3.1 Given, W is a non singular matrix

↳ $Wx = 0$ only has the trivial sol^n $x = 0$

$$\|x\|_W = \|Wx\|$$

For this to be a vector norm :

(i) $\|x\|_W = \|Wx\| \geq 0$ and $\|Wx\| = 0$ only if $Wx = 0$, i.e., $x = 0$ (since W is non-singular)

(ii) $\|x+y\|_W = \|W(x+y)\|$

$= \|Wx + Wy\|$

since $\|\cdot\|$ is a norm

$\leq \|Wx\| + \|Wy\| = \|x\|_W + \|y\|_W$

(iii) $\|\alpha x\|_W = \|\alpha W x\| = |\alpha| \, \|Wx\|$ (since $\|\cdot\|$ is a norm)

$= |\alpha| \, \|x\|_W$

Hence, proved.

3.2 $\rho(A) \rightarrow$ spectral radius of A

i.e., $\rho(A) = \max\limits_{\lambda} \{ |\lambda| : \lambda \text{ is an eigenvalue of } A \}$

For any eigenvalue λ and corresponding eigenvector x,

$$Ax = \lambda x$$

$$\Rightarrow \quad |\lambda| \, \|x\| = \|Ax\|$$

$$\Rightarrow \quad |\lambda| = \frac{\|Ax\|}{\|x\|} \leq \sup_{y} \frac{\|Ay\|}{\|y\|}$$

by definition of induced matrix norm

3.3 (a) $\|x\|_\infty = \max\limits_{1 \leq i \leq m} |x_i| = |x_j|$ (suppose x_j is max)

$$\|x\|_2 = \left(\sum_{i=1}^{m} |x_i|^2 \right)^{1/2} = \left(|x_j|^2 + \sum_{i \neq j} |x_i|^2 \right)^{1/2} \geq |x_j|$$

$$\therefore \quad \|x\|_\infty \le \|x\|_2$$

The equality is achieved when $\sum_{i \neq j} |x_i|^2 = 0$, i.e.,

$\forall i \neq j, \ |x_i| = 0$

$\rightarrow$ x has only 1 non-zero element. $(x = \alpha e_i)$

(b) $\|x\|_2 = \left(\sum_{i=1}^{m} |x_i|^2 \right)^{1/2} = \left(|x_1|^2 + |x_2|^2 + \dots + |x_m|^2 \right)^{1/2}$

$\le \left(\max_i |x_i|^2 + \max_i |x_i|^2 + \dots + \max_i |x_i|^2 \right)^{1/2}$

$= \sqrt{m} \ \|x\|_\infty$

Equality holds if $\forall i, \ |x_i|^2 = \max_{1 \le i \le m} |x_i|^2$

i.e., $x_i = \pm k \quad \forall \ 1 \le i \le m$

for some $k \in \mathbb{C}$.

(c) $\|A\|_\infty = \max_{1 \le i \le m} \|a_i^*\|_1$ (maximum row sum)

From result in (a), $\|Ax\|_\infty \le \|Ax\|_2$

From result in (b), $\|x\|_\infty \ge \frac{1}{\sqrt{n}} \|x\|_2$

$$\therefore \quad \frac{\|Ax\|_\infty}{\|x\|_\infty} \le \frac{\|Ax\|_2}{\frac{1}{\sqrt{n}} \|x\|_2}$$

Taking supremum on both sides,

$$\boxed{\|A\|_\infty \le \sqrt{n} \ \|A\|_2}$$

(d) $\|Ax\|_2 \le \sqrt{m} \ \|Ax\|_\infty$ (from (b))

$\|x_\infty\| \le \|x\|_2$ (from (a))

$$\therefore \quad \frac{\|Ax\|_2}{\|x\|_2} \leq \frac{\sqrt{m}\,\|Ax\|_\infty}{\|x\|_\infty}$$

Taking supremum on both sides :

$$\boxed{\|A\|_2 \leq \sqrt{m}\,\|A\|_\infty}$$

3.4 (a) $B = U A W$

where U is an identity matrix containing only the rows which we want to keep in B

V is an identity matrix containing only columns which we want to keep in B.

(b) Since U and V only contain 0s and 1s,

$$\|U\|_p \leq 1 \quad , \quad \|V\|_p \leq 1$$

From upper bound on $\|UAW\|_p \leq \|U\|_p \|A\|_p \|W\|_p$

$$\Rightarrow \quad \|B\|_p \leq \|A\|_p \, .$$

3.5 $\|E\|_F = \|uv^*\|_F$

$$= \sqrt{\sum_{i,j} |u_j v_i|^2} = \sqrt{\sum_i |v_i|^2 \sum_j |u_j|^2}$$

$$= \|u\|_2 \|v\|_2 \qquad \therefore \text{Proved.}$$

3.6 $\|x\|' = \sup_{\|y\|=1} |y^* x|$

Dual norm

(a) For $\|\cdot\|'$ to be a vector norm :

(i) $\|x\|' = \sup_{\|y\|=1} |y^* x| \geq 0$ (since we take absolute value) and $\|x\|' = 0 \Rightarrow x = 0$

(ii) $$\|x+y\|' = \sup_{\|a\|=1} |a^*(x+y)|$$

$$= \sup_{\|a\|=1} |a^* x + a^* y|$$

$$\leq \sup_{\|a\|=1} (|a^* x| + |a^* y|)$$

$$\leq \sup_{\|a\|=1} |a^* x| + \sup_{\|a\|=1} |a^* y|$$

$$= \|x\|' + \|y\|'$$

(iii) $$\|\alpha x\|' = \sup_{\|y\|=1} |y^* \alpha x|$$

$$= |\alpha| \sup_{\|y\|=1} |y^* x| = |\alpha| \|x\|'$$

$\therefore$ $\|\cdot\|'$ is a vector norm.

(b) $x, y \in \mathbb{C}^m$, $\|x\| = \|y\| = 1$

Given $x \in \mathbb{C}^m$, $\exists$ $z \in \mathbb{C}^m$, $z \neq 0$ s.t. $|z^* x| = \|z\|' \|x\|$.

Let $B = y z^*$ and we want $Bx = y$ & $\|B\| = 1$

$$Bx = (y z^*) x = y \underbrace{(z^* x)}$$

we want to compute this term.

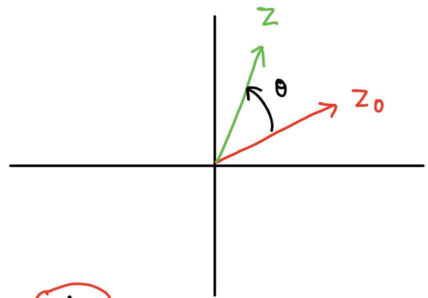
Not necessarily the same z.

Let z_0 be such that for the given x,

$$|z_0^* x| = \|z_0\|' \|x\| = \|z_0\|' \quad (\text{since } \|x\| = 1)$$

We know that for any vector v, $\frac{v}{\|v\|}$ gives $\arg(v)$.

Let z be such that



$$\frac{z}{\|z\|'} = e^{i\theta}\,\frac{z_0}{\|z_0\|'}$$

$$\Rightarrow \quad z = e^{i\theta}\,\frac{z_0}{\|z_0\|'}$$

where $\|z\|' = 1$.

$$z^* x = e^{-i\theta}\,\frac{z_0^*}{\|z_0\|'}\,x = \frac{|z_0^* x|}{\|z_0\|'} = \|x\| = 1$$

Since we have taken adjoint of vector.

$$\therefore \quad Bx = y\,(z^* x) = y.$$

For the second part, we have:

$$\|B\| = \sup_{\|a\|=1} \|Ba\| = \sup_{\|a\|=1} \|y z^* a\|$$

$$= \sup_{\|a\|=1} \|y\,(\overset{=1}{\cancel{z^* a}})\|$$

$$= \|y\| = 1.$$ Hence, proved