Thesis, Chap 1: pair approximation: the 4 state model

The model describes an arid ecosystem as a lattice-structured habitat. Each site can be in four different states: occupied by a nurse, occupied by a protégée, empty or degraded. The transition rates between states are as follows:

- $w_{o,p} = (\delta_p \rho_p + (1-\delta_p) q_{p|o})(b_p - c_p \rho_p - c_{np}\rho_n - g(1-q_{n|o}n))$
- $w_{o,n} = (\delta_n \rho_n + (1-\delta_n) q_{n|o})(b_n - c_n \rho_n - c_{pn}\rho_p - g q_{p|o} p)$
- $w_{n,o} = m_n$
- $w_{p,o} = m_p$
- $w_{o,-} = d$
- $w_{-,o} = r + q_{n|-} f + q_{p|-} f$

We intend to write the pair approximations corresponding to that model:

- There 4 states (n, p, 0, $-$), so 4 singleton variables: ρ_n, ρ_p, ρ_0, ρ_-.
- There are ten distinct doublets (ρ_{nn}, ρ_{n0}, ρ_{np}, ρ_{n-}, ρ_{pp}, ρ_{p0}, ρ_{p-}, ρ_{00}, ρ_{0-}, ρ_{--}) since $\rho_{\sigma\sigma'} = \rho_{\sigma'\sigma}$.
- There are five conservation equations:

1. $\rho_n + \rho_p + \rho_0 + \rho_- = 1$
2. $\rho_n = \rho_{nn} + \rho_{n0} + \rho_{np} + \rho_{n-}$
3. $\rho_0 = \rho_{00} + \rho_{n0} + \rho_{p0} + \rho_{0-}$
4. $\rho_p = \rho_{pp} + \rho_{p0} + \rho_{np} + \rho_{p-}$
5. $\rho_- = \rho_{--} + \rho_{0-} + \rho_{n-} + \rho_{p-}$

So, we need $10 + 4 - 5 = 9$ equations to solve this system. We chose: $\frac{d\rho_{nn}}{dt}$, $\frac{d\rho_{n-}}{dt}$, $\frac{d\rho_{np}}{dt}$, $\frac{d\rho_{pp}}{dt}$, $\frac{d\rho_{p-}}{dt}$, $\frac{d\rho_{--}}{dt}$, $\frac{d\rho_n}{dt}$, $\frac{d\rho_p}{dt}$, $\frac{d\rho_-}{dt}$.
The objective is to write those 9 equations as a function of the 9 chosen variables only.

Equations of the singleton variables:

$$\begin{aligned}
\frac{d\rho_p}{dt} =& \rho_0(\delta_p \rho_p + (1-\delta_p) q_{p|o})(b_p - c_p \rho_p - c_{np}\rho_n - g(1-q_{n|o}n)) \\
& - \rho_p m_p \\
\frac{d\rho_n}{dt} =& \rho_0(\delta_n \rho_n + (1-\delta_n) q_{n|o})(b_n - c_n \rho_n - c_{pn}\rho_p - g q_{p|o} p) \\
& - \rho_n m_n \\
\frac{d\rho_-}{dt} =& \rho_0 d - \rho_-(r + f q_{n|-} + f q_{p|-})
\end{aligned}$$

Replacing the variables here below by their expressions in the above equations:

- $\rho_o = 1 - \rho_p - \rho_n - \rho_-$
- $q_{p|o} = \frac{\rho_{po}}{\rho_o} = \frac{\rho_p - \rho_{pp} - \rho_{pn} - rho_{p-}}{1-\rho_p-\rho_n-\rho_-}$
- $q_{n|o} = \frac{\rho_{no}}{\rho_o} = \frac{\rho_n - \rho_{nn} - \rho_{pn} - rho_{n-}}{1-\rho_p-\rho_n-\rho_-}$
- $q_{n|-} = \frac{\rho_{n-}}{\rho_-}$
- $q_{p|-} = \frac{\rho_{p-}}{\rho_-}$

Equations of the doublets:

$$
\begin{aligned}
\frac{d\rho_{nn}}{dt} =& 2\rho_{n0}(\delta_n\rho_n + \frac{(1-\delta_n)}{z} + (1-\delta_n)\frac{(z-1)}{z}q_{n|0})(b_n - c_n\rho_n - c_{pn}\rho_p - \frac{z-1}{z}gq_{p|o}p) \\
&- 2\rho_{nn}m_n \\
\frac{d\rho_{pp}}{dt} =& 2\rho_{po}(\delta_p\rho_p + \frac{(1-\delta_p)}{z} + (1-\delta_p)\frac{(z-1)}{z}q_{p|o})(b_p - c_p\rho_p - c_{np}\rho_n - g(1 - \frac{(z-1)}{z}q_{n|o}n)) \\
&- 2\rho_{pp}m_p \\
\frac{d\rho_{--}}{dt} =& 2\rho_{0-}d - 2\rho_{--}(r + f\frac{z-1}{z}(q_{n|-} + q_{p|-})) \\
\frac{d\rho_{n-}}{dt} =& \rho_{no}d + \rho_{o-}(\delta_n\rho_n + (1-\delta_n)\frac{(z-1)}{z}q_{n|o})(b_n - c_n\rho_n - c_{pn}\rho_p - g\frac{(z-1)}{z}q_{p|o}p) \\
&- \rho_{n-}(m_n + r + \frac{f}{z} + \frac{(z-1)}{z}f(q_{n|-} + q_{p|-})) \\
\frac{d\rho_{np}}{dt} =& \rho_{n0}(\delta_p\rho_p + (1-\delta_p)\frac{(z-1)}{z}q_{p|o})(b_p - c_p\rho_p - c_{np}\rho_n - g(1 - \frac{n}{z} - \frac{(z-1)}{z}q_{n|o}n)) \\
&+ \rho_{p0}(\delta_n\rho_n + (1-\delta_n)\frac{(z-1)}{z}q_{n|o})(b_n - c_n\rho_n - c_{pn}\rho_p - \frac{gp}{z} - g\frac{z-1}{z}q_{p|o}p) \\
&- \rho_{np}(m_p + m_n) \\
\frac{d\rho_{p-}}{dt} =& \rho_{po}d + \rho_{o-}(\delta_p\rho_p + (1-\delta_p)\frac{(z-1)}{z}q_{p|o})(b_p - c_p\rho_p - c_{np}\rho_n - g(1 - \frac{(z-1)}{z}q_{n|o}n)) \\
&- \rho_{p-}(m_p + r\frac{f}{z} + f\frac{(z-1)}{z}(q_{p|-} + q_{n|-})) \\
\frac{d\rho_{--}}{dt} =& 2\rho_{0-}d - 2\rho_{--}(r + f\frac{z-1}{z}(q_{n|-} + q_{p|-})
\end{aligned}
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Using the expression of the variables given in the previous part and here below:

- $\rho_{0-} = \rho_- - \rho_{--} - \rho_{n-} - \rho_{p-}$
- $q_{n|p} = \frac{\rho_{pn}}{\rho_p}$
- $q_{p|n} = \frac{\rho_{pn}}{\rho_n}$
- $\rho_{po} = \rho_p - \rho_{pp} - \rho_{np} - \rho_{p-}$
- $\rho_{no} = \rho_n - \rho_{nn} - \rho_{np} - \rho_{n-}$