Continuous and Discrete Wavelet Transform and Multiscale Analysis

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Abstract. Wavelets have undergone a rapid growth in the last three decades both in mathematical research and applications. The wavelet analysis was developed as a new tool for data analysis in fields like geophysics, signal processing, data compression, regularization of inverse problems, fluid mechanics, etc. In this paper we review some aspects of continuous and discrete wavelet transform. We start by reviewing basic aspects of Fourier analysis. Then, we present the continuous wavelet transform which uses locally confined little waves, instead the trigonometric functions in the Fourier theory, as analyzing functions. We will show that translating and scaling allows for a frequency resolution at arbitrary position. The continuous theory allows an elegant introduction to the Discrete Wavelet Transform, through the concept of wavelet frame. This is the starting point to present wavelet basis and multiscale analysis. We include also an example of discrete wavelet based on Haar functions and present some results in multidimensional wavelet analysis.

1 Introduction

The continuous wavelet theory, as well as the Fourier one, can be viewed in the context of mathematical operators that map functions from a functional space to another one. The most famous is the Fourier Transform (FT) which transforms a signal (function) that exists in the time (or space) domain to the frequency domain [2].

The FT accomplishes this task through a kernel composed by sine and cosine waveforms. This is the origin of the main disadvantage of FT for signal analysis: *if a signal changes at a specific time, its transform changes everywhere and a simple inspection of the transformed signal does not reveal the position of the alteration* [7]. The wavelet framework can overcome this problem.

The Continuous Wavelet Transform (CWT) addresses this issue through the scalar product of the original signal with a translated and dilated version of a locally confined function, called *wavelet* . Therefore, the CWT of a function depends on two parameters, b for translation and a for dilation. The parameter b shifts the wavelet so that local information around time $t = b$ is contained in the transformed function. The parameter a controls the *window* size in which the signal analysis must be performed. In this way, the obtained functional representation can overcomes the missing localization property of the FT analysis [7].

The concept of basis functions and scale-varying basis functions is fundamental to understanding wavelets. Historically, such theory received a great impact in the 1930s, when several groups working independently had established some foundations of such theory and its applications. For instance, by using the scale-varying Haar basis function, the physicist Paul Levy, investigated at that time the Brownian motion. He found the Haar basis function superior to the Fourier basis functions for that study [5].

In 1980, Grossman and Morlet, a physicist and an engineer, broadly defined wavelets in the context of quantum physics. In 1985, Stephane Mallat discovered some relationships between quadrature mirror filters, pyramid algorithms, and orthonormal wavelet bases. A couple of years later, Ingrid Daubechies used Mallat's worked to perform perhaps the most elegant development in this field. The set of orthonormal wavelet basis functions constructed has become a very remarkable work for wavelet applications since then [5].

In this paper we review some aspects of continuous and discrete wavelet theory. We start with basic elements of Fourier analysis. Then, we discuss the continuous wavelet transform (CWT) which uses locally confined little waves as analyzing functions. We clarify the fact that translating and scaling allows for a frequency resolution at arbitrary position. Therefore, we present some results about time-frequency analysis and the inverse of the CWT.

The continuous theory allows an elegant introduction to the Discrete Wavelet Transform (DWT), through the concept of wavelet frame. The key idea is that the representation of a function through the inverse CWT, a double integral in (a, b) space, is redundant. Therefore, the integral representation can be replaced by a double sum without loss of information. So, following [7], we review wavelet frames and the reconstruction of an input function from a discretization of the inverse CWT. The main consequences are the (orthonormal) wavelet basis and multiscale analysis.

We also describe an example of discrete wavelet based on Haar functions and present some results in multidimensional wavelet analysis. The presentation is organized based on the references [7, 12, 1]. There are many sites that can be consulted to help the beginners in wavelet theory [13, 5, 6, 3, 13]. Besides, toolboxes can be found for the MatLab package [9, 11].

2 Fourier Analysis Revised

In this section we review some important results about continuous Fourier Analysis. A special attention will be given to the connections between the Fourier Transform and Fourier Series representation of a signal. Thus, let us begin with usual definitions in this area. For simplicity, we restrict our discussion to the one dimensional case. Generalizations to higher-dimensional cases are straightforward. In this section our presentation follows the reference [1].

Throughout this text, all functions f are defined on the real line, with values in the complex domain C, $f : \Re \to C$. Besides, they are assumed to be piecewise continuous and having the integral:

$$\int_{-\infty}^{\infty} |f(x)|^p \, dx, \tag{1}$$

finite. The set of such functions is denoted by $L^P(\Re)$. In this set, the p-norm and the inner product are well defined by:

$$\|f\|_p = \left\{ \int_{-\infty}^{\infty} |f(x)|^p \, dx \right\}^{1/p}, \quad 1 \leq p < \infty, \tag{2}$$

$$\langle f, g \rangle = \int_{-\infty}^{\infty} f(x) \overline{g(x)} dx, \quad f, g \in L^2(\Re). \tag{3}$$

Definition 1: Given a function $f \in L^1(\Re)$, its Fourier Transform is defined by:

$$\widehat{f}(\omega) = (F f)(\omega) = (2\pi)^{-1/2} \int_{-\infty}^{\infty} f(x) \exp(-j\omega x) \, dx. \tag{4}$$

Definition 2: Given the Fourier transform $\widehat{f}$, the transformed function f can be obtained by the Inverse Fourier Transform, given as follows:

$$f(x) = \left(F^{-1} \widehat{f}\right)(x) = (2\pi)^{-1/2} \int_{-\infty}^{\infty} \widehat{f}(\omega) \exp(jx\omega) \, d\omega. \tag{5}$$

Definition 3: Let f and g be functions in $L^1(\Re)$. Then the *convolution* of f and g is also an $L^1(\Re)$ function h which is defined by:

$$h(x) = (f * g)(x) = \int_{-\infty}^{\infty} f(x - y) \, g(y) \, dy. \tag{6}$$

Theorem 1: Any function $f \in L^2(-\pi, \pi)$ has a Fourier series representation given by:

$$f(x) = \sum_{n=-\infty}^{+\infty} a_n \exp(jxn). \tag{7}$$

Dem: See [2].

Among the properties of the Fourier Transform, the following ones have a special place for signal processing techniques, according to our discussion presented in [4].

Property 1: If the derivative df/dx of f exists and is in $L^1(\Re)$, then:

$$\left[F\left(\frac{df}{dx}\right)\right](\omega) = \widehat{f'}(\omega) = j\omega\widehat{f}. \tag{8}$$

Property 2: $\widehat{f} \to 0$, as $\omega \to \infty$ or $\omega \to -\infty$.

For the proofs, see [1], pp. 25.

Property 3 (Convolution Theorem): The Fourier transform of the convolution of two functions is the product of their Fourier transforms, that is:

$$h(x) = (f * g)(x) \Leftrightarrow \widehat{h}(\omega) = (2\pi)^{1/2}\,\widehat{f}(\omega)\,\widehat{g}(\omega). \tag{9}$$

Dem: See [2].

An important aspect for signal (and image) processing is the relationship between the coefficients a_n in the series (7) and the Fourier transform defined by the integral (4), in the case of a function $f \in L^2(-\pi,\pi)$. In order to find this relationship, let us remember that, from the orthogonality of the functions $\{\exp(jxn), n \in Z\}$, it is easy to show what the coefficients a_n can be obtained by the expression:

$$a_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\exp(-jxn)\,dx. \tag{10}$$

Now, we must observe that this expression is equivalent to:

$$a_n = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(x)\exp(-jxn)\,dx, \tag{11}$$

once $f \in L^2(-\pi,\pi)$.

Thus, using the Fourier transform definition in expression (4), we get:

$$a_n = \widehat{f}(n). \tag{12}$$

This result shows the connections between Fourier series and the Fourier Transform. A more general version of it is used in [4] to discussion Fourier analysis in the context of signal processing.

3 The Continuous Wavelet Transform (CWT)

A function $\psi \in L^2$ which satisfies the admissibility condition

$$0 < c_\psi := 2\pi\int_{\Re}\frac{\left|\widehat{\psi}(\omega)\right|^2}{|\omega|}d\omega < \infty$$

is called a **wavelet**.

The Continuous Wavelet Transform (CWT) L_ψ of a function $f \in L^2(\Re)$ with respect to the wavelet ψ is defined as:

$$L_\psi f(a,b) = \frac{1}{\sqrt{c_\psi}}|a|^{-1/2}\int_{\Re} f(t)\psi\left(\frac{t-b}{a}\right)dt \tag{13}$$

$a \in \Re\backslash\{0\}, b \in \Re$.

Lemma 1: Let the k times, $k \geq 1$, differentiable function φ be given, with $\varphi, \varphi^{(k)} \in L^2(\Re)$ and $\varphi \neq 0$. Then,

$$\psi(x) := \varphi^{(k)}(x)$$

is a wavelet.

Examples

1. Haar Wavelet: Let
$$\psi\left(t\right)=\begin{cases} 1: 0\leq t<1/2 \\ -1: 1/2\leq t<1 \\ 0: otherwise. \end{cases} \tag{14}$$
Then the Fourier transform of ψ, with the help of the sinc function:
$$sinc(x)=\frac{\sin\left(x\right)}{x} \tag{15}$$
is given by:
$$\widehat{\psi}\left(\omega\right)=ie^{-i\omega/2}\sin\left(\omega/4\right)sinc\left(\omega/4\right)/\sqrt{2\pi},$$
so $\left|\widehat{\psi}\right|$ is an even function with $c_{\psi}=2\ln 2$ and an absolute global maximum at $\omega_0=\pm 4.6622$. The function ψ is the Haar wavelet. It is perhaps the most traditional example of wavelet.

2. Mexican Hat: Let $\varphi\in L^1\left(\Re\right)$ be a continuously differentiable function and $\psi=\varphi\prime\in L^2$. Then ψ satisfies the admissibility condition. An important example of this type is the Mexican hat:
$$\psi\left(x\right)=-\frac{d^2}{dx^2}e^{-x^2/2}=\left(1-x^2\right)e^{-x^2/2}=\left(1-x^2\right)e^{-x^2/2}$$

Corollary 1 Let the function $0\neq\psi\in L^2\left(\Re\right)$ have a compact support. Then the following are equivalent:
(i) The mean value $\int_{\Re}\psi(t)dt$ of ψ is zero.
(ii) The function ψ is a wavelet.
Proof: See [7].
Theorem 2 (Inversion): We can demonstrate the following reconstruction formula:
$$f\left(t\right)=c_{\psi}^{-1/2}\int_{\Re}\int_{\Re}\left\{L_{\psi}f(a,b)\right\}\left|a\right|^{-1/2}\psi\left(\frac{t-b}{a}\right)\frac{dadb}{a^2}$$
Proof: See [7].

4 Time-Frequency function

Let the center t^* and radius Δ_{ψ} of a window function $\psi=\psi\left(t\right)$ are defined to be:
$$t^*:=\frac{1}{\left\|\psi\right\|_2^2}\int_{-\infty}^{\infty}t\,\left|\psi\left(t\right)\right|^2dt, \tag{16}$$
$$\Delta_{\psi}=\frac{1}{\left\|\psi\right\|_2}\left\{\int_{-\infty}^{\infty}\left(t-t^*\right)^2\,\left|\psi\left(t\right)\right|^2dt\right\}^{1/2}. \tag{17}$$
If $t^*<\infty$ and $\Delta_{\psi}<\infty$, we say that the signal ψ is *localized* about the point t^* with the time window $\left[t^*-\Delta_{\psi},t^*+\Delta_{\psi}\right]$. The time window corresponding to the wavelet transform kernel:
$$\psi_{a,b}\left(t\right)=\psi\left(\frac{t-b}{a}\right), \tag{18}$$
can be computed as follows. Consider that,

$$x = (t-b)/a \Longrightarrow t = xa + b \quad and \quad dt = adx. \tag{19}$$

Therefore:

$$\|\psi_{a,b}\|_2^2 = \int_{-\infty}^{\infty} \left|\psi\left(\frac{t-b}{a}\right)\right|^2 dt = \int_{-\infty}^{\infty} |\psi(x)|^2 \, adx = a\,\|\psi\|_2^2\,. \tag{20}$$

The variable change in expression (19) can be used to compute the center of the time window for function (18), given by:

$$\widetilde{t} = \frac{1}{\|\psi_{a,b}\|_2^2} \int_{-\infty}^{\infty} t\; |\psi_{a,b}(t)|^2 \, dt \tag{21}$$

Using expression (19) and the result (20) it is possible to rewriting the equation (21) as:

$$\begin{aligned}
\widetilde{t} &= \frac{1}{a\,\|\psi\|_2^2} \int_{-\infty}^{\infty} (xa+b)\; |\psi(x)|^2 \, adx \\
\widetilde{t} &= \frac{1}{a\,\|\psi\|_2^2} \left[a^2 \int_{-\infty}^{\infty} x\,|\psi(x)|^2 \, dx + ba \int_{-\infty}^{\infty} |\psi(x)|^2 \, dx\right]
\end{aligned}$$

$$\widetilde{t} = a^2 \underbrace{\frac{1}{a} \frac{1}{\|\psi\|_2^2} \int_{-\infty}^{\infty} x\,|\psi(x)|^2 \, dx}_{t^*} + ba \frac{1}{a\,\|\psi\|_2^2} \underbrace{\int_{-\infty}^{\infty} |\psi(x)|^2 \, dx}_{\|\psi\|_2^2}$$

$$\widetilde{t} = at^* + b \tag{22}$$

Following expression (17) the radius of the time window is given by:

$$\widetilde{\Delta}_\psi = \frac{1}{\|\psi_{a,b}(t)\|_2} \left\{ \int_{-\infty}^{\infty} \left(t - \widetilde{t}\right)^2 \; |\psi_{a,b}(t)|^2 \, dt \right\}^{1/2} \tag{23}$$

Substituting $\widetilde{t}$ by equation (22) and using result (20) we can rewrite the equation (23) as follows:

$$\widetilde{\Delta}_\psi = \frac{1}{\sqrt{|a|}\,\|\psi\|_2} \left\{ \int_{-\infty}^{\infty} (ax + b - at^* - b)^2 \; |\psi(x)|^2 \, adx \right\}^{1/2}$$

Therefore:

$$\widetilde{\Delta}_\psi = \frac{\sqrt{|a|}}{\sqrt{|a|}\,\|\psi\|_2} \left\{ \int_{-\infty}^{\infty} a^2 (x - t^*)^2 \; |\psi(x)|^2 \, dx \right\}^{1/2}$$

$$\widetilde{\Delta}_\psi = |a| \underbrace{\frac{1}{\|\psi\|_2} \left\{ \int_{-\infty}^{\infty} (x - t^*)^2 \; |\psi(x)|^2 \, dx \right\}^{1/2}}_{\Delta\psi}$$

$$\widetilde{\Delta}_\omega = |a|\,\Delta_\psi \tag{24}$$

Then, by expressions (22) and (24) the "time window" of the function $\psi_{a,b}(t)$ is:

$$[b + at^* - a\Delta_\psi, b + at^* + a\Delta_\psi] \tag{25}$$

The frequency window for the function $\widehat{\psi}$ (the Fourier transform of ψ) is defined analogous to expressions (16)-(17):

$$\omega^* = \frac{1}{\left\|\widehat{\psi}\right\|_2^2} \int_{-\infty}^{\infty} x \left|\widehat{\psi}(x)\right|^2 dx, \tag{26}$$

$$\Delta_{\widehat{\psi}} = \frac{1}{\left\|\widehat{\psi}\right\|_2} \left\{ \int_{-\infty}^{\infty} (x - \omega^*)^2 \left|\widehat{\psi}(x)\right|^2 dx \right\}^{1/2}. \tag{27}$$

From the Prancherel's theorem it follows that CWT can computed in the frequency domain by:

$$(W_\psi f)(a, b) = \frac{|a|^{1/2}}{\sqrt{c_\psi}} \int_{-\infty}^{\infty} \widehat{f}(\omega) e^{ib\omega} \overline{\widehat{\psi}(a\omega)} d\omega. \tag{28}$$

Therefore, we must compute the frequency window for $\widehat{\psi}(a\omega)$. Lets introduce the notation:

$$\widehat{\psi}_a(\omega) = \widehat{\psi}(a\omega). \tag{29}$$

Therefore, expression (26) becomes:

$$\widetilde{\omega^*} = \frac{1}{\left\|\widehat{\psi}_a\right\|_2^2} \int_{-\infty}^{\infty} \omega \left|\widehat{\psi}_a(\omega)\right|^2 d\omega \tag{30}$$

Firstly, changin the integration variable:

$$x = a\omega \Longrightarrow \omega = x/a \quad and \quad d\omega = \frac{dx}{a} \tag{31}$$

we get:

$$\left\|\widehat{\psi}_a\right\|_2^2 = \int_{-\infty}^{\infty} \left|\widehat{\psi}(a\omega)\right|^2 d\omega = \int_{-\infty}^{\infty} \left|\widehat{\psi}(x)\right|^2 \frac{dx}{a} = \frac{1}{a} \left\|\widehat{\psi}\right\|_2^2. \tag{32}$$

By substituting this result in expression (30) and changing variables following expression (31) we re-write $\widetilde{\omega^*}$ as:

$$\widetilde{\omega^*} = \frac{a}{\left\|\widehat{\psi}\right\|_2^2} \left[\int_{-\infty}^{\infty} \frac{x}{a} \left|\widehat{\psi}(x)\right|^2 \frac{dx}{a} \right]. \tag{33}$$

Therefore,

$$\widetilde{\omega^*} = \frac{1}{a} \underbrace{\frac{1}{\left\|\widehat{\psi}(x)\right\|_2^2} \int_{-\infty}^{\infty} x \left|\widehat{\psi}(x)\right|^2 dx}_{\omega^*}.$$

So,

$$\widetilde{\omega^*} = \frac{1}{a} \omega^* \tag{34}$$

From expression (27) the radius of the frequency window for function $\widehat{\psi}_a$ defined by equation (29) is:

$$\widetilde{\Delta_{\widehat{\psi}}} = \frac{1}{\left\|\widehat{\psi}_a\right\|_2} \left\{ \int_{-\infty}^{\infty} \left(\omega - \widetilde{\omega^*}\right)^2 \left|\widehat{\psi}\left(a\omega\right)\right|^2 d\omega \right\}^{1/2}. \tag{35}$$

Again, by replacing $\left\|\widehat{\psi}_a\right\|_2$ by expression (32), using result (34) and changing the integration varible following expression (31) we get:

$$\widetilde{\Delta_{\widehat{\psi}}} = \frac{\sqrt{a}}{\left\|\widehat{\psi}\right\|_2} \left\{ \int_{-\infty}^{\infty} \left(\frac{x}{a} - \frac{1}{a}\omega^*\right)^2 \left|\widehat{\psi}\left(x\right)\right|^2 \frac{dx}{a} \right\}^{1/2}, \tag{36}$$

which finally renders:

$$\widetilde{\Delta_{\widehat{\psi}}} = \frac{\sqrt{a}}{\sqrt{a^3}} \underbrace{\frac{1}{\left\|\widehat{\psi}\right\|_2} \left\{ \int_{-\infty}^{\infty} \left(x - \omega^*\right)^2 \left|\widehat{\psi}\left(x\right)\right|^2 dx \right\}^{1/2}}_{\Delta\widehat{\psi}}.$$

Therefore:

$$\widetilde{\Delta_{\widehat{\psi}}} = \frac{1}{|a|}\Delta_{\widehat{\psi}}. \tag{37}$$

Therefore, the results (34) and (37) show that the function $\widehat{\psi}_a$ can be localized in the frequency domain by the "frequency window":

$$\left[\frac{\omega^*}{a} - \frac{1}{|a|}\Delta_{\widehat{\psi}}, \frac{\omega^*}{a} + \frac{1}{|a|}\Delta_{\widehat{\psi}}\right] \tag{38}$$

If we put together the results (25) and (38) we get the "time-frequency window":

$$\left[b + at^* - a\Delta_{\psi}, b + at^* + a\Delta_{\psi}\right] \quad \times \quad \left[\frac{\omega^*}{a} - \frac{1}{|a|}\Delta_{\widehat{\psi}}, \frac{\omega^*}{a} + \frac{1}{|a|}\Delta_{\widehat{\psi}}\right] \tag{39}$$

The following theorem helps to interprete the time-frequency window respect to the scale and translation parameters a, b.

Theorem 3: (Heisenberg' Uncertainty Relation)

Let $g \in L^2\left(\Re\right)$ such that $\|g\|_2 = 1$. Then:

$$\left[\int_{-\infty}^{\infty} \left(t - t_0\right)^2 \left|g\left(t\right)\right|^2 dt\right] \cdot \left[\int_{-\infty}^{\infty} \left(\omega - \omega_0\right)^2 \left|\widehat{g}\left(\omega\right)\right|^2 d\omega\right] \geq \frac{1}{4}. \tag{40}$$

Proof: [7]

We must consider a graphical interpretation of the time-window frequency. In the Figure 1 the time-frequency window is pictured for two pairs of parameters $\left(a_1, b_1\right)$ and $\left(a_2, b_2\right)$. Frow expression (39) we see that the center of the time-windows should be $\left(b_1 + a_1 t^*, \frac{\omega^*}{a_1}\right)$ and $\left(b_2 + a_2 t^*, \frac{\omega^*}{a_2}\right)$, respectively. Moreover, if $a > 0$ the area of the windows are $4a\Delta_{\psi}\frac{1}{a}\Delta_{\widehat{\psi}} = 4\Delta_{\psi}\Delta_{\widehat{\psi}} \geq 1$ from the Heisenberg' uncertainty relation. So, the window area is the same for both $\left(a_1, b_1\right)$ and $\left(a_2, b_2\right)$. However, if $a_2 > a_1 > 0$, then the time localization for the pair $\left(a_1, b_1\right)$ is bether then for $\left(a_2, b_2\right)$ because $a_1\Delta_{\psi} < a_2\Delta_{\psi}$. On the other hand, the localization in the frequency space for the pair $\left(a_1, b_1\right)$ is worst then for $\left(a_2, b_2\right)$ because $\frac{1}{|a_1|}\Delta_{\widehat{\psi}} > \frac{1}{|a_2|}\Delta_{\widehat{\psi}}$.

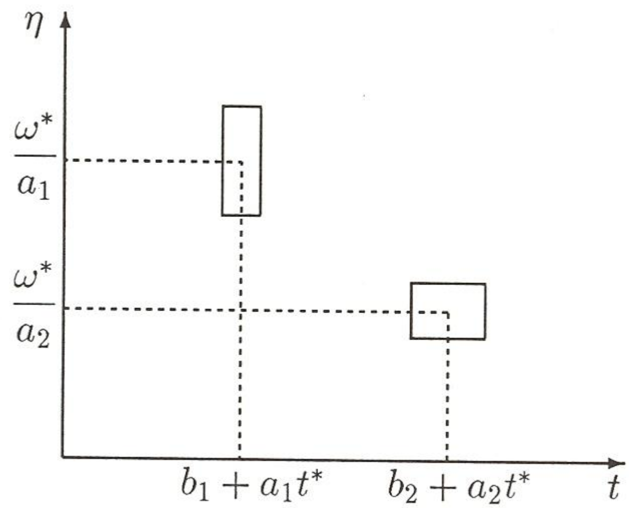

Figure 1: Time-frequency window (Reprinted from [1]).

5 Wavelet Frames

According to the inverse of the CWT (Theorem 2), a function $f \in L^2(\Re)$ has the representation:

$$f(x) = \frac{1}{\sqrt{c_\psi}} \int\limits_{\Re} \int\limits_{\Re} L_\psi f(a,b) \frac{1}{\sqrt{|a|}} \psi\left(\frac{x-b}{a}\right) \frac{dadb}{a^2} \tag{41}$$

with the wavelet transform L_ψ respect to the wavelet ψ. Is it necessary to know $L_\psi f$ at every point $(a,b) \in \Re\setminus\{0\} \times \Re$ in order to get f back? The answer is negative. That means, the integral representation (41) is redundant and consequently, under some constraints, the integral can be replaced by a double sum without any loss of information.

As an example we consider the lattice:

$$\{(a_0^m, nb_0a_0^m) \mid m,n \in Z\} \subset \Re\setminus\{0\} \times \Re \tag{42}$$

with $a_0 > 1, b_0 > 0$ and the corresponding set of functions

$$\left\{\psi_{m,n}^{(a_0,b_0)}(x) = a_0^{-m/2} \psi\left(a_0^{-m}x - nb_0\right) \mid m,n \in Z\right\} \tag{43}$$

Definition 4: 2.1.1

Let $a_0 > 1, b_0 > 0$ and $\psi \in L^2(\Re)$. The system of functions $\left\{\psi_{m,n}^{(a_0,b_0)} \mid m,n \in Z\right\}$, cf.(43), is a wavelet frame for $L^2(\Re)$ if there exist constants $A, B > 0$ such that

$$A\|f\|_{L^2}^2 \leq \sum_{m\in Z}\sum_{n\in Z} \left|\left\langle \psi_{m,n}^{(a_0,b_0)}, f\right\rangle_{L^2}\right| \leq B\|f\|_{L^2}^2 \tag{44}$$

We say that the triple (ψ, a_0, b_0) generates the frame. The constants A and B are called the bound of the frame. The frame is tight if $A = B$.

Lemma 2

A tight frame (ψ, a_0, b_0) with bounds $A = B = 1$ generates an orthonormal basis of $L^2(\Re)$, if ψ is standardized, i.e. $\|\psi\|_{L^2} = 1$.

Proof: The proof is divided into three parts:

(i) Parseval's identity

$$\|f\|_{L^2}^2 = \sum_{m,n\in Z} \left|\left\langle \psi_{m,n}^{(a_0,b_0)}, f\right\rangle_{L^2}\right|^2$$

comes immediately from (44).
(ii) The standardization

$$\left\|\psi_{m,n}^{(a_0,b_0)}\right\|_{L^2} = \|\psi\|_{L^2} = 1$$

follows inevitably from the construction of $\psi_{m,n}^{(a_0,b_0)}$ from ψ using $L^2 - unitary$ operators.
(iii) The orthogonality of $\left\{\psi_{m,n}^{(a_0,b_0)} \mid m,n\right\}$ is based on the fact that, for $\mu,\nu \in Z$ arbitrary we have:

$$\begin{aligned} 1 &= \left\|\psi_{\mu,\nu}^{(a_0,b_0)}\right\|_{L^2} = \sum_{m,n\in Z}\left|\left\langle\psi_{\mu,\nu}^{(a_0,b_0)},\psi_{m,n}^{(a_0,b_0)}\right\rangle_{L^2}\right|^2 \\ &= \|\psi\|_{L^2}^2 + \sum_{\substack{m,n\in Z \\ (m,n)\neq(\mu,\nu)}}\left|\left\langle\psi_{\mu,\nu}^{(a_0,b_0)},\psi_{m,n}^{(a_0,b_0)}\right\rangle_{L^2}\right|^2 \end{aligned}$$

which implies $\left\langle\psi_{\mu,\nu}^{(a_0,b_0)},\psi_{m,n}^{(a_0,b_0)}\right\rangle_{L^2} = \delta_{\mu,m}\delta_{\nu,n}$.

6 Multiscale Analysis

Definition 5: A multiscale analysis (MSA) of $L^2(\Re)$ is an increasing sequence of closed subspaces, called **scale spaces**, $V_m \subset L^2(\Re)$:

$$\{0\} \subset \cdots \subset V_2 \subset V_1 \subset V_0 \subset V_{-1} \subset V_{-2} \subset \cdots \subset L^2(\Re) \tag{45}$$

such that the following are true [7]:

$$\overline{\bigcup_{m\in Z} V_m} = L^2(\Re), \tag{46}$$

$$\bigcap_{m\in Z} V_m = \{0\}, \tag{47}$$

$$f(x) \in V_m \Longleftrightarrow f(2^m x) \in V_0 \tag{48}$$

There is a function $\varphi \in L^2(\Re)$ whose interger translates generate a Riesz basis of V_0 (that mean, V_0 is the closure of the set $span\{\varphi_{m,k} \mid k \in Z\}$):

$$V_0 = \overline{span\{\varphi(x-k) \mid k \in Z\}}$$

and

$$A\sum_{k\in Z} c_k^2 \leq \left\|\sum_{k\in Z} c_k\varphi(\cdot - k)\right\|_{L^2}^2 \leq B\sum_{k\in Z} c_k^2 \tag{49}$$

for all $\{c_k\}_{k\in z} \in l^2(Z)$. A and B are positive constants.

7 Remark

(a) Conditions (46) and (47) are satisfied by many families $\{V_m\}_{m\in Z}$. Property (48) is the special feature of an MSA: the spaces V_m are scaled versions of the basic space V_0, which is spanned by the translations of φ, the scaling function (49).

For $m \longrightarrow \infty$ the functions in V_m are dilated, i.e. their details enlarged. If m tends to $-\infty$, then the spaces V_m contain smaller and smaller structures. The limits

$$\lim_{m\longrightarrow+\infty} \|P_m f\|_{L^2} = 0, \tag{50}$$

$$\lim_{m\longrightarrow-\infty} \|P_m f - f\|_{L^2} = 0, \tag{51}$$

make this interpretation precise. P_m denotes the orthogonal projector onto V_m. In this way, we say that: $P_m f$ is the reperesentation of f on the 'scale' V_m and contains all details of f up to the size 2^m.

(b) The relation (49) implies V_0 is invariant under interger translation

$$f \in V_0 \Longleftrightarrow f(x-k) \in V_0 \quad \text{for } k \in Z. \tag{52}$$

Whith (48) it follows that

$$f \in V_m \Longleftrightarrow f(x-2^m k) \in V_m \quad \text{for } k \in Z. \tag{53}$$

(c) The space V_m is spanned by the functions:

$$\varphi_{m,k}(x) := 2^{-m/2}\varphi\left(2^{-m}x-k\right), \tag{54}$$

$$V_m = \overline{span\left\{\varphi_{m,k} \mid k \in Z\right\}}. \tag{55}$$

This is based on (48) and (49). The functions in (54) all have the same L^2-norm $\|\varphi_{m,k}\|_{L^2} = \|\varphi\|_{L^2}$. We shall remember that, according to Lemma 2, a tight frame (φ, a_0, b_0) with bounds $A = B = 1$ generates an orthonormal basis of $L^2(\Re)$, if φ is standardized, i.e. $\|\varphi\|_{L^2} = 1$. By observing (54) and comparing with the frame definition, we notice that $a_0 = 2$ and $b_0 = 1$.

Scaling Equation: The scaling function φ satisfies a scaling equation, i.e. there is a sequence $\{h_k\}_{k\in Z}$ of real numbers such that:

$$\varphi(x) = \sqrt[2]{2}\sum_{k\in Z} h_k \varphi(2x-k). \tag{56}$$

Proof: This follows from the fact that $\varphi \in V_0 \subset V_{-1} = \overline{span\left\{\sqrt[2]{2}\varphi(2x-k) \mid k \in Z\right\}}$.

8 Wavelet Spaces

The key for the construction of both orthogonal wavelet bases and fast algorithms lies in equation (56). In this way, we denote by W_m the orthogonal complement of V_m in V_{m-1}, that means:

$$V_{m-1} = W_m \oplus V_m,\ V_m \perp W_m. \tag{57}$$

If Q_m is the orthogonal projector of $L^2(\Re)$ in W_m and P_{m-1} denotes the orthogonal projector onto V_{m-1} (likewise in expression (50)-(51)), then, decomposition (57) means:

$$P_{m-1} = Q_m + P_m. \tag{58}$$

From (57) it is clear that:

$$V_{m-1} = W_m \oplus \left(W_{m+1} \oplus V_{m+1}\right),\ V_{m+1} \perp W_{m+1}, \tag{59}$$

and so on. Therefore, using the MSA definition, it follows that:

$$V_{m-1} = \bigoplus_{j\geq m} W_j \tag{60}$$

and so:

$$L^2(\Re) = \bigoplus_{j \in Z} W_j. \tag{61}$$

The spaces W_m, with $m \in Z$, are named **wavelet spaces**.

9 Wavelet and Scaling Equation

Let $\{V_m\}_{m \in Z}$ be an MSA generated by the orthogonal scaling function $\varphi \in V_0$ and the function $\psi \in V_{-1}$, defined by

$$\psi(x) = \sqrt{2} \sum_{k \in Z} g_k \varphi(2x - k) = \sum_{k \in Z} g_k \varphi_{-1,k}(x), \tag{62}$$

$$g_k = (-1)^k h_{1-k}, \tag{63}$$

where $\{h_k\}_{k \in Z}$ are the coefficients of the scaling equation (56).

Theorem 4: The function ψ defined by expressions (63)-(62) has the following properties [7]:

(i) $\left\{\psi_{m,k}(x) = 2^{-m/2}\varphi\left(2^{-m}x - k\right) \mid k \in Z\right\}$ is an orthonormal basis for W_m,

(ii)$\{\psi_{m,k} \mid m, \quad k \in Z\}$ is an orthonormal basis for $L^2(\Re)$,

(iii) ψ is a wavelet with $c_\psi = 2\pi \int_{\Re} |\omega|^{-1} \left|\hat{\psi}(w)\right|^2 dw = 2\ln 2$.

10 Example of MSA: Haar Wavelet

Let the Haar scaling function given by:

$$\varphi(x) = \begin{cases} 1, & 0 \le x < 1 \\ 0, & otherwise \end{cases} \tag{64}$$

The space V_m is defined by expressions (54)-(55). In this section we study the MSA generated through the Haar scaling function.

Firstly, we shall observe that, from equation (54), the support a function $\varphi_{m,k} \in V_m$ is the interval $[2^m k, 2^m(k+1))$. For instance, if set $m = 0$ in expression (54) we obtain:

$$\varphi_{0,k}(x) := \varphi(x - k) \tag{65}$$

Now, when varying the k, we get V_0;

.

.

.

$k = -2$

$\varphi_{0,-2}(x) := \varphi(x+2) = \begin{cases} 1, & -2 \le x < -1 \\ 0, & otherwise \end{cases}$

$k = -1$

$\varphi_{0,-1}(x) := \varphi(x+1) = \begin{cases} 1, & -1 \le x < 0 \\ 0, & otherwise \end{cases}$

$k = 0$

$\varphi_{0,0}(x) := \varphi(x) = \begin{cases} 1, & 0 \le x < 1 \\ 0, & otherwise \end{cases}$

$k = 1$

$\varphi_{0,1}(x) := \varphi(x-1) = \begin{cases} 1, & 1 \le x < 2 \\ 0, & otherwise \end{cases}$

$k = 2$

$\varphi_{0,2}(x) := \varphi(x-2) = \begin{cases} 1, & 2 \le x < 3 \\ 0, & otherwise \end{cases}$

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.
Therefore, we notice that the support of $\varphi_{0,k}$ is $[k, k+1)$.

Remark 1: The support of $\varphi_{m,k} \in V_m$ is the interval $[2^m k, 2^m(k+1))$.

Remark 2: The system de function $\{\varphi_{m,k} \mid k \in Z\}$ is a orthonormal basis of V_m.

The family $\{\cdots, \varphi_{0,-2}(x), \varphi_{0,-1}(x), \varphi_{0,0}(x), \varphi_{0,1}(x), \varphi_{0,2}(x), \cdots\}$ so defined generates an MSA. In fact, we shall firstly observe that projector P_0 in the range $[k, (k+1))$ is given by:

$$P_o f \mid_{[k,(k+1)[} = \int_k^{k+1} f(x)\varphi(x-k)\,dx = \int_k^{k+1} f(x)dx$$

As the system de function $\{\varphi_{o,k} \mid k \in Z\}$ is a orthonormal basis of V_0, the expansion of $P_0 f$ using this basis

$$P_0 f = \sum_{k \in Z} c_k^0(f)\varphi_{0,k}$$

is such that:

$$c_k^0(f) = \langle P_0 f, \varphi_{0,k}\rangle_{L^2} = \langle f, \varphi_{0,k}\rangle_{L^2} = \int_k^{k+1} f(x)dx$$

We now explore the difference between $P_0 f$ and the next coarser aproximation $P_1 f$. The scaling equation (56) for this example is:

$$\varphi(x) = \sqrt{2}\left(\frac{1}{\sqrt{2}}\varphi(2x) + \frac{1}{\sqrt{2}}\varphi(2x-1)\right). \tag{66}$$

We can verify this equality through as follows. In this case we consider $m = -1$, for $\varphi \in V_0 \subset V_{-1}$. Thus (54) becomes:

$$\varphi_{-1,k}(x) := 2^{1/2}\varphi(2x-k),$$

and so, by varying the index k we get:

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.
.

$k = -1$

$\varphi_{-1,-1}(x) := 2^{1/2}\varphi(2x+1) = \begin{cases} 1, & -1/2 \le x < 0 \\ 0, & otherwise \end{cases}$

$k = 0$

$\varphi_{-1,0}(x) := 2^{1/2}\varphi(2x) = \begin{cases} 1, & 0 \le x < 1/2 \\ 0, & otherwise \end{cases}$

$k = 1$

$\varphi_{-1,1}(x) := 2^{1/2}\varphi(2x-1) = \begin{cases} 1, & 1/2 \le x < 1 \\ 0, & otherwise \end{cases}$

$k = 2$

$\varphi_{-1,2}(x) := 2^{1/2}\varphi(2x-2) = \begin{cases} 1, & 1 \le x < 3/2 \\ 0, & otherwise \end{cases}$

.
.
.

From these expressions and the definition of φ in equation (64) we observe that

$$\varphi(x) \cdot \varphi(2x-k) \neq 0 \quad \Leftrightarrow \quad k = 0 \quad or \quad k = 1.$$

So, equation (56) becomes:

$$\varphi\left(x\right)=\sqrt{2}\left(h_{0}2^{1/2}\varphi\left(2x\right)+h_{1}2^{1/2}\varphi\left(2x-1\right)\right). \tag{67}$$

To find the coeffiecents h_0 and h_1, we go back to (67) and we isolated the coefficients h_0, h_1:

$$\begin{aligned}\left\langle\varphi\left(x\right),\varphi\left(2x\right)\right\rangle &= \sqrt{2}h_{0} \quad\Rightarrow\quad h_{0}=\frac{1}{\sqrt{2}}\left\langle\varphi\left(x\right),\varphi\left(2x\right)\right\rangle, \\ \left\langle\varphi\left(x\right),\varphi\left(2x-1\right)\right\rangle &= \sqrt{2}h_{1} \quad\Rightarrow\quad h_{1}=\frac{1}{\sqrt{2}}\left\langle\varphi\left(x\right),\varphi\left(2x-1\right)\right\rangle\end{aligned} \tag{68}$$

Now, we will calculate the inner products:

$$\left\langle\varphi\left(x\right),\varphi\left(2x\right)\right\rangle=2^{1/2}\int_{0}^{1/2}\underbrace{\varphi\left(x\right)\varphi\left(2x\right)}_{1}dx=2^{1/2}\left[\frac{1}{2}-0\right]=\frac{\sqrt{2}}{2}=\frac{1}{\sqrt{2}} \tag{69}$$

$$\left\langle\varphi\left(x\right),\varphi\left(2x-1\right)\right\rangle=2^{1/2}\int_{1/2}^{1}\underbrace{\varphi\left(x\right)\varphi\left(2x-1\right)}_{1}dx=2^{1/2}\left[1-\frac{1}{2}\right]=\frac{\sqrt{2}}{2}=\frac{1}{\sqrt{2}} \tag{70}$$

Replacing expressions (69) and (70) in (68) we obtain:

$$h_{0}=\frac{1}{\sqrt[2]{2}}.\frac{1}{\sqrt{2}}=\frac{1}{2}$$

$$h_{1}=\frac{1}{\sqrt[2]{2}}.\frac{1}{\sqrt{2}}=\frac{1}{2}$$

Therefore, expression (67) takes the final form:

$$\varphi\left(x\right)=\sqrt{2}\left(\frac{1}{\sqrt{2}}\varphi\left(2x\right)+\frac{1}{\sqrt{2}}\varphi\left(2x-1\right)\right), \tag{71}$$

which is identical to expression (66).

Lemma 4: The general scaling equation has the form:

$$\varphi_{m+1,k}=\frac{1}{\sqrt{2}}\left(\varphi_{m,2k}+\varphi_{m,2k+1}\right). \tag{72}$$

Proof: To prove this equation, we must show that:

$$\left\langle\varphi_{m+1,k},\varphi_{m,2k}\right\rangle=\frac{1}{\sqrt{2}} \tag{73}$$

$$\left\langle\varphi_{m+1,k},\varphi_{m,2k+1}\right\rangle=\frac{1}{\sqrt{2}} \tag{74}$$

$$\left\langle\varphi_{m+1,k},\varphi_{m,s}\right\rangle=0, \quad otherwise. \tag{75}$$

By remembering that:

$$\varphi_{m+1,k}\left(x\right):=2^{-(m+1)/2}\varphi\left(2^{-(m+1)}x-k\right),$$

$$\varphi_{m,2k}\left(x\right):=2^{-m/2}\varphi\left(2^{-m}x-k\right),$$

$$\varphi_{m,2k+1}\left(x\right):=2^{-m/2}\varphi\left(2^{-m}x-(2k+1)\right),$$

we can compute the inner products:

$$\langle \varphi_{m+1,k}, \varphi_{m,2k} \rangle = 2^{-2m-1/2} \underbrace{\left\langle \varphi\left(2^{-(m+1)}x - k\right), \varphi\left(2^{-m}x - k\right) \right\rangle}_{(I)} \tag{76}$$

$$\langle \varphi_{m+1,k}, \varphi_{m,2k+1} \rangle = 2^{-2m-1/2} \underbrace{\left\langle \varphi\left(2^{-(m+1)}x - k\right), \varphi\left(2^{-m}x - (2k+1)\right) \right\rangle}_{(II)} \tag{77}$$

where (I) and (II) are given by:

$$\begin{aligned}
\underbrace{\left\langle \varphi\left(2^{-(m+1)}x - k\right), \varphi\left(2^{-m}x - k\right) \right\rangle}_{(I)} &= \int_{2k/2^{-m}}^{1+2k/2^{-m}} \underbrace{\varphi\left(2^{-(m+1)}x - k\right), \varphi\left(2^{-m}x - k\right)}_{1} dx = \\
\underbrace{\left\langle \varphi\left(2^{-(m+1)}x - k\right), \varphi\left(2^{-m}x - k\right) \right\rangle}_{(I)} &= \left[\frac{1+2k}{2^{-m}} - \frac{2k}{2^{-m}}\right] = \frac{1}{2^{-m}}
\end{aligned}$$

$$\begin{aligned}
\underbrace{\left\langle \varphi\left(2^{-(m+1)}x - k\right), \varphi\left(2^{-m}x - (2k+1)\right) \right\rangle}_{(II)} &= \int_{1+2k/2^{-m}}^{2+2k/2^{-m}} \underbrace{\varphi\left(2^{-(m+1)}x - k\right), \varphi\left(2^{-m}x - (2k+1)\right)}_{1} dx = \\
\underbrace{\left\langle \varphi\left(2^{-(m+1)}x - k\right), \varphi\left(2^{-m}x - (2k+1)\right) \right\rangle}_{(II)} &= \left[\frac{2+2k}{2^{-m}} - \frac{1+2k}{2^{-m}}\right] = \frac{1}{2^{-m}}
\end{aligned}$$

Replacing the values of (I) and (II) in expressions (76) and (77), respectively, we find:

$$\langle \varphi_{m+1,k}, \varphi_{m,2k} \rangle = 2^{-2m-1/2} \times \frac{1}{2^{-m}} = \frac{1}{\sqrt{2}},$$

$$\langle \varphi_{m+1,k}, \varphi_{m,2k+1} \rangle = 2^{-2m-1/2} \times \frac{1}{2^{-m}} = \frac{1}{\sqrt{2}},$$

which complete the demonstration of results (73)-(74). (Expression (75):

$$\varphi_{m+1,k} = \frac{1}{\sqrt{2}} \left(\varphi_{m,2k} + \varphi_{m,2k+1}\right).$$

As a consequence, we have:

$$c_k^{m+1}(f) = \frac{1}{\sqrt{2}} \left(c_{2k}^m(f) + c_{2k+1}^m(f)\right). \tag{78}$$

The projection $P_{m+1}f$ turns out to be an averaged version of $P_m f$. For the difference $P_m f - P_{m+1}f$ we get

$$\begin{aligned}
P_m f - P_{m+1} f &= \sum_{k \in Z} c_k^m(f) \varphi_{m,k} - \sum_{k \in Z} c_k^{m+1}(f) \varphi_{m+1,k} \\
&= \sum_{k \in Z} c_k^m(f) \varphi_{m,k} - \frac{1}{2} \sum_{k \in Z} (c_{2k}^m + c_{2k+1}^m)(\varphi_{m,2k} + \varphi_{m,2k+1}) \\
&= \sum_{k \in Z} (c_{2k}^m \varphi_{m,2k} + c_{2k+1}^m \varphi_{m,2k+1}) - \frac{1}{2} \sum_{k \in Z} (c_{2k}^m + c_{2k+1}^m)(\varphi_{m,2k} + \varphi_{m,2k+1}) \\
&= \frac{1}{2} \sum_{k \in Z} (c_{2k}^m - c_{2k+1}^m)(\varphi_{m,2k} - \varphi_{m,2k+1})
\end{aligned}$$

From the scaling equation (72) and the theorem 4 we get:

$$\psi\left(x\right)=\varphi\left(2x\right)-\varphi\left(2x-1\right)=\begin{cases}1, & 0\leq x<1/2\\ -1, & 1/2\leq x<1\\ 0, & otherwise\end{cases}$$

which is just the Haar Wavelet. More generally:

$$\frac{1}{\sqrt{2}}\left(\varphi_{m,2k}-\varphi_{m,2k+1}\right)=\psi_{m+1,k}. \tag{79}$$

The system of function $\{\psi_{m,k}\mid k\in Z\}$, defined in the theorem 4 makes up an orthonormal basis of the space W_m.

11 Fast Wavelet Transform

In this section we introduce the basic algorithms for the fast computation of the discrete wavelet transform. It is performed iun the context of multiscale analysis and by applying the scaling equation.

We consider a function f in V_0, the basic space of a multiscale analysis, as well as an orthogonal scaling function $\varphi\in V_0$. From Definition 5 we know f has the expansion:

$$f(x)=\sum_{k\in Z}c_k^0\varphi\left(x-k\right) \tag{80}$$

with expansion coefficients

$$c^0=\left\{c_k^0\mid k\in Z\right\}.$$

As before we let ψ denote the orthogonal wavelet corresponding to the scaling function φ, which generates an orthonormal basis:

$$\left\{\psi_{m,k}=2^{-m/2}\varphi\left(2^{-m}\cdot-k\right)\mid m,k\in Z\right\},$$

of $L^2(R)$, acoording to the theorem 4.

Now, we can compute the discrete wavelet transform, which is given by the scalar products:

$$\sqrt{c_\psi}L_\psi f\left(2^m,2^mk\right)=\left\langle f,\psi_{m,k}\right\rangle_{L^2},\quad m\in N_0,k\in Z.$$

For this we use the following notation:

$$d_k^m=\left\langle f,\psi_{m,k}\right\rangle_{L^2},\quad d^m=\{d_k^m\mid k\in Z\}\quad l^2\left(Z\right),$$
$$c_k^m=\left\langle f,\varphi_{m,k}\right\rangle_{L^2},\quad c^m=\{c_k^m\mid k\in Z\}\quad l^2\left(Z\right).$$

So, with the of the scaling equations (56) and (62), we get:

$$d_k^m=\left\langle f,\psi_{m,k}\right\rangle_{L^2}=\sum_{l\in Z}g_l\left\langle f,\varphi_{m-1,2k+l}\right\rangle_{L^2}=\sum_{l\in Z}g_{l-2k}c_l^{m-1},$$

$$c_k^m=\left\langle f,\varphi_{m,k}\right\rangle_{L^2}=\sum_{l\in Z}h_l\left\langle f,\varphi_{m-1,2k+l}\right\rangle_{L^2}=\sum_{l\in Z}h_{l-2k}c_l^{m-1}.$$

Thus the decomposition algorithm is already completed: starting with the sequence c^0 we can compute the discrete wavelet decomposition recursively via discrete convolutions. It is important to observe tha and all operations are carried out discretely on the sequences of coefficients c^m and d^m.

This process can be better expressed with the help of the decomposition operators H and G, defined by:

$$H:l^2\left(Z\right)\longrightarrow l^2\left(Z\right)$$

$$c \longmapsto Hc = c *_2 h = \left\{ (Hc)_k = \sum_{l \in Z} h_{l-2k} c_l \right\}, \tag{81}$$

$$G : l^2(Z) \longrightarrow l^2(Z)$$

$$c \longmapsto Gc = c *_2 g = \left\{ (Gc)_k = \sum_{l \in Z} g_{l-2k} c_l \right\}, \tag{82}$$

where $h = \{h_k \mid k \in Z\}$ and $g = \{g_k \mid k \in Z\}$ are, respectivelly, the sequences of scaling and wavelet coefficients.

We shall observe that we are not using the standard discrete convolution - in the convolutions above it enters only the second index. That is way we use the symbol $*_2$ for this operation.

We now turn to the reconstruction of the initial sequence c^0 from the computed sequences of coefficients $\left\{c^M, d^m \mid m = 1, ..., M\right\}$, through the following algorithm due to Mallat in [8].

Fast Wavelet Transform
Input: $c^0 = \{c_k \mid k \in Z\}$
M levels of decomposition (number of scales)

Calculate for $m = 1, ..., M$
$d^m = Gc^{m-1}$
$c^m = Hc^{m-1}$

Output: c^M
$d^m, m = 1, ..., M$

$$\begin{array}{ccccccccc} c^0 & \xrightarrow{H} & c^1 & \xrightarrow{H} & c^2 \cdots\cdots & c^{M-1} & \xrightarrow{H} & c^M \\ & \searrow & & \searrow & & & \searrow & \\ & d^1 & & d^2 & & \cdots\cdots\cdots\cdots & & d^M \end{array}$$

As an example, we consider the reconstruction of the sequence c^0 from c^1 and d^1. We use the orthogonal decomposition of V_0 into the two subspaces V_1 and W_1 and the scaling equation. Then we have:

$$\begin{aligned} \sum_{k \in Z} c_k^0 \varphi_{0,k} &= \sum_{k \in Z} c_j^1 \varphi_{1,j} + \sum_{k \in Z} d_j^1 \psi_{1,j} \\ &= \sum_{j \in Z} c_j^1 \sum_{l \in Z} h_l \varphi_{0,2j+k} + \sum_{j \in Z} d_j^1 \sum_{l \in Z} g_l \varphi_{0,2j+k}, \end{aligned}$$

Comparing coefficients gives:

$$c_k^0 = \sum_{l \in Z} c_j^1 h_{k-2l} + \sum_{l \in Z} d_l^1 g_{k-2l.} \tag{83}$$

Observe that, by keeping c^M and $d^m, m = 1, ..., M$, we can recursively reconstruct the sequence c^0 (Fast Wavelet Reconstruction)

12 MSA for Periodic Data

We consider a function f in V_0, the basic space of a multiscale analysis of an orthogonal scaling function φ. However, in this section, let us consider periodic scaling and the space V_0 with period L [12]:

$$V_0 = \left\{ \sum_{k \in Z} c_k^0 \varphi_{0,k}(x) ; \quad c_k^0 = c_{k+L}^0, \quad k \in Z \right\}.$$

Such formulation is useful when considering finite sequences $\left\{c_1^0, c_2^0, \cdots, c_L^0\right\}$ as input. Therefore:

$$c_k^1 = \langle f, \varphi_{1,k}(x)\rangle = \left\langle \sum_{s\in Z} c_s^0 \varphi_{0,s}(x), \varphi_{1,k}(x)\right\rangle =$$

$$\sum_l h_l \left\langle \sum_{s\in Z} c_s^0 \varphi_{0,s}(x), \varphi_{0,2k+l}(x)\right\rangle = \sum_l h_l \sum_{s\in Z} c_s^0 \langle \varphi_{0,s}(x), \varphi_{0,2k+l}(x)\rangle =$$

$$\sum_l h_l c_{2k+l}^0.$$

So,

$$c_{k+\frac{L}{2}}^1 = \sum_l h_l c_{2\left(k+\frac{L}{2}\right)+l}^0 = \sum_l h_l c_{2k+l+L}^0 = \sum_l h_l c_{2k+l}^0 = c_k^1.$$

Analogously for d_k^1, we find:

$$d_k^1 = \sum_l g_l c_{2k+l}^0.,$$

and so, we also get:

$$d_{k+\frac{L}{2}}^1 = d_k^1.$$

Therefore, we can write:

$$V_0 \quad \simeq \quad \Re^L$$

$$V_1 \oplus W_1 \quad \simeq \quad \Re^{L/2} \oplus \Re^{L/2}$$

The whole discrete wavelet transform for periodic data can be given as follows:

- Choose an orthogonal scaling function $\varphi \in L^2(\Re)$ whose interge translates generate an Riesz bais of V_0, i.e.

$$V_0 = \overline{span\left\{\varphi(x-k) \mid k \in Z\right\}} \tag{84}$$

and satisfies relation (49).

- Solve the scaling equation; that is, find h_k such that:

$$\varphi(x) = \sqrt{2}\sum_{k\in Z} h_k \varphi(2x-k).$$

- Generate the wavelet $\psi \in V_{-1}$:

$$\psi(x) = \sqrt{2}\sum_{k\in Z}(-1)^k h_{-1,k}\varphi(2x-k) = \sum_{k\in Z}(-1)^k h_{-1,k}\varphi_{-1,k}(x), \tag{85}$$

- Given an intput $c_1^0, c_2^0, \cdots, c_L^0$, then, we compute

$$c_k^1 = \sum_l h_l c_{2k+l}^0. \tag{86}$$

$$d_k^1 = \sum_l g_l c_{2k+l}^0 \cdot, \tag{87}$$

- Now, take $c_{k_1}^1, c_{k_2}^1, \cdots, c_{k_{L/2}}^1$ as the input and compute expressions (86)-(87) to get $\{c_k^2, d_k^2\}$, with period $L/4$. Proceed until level M, with period $(2^{-M}) \cdot L$.

13 Multidimensional Multiscale Analysis

The multidimensional MSAs givens the foundation for the use of wavelets in image analysis and image compression. Likewise in the one-dimensional case, a multiscale analysis of $L^2(\Re^n)$ consists of an increasing sequence of closed subsets $\{V_m^n\}_{m\in Z}$ of $L^2(\Re^n)$ such that:

$$\overline{\bigcup_{m\in Z} V_m^n} = L^2(\Re^n) \text{ and } \bigcap_{m\in Z} V_m^n = \{0\}.$$

Now, as we have n dimensions, we must consider translations $k \in Z^n$ and regular matrices D, the dilation matrices, such that:

$$f(x) \in V_m^n \iff f(Dx) \in V_{m-1}^n$$

The basic space V_0 is again produced by a scaling function $\varphi \in L^2(\Re^n)$,

$$\{\varphi(x-k) \mid k \in Z^n\} \text{ gives a Riesz basis of } V_0^n.$$

Besides, the family of functions:

$$\left\{\varphi_{m,k}(x) = |\det D|^{-m/2} \varphi\left(D^{-m}x - k\right) \mid k \in Z^n\right\}$$

is an orthonormal basis of V_m^n, because we may assume, without any loss of generality, the orthonormality of $\{\varphi(x-k) \mid k \in Z^n\}$ and

$$\int_{\Re^n} \varphi(x) \quad dx \neq 0.$$

In accordance with the philosophy of the multiscale decomposition, the dilation matrix should dilate in every direction, i.e. the eigenvalues of D all have modulus greater than 1,

$$\lambda \in \sigma(D) \implies |\lambda| > 1.$$

Moreover, D should have integer entries, which is equivalent to:

$$DZ^n \subset Z^n.$$

Multidimensional wavelets are those functions which span the orthogonal complement of V_0^n in V_{-1}^n. The following theorem sets the backgroung for the construction of the Multidimensional wavelet space.

Theorem 5: Let $\{V_m^n\}_{m\in Z}$ be an MSA with dilation matrix D. Then there exist $|\det D| - 1$ wavelets

$$\psi_1, \psi_2, \cdots, \psi_{|\det D|-1} \in V_{-1}^n,$$

that generate an orthonormal basis of the orthogonal complement of V_m^n in V_{m-1}^n, in the sense that:

$$W_{m,j}^n = \overline{span\left\{\psi_{m,j,k}(x) = |\det D|^{-m/2} \psi_j\left(D^{-m}x - k\right) \mid k \in Z^n\right\}}, \quad j = 1, ..., \det A - 1, \quad m \in Z. \tag{88}$$

Besides, the set of functions:

$$\left\{\psi_{m,j,k}\left(x\right)=\left|\det D\right|^{-m/2}\psi_j\left(D^{-m}x-k\right)\mid j=1,...,\det D-1,m\in Z,k\in Z^n\right\}$$

is an orthonormal basis of $L^2\left(\Re^n\right)$.

Proof: See [10].

Therefore, this theorem allows to obtain an orthogonal decomposition of V_m^n in $|\det D|$ subspaces:

$$V_m^n=\overset{|\det D|-1}{\underset{j=1}{\oplus}}W_{m+1,j}^n\oplus V_{m+1}^n \tag{89}$$

For instance, the orthogonal decomposition of V_{-1}^n in $|\det D|$ subspaces is given by:

$$V_{-1}^n=\bigoplus_{j=1}^{|\det D|-1}W_{0,j}^n\oplus V_0^n, \tag{90}$$

where the spaces $W_{0,j}^n$ are spanned by

$$W_{0,j}^n=\overline{span\left\{\psi_j\left(x-k\right)\mid k\in Z^n\right\}.} \tag{91}$$

14 Tensor Wavelets

As an example we consider the tensor product wavelets for $n=2$. Tensor wavelets in $L^2\left(\Re^2\right)$ are based on a dilation matrix given by:

$$D=\begin{pmatrix}2&0\\0&2\end{pmatrix} \tag{92}$$

and constructed through the tensor product of the corresponding scaling functions:

$$\varphi\left(z\right)=\varphi_1\left(x\right)\varphi_2\left(y\right), \tag{93}$$

where $z=(x,y)$. The main result in this area is the following lemma, which is a consequence of theorem 5.

Lemma 5: Let $\varphi\in$ be the tensor scaling function $\varphi\left(z\right)=\varphi_1\left(x\right)\varphi_2\left(y\right)$. Then, $\varphi\in L^2\left(\Re^2\right)$ satisfies the scaling equation:

$$\varphi\left(z\right)=2\sum_{k\in Z^2}h_k\varphi\left(Dz-k\right), \tag{94}$$

with:

$$h_k=h_{k_1}^1h_{k_2}^2,\quad k=\left(k_1,k_2\right).$$

In this case, there are $|\det D|-1=3$ wavelets, say ψ^1,ψ^2 and ψ^3. We can show that these can be generated through the one dimensional orthogonal wavelets ψ_1,ψ_2 generated by scaling functions φ_1 and φ_2, as follows [7]:

$$\psi^1\left(z\right)=\psi_1\left(x\right)\varphi_2\left(y\right), \tag{95}$$

$$\psi^2\left(z\right)=\varphi_1\left(x\right)\psi_2\left(y\right), \tag{96}$$

$$\psi^3\left(z\right)=\psi_1\left(x\right)\psi_2\left(y\right). \tag{97}$$

Then,

$$\left\{\psi_{m,i,k}\left(z\right)=4^{-m/2}\psi^i\left(2^{-m}z-k\right)\mid i=1,2,3,m\in Z,k\in Z^2\right\} \tag{98}$$

is an orthonormal basis of $L^2\left(\Re^2\right)$.

Two-dimensional scaling functions and wavelets of the form (93) or (95) are called $separable$. Their advantage is quite obvious: they are easy to derive from one-dimensional MSAs. Because of their construction the wavelets in (95) are *anisotropic*, i.e. they prefer certain directions (the x and y directions as well as diagonals). This is desired for edge detection of digital images but it is unattractive for the purpose of data compression because this application demands isotropy and a smaller number of wavelets.

15 Multiresolution in $\mathbf{L}^2(\Re^2)$ By Tensor Wavelets

Let $\left\{V_m^2\right\}_{m\in Z}$ a sequence of subspaces in $L^2\left(\Re^2\right)$. The approximation of a signal $f(x,y)$ in the level of resolution m is equal to its prejection in the vectorial space V_m^2.

In the case of tensor wavelets theory, we can show that the tensor scale function $\varphi\left(x\right)$, defined by expression (93), generates a orthonormal basis to esch space V_m^2, given by:

$$\left\{\varphi_{m,k}\left(x\right)=\left|\det D\right|^{-m/2}\varphi\left(D^{-m}x-k\right)k\in Z^2\right\} \tag{99}$$

where, D is the dilation matrix defined in equation (92). Therefore, expression (99) becomes:

$$\left\{\varphi_{m,k}\left(x,y\right)=2^{-m}\varphi\left(2^{-m}x-k_x,2^{-m}y-k_y\right)k\in Z^2\right\}. \tag{100}$$

Once $\varphi\left(x,y\right)=\varphi_1\left(x\right)\varphi_2\left(y\right)$

$$\varphi_{m,k}\left(x\right):=2^{-m/2}\varphi\left(2^{-m}x-k\right)= \tag{101}$$

$$2^{-m}\varphi\left(2^{-m}x-k_x,2^{-m}y-k_y\right)=\left[2^{-m/2}\varphi_1\left(2^{-m}x-k_x\right)\right]\cdot\left[2^{-m/2}\varphi_2\left(2^{-m}y-k_y\right)\right].$$

If φ_1 and φ_2 are the same scale function, say φ, then:

$$2^{-m}\varphi\left(2^{-m}x-k_x,2^{-m}y-k_y\right)=\varphi_{m,k_x}\left(x\right)\cdot\varphi_{m,k_y}\left(y\right),\quad\left(k_x,k_y\right)\in Z^2.$$

The projection of a image function $f\left(x,y\right)\in L^2\left(\Re^2\right)$ in the space V_m^2 is given by.

$$P_mf\left(x,y\right)=\sum_{k_x=-\infty}^{\infty}\sum_{k_y=-\infty}^{\infty}\left\langle f\left(u,v\right),\varphi_{m,k_x,k_y}\left(u,v\right)\right\rangle\varphi_{m,k_x,k_y}\left(x,y\right) \tag{102}$$

So, the discrete approximation of a signal $f\left(x,y\right)$ in the level of resolution m is caracterized by the following sequence of inner products:

$$P_mf=\left(\left\langle f\left(x,y\right),\varphi_{m,k_x}\left(x\right)\varphi_{m,k_y}\left(y\right)\right\rangle\right)_{\left(k_x,k_y\right)\in Z^2} \tag{103}$$

16 Details in $L^2\left(\Re^2\right)$

The details in the level of resolution m, is equal to the orthogonal projection of the signal in subspace W_m^2 which is the orthogonal complement of V_m^2 in V_{m-1}^2.

By theorem 5, it is possible to built an orthogonal basis for W_m^2, because there exist $\left|\det D\right|-1$ wavelets, $\left\{\psi_1,\psi_2\cdots\psi_{\left|\det A\right|-1}\right\}\in V_m^2$ that generate the orthonormal basis of the orthogonal complement of V_m^2 in V_{m-1}^2. So, consider the orthonormal basis of $L^2\left(\Re^2\right)$ given by:

$$\left\{\psi_{j,m,k}\left(x,y\right)=2^{-m}\psi_j\left(2^{-m}x-k_x,2^{-m}y-k_y\right)j=1,\cdots,\left|\det D\right|-1,\quad m\in Z\quad e\quad k\in Z^2\right\} \tag{104}$$

According to expression (89), we can obtain an orthogonal decomposition of V_m^2 in $|\det D|$ subspaces:

$$V_m^2 = \overset{|\det D|-1}{\underset{j=1}{\oplus}} W_{m+1,j}^2 \oplus V_{m+1}^2$$

Where, by expression (88), the space $W_{m+1,j}^2$ is generated by:

$$W_{m+1,j}^2 = \overline{span\left\{\psi_{j,m+1,k}\left(x,y\right) \mid k \in Z^2\right\}}$$

If φ_1 and φ_2 are the same scale function and ψ_1, ψ_2 are the same $1D$ wavelet function then expression (95) becomes:

$$\psi^1\left(x,y\right) = \psi\left(x\right)\varphi\left(y\right), \quad \psi^2\left(x,y\right) = \varphi\left(x\right)\psi\left(y\right), \quad \psi^3\left(x,y\right) = \psi\left(x\right)\psi\left(y\right). \tag{105}$$

Therefore, the loss of information when going from $P_m f$ to $P_{m+1} f$ is equal to the orthogonal projection of $f\left(x,y\right)$ into $W_{m+1}^2 = \overset{|\det D|-1}{\underset{j=1}{\oplus}} W_{m+1,j}^2$, which is given by:

$$\begin{array}{rcl} D_m^1 f &=& \left(\left\langle f\left(x,y\right), 2^{-m}\psi^1\left(2^{-m}x - k_x, 2^{-m}y - k_y\right)\right\rangle\right) \\ D_m^2 f &=& \left(\left\langle f\left(x,y\right), 2^{-m}\psi^2\left(2^{-m}x - k_x, 2^{-m}y - k_y\right)\right\rangle\right) \\ D_m^3 f &=& \left(\left\langle f\left(x,y\right), 2^{-m}\psi^3\left(2^{-m}x - k_x, 2^{-m}y - k_y\right)\right\rangle\right). \end{array} \tag{106}$$

By using expressios (105), we can write these equations as:

$$\begin{array}{rcl} D_m^1 f &=& \left(\left\langle f\left(x,y\right), 2^{-m}\psi\left(2^{-m}x - k_x\right)\varphi\left(2^{-m}y - k_y\right)\right\rangle\right) \\ D_m^2 f &=& \left(\left\langle f\left(x,y\right), 2^{-m}\varphi\left(2^{-m}x - k_x\right)\psi\left(2^{-m}y - k_y\right)\right\rangle\right) \\ D_m^3 f &=& \left(\left\langle f\left(x,y\right), 2^{-m}\psi\left(2^{-m}x - k_x\right)\psi\left(2^{-m}y - k_y\right)\right\rangle\right) \end{array} \tag{107}$$

16.1 Example de MSR Separable: Bi-dimensional Case

We can show that in $L^2\left(\Re^2\right)$ the projections $P_m f$ e $D_m^i f$ can be calculated with a separated filtering operation in each direction. So, let $\left(\psi_{m;\beta_x:\beta_y}^i\right)_{1\leq i\leq 3}$ be a set of separable wavelet, as stated in the theorem 5. Each wavelet $\psi_{m;k_x:k_y}^i\left(x,y\right)$ belongs to the space $W_m^2 \subset V_{m-1}^2$. So, we can expand $\psi_{m;k_x:k_y}^i\left(x,y\right)$ in the orthogonal basis $\varphi_{m-1;k_x:k_y}^i\left(x,y\right)$ of space V_{m-1}^2,ie,

$$\psi_{m;\beta_x:\beta_y}^i\left(x,y\right) = \sum_{k_x=-\infty}^{\infty}\sum_{k_y=-\infty}^{\infty}\left\langle\psi_{m;\beta_x:\beta_y}^i\left(u,v\right), \varphi_{m-1;k_x:k_y}\left(u,v\right)\right\rangle\varphi_{m-1;k_x:k_y}\left(x,y\right) \tag{108}$$

By introducing new variables n_x and n_y:

$$\psi_{m;\beta_x:\beta_y}^i\left(x,y\right) = \sum_{k_x=-\infty}^{\infty}\sum_{k_y=-\infty}^{\infty}\left\langle\psi_{m;\beta_x:\beta_y}^i\left(u,v\right), \varphi_{m-1;k_x-2n_x:k_y-2n_y}\left(u,v\right)\right\rangle\varphi_{m-1;k_x:k_y}\left(x,y\right). \tag{109}$$

Expression (109) can be interpreted as a filtering process with the bi-dimensional impulse response:

$$g^i\left(n,p\right) = \left\langle\psi_{m;\beta_x:\beta_y}^i\left(u,v\right), \varphi_{m-1;n:p}\left(u,v\right)\right\rangle \tag{110}$$

where i specifies the index of the wavelet (see equation (105)), $n = k_x - 2n_x$, $p = k_y - 2n_y$ and $1 \leq i \leq 3$. Therefore:

$$\begin{aligned} g^1(n,p) &= \left\langle \psi^1_{m;\beta_x:\beta_y}(u,v), \varphi_{m-1;n:p}(u,v) \right\rangle = \\ \left\langle \varphi_{m:\beta_x}(u)\,\psi_{m;\beta_y}(v), \varphi_{m-1;n}(u)\,\varphi_{m-1;p}(v) \right\rangle &= \\ \left\langle \varphi_{m:\beta_x}(u), \varphi_{m-1;n}(u) \right\rangle \left\langle \psi_{m;\beta_y}(v), \varphi_{m-1;p}(v) \right\rangle &= h(n)g(p) \end{aligned} \tag{111}$$

$$\begin{aligned} g^2(n,p) &= \left\langle \psi^2_{m;\beta_x:\beta_y}(u,v), \varphi_{m-1;n:p}(u,v) \right\rangle = \\ \left\langle \psi_{m;\beta_y}(u)\,\varphi_{m:\beta_x}(v), \varphi_{m-1;n}(u)\,\varphi_{m-1;p}(v) \right\rangle &= \\ \left\langle \psi_{m;\beta_y}(u)\,\varphi_{m-1;n}(u), \right\rangle \left\langle \varphi_{m:\beta_x}(v), \varphi_{m-1;p}(v) \right\rangle &= g(n)h(p) \end{aligned} \tag{112}$$

$$\begin{aligned} g^3(n,p) &= \left\langle \psi^3_{m;\beta_x:\beta_y}(u,v), \varphi_{m-1;n:p}(u,v) \right\rangle = \\ \left\langle \psi_{m:\beta_x}(u)\,\psi_{m;\beta_y}(v), \varphi_{m-1;n}(u)\,\varphi_{m-1;p}(v) \right\rangle &= \\ \left\langle \psi_{m:\beta_x}(u), \varphi_{m-1;n}(u) \right\rangle \left\langle \psi_{m;\beta_y}(v), \varphi_{m-1;p}(v) \right\rangle &= g(n)g(p) \end{aligned} \tag{113}$$

So, by using expressions (111)-(113), we can write the projection of $f(x,y)$ in the wavelet spaces, given by the inner product between $f(x,y)$ and expression (109), as follows:

$$\left\langle f(x,y), \psi^1_{m;\beta_x:\beta_y}(x,y) \right\rangle = \sum_{k_x=-\infty}^{\infty} h(n) \sum_{k_y=-\infty}^{\infty} g(p) \left\langle f(x,y), \varphi_{m-1;k_x:k_y}(x,y) \right\rangle = \tag{114}$$

$$\sum_{k_x=-\infty}^{\infty} h(k_x - 2n_x) \sum_{k_y=-\infty}^{\infty} g(k_y - 2n_y) \left\langle f(x,y), \varphi_{m-1;k_x:k_y}(x,y) \right\rangle =$$

$$\sum_{k_x=-\infty}^{\infty} h(k_x - 2n_x) \sum_{k_y=-\infty}^{\infty} g(k_y - 2n_y)\, c^{m-1}_{k_x:k_y},$$

$$\left\langle f(x,y), \psi^2_{m;\beta_x:\beta_y}(x,y) \right\rangle = \sum_{k_x=-\infty}^{\infty} g(n) \sum_{k_y=-\infty}^{\infty} h(p) \left\langle f(x,y), \varphi_{m-1;k_x;k_y}(x,y) \right\rangle = \tag{115}$$

$$\sum_{k_x=-\infty}^{\infty} g(k_x - 2n_x) \sum_{k_y=-\infty}^{\infty} h(k_y - 2n_y) \left\langle f(x,y), \varphi_{m-1;k_x;k_y}(x,y) \right\rangle =$$

$$\sum_{k_x=-\infty}^{\infty} g(k_x - 2n_x) \sum_{k_y=-\infty}^{\infty} h(k_y - 2n_y)\, c^{m-1}_{k_x;k_y},$$

$$\left\langle f(x,y), \psi^3_{m;\beta_x:\beta_y}(x,y) \right\rangle = \sum_{k_x=-\infty}^{\infty} g(n) \sum_{k_y=-\infty}^{\infty} g(p) \left\langle f(x,y), \varphi_{m-1;k_x:k_y}(x,y) \right\rangle = \tag{116}$$

$$\sum_{k_x=-\infty}^{\infty} g(k_x - 2n_x) \sum_{k_y=-\infty}^{\infty} g(k_y - 2n_y) \left\langle f(x,y), \varphi_{m-1;k_x:k_y}(x,y) \right\rangle =$$

$$= \sum_{k_x=-\infty}^{\infty} g(k_x - 2n_x) \sum_{k_y=-\infty}^{\infty} g(k_y - 2n_y)\, c^{m-1}_{k_x;k_y}.$$

Therefore, by comparing equations (107) with the expressions (114)-(116) we conclude that the details $D^i_m f$ can be calculated through successive filtering in both directions. The filter used in each direction, denoted above by $g(.)$ and $h(.)$, are defined by expressions (111)-(113).

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