ECEN 2703: Discrete Mathematics

Assignment #5

1. [10 points.] Russell's paradox (see the slides for sets and functions) is about the set R of all sets that do not contain themselves. The paradox arises when we ask whether R contains itself.

   Russell gave another popular version of the paradox, known as the *Barber's paradox*: In a village lives a man who is a barber. He shaves all the men of the village who do not shave themselves and no one else. Does the barber shave himself?

   Once we understand the pattern, it's not difficult to create our own version of the paradox. Here's an example: in a small town lives a dog owner who runs a dog-walking business. She walks all the dogs in town that are not walked by their owners and no other dog. Does the dog-walker walk her dog?

   Generalizing, we have a two-place predicate, $P(x, y)$, which is interpreted as "$x \in y$," "y shaves x," or "y walks x's dog" in the three forms of the paradox we have considered. The state of affairs is then described, in all three cases, by the following sentence of predicate logic:

   $$\exists y \, \forall x \, \big(P(x, y) \leftrightarrow \neg P(x, x)\big) \ .$$

   Your task in this exercise is to show that the sentence above is *contradictory*. All you need to do is to apply two inference rules: existential and universal instantiation.

2. [10 points.] The following argument is clearly bogus. What is the mistake?

   Let the domain of discourse be the integers. It is the case that

   $$\forall x \, \exists y \, (x \neq y) \ .$$

   We now apply the rule of existential instantiation to $\exists y \, (x \neq y)$ to obtain

   $$\forall x \, (x \neq c) \ .$$

   This instantiation is valid because c does not appear elsewhere in our formula. We now apply universal instantiation to obtain

   $$c \neq c \ .$$

This step is valid because the choice of c is unrestricted in universal instantiation. Well, $c \neq c$ is clearly contradictory. Where did we go astray in the argument above?

3. [10 points.] Rewrite the following predicate-logic sentence,

$$\forall \varepsilon > 0 \, \exists \delta > 0 \, \forall x \, (|x| < \delta \rightarrow |f(x)| < \varepsilon) \ ,$$

so that it does not use any quantifiers with restricted domains.

4. [10 points.] Write a Python script that uses Z3 to prove that this sentence of predicate logic from the slides,

$$\forall x \, \exists y \, \big(x < y \wedge \forall z \, (x \geq z \vee z \geq y)\big)$$

is true when the domain of discourse is the integers.

Note that the formula does not contain any constant symbols, and the interpreted relation symbols ($<$, $\geq$) do not let Z3 figure out whether the domain should be the integers or the reals. However, declaring the quantification variables with `Int` gives Z3 enough information. (See the Example section in the slides on predicate logic.) Recall that you should check the truth of our sentence by checking that its negation is not satisfiable.

Submit your code to Canvas. Your script should print "Valid." if it proves that the formula is indeed valid. Z3 should take almost no time. If it runs for a while, stop the process, and look for something amiss in your script.