Chapter 3 : Directed graphs, and Coloring

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Outline

1. Directed graphs
2. Coloring of graphs

More details on this chapter can be found in Sect 1.4 and Sect. 5.1 of D. B. West's book.

Outline

1. Directed graphs

2. Coloring of graphs

Definition

A **directed graph** or **digraph** G is a triple consisting of a vertex set $V(G)$, an edge set $E(G)$, and a function assigning each edge an ordered pair of vertices.
The first vertex of the ordered pair is the **tail** of the edge, and the second is the **head** ; together, they are the **endpoints**. We say that an edge is an edge **from** its tail **to** its head.

Definition

In a digraph, a **loop** is an edge whose endpoints are equal. **Multiple edges** are edges having the same ordered pair of endpoints. A digraph is **simple** if each ordered pair is the head and tail of at most one edge ; one loop may be present at each vertex. In a simple digraph, we write uv for an edge with tail u and head v. If there is an edge from u to v, then v is a successor of u, and u is a predecessor of v. We write $u \to v$ for "there is an edge from u to v".

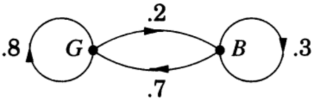
Example : Digraph of a Markov chain

Consider a system that can be in n possible states $s_1, s_2, \cdots, s_n$. After being in the state s_i, the system will go to state s_j with probability P_{ij}. Note that for each state s_i, we have $\sum_j P_{ij} = 1$. The digraph G representing this Markov chain is such that :

- $V(G) = \{s_1, s_2, \cdots, s_n\}$
- $s_i s_j \in E(G)$ iff $P_{ij} > 0$, in this case the edge $s_i s_j$ has a weight P_{ij}.

Definition : We say that a graph (or digraph) G is **weighted** if we associate to each edge a value, called a **weight**.

Example : Suppose that weather has two states : good (G) and bad (B). Air masses move slowly enough that tomorrow's weather tends to be like today's. In most places, storms don't linger long, so we might have transition probabilities as follows :

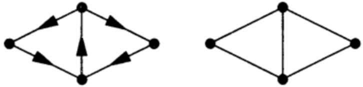


Definition

A digraph is a **path** if it is a simple digraph whose vertices can be linearly ordered so that there is an edge with tail u and head v if and only if v immediately follows u in the vertex ordering. A **cycle** is defined similarly using an ordering of the vertices on a circle.

Definition

The **underlying graph** of a digraph D is the graph G obtained by treating the edges of D as unordered pairs ; the vertex set and edge set remain the same, and the endpoints of an edge are the same in G as in D, but in G they become an unordered pair.



D G

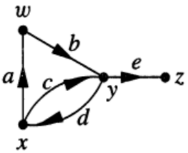
Definition

The definitions of **subgraph**, **isomorphism** and **decomposition** are the same for graphs and digraphs.
In the adjacency matrix $A(G)$ of a digraph G, the entry in position i, j is the number of edges from v_i to v_j.
In the incidence matrix $M(G)$ of a loopless digraph G, we set $m_{ij} = 1$ if v_i is the tail of e_j and $m_{ij} = -1$ if v_i is the head of e_j.

Example :

$$\begin{array}{c} \\ w \\ x \\ y \\ z \end{array}\begin{array}{c} \begin{array}{cccc} w & x & y & z \end{array} \\ \begin{pmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \end{array}$$

$A(G)$



G

$$\begin{array}{c} \\ w \\ x \\ y \\ z \end{array}\begin{array}{c} \begin{array}{ccccc} a & b & c & d & e \end{array} \\ \begin{pmatrix} -1 & +1 & 0 & 0 & 0 \\ +1 & 0 & +1 & -1 & 0 \\ 0 & -1 & -1 & +1 & +1 \\ 0 & 0 & 0 & 0 & -1 \end{pmatrix} \end{array}$$

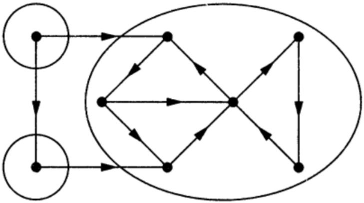
$M(G)$

Definition

A digraph is **weakly connected** if its underlying graph is connected.
A digraph is **strongly connected** or **strong** if for each ordered pair u, v of vertices, there is a path from u to v.
The **strong components** of a digraph are its maximal strong subgraphs.

Examples :

- As a digraph, an n-vertex path has n strong components, but a cycle has only one strong component.
- In the diagraph below, the three circled subdigraphs are the strong components :



Vertex degrees

Definition

Let v be a vertex in a digraph. The **outdegree** $d^+(v)$ is the number of edges with tail v. The **indegree** $d^-(v)$ is the number of edges with head v. The **out-neighborhood** or **successor set** $N^+(v)$ is $\{x \in V(G) : v \to x\}$. The **in-neighborhood** or **predecessor set** $N^-(v)$ is $\{x \in V(G) : x \to v\}$. The minimum and maximum indegree are $\delta^-(G)$ and $\Delta^-(G)$; for outdegree we use $\delta^+(G)$ and $\Delta^+(G)$.

Proposition

In a digraph G, $\sum_{v \in V(G)} d^+(v) = |E(G)| = \sum_{v \in V(G)} d^-(v)$.

Proof : Every edge has exactly one tail and exactly one head.

Remark : In a graph, the minimum and maximum degree are denoted $\delta(G)$ and $\Delta(G)$, and we have $\sum_{v \in V(G)} d(v) = 2|E(G)|$. $\Delta(G)$ is also called the degree of the graph.

Eulerian digraphs

The definitions of **trail**, **walk**, **circuit**, and the **connection relation** are the same in graphs and digraphs when we list edges as ordered pairs of vertices.

Definition

An **Eulerian trail** in a digraph (or graph) is a trail containing all edges.
An **Eulerian circuit** is a closed trail containing all edges.
A digraph is **Eulerian** if it has an Eulerian circuit.

Characterization of an Eulerian digraph

Lemma

If G is a digraph with $\delta^+(G) \geq 1$, then G contains a cycle. The same conclusion holds when $\delta^-(G) \geq 1$.

Proof : Consider a maximal path in G.

Theorem

A digraph is Eulerian if and only if $d^+(v) = d^-(v)$ for each vertex v and the underlying graph has at most one nontrivial component.

Outline

1. Directed graphs
2. Coloring of graphs

Definition

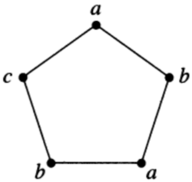
A k-**coloring** of a graph G is a labeling $f : V(G) \to S$, where $|S| = k$.
The labels are **colors**; the vertices of one color form a **color class**.
A k-coloring is **proper** if adjacent vertices have different labels. A graph is k-**colorable** if it has a proper k-coloring.
The **chromatic number** $\chi(G)$ is the least k such that G is k-colorable.
If $\chi(G) = k$, we say that G is k-**chromatic**.

Remark

In a proper coloring, each color class is an independent set, so G is k-colorable if and only if $V(G)$ is the union of k independent sets. Thus "k-colorable" and "k-partite" have the same meaning.

Example

Since a graph is 2-colorable if and only if it is bipartite, and C_5 is not bipartite (because it contains an odd cycle), then C_5 has a chromatic number at least 3. Since C_5 is 3-colorable, as shown below, it has a chromatic number exactly 3.



Lower bounds on $\chi(G)$

Recall : The independence number of a graph G, written $\alpha(G)$, is the biggest size of an independent subset of $V(G)$.

Definition

The **clique number** of a graph G, written $\omega(G)$, is the maximum size of a clique in G.

Proposition

For every graph G with n vertices, we have

$$\chi(G) \geq \omega(G) \text{ and } \chi(G) \geq \frac{n}{\alpha(G)}.$$

Proof :

An upper bound on $\chi(G)$

Algorithm : (Greedy coloring)
The greedy coloring relative to a vertex ordering $\{v_1, \cdots, v_n\}$ of $V(G)$ by the color set S is obtained by coloring vertices in the order $v_1, \cdots, v_n$, assigning to v_i the color with the smallest index in S that was not already used to color the neighbors of v_i that were already colored.

Proposition

For every graph G, we have

$$\chi(G) \leq \Delta(G) + 1.$$

Recall that $\Delta(G)$ is the degree of the graph G (i.e. the largest degree of its vertices).

Proof : The greedy coloring doesn't use more that $\Delta(G) + 1$ colors.

Remark : Note that the bound $\Delta(G) + 1$ is optimal in the case of complete graphs and cycles.