Algorithmic Foundations : Graph Algorithms

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Why Graphs ?

Graphs have applications in many areas, including :

- Social networks, air networks, road networks, energy networks (electricity, gas, water,...)
- Mathematics : graphs are useful is many areas of mathematics (geometry, dynamic systems, operations research,...)
- Computer science : links between web pages, networks of communication, data organization, map networks, ...
- Statistical physics (interactions between parts of a system), study of molecules, ...

Some of the objectives of the course :

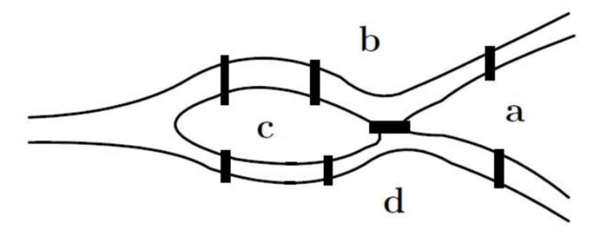
- Understand the structure of graphs
- Learn the techniques used to analyze problems in graph theory
- Learn algorithms on trees and graphs

Chapter 1 : Fundamental concepts about graphs

Outline

The Königsberg Bridge Problem

The city of Königsberg was located on the Pregel river in Prussia which divides it into four regions. These regions are connected together using seven bridges, as shown in the following drawing :

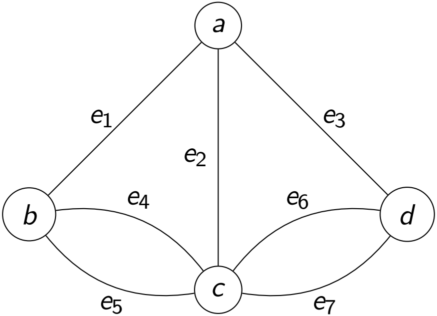


Citizens wondered if they could take a walk in the city while crossing each bridge exactly once.

The resolution of this problem was given by Leonhard Euler in 1736.

The Königsberg Bridge Problem

To simplify this problem, we reduce the drawing to the following diagram :



The four regions are represented by vertices and the seven bridges by edges.

Answer : Such walk is not possible.
Proof :

Definition

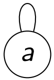
A **graph** G is a triple consisting of a **vertex set** $V(G)$, an **edge set** $E(G)$, and a relation that associates with each edge two vertices (not necessarily distinct) called its **endpoints**.

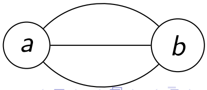
Example

In the graph showed above, the vertex set is $\{a, b, c, d\}$, the edge set is $\{e_1, e_2, e_3, e_4, e_5, e_6, e_7\}$, and the assignment of endpoints to edges can be read from the diagram.

Definition

A **loop** is an edge whose endpoints are equal. **Multiple edges** are edges having the same pair of endpoints.





Definition

A **simple graph** is a graph without loops and multiple edges.
A graph is **loopless** means that multiple edges are allowed but loops are not.

Example

The graph of the Königsberg bridge problem is loopless but it is not simple.

Remark

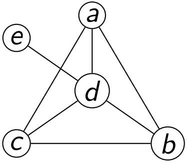
We specify a simple graph G by its vertex set $V(G)$ and edge set $E(G)$. The edge set is treated as a set of unordered pairs of vertices. We write $e = uv$ (or $e = vu$) for an edge e with endpoints u and v.

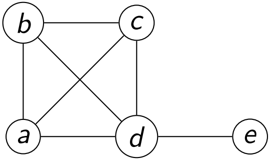
Definition

We say that the vertices u and v are **adjacent** (or **neighbors**) if they are the endpoints of the same edge.

Example

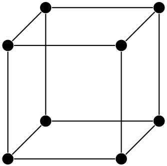
The two following drawings show the same simple graph. The vertex set is $\{a, b, c, d, e\}$ and the edge set is {ab, ac, ad, bc, bd, cd, de} :





Remark

The terms "vertex" and "edge" arise from solid geometry. For example the following 3-d cube has vertices linked by edges.



Outline

Acquaintance relations and subgraphs

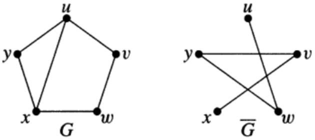
Question

Does every set of six people contain three mutual acquaintances or three mutual strangers ?

We can model this problem using a simple graph with a vertex for each person and an edge for each acquainted pair.
Note that the "nonacquaintance" relation yields another graph with the "complementary" set of edges.

Definition

The **complement** $\bar{G}$ of a simple graph G is the simple graph with vertex set $V(G)$ defined by $uv \in E(\bar{G})$ if and only if $uv \notin E(G)$.
A **clique** in a graph is a set of pairwise adjacent vertices.
An **independent set** (or **stable set**) in a graph is a set of pairwise nonadjacent vertices.



In the graph G, $\{u, x, y\}$ is a clique of size 3 and $\{u, w\}$ is an independent set of size 2, and these are the largest such sets.

Remark

In the complement graph $\bar{G}$, cliques become independent sets, and independent sets become cliques.

Reformulation of the above question : Is it true that every 6-vertex graph has a clique of size 3 or an independent set of size 3 ?
Answer : (As exercice)

Notice that by deleting the edge ux from the above graph G, we get a 5-vertex graph having no clique or independent set of size 3. So the answer is no for 5-vertex graphs.

Job assignments and bipartite graphs

Question

We have m jobs, and n people, but not all people are qualified for all jobs. Can we fill the jobs with qualified people ?

We model this using a simple graph G with vertices for the jobs and people ; job j is adjacent to person p if p can do j.

So the question is to find m pairwise disjoint edges in G.

Definition

A graph G is **bipartite** if $V(G)$ is the union of two disjoint independent sets called **partite sets** of G.

Scheduling and graph coloring

Question

Suppose we must schedule Parliament committee meetings into designated weekly time periods. We cannot assign two committees to the same time if they have a common member. How many different time periods do we need ?

Modeling : We create a vertex for each committee, with two vertices adjacent when the two committees have a common member. We must assign labels (time periods) to the vertices such that adjacent vertices receive different labels.

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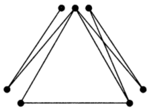

We can use one color (label) for each of the three independent sets.

The members of a clique must receive distinct labels $\Rightarrow$ the minimum number of colors (time periods) is three.

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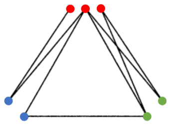

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Definition

A **coloring** of a graph is an assignment of a color to each vertex such that adjacent vertices receive different colors.
The **chromatic number** of a graph G, written $\chi(G)$, is the minimum number of colors needed to color G.
A graph G is k-**partite** if $V(G)$ can be expressed as the union of k (possibly empty) independent sets.

So the above question is asking about the chromatic number of the graph and the corresponding coloring.

Remark

- The k-partite notion generalizes the idea of bipartite graphs, which are 2-partite.
- A graph is k-partite if and only if its chromatic number is at most k.

Routes in road networks

Modeling :

We can model a road network using a graph where :

- Vertices are intersections,
- Edges are road segments between intersections,
- We can assign edge weights to measure distance or travel time.

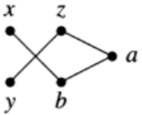
Question

How can we find the shortest route from a point x to a point y ?

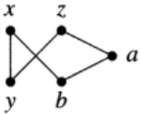
Definition

A **path** is a simple graph whose vertices can be ordered so that two vertices are adjacent if and only if they are consecutive in the list.
A **cycle** is a graph with an equal number of vertices and edges whose vertices can be placed around a circle so that two vertices are adjacent if and only if they appear consecutively along the circle.

Example : a path



a cycle



Definition

A **subgraph** of a graph G is a graph H such that $V(H) \subset V(G)$ and $E(H) \subset E(G)$ and the assignment of endpoints to edges in H is the same as in G. We write $H \subset G$ and say that "G contains H".
A graph G is **connected** if each pair of vertices in G is connected by a path ; otherwise, G is **disconnected**.

Outline

Matrix representation

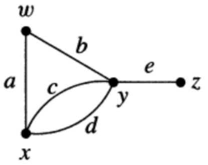
Definition

Let G be a loopless graph with vertex set $V(G) = \{v_1, \ldots, v_n\}$ and edge set $E(G) = \{e_1, \cdots, e_m\}$.
The **adjacency matrix** of G, written $A(G)$, is the n-by-n matrix in which entry a_{ij} is the number of edges in G with endpoints $\{v_i, v_j\}$.
The **incidence matrix** $M(G)$ is the n-by-m matrix in which entry m_{ij} is 1 if v_i is an endpoint of e_j and otherwise is 0.

Example :

$$\begin{array}{c} \\ w \\ x \\ y \\ z \end{array} \begin{array}{c} \begin{array}{cccc} w & x & y & z \end{array} \\ \begin{pmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 2 & 0 \\ 1 & 2 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \end{array}$$

$A(G)$



G

$$\begin{array}{c} \\ w \\ x \\ y \\ z \end{array} \begin{array}{c} \begin{array}{ccccc} a & b & c & d & e \end{array} \\ \begin{pmatrix} 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \end{array}$$

$M(G)$

Definition

If vertex v is an endpoint of edge e, we say that v and e are **incident**. The **degree** of vertex v (in a loopless graph) is the number of incident edges to v. We denote the degree of v by $\deg(v)$.

Remark

- An adjacency matrix is determined by a vertex ordering.
- Every adjacency matrix is symmetric ($a_{ij} = a_{ji}$ for all i, j).
- An adjacency matrix of a simple graph G has entries 0 or 1, with 0s on the diagonal.
- The degree of v is the sum of the entries in the row for v in either $A(G)$ or $M(G)$.

Graph isomorphism

Definition

An **isomorphism** from a simple graph G to a simple graph H is a bijection $f : V(G) \to V(H)$ such that $uv \in E(G)$ if and only if $f(u)f(v) \in E(H)$. We say "G is-isomorphic to H", written $G \cong H$, if there is an isomorphism from G to H.

Example : The following graphs G and H are isomorphic :

Proposition

The isomorphism relation is an equivalence relation on the set of (simple) graphs.

Proof :

Recall

An equivalence relation partitions a set into equivalence classes ; two elements satisfy the relation if and only if they lie in the same class.

Definition

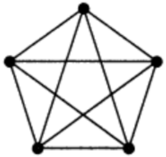
An **isomorphism class** of graphs is an equivalence class of graphs under the isomorphism relation.
We use the expression "unlabeled graph" to mean an isomorphism class of graphs.

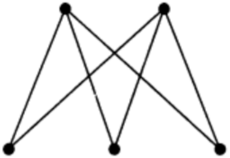
Definition

The (unlabeled) path and cycle with n vertices are denoted P_n and C_n, respectively.
A **complete graph** is a simple graph whose vertices are pairwise adjacent ; the (unlabeled) complete graph with n vertices is denoted K_n.
A **complete bipartite graph** (or **biclique**) is a simple bipartite graph such that two vertices are adjacent if and only if they are in different partite sets. When the sets have sizes r and s, the (unlabeled) biclique is denoted $K_{r,s}$.

Example :

K_5





$K_{2,3}$

Outline

Decomposition of graphs

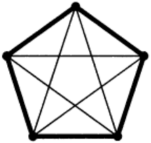
Definition

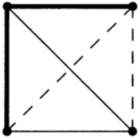
A **decomposition** of a graph is a list of subgraphs such that each edge appears in exactly one subgraph in the list.

Definition and Remark

A graph is **self-complementary** if it is isomorphic to its complement. An n-vertex graph H is self-complementary if and only if K_n has a decomposition consisting of two copies of H.

Example : We can decompose K_5 into two 5-cycles, and thus the 5-cycle is self-complementary (left figure).
We can decompose K_4 using three copies of P_3 (right figure).



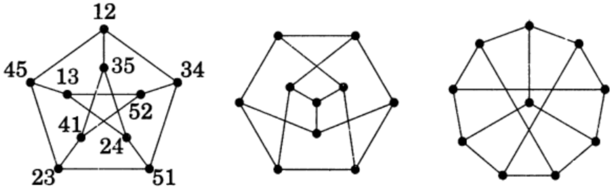


Petersen graph

Definition

The **Petersen graph** is the simple graph whose vertices are the 2-element subsets of a 5-element set and whose edges are the pairs of disjoint 2-element subsets.

Here, we take $[5] = \{1, 2, 3, 4, 5\}$ as our 5-element set, we write the pair $\{a, b\}$ as ab or ba.
In the following figure, we have three ways to draw the Petersen graph :



The Petersen graph consists of two disjoint 5-cycles plus edges that pair up vertices on the two 5-cycles.

Proposition

If two vertices are nonadjacent in the Petersen graph, then they have exactly one common neighbor.

Proof :

Definition

The **girth** of a graph with a cycle is the length of its shortest cycle. A graph with no cycle has an infinite girth.

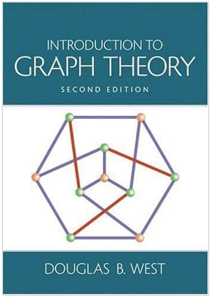
Lemma

The girth of the Petersen graph is equal to 5.

Proof :

Book references

Introduction to graph theory [2nded], by Douglas B. West (2001) :



(See Sect 1.1 of West's book for more details on our chapter 1.)

Algorithms on trees and graphs [2nded], by Gabriel Valiente (2021) :