_{n=1}^{\infty} \sin \frac{n\pi x}{L} \left(A_n \sin \frac{n\pi ct}{L} + B_n \cos \frac{n\pi ct}{L} \right).$$

To find $y(x, t)$ given a set of initial conditions, substitute the initial conditions into the general solution, find equations involving the unknown coefficients A_n and B_n, then calculate A_n and B_n using the formula for the Fourier sine series coefficients, Eq. (12.4).

At $t = 0$ the displacement of the string is zero, so

$$\begin{aligned} 0 &= \sum_{n=1}^{\infty} B_n \sin \frac{n\pi x}{L} \\ &\Rightarrow B_n = 0. \end{aligned}$$

At $t = 0$ the initial velocity is v, so

$$v = \sum_{n=1}^{\infty} \left(\frac{A_n n\pi c}{L} \right) \sin \frac{n\pi x}{L}. \tag{12.11}$$

This is just the Fourier series representation of a constant, v. So,

$$\left(\frac{A_n n\pi c}{L} \right) = \frac{2}{L} \int_0^L v \sin \frac{n\pi x}{L} dx.$$

Then, using the result from Eq. (12.5),

$$\left(\frac{A_n n\pi c}{L} \right) = \begin{cases} 0 & \text{if } n \text{ is even} \\ \frac{4v}{n\pi} & \text{if } n \text{ is odd} \end{cases}$$

$$\Rightarrow A_n = \begin{cases} 0 & \text{if } n \text{ is even} \\ 4vL/(n^2\pi^2 c) & \text{if } n \text{ is odd.} \end{cases}$$

Once we have calculated the A_n and B_n we can write down the full solution $y(x, t)$ that describes the displacement of the string as a function of x and t. In the case when the initial displacement is 0 and the initial velocity is v,

$$y(x,t) = \sum_{n=1}^{\infty} \frac{4vL}{n^2\pi^2 c} \sin \frac{n\pi x}{L} \sin \frac{n\pi ct}{L}. \tag{12.12}$$

Exercise 12.3. The initial displacement of a string of length L fixed at its end points is given by $y(x,0) = \alpha x$, where α is a constant. The initial velocity is zero. Find the solution for $y(x,t)$ as an infinite series.

Exercise 12.4. A string is fixed at its end points at $x = 0$ and $x = L$. If the initial displacement is $y(x,0) = \sin(\pi x/L)$ and the initial velocity is $y_t(x,0) = x$, find the solution for $y(x,t)$ as an infinite series.

Exercise 12.5. A string fixed at its end points is released from rest with initial displacement $y(x,0) = \exp(-\alpha^2(x - L/2)^2)$ where $\alpha \gg 1/L$. Find the displacement, $y(x,t)$ at time t.

Hint: If αL is large then

$$\int_0^L e^{-\alpha^2(x-L/2)^2} \sin\frac{n\pi x}{L} \approx \frac{\sqrt{\pi}}{\alpha} e^{-n^2\pi^2/4L^2\alpha^2} \sin\frac{n\pi}{2}.$$

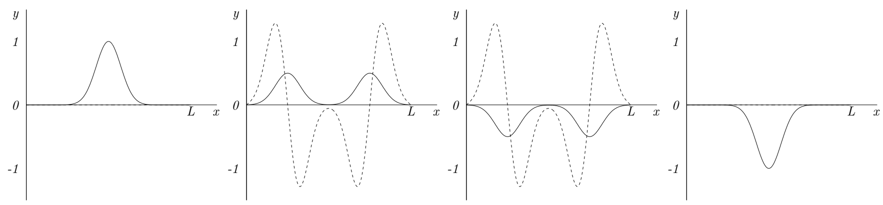


Figure 12.5: Wave from exercise 12.5. From left to right the panels show the displacement (solid line) and transverse velocity (dashed line) of the string at $t = 0$, $t = L/4c$, $t = 3L/4c$ and $t = L/c$.

Other differential equations

We can use the techniques from section 12.2.1 to find the solutions to other differential equations.

As a brief example, let us consider the solution to Laplace equation for the electrostatic potential on a metal plate, $\nabla^2\phi(x,y) = 0$. Imagine the potential, $\phi(x,y) \to 0$ as $y \to \infty$ and is set to zero at $x = 0$ and $x = L$, then

$$\phi(x,y) = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{L} e^{-n\pi y/L}.$$

If the potential at $y = 0$ has the form $p(x)$, then

$$y(x,0) = p(x) = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{L},$$

and we can calculate the coefficients simply by applying the formula for the Fourier sine series coefficients,

$$B_n = \frac{2}{L}\int_0^L p(x)\sin\frac{n\pi x}{L}dx.$$

Exercise 12.6. The electric potential $\phi(x,y)$ has boundary conditions, $\phi(x,0) = x^2$, $\phi(0,y) = 0$, $\phi(L,y) = 0$, and $\phi(x,y) \to 0$ as $y \to \infty$. Show that

$$\phi(x,y) = \sum_{n=1}^{\infty}\left(\frac{2L^2}{n\pi}(-1)^{n+1} + \frac{4L^2}{n^3\pi^3}((-1)^n - 1)\right)\sin\frac{n\pi x}{L}e^{-n\pi y/L}. \quad (12.13)$$

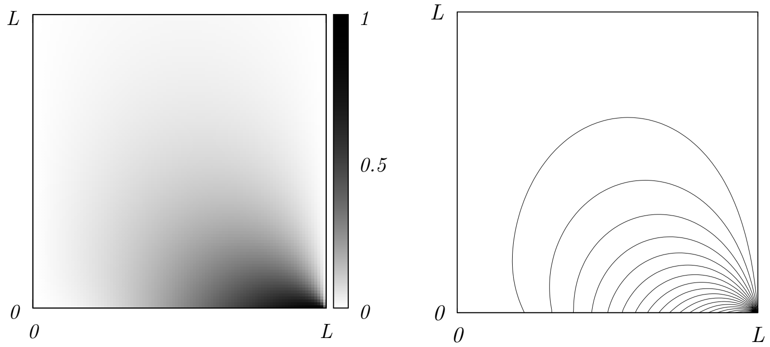


Figure 12.6: $\phi(x,y)$ from Eq. (12.13). Left: colour map of the electric potential. Right: equipotential lines.

12.3 Fourier half range cosine series

Another representation of a function defined in the range $0 \le x < L$ uses cosine instead of sine functions. This is equally valid: within the range $0 \le x < L$ both sines and cosines form mathematically *complete sets*. This means we can expand any function within this range in terms of either sines, or cosines.

The cosine representation of a function, $f(x)$ is,

$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{L}. \quad (12.14)$$

Fourier cosine series coefficients

The coefficients a_n, $n = 0, 1, 2, \ldots$ are given by,

$$a_n = \frac{2}{L}\int_0^L f(x)\cos\frac{n\pi x}{L}dx. \tag{12.15}$$

Usually it is much easier to calculate a_0 separately from all the other a_n. Then a_0 is simply,

$$a_0 = \frac{2}{L}\int_0^L f(x)dx. \tag{12.16}$$

Note that $\frac{1}{2}a_0$ is simply the average of the function, $f(x)$ in the range $0 \leq x < L$.

Derivation of Fourier cosine series coefficients

The derivation for a_n is along similar lines to that of the Fourier sine series coefficients. First multiply both sides of Eq. (12.14) by $\cos(m\pi x/L)$ and integrate from 0 to L,

$$\int_0^L f(x)\cos\frac{m\pi x}{L}dx = \frac{1}{2}a_0\int_0^L \cos\frac{m\pi x}{L}dx + \sum_{n=1}^{\infty} a_n \int_0^L \cos\frac{n\pi x}{L}\cos\frac{m\pi x}{L}dx. \tag{12.17}$$

From the standard integral (A.3) all the terms in the sum on the right of Eq. (12.17) are zero except for the one with $m = n$. Then, if $m = n$,

$$\int_0^L f(x)\cos\frac{m\pi x}{L}dx = a_m\int_0^L \cos^2\frac{m\pi x}{L} = \frac{L}{2}a_m,$$

and by rearranging and replacing m by n we have the result in Eq. (12.15).

If we choose $m = 0$ in Eq. (12.17) then $\cos(m\pi x/L) = 1$ and all the terms in the sum go to zero. We are left with,

$$\int_0^L f(x)dx = \frac{1}{2}a_0\int_0^L dx = \frac{L}{2}a_o,$$

which, after rearranging is identical to the result for a_0 in Eq. (12.16).

Using the Fourier cosine series results

Exercise 12.7. A function $f(x) = x$ in $0 \le x < \pi$ is expanded as a Fourier cosine series. Calculate the coefficients a_0 and a_n, and show that,

$$x = \frac{\pi}{2} - \frac{4}{\pi} \sum_{n,\text{odd}} \frac{1}{n^2} \cos nx. \tag{12.18}$$

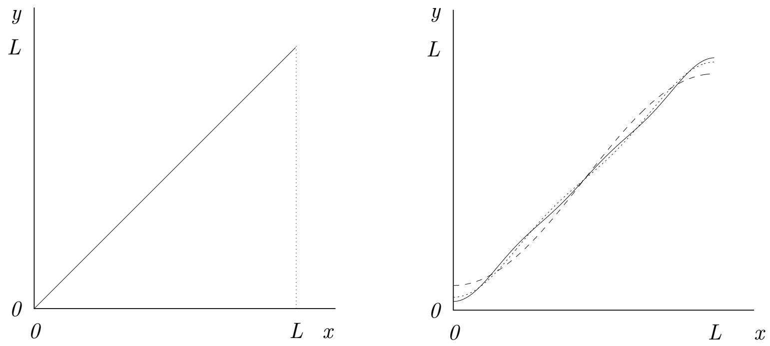


Figure 12.7: Left: the function, $f(x) = x$ in $0 \le x < \pi$. Right: the first three partial sums of its Fourier cosine series, dashed line $f_1 = \pi/2 - 4\cos(x)/\pi$, dotted line $f_2 = f_1 - 4\cos(3x)/(9\pi)$, solid line $f_3 = f_2 - 4\cos(5x)/(25\pi)$.

12.4 Numerical series

Both sine and cosine Fourier representations can be used to derive useful numerical series results for constants. Take, for example, the result from Eq. (12.18). Writing out the first few terms in the cosine series for x defined between 0 and π we have,

$$x = \frac{\pi}{2} - \frac{4}{\pi}\left(\cos x + \frac{1}{9}\cos 3x + \frac{1}{25}\cos 5x + \dots\right).$$

This equation is true for any x where $0 \le x < \pi$. We can substitute a particular value for x into both the left and right hand sides and the equality will still hold. Lets choose, for example, $x = 0$. Then, all the $\cos nx = 1$ and

$$0 = \frac{\pi}{2} - \frac{4}{\pi}\left(1 + \frac{1}{9} + \frac{1}{25} + \dots\right)$$

$$\Rightarrow \pi^2 = 8\left(1 + \frac{1}{9} + \frac{1}{25} + \dots\right),$$

is a series expansion for π^2.

Exercise 12.8. A function, $f(x)$, is defined by

$$f(x) = \begin{cases} 1 & 0 \le x < L/2 \\ 0 & \mathrm{L}/2 \le x < L. \end{cases}$$

Expand $f(x)$ as a Fourier cosine series and show that $a_n = 0$ if n is even and that, if n is odd,

$$a_n = \frac{2}{n\pi}(-1)^{(n-1)/2}. \tag{12.19}$$

Write down the cosine series for $f(x)$ in $0 \le x < L$ and deduce that

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7}.$$

12.5 Periodic extension of Fourier series

So far we have examined the sine and cosine Fourier series representations of functions within a limited range, $0 \le x < L$. However, both sine and cosines repeat periodically. So, if we plot the Fourier series representations outside of this range we will get functions that repeat periodically with wavelength $2L$.

Outside the given finite range, the Fourier series of $f(x)$ represents a periodic extension of the function with $f(x + 2L) = f(x)$.

To understand the periodic extension of Fourier series it is important to first understand the symmetry of *even* and *odd* functions. There is a short revision section on even and odd functions in the appendix.

Even and odd symmetry of periodic functions

Sine waves are odd, so any Fourier sine series representation of a periodic function must have odd symmetry. Similarly, cosine waves are even, so any Fourier cosine series representation of a periodic function must have even symmetry.

Fig. 12.8 shows sine and cosine representations of $f(x)$. Earlier we saw that, within $0 \le x < L$ we could expand the function $f(x) = x$ as either a sine series, *or* as a cosine series (Fig. 12.8: left). However, because sines and cosines have different symmetry, when we expand the range we obtain different shape waveforms for each of the series: the 'sawtooth' wave for the sine series (Fig. 12.8: centre) has odd symmetry, while the 'triangle' wave for the cosine series (Fig. 12.8: left) has even symmetry. For both periodic extensions, $f(x) = f(x + 2L)$. For the sine series we also have $f(x) = -f(-x)$ and, for the cosine series, $f(x) = f(-x)$.

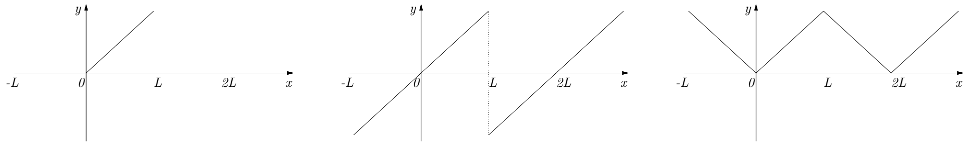


Figure 12.8: Left: Fourier sine or cosine representation of $f(x) = x$ within $0 \le x < L$. Center: periodic extension of sine series representation of $f(x)$. Right: periodic extension of cosine series representation of $f(x)$.

Exercise 12.9. A function is defined by $f(x) = x^2$, for $0 \le x < L$. Sketch the Fourier sine and cosine series representations of $f(x)$ in $-L \le x < 3L$.

Exercise 12.10. Within $0 \le x < L$, $f(x) = x$ can be expanded as a sine series, Eq. (12.7) or a cosine series Eq. (12.18). What function does each of these series represent in the range $-L \le x < 0$?

Figure 12.9 shows three examples of waveforms that are common in signal analysis. We can obtain a square wave by expanding the range of the Fourier half range sine series representation of $f(x) = 1$. We can similarly obtain a triangle wave from the Fourier half range cosine series representation of $f(x) = x$ from exercise 12.7. But what happens if we consider a function, $f(x)$, defined in the *full range* from $x = -L$ to $x = L$? If $f(x)$ is neither even or odd then we cannot expand it as only a sum of sine waves or only a sum of cosine waves. Instead we must use a Fourier full range series (section 12.6).

The rectified half wave (Fig. 12.9: right) is one such function. This is obtained from the full series representation of a function defined by $f(x) = 0$ where $-L \le x < 0$, $f(x) = \sin(\pi x/L)$ where $0 \le x < L$.

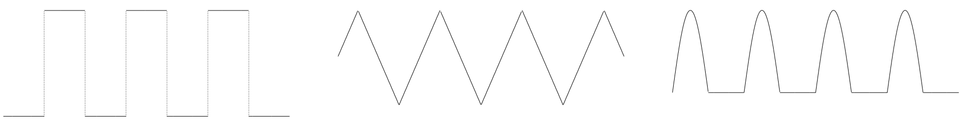


Figure 12.9: Some common waveforms: Square wave (left), triangle wave (centre), rectified half-wave (right).

12.6 Fourier full range series

The range 0 to $2L$ (or, alternatively, $-L$ to L) is called the *full range* because it contains a full wavelength of the periodic function. In this section we will mostly use the range $-L$ to L however, both 0 to $2L$, and $-L$ to L are exactly equivalent.

In the range $-L$ to L neither sine or cosine waves form a complete set. If a function defined between $-L$ and L has odd symmetry it can be represented as a sine series. If a function has even symmetry it can be represented as a cosine series. However, in the general case, to represent a function of arbitary symmetry, we need to include *both* sine and cosine terms in the representation.

12.6.1 Fourier full range series formula

The full range Fourier series for $f(x)$ in the range $-L \leq x < L$ is,

$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi x}{L} + b_n \sin\frac{n\pi x}{L}\right). \tag{12.20}$$

The formulae for the Fourier series full range coefficients can be derived in a similar way to the formulae for the sine and cosine half range coefficients. We have,

$$\begin{aligned} a_0 &= \frac{1}{L}\int_{-L}^{L} f(x)dx \\ a_n &= \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L}dx \\ b_n &= \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{n\pi x}{L}dx. \end{aligned} \tag{12.21}$$

Note: If the full range is defined to be between 0 to $2L$ the formulae remain the same except that the limits of the integration go from 0 to $2L$.

Exercise 12.11. A function $f(x) = 1 + x$ within $-\pi \leq x < \pi$. Calculate the Fourier full range series of $f(x)$.

Exercise 12.12. Calculate the full range Fourier series for a 'sawtooth' wave, $f(x) = x$, $-\pi \leq x < \pi$. Explain why the series is the same as the half range sine representation in exercise 12.1. By writing out the result for an appropriately chosen value of x, show that

$$\sum_{m=0}^{\infty}\frac{(-1)^m}{2m+1} = \frac{\pi}{4}.$$

12.7 Complex form of Fourier Series

Instead of Eq. (12.20) we could equally well write the complex form,

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{in\pi x/L}, \tag{12.22}$$

where,

$$c_n = \frac{1}{2L}\int_{-L}^{L} f(x)e^{-in\pi x/L}dx. \tag{12.23}$$

Sometimes this form is more convenient than the sine and cosine forms.

Derivation of complex series results

Recalling that sines and cosines can be written in terms of complex exponentials we can obtain Eq. (12.22) directly from Eq. (12.20).

$$\begin{aligned} f(x) &= a_0 + a_1\cos\frac{\pi x}{L} + b_1\sin\frac{\pi x}{L} + a_2\cos\frac{2\pi x}{L} + b_2\sin\frac{2\pi x}{L} + \ldots \\ &= a_0 + a_1\frac{e^{i\pi x/L}+e^{-i\pi x/L}}{2} + b_1\frac{e^{i\pi x/L}-e^{-i\pi x/L}}{2i} \\ &\quad + a_2\frac{e^{i2\pi x/L}+e^{-i2\pi x/L}}{2} + b_2\frac{e^{i2\pi x/L}-e^{-i2\pi x/L}}{2i} + \ldots. \end{aligned}$$

Collecting together exponentials with the same powers we have,

$$\begin{aligned} f(x) &= \ldots + (a_2/2 - b_2/(2i))e^{-i2\pi x/L} + (a_1/2 - b_1/(2i))e^{-i\pi x/L} + a_0 \\ &\quad + (a_1/2 + b_1/(2i))e^{i\pi x/L} + (a_2/2 + b_2/(2i))e^{i2\pi x/L} + \ldots \\ &= \ldots + c_{-2}e^{-i2\pi x/L} + c_{-1}e^{-i\pi x/L} + c_0 + c_1 e^{i\pi x/L} + c_2 e^{i2\pi x/L} + \ldots, \end{aligned}$$

which is identical to Eq. (12.22).

To find the formula for the complex coefficients, c_n, multiply both sides of Eq. (12.22) by $e^{-i\pi mx/L}$ and integrate,

$$\int_{-L}^{L} f(x)e^{-im\pi x/L}dx = \sum_{n=-1}^{\infty} c_n \int_{-L}^{L} e^{i(n-m)\pi x/L}dx.$$

The integral on the right is a standard integral, Eq. (A.7). Using this result we find,

$$\int_{-L}^{L} f(x)e^{-im\pi x/L}dx = \sum_{n=-\infty}^{\infty} c_n 2L\delta_{nm}, \tag{12.24}$$

then rearranging Eq. (12.24) for c_n we obtain the result in Eq. (12.23).

Exercise 12.13. If $f(x) = 1 + x$, $-\pi \leq x < \pi$ show that the complex Fourier series coefficients are given by

$$c_n = \delta_{n0} + \frac{(-1)^{n+1}L}{in\pi}.$$

Exercise 12.14. A function $f(x) = \exp(px)$ in $-\pi \leq x < \pi$. Expand $f(x)$ as a sum of complex exponentials to show that

$$f(x) = \sum_{n=-\infty}^{\infty} \frac{(-1)^n}{\pi(p - in)} \sinh(p\pi) e^{inx}.$$

Example 12.3. Show that the result for the complex Fourier series in exercise 12.13 is equal to that obtained in exercise 12.11.

The complex Fourier series for $f(x) = 1 + x$ in $-L \leq x < L$, is

$$\begin{aligned} f(x) &= \sum_{n=-\infty}^{\infty} \left(\delta_{n0} + \frac{(-1)^{n+1}L}{in\pi} \right) e^{in\pi x/L}, \\ &= 1 + \sum_{n=-\infty}^{-1} \frac{(-1)^{n+1}L}{in\pi} e^{in\pi x/L} + \sum_{n=1}^{\infty} \frac{(-1)^{n+1}L}{in\pi} e^{in\pi x/L}. \end{aligned}$$

Then, replacing n by $-n$ in the first summation, and factorising out $-L/\pi$ we have,

$$\begin{aligned} f(x) &= 1 + \frac{-L}{\pi} \sum_{n=1}^{\infty} \left[\frac{(-1)^{-n}}{i(-n)} e^{i(-n)\pi x/L} + \frac{(-1)^n}{in} e^{in\pi x/L} \right], \\ &= 1 + \frac{-L}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^n}{in} \left[e^{in\pi x/L} - e^{-in\pi x/L} \right], \\ &= 1 + \sum_{n=1}^{\infty} \frac{2(-1)^{n+1}L}{n\pi} \sin \frac{n\pi x}{L}, \end{aligned} \tag{12.25}$$

where, in the last step, we have used the fact that $e^{in\pi x/L} - e^{-in\pi x/L} = 2i \sin(n\pi x/L)$.

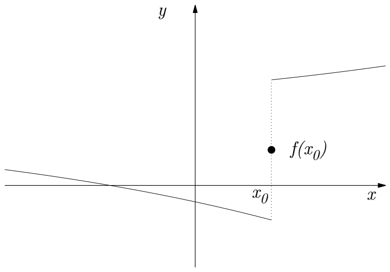
12.8 Properties of Fourier series

Here we quote without proof some facts about Fourier series that you should know.

General properties

a) Except for some pathological functions which do not occur in physical problems, we can *always* expand a function, $f(x)$, defined in a finite interval as a Fourier series which will converge with sum $f(x)$ at all points at which $f(x)$ is continuous.

b) If $f(x)$ has a discontinuity at $x = x_0$ its Fourier series will converge to the average of the limit from the left and the limit from the right.



The Fourier series of a discontinuous function sums to a value midway along the discontinuity.

$$f(x_0) \to \lim_{\epsilon \to 0} \frac{1}{2} \left(f(x_0 + \epsilon) + f(x_0 - \epsilon) \right).$$

c) A Fourier series can be integrated term by term: the resulting series always converges to $\int f(x)dx$.

d) Term by term differentiation of a Fourier series *may* produce a divergent series. If the series produced by differentiation does converge then it is the Fourier series for $f'(x)$.

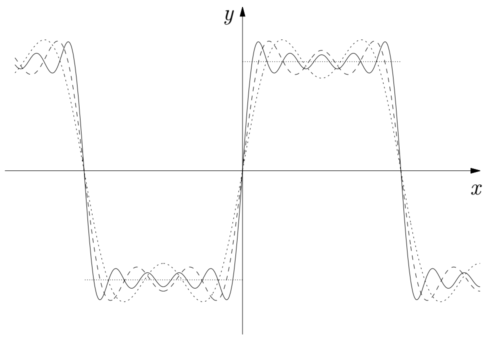
Convergence of series

The properties above apply to the converged Fourier series containing an infinite number of terms. In practice we often calculate the sum of only a finite number of the terms in the series which we use as an approximation. It is important to know when this is likely to give a good approximation.

a) If $f(x)$ or its periodic extension has *discontinuities* we expect a_n, b_n, c_n to be of order $1/n$ and the convergence is *slow.*

b) If $f(x)$ or its periodic extension is *continuous* we expect a_n, b_n, c_n to be of order $1/n^2$ and the convergence is *rapid.*

c) The Gibbs phenomenom. At a discontinuity (e.g. at x_0) convergence of a Fourier series is slow and a finite sum of N terms will persistently under- or over-estimate $f(x)$ near x_0. The size of the overshoot (or undershoot) does *not* tend to zero as $N \to \infty$, but region of the overshoot (or undershoot) does become narrower as the series converges.



The figure on the left shows the partial Fourier sine series of $f(x) = 1$, $0 \leq x < \pi$ with $n_{max} = 3$ (dotted), $n_{max} = 5$ dashed and $n_{max} = 25$ (solid line) terms.

Exercise 12.15. By using the results above briefly explain why the convergence of the half range cosine series representation of x (Eq. 12.18) is much faster than that of the sine series representation (Eq. 12.7).

Exercise 12.16. State whether each of the following functions, defined in the range $-\pi \leq x < \pi$, can be expanded as (a) a Fourier sine series, (b) a cosine series, or (c) a Fourier series containing both sine and cosine terms: (i) x^3; (ii) $x^2 + x$; (iii) $\exp(x)$; (iv) $|x|$.

12.9 Introduction to Fourier Transforms

So far, we have seen that we can express an arbitrary periodic function as a Fourier series, a sum of harmonic (sine and cosine) waves.

The *Fourier transform* gives us an analogous way to represent a general function that is not periodic. Fourier transforms are used in an enormous range of pure and applied science, including information processing, electronics and communications.

12.9.1 Fourier integrals and transforms

Fourier integrals

The Fourier series of a function $f(x)$ with period $\lambda = 2L$ can be written as,

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{in\pi x/L} = \sum_{n=-\infty}^{\infty} c_n e^{2in\pi x/\lambda}. \qquad (12.26)$$

So, what happens if the function is non-periodic? A non-periodic function is equivalent to a periodic function in the limit that $\lambda \to \infty$. So, let us consider what happens to 12.26 when λ becomes large.

First we define a new variable $k = 2\pi n/\lambda$. Then we can think of the Fourier coefficients, c_n as a function, $\tilde{f}(k) = c_{k\lambda/2\pi}$ defined on a line of points in k-space. The distance between the points in k-space is $\Delta_k = 2\pi/\lambda$. Then, using a technique often used in quantum mechanics and statistical physics, we can rewrite the sum in Eq. (12.26) as,

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{ikx} = \sum_{n=-\infty}^{\infty} \tilde{f}(k) e^{ikx} = \frac{\lambda}{2\pi} \sum_{n=-\infty}^{\infty} \tilde{f}(k) e^{ikx} \Delta_k.$$

In the limit as $\lambda \to \infty$, $\Delta_k \to 0$ and we can approximate the sum over k-points as an integral dk,

$$f(x) = \frac{\lambda}{2\pi} \int_{-\infty}^{\infty} \tilde{f}(k) e^{ikx} dk.$$

Finally, if we define $F(k) = \lambda \tilde{f}(k)$ we can write,

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(k) e^{ikx} dk. \tag{12.27}$$

This is a Fourier integral. It is a representation of an arbitrary (non-periodic) function, $f(x)$, in terms of simple harmonics.

Fourier transforms

To obtain a result for $F(k)$ in Eq. (12.27) we simply apply the formula (Eq. (12.23) for the Fourier series coefficients, $c_n = c_{k\lambda/2\pi}$, in the limit as $\lambda \to \infty$,

$$F(k) = \lim_{\lambda \to \infty} \lambda c_{kL/2\pi} = \int_{-\infty}^{\infty} f(x) e^{-ikx} dx. \tag{12.28}$$

$F(k)$ is called the Fourier transform of $f(x)$. The two functions, $F(k)$ and $f(x)$ are called a Fourier transform pair. $f(x)$ is the inverse Fourier transform of $F(k)$.

Unfortunately there is no standard definition in the literature of what constitutes a transform and what constitutes an inverse transform. The only requirement is that one of Eqs. (12.27) and (12.28) contains e^{-ikx} and one contains e^{ikx}. Similarly there is no set convention for the constant factors in front of these integrals. We have used a factor $1/2\pi$ on the inverse transform and 1 on the transform, but you will often see the opposite of this, or sometimes $1/\sqrt{2\pi}$ is used in front of both. This means that, when reading the literature, you should be careful to identify the conventions used.

The argument of the Fourier transform, k, has units that are the reciprocal of the dimensions of the variable x. We will often use x to denote position, in which case k is the wavenumber. Similarly, if the variable is a time, t, the transform variable will be a frequency, often denoted by ω. Physicists often talk about using Fourier transforms to transform from real space to reciprocal or k-space, or from the time-domain to the frequency-domain.

Calculating Fourier transforms and integrals

Fourier transforms and integrals are ordinary integrals that can be evaluated in the usual way. We will consider a specific example: the Fourier transform of a Gaussian.

Example 12.4. Find the Fourier transform of $f(x) = \exp(-x^2/\sigma^2)$. From Eq. (12.28) we have,

$$F(k) = \int_{-\infty}^{\infty} e^{-x^2/\sigma^2} e^{-ikx} dx = \int_{-\infty}^{\infty} e^{-(x^2/\sigma^2 + ikx)} dx.$$

This integral is performed with a standard trick. First, we complete the square in the argument of the exponential,

$$x^2/\sigma^2 + ikx = (x/\sigma + ik\sigma/2)^2 + k^2\sigma^2/4.$$

Then,

$$\begin{aligned} F(k) &= e^{-k^2\sigma^2/4} \int_{-\infty}^{\infty} e^{-(x/\sigma + ik\sigma/2)^2} dx, \\ &= \sigma e^{-k^2\sigma^2/4} \int_{-\infty}^{\infty} e^{-x'^2} dx', \end{aligned} \tag{12.29}$$

where we have changed variable from x to $x' = x/\sigma - ik\sigma/2$, so $dx' = dx/\sigma$. In this particular case, the change of variable leaves the limits on the integral unchanged at $-\infty$ and ∞. Then, using the fact that $\int_{-\infty}^{\infty} \exp(-x^2) dx = \sqrt{\pi}$, we find that the Fourier transform of a Gaussian is,

$$F(k) = \sigma\sqrt{\pi} e^{-k^2\sigma^2/4}. \tag{12.30}$$

So the Fourier transform of a Gaussian of half width σ is a Gaussian of half-width $2/\sigma$. ie. a wide Gaussian function in real space transforms to a narrow Gaussian function in k-space and vice-versa.

Exercise 12.17. Show that the Fourier transform of the function defined by $f(x) = a$ for $|x| \leq L$, $f(x) = 0$ for $|x| > L$ is

$$F(k) = \frac{2a \sin kL}{k}.$$

Exercise 12.18. Find the inverse Fourier transform of $F(k) = \exp(-k^2/4)$.

12.9.2 Convolutions

In later units you will study in detail the properties of Fourier transforms, including their application in calculating convolutions and correlations. Here we will briefly introduce a useful result - the convolution theorem.

A convolution integral has the general form

$$C(x) = \int_{-\infty}^{\infty} f(x - x')g(x')dx'. \tag{12.31}$$

This integral relates an *output* function, $C(x)$, to the *input* $g(x)$ and the *response* $f(x)$. Figure 12.10 shows an example from imaging processing where we degrade an initially sharp image with a Gaussian 'blur'. The process is illustrated in one dimension in Fig. 12.11: we convolve the input, $g(x)$, with a Gaussian response function, $f(x) = \exp(-x^2)$, to give the new blurred output $C(x) = \int_{-\infty}^{\infty} f(x - x')g(x')dx'$. You will often see a similar effect in experimental physics where some initially sharp 'input' signal will suffer from Gaussian broadening as a result of an imprecise experimental response.

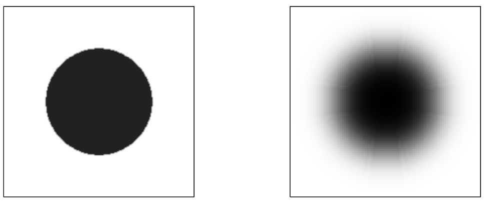


Figure 12.10: Image processing with convolutions. A sharp image of a circle (left) is convolved with a Gaussian response function to give the blurred image (right).

Convolution Theorem

The convolution theorem is a useful relation between the Fourier components of the input and output functions. According to the convolution theorem,

$$C(k) = F(k)G(k). \tag{12.32}$$

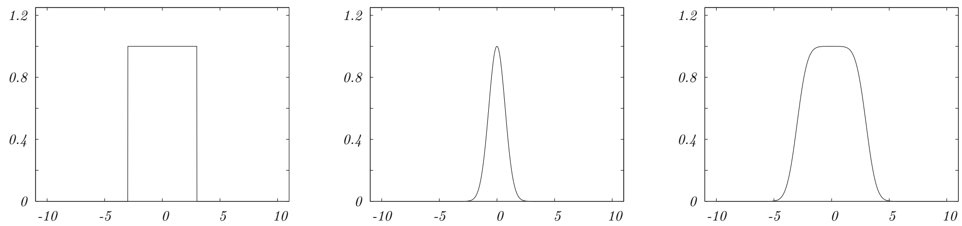


Figure 12.11: Example convolution. Left: sharp 'input' signal, $g(x) = 1, |x| < 3$. Centre: Gaussian response function, $f(x) = \exp(-x^2)$. Right: blurred output signal, $C(x) = \int_{-\infty}^{\infty} f(x-x')g(x')dx'$.

To prove this relation we simply calculate the Fourier transform of the output function $C(x)$,

$$C(k) = \int_{-\infty}^{\infty} C(x)e^{-ikx}dx = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x-x')g(x')e^{-ikx}dx'dx.$$

Putting $u = x - x'$ and using u as the integration variable instead of x we have,

$$\begin{aligned} C(k) &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(u)g(x')e^{-ik(x'+u)}dx'du \\ &= \int_{-\infty}^{\infty} f(u)e^{-iku}du \int_{-\infty}^{\infty} g(x')e^{ikx'}dx' \\ &= F(k)G(k). \end{aligned}$$

Fourier Deconvolution

The convolution theorem tells us that convolutions in real space are simply multiplications in reciprocal space. This is very useful if we want to *deconvolve* a signal.

For example, let us imagine that, in an experiment, we want to measure some input $g(x)$. However, because of our experimental response, $f(x)$, we actually measure a degraded output signal, $C(x)$. How do we get back to the actual input that we would like to measure? We could try to solve the integral equation (12.31) but it is much easier to make use of the convolution theorem.

In reciprocal space we can find $G(k)$ by dividing $C(k)$ by $F(k)$. Then we can easily obtain $g(x)$ with an inverse transform,

$$g(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{C(k)}{F(k)}e^{ikx}dk. \qquad (12.33)$$

Figure 12.12 shows an illustration of this process. An initially blurred image is sharpened by deconvolving the image with a Gaussian response function.

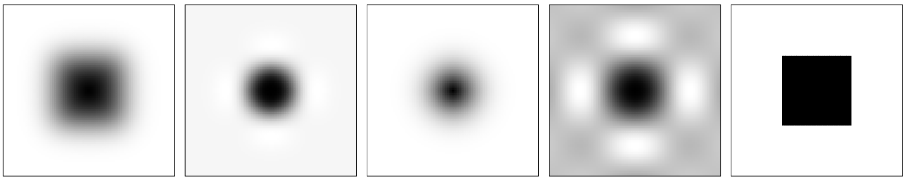


Figure 12.12: Sharpening of an image by deconvolution. From left to right: original blurred image $C(x, y)$; Fourier transform of blurred image, $C(k_x, k_y)$; response function in reciprocal space, $F(k_x, k_y)$; input function in reciprocal space, $G(k_x, k_y) = C(k_x, k_y)/F(k_x, k_y)$; decovolved image, $g(x, y)$.

Exercise 12.19. Write down the analogous results for the convolution integral and convolution theorem, in the time and frequency domain.

Exercise 12.20. In reciprocal space an experimental signal that has been broadened with a Gaussian response function, $F(k) = \sqrt{\pi}\exp(-ak^2/4)$, has the form $C(k) = (2\sqrt{\pi}/k)\exp(-ak^2/4)\sin k$. Use the convolution theorem to write down the input signal, $G(k)$, in reciprocal space, then transform this into real space to find the original form of the input, $g(x)$. *Hint*: You can often find a Fourier transform simply by recognising that it is related to a known inverse transform.

12.10 Revision points

After completing this unit you should be able to:

a) Appreciate how an arbitrary periodic function can be represented by an infinite summation of sines and cosines, the *Fourier series* of a function.

b) Use the Fourier series representation to find the solution of partial differential equations, including the wave equation, satisfying given boundary conditions.

c) Distinguish between *full range* (both complex and real form), and *half range sine* and *cosine* Fourier representations of a given function.

d) Write down the formulae for the Fourier series coefficients in each case.

e) Find the Fourier half range and full range series representations of a given function in a finite interval.

f) Understand the periodic extensions of Fourier series.

g) Appreciate how an arbitrary non-periodic function can by represented by a *Fourier transform*.

h) Calculate the Fourier transform and inverse transform of a given function.

i) Prove the convolution theorem from the convolution integral.

12.11 Problems

Exercise 12.21. A function $f(\theta) = \theta^3$ in $0 \leq \theta < \pi$ is expanded in (a) a Fourier sine series and (b) a Fourier cosine series. Sketch the form of the function represented by these series in the range $-\pi < \theta < 3\pi$.

Exercise 12.22. A function $f(\theta)$, $\pi \leq \theta < \pi$ is known to have a Fourier series of the form

$$f(\theta) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} \cos n\theta.$$

Show that $f(\theta) = f(-\theta)$.

Exercise 12.23. Find the Fourier series for $f(x) = x^2$, $-\pi \leq x < \pi$ and show that

$$\frac{\pi^2}{12} = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2}$$

Exercise 12.24. Find the Fourier series for the function

$$f(\theta) = \begin{cases} \alpha, & -\pi \leq \theta < 0 \\ \beta, & 0 \leq \theta < \pi, \end{cases}$$

where α and β are constants. *Hint*: the neatest approach is to write $f(\theta)$ as the sum of a symmetric function and an antisymmetric one.

By setting $\theta = \pi/2$ show that,

$$\sum_{n=0}^{\infty} = \frac{(-1)^n}{2n+1} = \frac{\pi}{4}$$

Exercise 12.25. Find the Fourier series for

$$f(t) = \begin{cases} -\sin t, & -\pi \leq t < 0 \\ \sin t, & 0 \leq t < \pi. \end{cases}$$

Hence show that

$$\sum_{n=2,4,6,\ldots}^{\infty} \frac{1}{n^2 - 1} = \frac{1}{2}.$$

Exercise 12.26. Expand the function $f(x) = x$ as a half range Fourier cosine series in the range $0 \leq x < L$. What function does this series represent in the interval $-L \leq x < L$?

Exercise 12.27. Expand the function $f(x) = \sin \pi x$ as a half range Fourier cosine series in the range $0 \leq x < 1$. By considering the series for a suitable value of x show that,

$$\sum_{p=1}^{\infty} \frac{(-1)^p}{4p^2 - 1} = \frac{1}{2} - \frac{\pi}{4}.$$

Exercise 12.28. Calculate the full range complex Fourier series representation of $f(x) = x + x^2$.

Exercise 12.29. Show that the Fourier transform of a function defined by $f(x) = \exp(-a^2 x)$ for $x \geq 0$, $f(x) = 0$ otherwise, is $1/(ik + a^2)$.

Exercise 12.30. Calculate the inverse Fourier transform of $F(k) = \exp(-k^2)$.

Appendix A

Additional results

A.1 Kronecker delta

The Kronecker delta is convenient mathematical shorthand. It is defined by

$$\delta_{nm} = \begin{cases} 1 & \text{if } n = m \\ 0 & \text{if } n \neq m. \end{cases} \tag{A.1}$$

A.2 Standard integrals

$$\int_0^L \sin\frac{n\pi x}{L} \sin\frac{m\pi x}{L} dx = \begin{cases} 0 & \text{if } m = n = 0 \\ \frac{L}{2}\delta_{mn} & \text{otherwise} \end{cases} \tag{A.2}$$

$$\int_0^L \cos\frac{n\pi x}{L} \cos\frac{m\pi x}{L} dx = \begin{cases} L & \text{if } m = n = 0 \\ \frac{L}{2}\delta_{mn} & \text{otherwise} \end{cases} \tag{A.3}$$

$$\int_{-L}^L \sin\frac{n\pi x}{L} \sin\frac{m\pi x}{L} dx = \begin{cases} 0 & \text{if } m = n = 0 \\ L\delta_{mn} & \text{otherwise} \end{cases} \tag{A.4}$$

$$\int_{-L}^L \cos\frac{n\pi x}{L} \cos\frac{m\pi x}{L} dx = \begin{cases} 2L & \text{if } m = n = 0 \\ L\delta_{mn} & \text{otherwise} \end{cases} \tag{A.5}$$

$$\int_{-L}^L \sin\frac{n\pi x}{L} \cos\frac{m\pi x}{L} dx = 0 \tag{A.6}$$

$$\int_{-L}^{L} e^{i(n-m)\pi x/L} dx = 2L\delta_{nm}, \tag{A.7}$$

A.2.1 Derivation of standard integrals

The results for the standard integrals Eqs. (A.2) to (A.6) can be found using the product formulae for trigonometric functions. For example, to obtain Eq. (A.2) we write,

$$\begin{aligned}
\int_0^L \sin\frac{m\pi x}{L}\sin\frac{n\pi x}{L}dx &= \frac{1}{2}\int_0^L \left(\cos\frac{(m-n)\pi x}{L} - \cos\frac{(n+m)\pi x}{L}dx\right) \\
&= \frac{1}{2}\frac{L}{(m-n)\pi}\left[\sin\frac{(m-n)\pi x}{L}\right]_0^L - \frac{1}{2}\frac{L}{(m+n)\pi}\left[\sin\frac{(m+n)\pi x}{L}\right]_0^L \\
&= 0 \text{ if } m \neq n, \text{ because all the sine terms are zero.}
\end{aligned}$$

If $m = n \neq 0$ then the integral becomes,

$$\begin{aligned}
\int_0^L \sin\frac{m\pi x}{L}\sin\frac{n\pi x}{L}dx &= \int_0^L \sin^2\frac{n\pi x}{L} \\
&= \frac{1}{2}\int_0^L \left(1 - \cos\frac{2n\pi x}{L}\right)dx \\
&= \frac{L}{2}.
\end{aligned}$$

To obtain Eq. (A.7) we have, if $n \neq m$,

$$\begin{aligned}
\int_{-L}^{L} \exp\left(\frac{i(n-m)\pi x}{L}\right)dx &= \left[\frac{L}{i(n-m)\pi}\exp\left(\frac{i(n-m)\pi x}{L}\right)\right]_{-L}^{L} \\
&= \frac{L}{i(n-m)\pi}\left[e^{i(n-m)\pi} - e^{-i(n-m)\pi}\right] \\
&= 0,
\end{aligned}$$

because $\exp(i(n-m)\pi) = \exp(-i(n-m)\pi) = \cos((n-m)\pi)$.

However, if $n = m$, then

$$\begin{aligned}
\int_{-L}^{L} \exp\left(\frac{i(n-m)\pi x}{L}\right)dx &= \int_{-L}^{L} dx \\
&= 2L.
\end{aligned}$$

A.3 Even and Odd functions

A.3.1 Definitions

If $f(-x) = f(x)$ the function, $f(x)$, is *even*. This means that $f(x)$ is symmetric about the y-axis. $f(x) = x^2$ and $f(x) = \cos(n\pi x/L)$ are examples of even functions.

If $g(-x) = -g(x)$ the function, $g(x)$, is *odd*. This means that $g(x)$ is anti-symmetric about the y-axis. $g(x) = x^3$ and $g(x) = \sin(n\pi x/L)$ are examples of odd functions.

A function may be neither even or odd. For example, both $h(x) = \exp(x)$ and $h(x) = x + \cos(x)$ are functions that are neither even or odd.

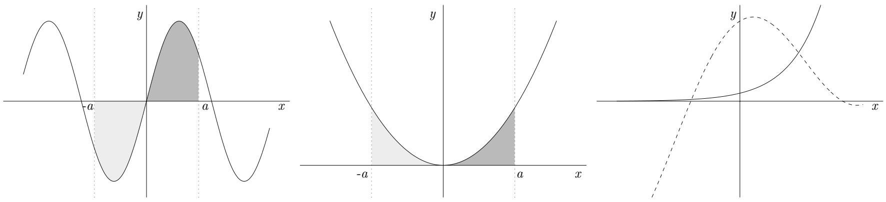


Figure A.1: Left: an example of an odd function, $f(x) = \sin 2x$. Centre: an example of an even function, $f(x) = x^2$. Right: Two functions, $h_1(x) = \exp(x)$ (solid line) and $h_2(x) = x + \cos x$ (dashed line) that are neither odd or even.

It is always possible to write a function as a sum of even and odd functions,

$$h(x) = f(x) + g(x),$$

where

$$f(x) = \frac{1}{2}(h(x) + h(-x)) \text{ is even,}$$

and

$$g(x) = \frac{1}{2}(h(x) - h(-x)) \text{ is odd.}$$

Exercise A.1. Show that $f(x)$ and $g(x)$ defined above are respectively even and odd.

A.3.2 Products of even and odd functions

The product of two even functions is always even. So, because $f_1(x) = x^2$ and $f_2(x) = \cos(x)$ are both even, then $h(x) = f_1(x)f_2(x) = x^2\cos(x)$ is even.

The product of two odd functions is always even. So, because $g_1(x) = x$ and $g_2(x) = \sin(x)$ are both odd, then $h(x) = g_1(x)g_2(x) = x\sin(x)$ is even.

The product of an even function and an odd function is always odd. So, because $f(x) = x^2$ is even and $g(x) = \sin(x)$ is odd, then $h(x) = f(x)g(x) = x^2\sin(x)$ is odd.

A.3.3 Integrals of even and odd functions

We can make use of symmetry to simplify the integrals of even and odd functions between limits that are symmetric about the origin. The integral of any odd function, $g(x)$ between $-a$ and a is simply zero. We can see this graphically in Fig. A.1: the area under the curve between $-a$ and 0 is equal and opposite to the area under the curve between 0 and a. For any even function we have $\int_{-a}^{0} f(x)dx = \int_0^a f(x)dx$. Again, this is illustrated graphically in Fig. A.1.

In general,

$$\int_{-a}^{a} h(x)dx = \begin{cases} 0 & \text{if } h(x) \text{ is odd} \\ 2\int_0^a h(x)dx & \text{if } h(x) \text{ is even.} \end{cases} \tag{A.8}$$

Exercise A.2. Derive the relations in Eq. (A.8). *Hint*: First split the integral into two, a part from $-a$ to 0 and a part from 0 to a, then change variable from x to $x' = -x$ in the region where $x < 0$.