33-777

Jan 23, 2020

Today

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- Celestial Mechanics
- Exoplanets

The Solar System

Any student of astrophysics needs to have a basic understanding of the Solar System, both the relavent scales and the basic physics governing the motions of solar system objects. This is important to build intuition for astrophysical systems, and because the Solar System serves as our prototypical example of a star + planets system.

The Solar System is composed of the Sun, planets, dwarf planets, asteroids, and comets. The Sun is by far the dominant constituent by mass,

$$1 \text{ solar mass} = 1\,M_\odot = 1.989\times10^{33}\text{ g}$$

The Sun accounts for $>99\%$ of the mass in the Solar System.

There are eight planets in the Solar System,

Planet	Mass	Sun-Planet Distance
Mercury	$0.055\ M_\oplus$	0.4 AU
Venus	$0.815\ M_\oplus$	0.7 AU
Earth	$1\ M_\oplus$	1 AU
Mars	$0.107\ M_\oplus$	1.5 AU
Jupiter	$318\ M_\oplus$	5.2 AU
Saturn	$95\ M_\oplus$	9.5 AU
Uranus	$14\ M_\oplus$	19.2 AU
Neptune	$17\ M_\oplus$	30.1 AU

Note that,

$$1 \text{ Earth Mass} = 1\ M_\oplus = 5.972 \times 10^{27}\ g$$

$$\text{Earth-Sun distance} = 1\ AU = 1.496 \times 10^{13}\ cm$$

The Solar System contains a number of dwarf planets.

object	mass	object-Sun distance
Ceres	$0.00015\ M_\oplus$	2.7 AU
Pluto	$0.0022\ M_\oplus$	39.5 AU
Eris	$0.0028\ M_\oplus$	67.7 AU
Sedna	$0.00022\ M_\oplus$	479.7 AU

These are only some of the known dwarf planets. According to the IAU, both planets + dwarf planets

- orbit the Sun
- obtain a nearly spherical shape
- not a satellite (i.e. Moon)

Dwarf planets differ from planets in that they have not "cleared the neighborhood" around their orbit.

The "size" of the Solar System is ill-defined. Comet West, a comet observed in 1976, perhaps has the largest known aphelion of any bound object at $\sim 70{,}000$ AU.

The Oort cloud is a proposed spherically distributed "cloud" of comet-like objects at a distance of $\sim 50{,}000$ AU. The nearest star to the Sun is Proxima Centauri at a distance of 1.3 pc away, $\sim 2.7 \times 10^5$ AU.

Celestial Mechanics

Celestial Mechanics is the field of astrophysics concerned with the motions of celestial bodies, primarily planets and stars. This is a very old field. Tycho Brahe (1546-1601) was one of the first to turn this into a quantitative science. Brahe made impressively precise "naked eye" observations of the apparant location(s) of stars and planets. He (and his assitants) used sextants and quandrants to make measurements accurate to $\sim 4'$.

Brahe's protégé, Johannes Kepler (1571-1630), used his observations to show the apparant planetary motions were most easily explained with a helio-centric model of the solar system.

Kepler's key insight is that planetary orbits were best described as ellipses, with the Sun placed at one focus of the ellipse. Kepler formulated three laws to describe these orbits:

1. planetary orbits are elliptical with the Sun at one focus and the other empty.
2. a line between the Sun and a planet sweeps out equal areas during equal time intervals.
3. $$\left(\frac{\text{orbital period}}{\text{years}}\right)^2 = \left(\frac{\text{semimajor axis}}{\text{AU}}\right)^3$$

Note that the third law is only valid for objects orbiting the Sun. In <u>general</u>, for bound 2-body systems,

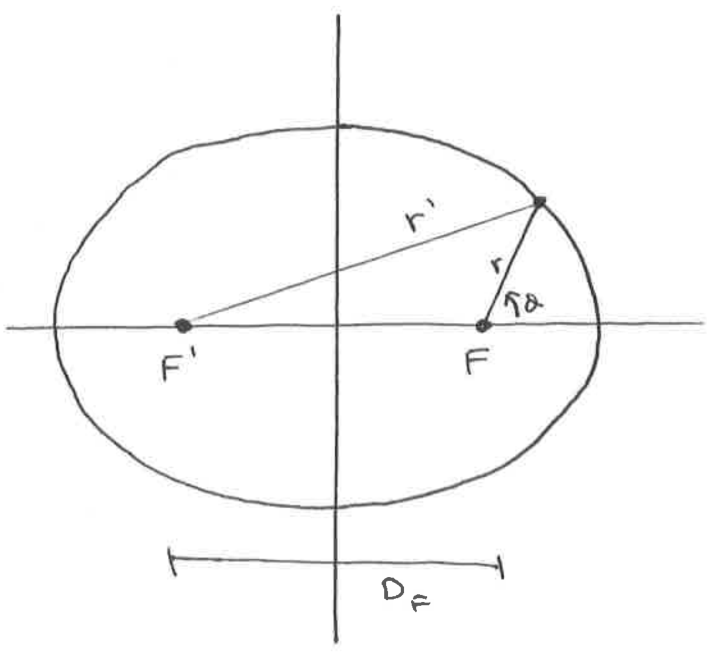
$$P^2 \propto a^3$$

orbital period

semimajor axis

It is worth reviewing gravitational 2-body dynamics. Later, when discussing star clusters and galaxies, we will deal with the much more complicated many body dynamics.

As empirically discovered by Kepler, binary (2-body) orbits are well described by elliptical orbits.



r : seperation between the object + focus F

r' : seperation between the object + focus F'

D_F : seperation between focii

For any object on the ellipse,

$$\text{const.} = l = r + r' + D_F$$

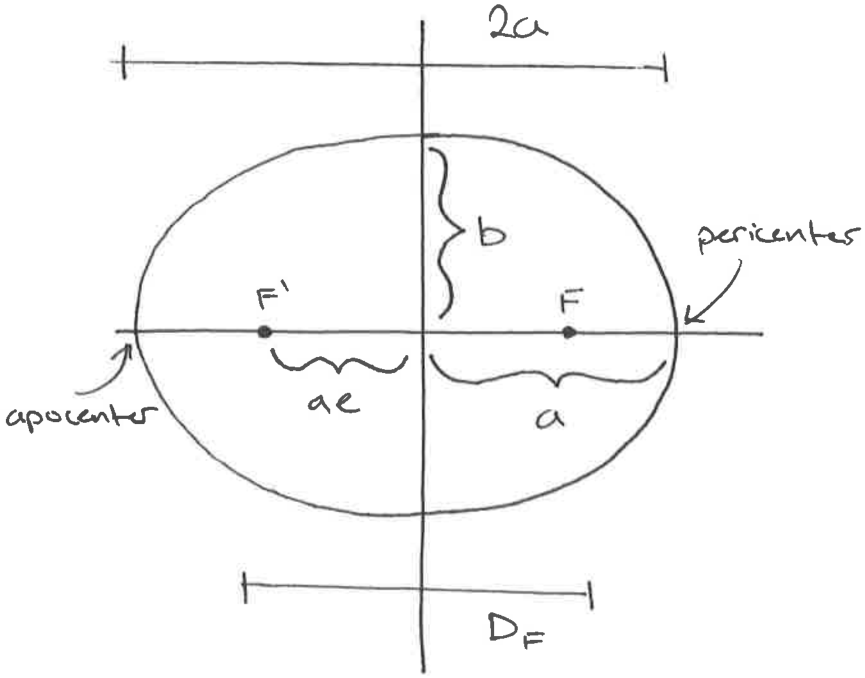
$$\Rightarrow r + r' = l - D_F$$

note

r, r' are variable

l, D_F are constants

Since d, D_F are a bit clunky, let's instead define the semi-major axis, a, and the eccentricity, e, to characterize the ellipse.



$$e \equiv \frac{D_F}{2a}$$

$$\Rightarrow ae = \frac{D_F}{2}$$

Consider the case where $d = 0$. This defines the pericenter of the orbit, i.e. the distance of closest approach. In the Solar System this is called perihelion. In this case,

$$r' = \frac{1}{2} d \quad \Rightarrow \quad d = 2r'$$

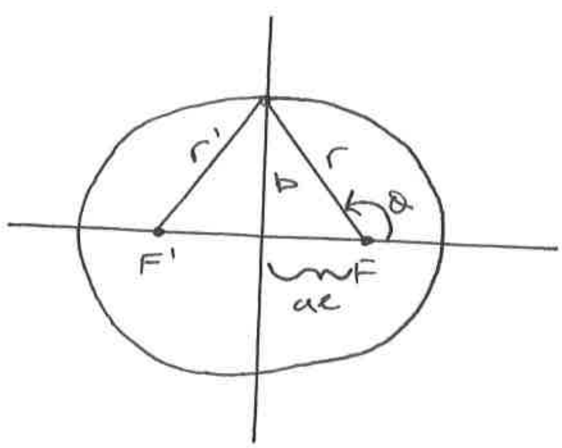
and

$$a = \frac{1}{2} D_F + r \quad \Rightarrow \quad D_F = 2(a - r)$$

Using the fact that l is constant,

$$r + r' = l - D_F$$
$$= 2r' - 2(a - r)$$
$$= -2a + 2(r + r')$$
$$\Rightarrow \quad r + r' = 2a$$

Now, consider an object on the semi-minor axis,



In this case,

$$r = r'$$

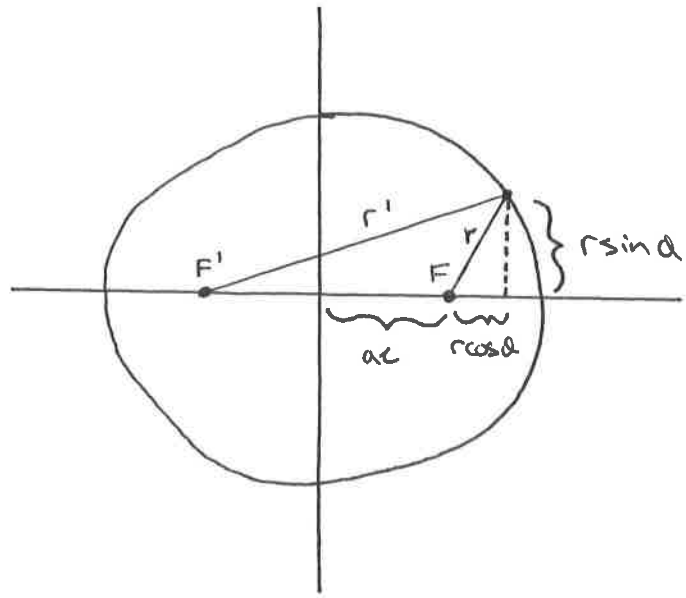
Using the previous result

$$\Rightarrow r = r' = a$$

From very simple geometric arguments,

$$a^2 = (ae)^2 + b^2$$
$$\Rightarrow \quad a^2(1 - e^2) = b^2$$
$$\Rightarrow \quad e = \left(1 - \frac{b^2}{a^2}\right)^{1/2}$$

Finally, we can work out the relationship between r and θ.



Again from simple geometry,

$$(r')^2 = (2ae + r\cos\theta)^2 + (r\sin\theta)^2$$

From an earlier result,

$$(r')^2 = (2a - r)^2$$

Combing these gives,

$$r = \frac{a(1-e^2)}{1 + e\cos\theta}$$

Example

With this last result, it is straight forword to calculate the distance of closest and furthest approach (pericenter + apocenter).

$$r_{peri}(\theta = 0) = \frac{a(1-e^2)}{1+e\cos\theta} = \frac{a(1-e^2)}{1+e}$$

$$= a(1-e)$$

$$r_{apo}(\theta = \pi) = \frac{a(1-e^2)}{1-e}$$

$$= a(1+e)$$

The Earth's orbital eccentricity is $e_\oplus = 0.0167$. How do r_{peri} and r_{apo} compare for the Earth?

$$\frac{r_{peri}}{r_{apo}} = \frac{a(1-e)}{a(1+e)} = \frac{1-e}{1+e} \simeq 0.97$$

A final note on this basic description of an orbit. A valid orbit need not be a closed ellipse. Parabolic and hyperbolic orbits are also valid. In these cases the binary system is marginally bound and unbound respectively.

orbit	eccentricity
circular	$e = 0$
elliptical	$0 < e < 1$
parabolic	$e = 1$
hyperbolic	$e > 1$

Newtonian Dynamics

As a practical matter, Kepler's description is most useful when $M_1 \gg M_2$, i.e. when one of the objects is much more massive than the other. In this case, the more massive object can be considered at rest. However, this is easily solved with the more generic Newtonian description.

In this case it is most convenient to consider the problem in the center of mass reference frame. For a multi-body system, the center-of-mass is defined as

$$\vec{r}_{cm} \equiv \frac{\sum_{i=1}^{N} m_i \vec{r}_i}{\sum_{i=1}^{N} m_i} = \frac{\sum_{i=1}^{N} m_i \vec{r}_i}{\underbrace{M}_{\substack{\text{total mass,} \\ M \equiv \sum_{i=1}^{N} m_i}}}$$

$$\vec{v}_{cm} \equiv \frac{d}{dt}(\vec{r}_{cm}) \Rightarrow \vec{v}_{cm} = \frac{\sum_{i=1}^{N} m_i \frac{d}{dt}(\vec{r}_i)}{M}$$

$$= \frac{\sum_{i=1}^{N} m_i \vec{v}_i}{M}$$

The center of mass position and velocity is simply the mass weighted average position and velocity of the constituent bodies.

Similarly, the center of mass momentum,

$$\vec{P}_{cm} = M\vec{v}_{cm} = \sum_{i=1}^{N} m_i \vec{v}_i$$

For an isolated system,

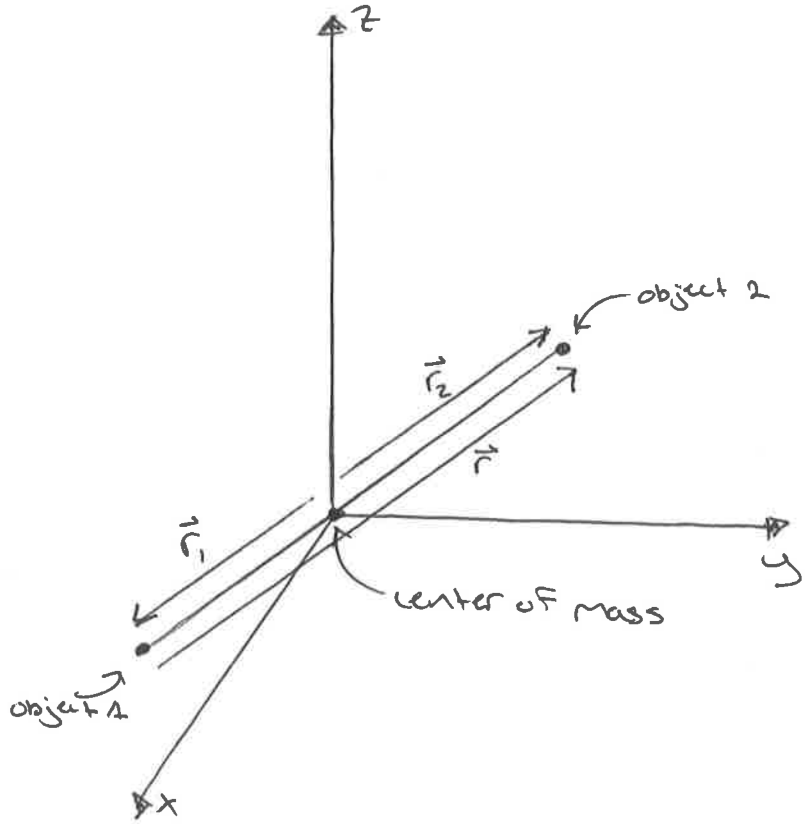
$$\frac{dP_{cm}}{dt} = \underbrace{\vec{F}_{Net}}_{\text{external force}} = 0$$

That is, $\vec{P}_{cm}$ and $\vec{v}_{cm}$ are constant.

Given this, a convenient choice for a coordinate system is one where the center-of-mass is at rest at the origin.

$$\Rightarrow \quad \vec{r}_{cm} = 0 \quad , \quad \vec{v}_{cm} = 0$$

Under these conditions, a complete solution to the 2-body problem can be computed.



The radial vector from object 1 to 2 is

$$\vec{r}_{12} \equiv \vec{r} = \vec{r}_2 - \vec{r}_1$$

By construction,

$$\vec{r}_{cm} = 0 = \frac{m_1 \vec{r}_1 + m_2 \vec{r}_2}{m_1 + m_2}$$

$$\Rightarrow m_1 \vec{r}_1 + m_2 \vec{r}_2 = 0$$

Using the definition of $\vec{r}$, we can rewrite $\vec{r}_1$, and $\vec{r}_2$ in terms of $\vec{r}$.

$$m_1\vec{r}_1 + m_2(\vec{r}_1 + \vec{r}) = 0$$

$$\Rightarrow \vec{r}_1(m_1+m_2) = -m_2\vec{r}$$

$$\Rightarrow \vec{r}_1 = \frac{-m_2}{m_1+m_2}\vec{r}$$

$$m_1(\vec{r}_2 - \vec{r}) + m_2\vec{r}_2 = 0$$

$$\Rightarrow \vec{r}_2(m_1+m_2) = m_1\vec{r}$$

$$\Rightarrow \vec{r}_2 = \frac{m_1}{m_1+m_2}\vec{r}$$

Thus, given the seperation between the two bodies, $\vec{r}$, if we know the masses m_1 and m_2, one can calculate $\vec{r}_1$ and $\vec{r}_2$.

Moving on, we can write down the total energy of the system

$$E = \underbrace{\frac{1}{2}m_1v_1^2 + \frac{1}{2}m_2v_2^2}_{\text{Kinetic}} - \underbrace{\frac{Gm_1m_2}{r}}_{\text{potential}}$$

$$= \frac{1}{2}m_1\left|\frac{d\vec{r}_1}{dt}\right|^2 + \frac{1}{2}m_2\left|\frac{d\vec{r}_2}{dt}\right|^2 - \frac{Gm_1m_2}{r}$$

Applying the relations for $\vec{r}_1$ and $\vec{r}_2$ in terms of $\vec{r}$ gives a convenient expression for the total energy.

$$E = \frac{1}{2}\frac{m_1 m_2}{(m_1+m_2)} v^2 - \frac{G m_1 m_2}{r}$$

where $\vec{v} = \frac{d\vec{r}}{dt}$

Given this result, it is convenient to define the reduced mass of the system,

$$\mu \equiv \frac{m_1 m_2}{m_1 + m_2}$$

Now the total energy is given by:

$$E = \frac{1}{2}\mu v^2 - \frac{G M \mu}{r}$$

Compare this result to the energy associated with a particle of mass m orbiting a stationary mass M $(M \gg m)$.

From this point it is relatively easy to derive Kepler's Laws. Here I will simply state the Newtonian forms.

1. Kepler's 1st law

$$r = \frac{L^2/\mu}{GM(1 + e\cos\theta)}$$

Here L is the total angular momentum of the system.

2. Kepler's 2nd law

$$\frac{dA}{dt} = \frac{1}{2}\frac{L}{\mu}$$

3. Kepler's 3rd law

$$P^2 = \frac{4\pi^2}{GM}a^3$$

Insolation Temperature

The effective surface temperature of the Sun is $T_\odot \simeq 5800$ K. In the end, the energy that heats the solar surface comes from nuclear fusion processes in the core (much more on this later). In general planets do not have signifigant energy sources. The reason planets, other solar system bodies, and exoplanets have surface temperatures significantly above the ambient temperature of space is that they absorb energy from their nearby star.

The equilibrium insolation temperature is the planetary surface temperature due to irradiation by the flux from its host star.

This can be calculated as follows.

The flux a distance, a, from a star is given by,

$$F = \frac{L_*}{4\pi a^2}$$

L_* : luminosity of the star

a : distance from star

Stars are an excellent approximation to a blackbody. Given this, the luminosity is given by,

$$L_* = 4\pi\sigma T_*^4 R_*^2$$

T_* : effective stellar surface temp.

R_* : stellar radius

Combining these two results gives the flux as,

$$F = \sigma T_*^4 \frac{R_*^2}{a^2}$$

The cross section of the planet that intercepts this flux is given by,

$$\pi R_p^2$$

R_p : planetary radius

The total luminosity absorbed by the planet is then,

$$P_i = F \pi R_p^2 (1 - \alpha)$$

where α is the albedo of the planetary surface.

If we assume the planet radiates as a blackbody and maintains an equilibrium temperature, then this energy must be radiated over the entire surface of the planet. This will be given by,

$$P_r = 4\pi R_p^2 \sigma T_p^4$$

By equating the absorbed and radiated power,

$$P_i = P_r$$

$$F \pi R_p^2 (1-\alpha) = 4\pi R_p^2 \sigma T_p^4$$

$$\frac{\sigma T_*^4 R_*^2 \pi R_p^2 (1-\alpha)}{a^2} = 4\pi R_p^2 \sigma T_p^4$$

$$T_p^4 = \frac{T_*^4 R_*^2 (1-\alpha)}{4a^2}$$

$$\Rightarrow T_p = \frac{T_*}{\sqrt{2}} \left(\frac{R_*}{a}\right)^{1/2} (1-\alpha)^{1/4}$$

If we assume $\alpha = 0$ (perfect absorber), the insolation temperature for the Earth at 1 AU

$$T_{\oplus} = 281 \text{ K} \simeq 7°\text{C}$$

This is not such a bad estimate. In reality, the Earth's albedo is > 0, the surface is not a uniform temperature, and the Earth's atmosphere significantly affects ground surface temperatures.

Similar calculations (allowing for atmospheric and albedo differences between planets) can be used to define a "goldilocks zone" around stars where planets may have liquid water.

Exoplanets

The planets interrior to Saturn (including Saturn itself) where known since ancient times. Uranus was discovered in 1781 by William Herschel (1738 - 1822). The last planet to be discovered in the solar system was Neptune, confirmed in 1846. Credit is generally given to Urbain Le Verrier who made predictions for Neptune's orbit from deviations in Uranus's orbit.

Only recently have planets been discovered beyond the solar system. The first confirmed exoplanets were detected in 1992 around a pulsar, PSR 1257+12. Three exoplanets were subsequently confirmed by inference in modelling the pulsation period anomalies. To date, more than 4,000 exoplanets have been confirmed.

Many exoplanet detection and characterization techniques are really just the application of Kepler's laws to other star planet systems. Here we will talk about a few of the major methods

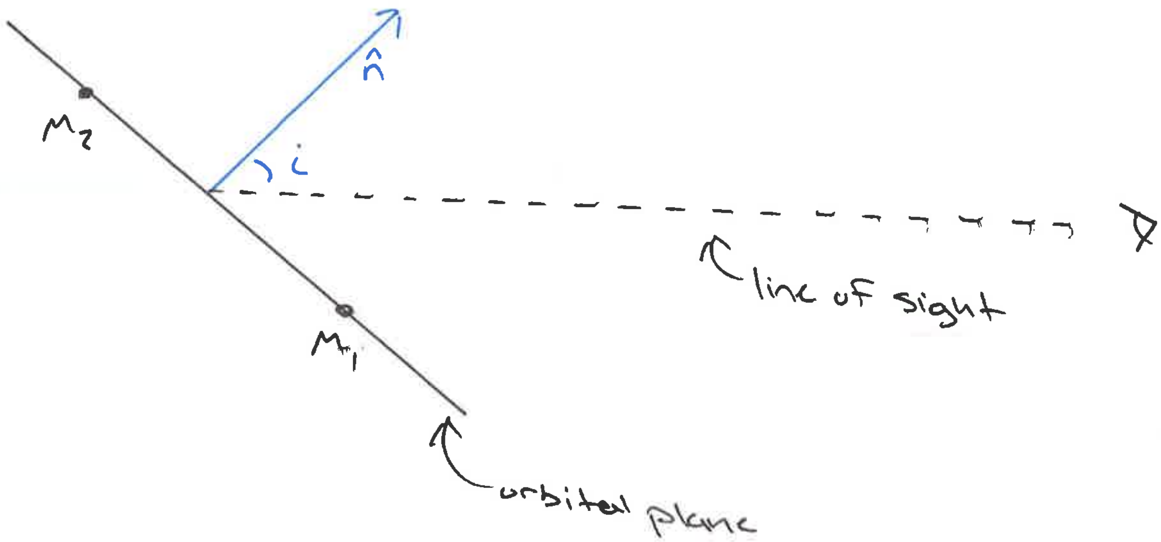
- radial velocity
- direct imaging
- transit
- Microlensing

Radial Velocity Method

This method relies on measuring doppler shift to infer the orbital velocity of one or more members in a system. The measured velocity amplitudes of the members of a binary system are related to the orbital velocity amplitude via

$$|v_{1,obs}| = |v_1|\sin(i) \quad , \quad |v_{2,obs}| = |v_2|\sin(i)$$

where i is the inclination angle.



For a circular orbit

$$|v_1| = \frac{2\pi r_1}{P_1}, \quad |v_2| = \frac{2\pi r_2}{P_2}, \quad P_1 = P_2 = P$$

From the measured velocities,

$$\frac{|v_{1,obs}|}{|v_{2,obs}|} = \frac{r_1}{r_2} = \frac{M_2}{M_1}$$

We can also express Kepler's 3rd law as,

$$(M_1 + M_2)\sin^3(i) = \frac{P\left(|v_{1,obs}| + |v_{2,obs}|\right)^3}{2\pi G}$$

For so called <u>spectroscopic binaries</u>, the mass of each object can be determined up to a factor of $\sin^3(i)$.

For <u>eclipsing binaries</u>, the systems are observed approximatly edge-on and $i \sim 90°$.

If the secondary object is too faint to observe, we can replace $v_{2,obs}$ in our re-expression of Kepler's 3rd law with

$$|v_{2,obs}| = |v_{1,obs}| \frac{M_1}{M_2}$$

$$\Rightarrow \quad (M_1 + M_2)\sin^3(i) = \frac{P|v_{1,obs}|^3 \left[1 + M_1/M_2\right]^3}{2\pi G}$$

$$\Rightarrow \quad \frac{M_2^3}{(M_1 + M_2)^2} \sin^3(i) = \frac{P|v_{1,obs}|^3}{2\pi G}$$

when $M_2 << M_1$, which is usually appropiate for planets, this simplifies to,

$$M_2 \sin(i) \simeq \left(\frac{P}{2\pi G}\right)^{1/3} |v_{1,obs}| M_1^{2/3}$$

From now on, I will use $M_p = M_2$ for the planet mass and $M_* = M_1$ for the host star's mass.

Example

For a Sun-Jupiter like system

$$|V_{*,obs}| \simeq M_p \sin(i) M_*^{-2/3} \left(\frac{P}{2\pi G}\right)^{-1/3}$$
$$= 31 \frac{m}{s} \times \sin(i)$$

The current state of the art allows doppler shifts in stars to be measured down to $\sim 1 \frac{m}{s}$.

2 Transit Method

This method relies on the planet-star system to form an eclipsing binary. This happens when the system is observed approximatly edge-on.

$$\sin(i) \sim 1 \Rightarrow i \simeq 90^\circ$$

In these type of systems, the planet will pass in front of and behind its host star. When the planet passes in front of its host star, the observed flux will decrease,

$$\frac{\Delta F}{F} = \left(\frac{r_p}{r_*}\right)^2$$

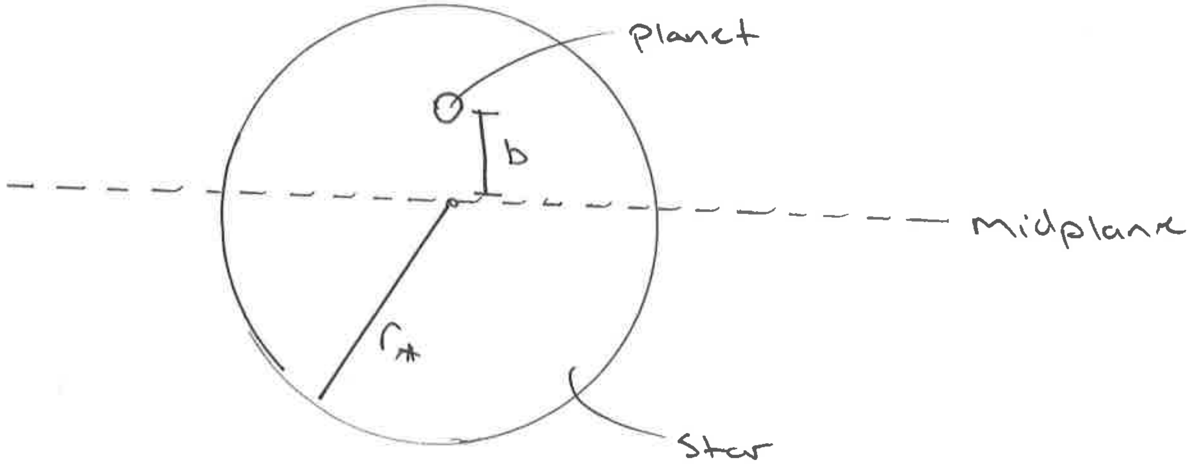
r_p: planet radius

r_*: star radius

The time a planet takes to complete a transit is then

$$t_{transit} = \frac{r_* P}{\pi a}\left(1 - b^2\right)^{1/2}$$

where b is the impact parameter, measured in r_*.



The impact parameter is related to the inclination angle by

$$\cos(i) = \frac{b r_*}{a}$$

$$\Rightarrow \quad b = \frac{a \cos(i)}{r_*}$$

If the properties of the host star are known, M_*, r_*, and $M_p \ll M_*$, the planet mass, size, and the inclination angle can be determined.

Direct Imaging

In principle, the most direct way to detect exoplanets is to directly image an exoplanetary system around a star. However, in practice, this is actually <u>very</u> challenging.

Ignoring the effects of atmospheric bluring, the largest optical telescopes ($D \sim 10$ m), operating in the infrared ($\lambda \sim 2\,\mu m$), can achcive diffraction limited seeing of

$$d_{min} = 1.22 \frac{\lambda}{D} = \frac{1.22(200 \times 10^{-9}\ m)}{(10\ m)} = 2.4 \times 10^{-7}$$

$$= 0.05''$$

1 AU subtends an angle greater or equal to this out to,

$$d = \frac{1\ AU}{0.05''} = 20\ pc$$

While this limits us to all but the nearest stars, the contrast between the star light and reflected light from the planet is very large.

$$\frac{f_p}{f_*} = \frac{\alpha \pi r_p^2}{4\pi a^2}$$

r_p : radius of planet
α : albedo of planet
a : semimajor axis of orbit

For a typical albedo of $\alpha = 0.5$ and a jupiter mass planet at 1 AU, the flux ratio is:

$$\frac{f_p}{f_*} = \frac{0.5}{4}\left(\frac{7\times10^9\ \text{cm}}{1.5\times10^{13}\ \text{cm}}\right)^2 \simeq 10^{-8}$$

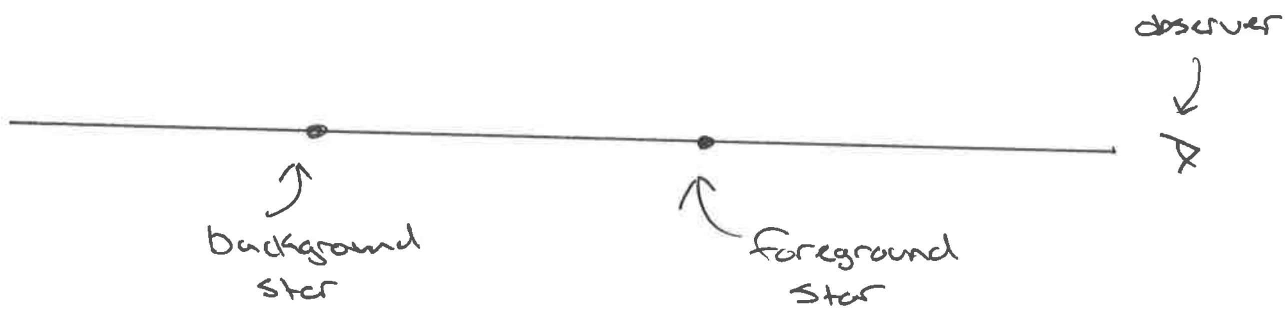
This makes detection particularly challenging. For example, even if the planet happened to fall within the first diffraction trough, slight imperfections in the optical system that scatters 1% of light will swamp the photons from the planet.

There are a few methods that try to overcome these challenges.

- use of <u>coronographs</u> to decrease the number of photons from the star
- search for planets with large orbits, although $f_p \propto \frac{1}{a^2}$
- search for "hot" planets that have detectable emitted photons

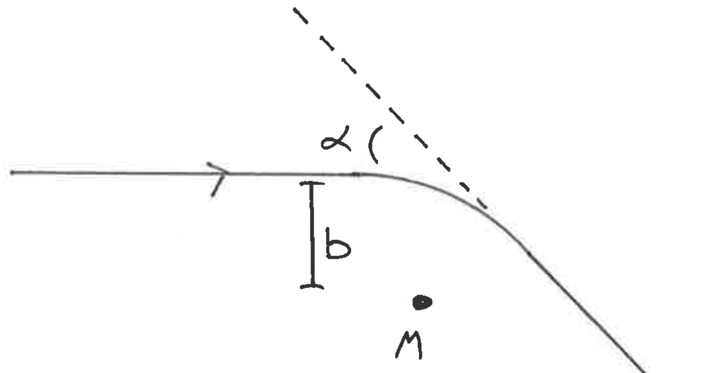
Gravitational Microlensing

Another sucessful exoplanet detection method is microlensing. This metod makes use of the chance alignment between an observer, a foreground star, and a background star.



A prediction of general relativity is that a light ray will be deflected in the presence of a massive object.

A light ray with an impact parameter, b, wrt a massive object, with mass M, will be deflected by an angle α.

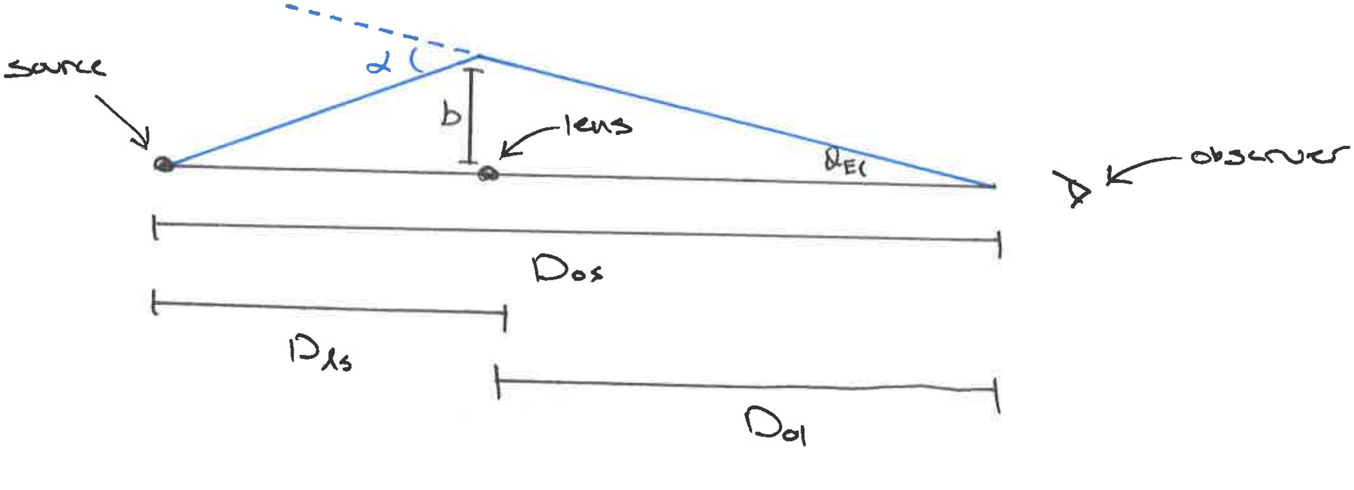


$$\alpha = \frac{4GM}{cb^2}$$

As long as the gravitational field is weak,

$$\frac{GM}{cb^2} << 1$$

and the distances between the source, the "lens", and the observer is >> b



The angle α_E defines the Einstein radius,

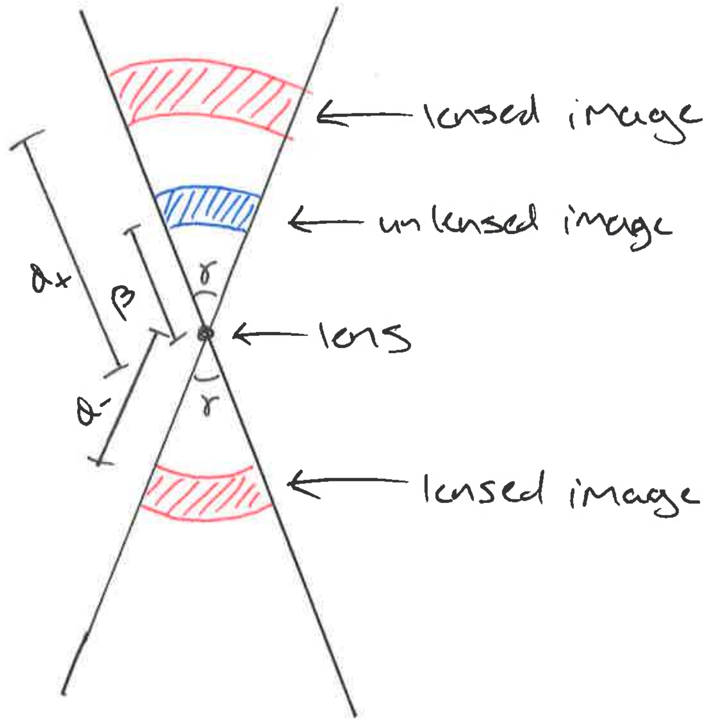
$$\alpha_E = \left(\frac{4GM}{c^2} \frac{D_{ls}}{D_{ol} D_{os}} \right)^{1/2}$$

The apparent angular location relative to the lens becomes slightly more complicated when the source is off-axis. In this case two images are produced at

$$\alpha_\pm = \frac{1}{2}\left[\beta \pm \left(\beta^2 + 4\alpha_E^2\right)^{1/2}\right]$$

where β is the angular distance off-axis of the source.

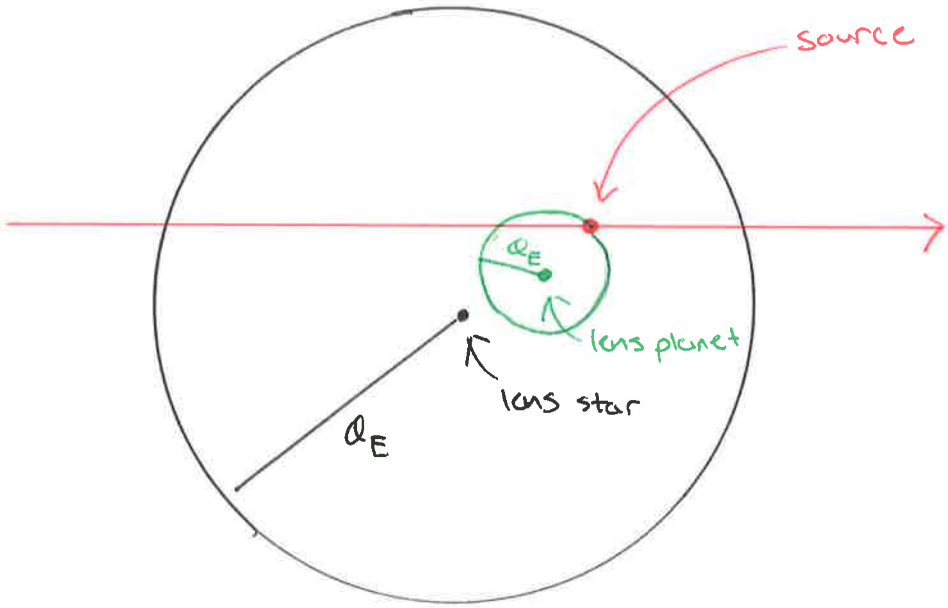
Gravitational lensing also affects shape and size of the source image. For a source that would unlensed subtend a tangential angle γ relative to the lens, the primary effect of lensing is to shift the location of the image while preserving γ.



This angular shift results in <u>magnification</u>, an increase in the angular size of the image. Because surface brightness is conserved this results in an amplification or brightening of the source.

$$a_{tot} = \frac{u^2 + 2}{u(u^2+4)^{1/2}} \qquad u \equiv \frac{\beta}{\theta_E}$$

Note interesting limits when $\beta = \theta_E$, $a = 1.34$, and when $\beta = 0$, $a = \infty$.



As the background (source) star moves through the einstein ring of the foreground (lens) star the source is magnified. The magnification is maximized when the angular seperation between the source and lens is minimized.

If the source passes close* to a planet orbiting the foreground star, the magnification will breifly increase.

* in projection