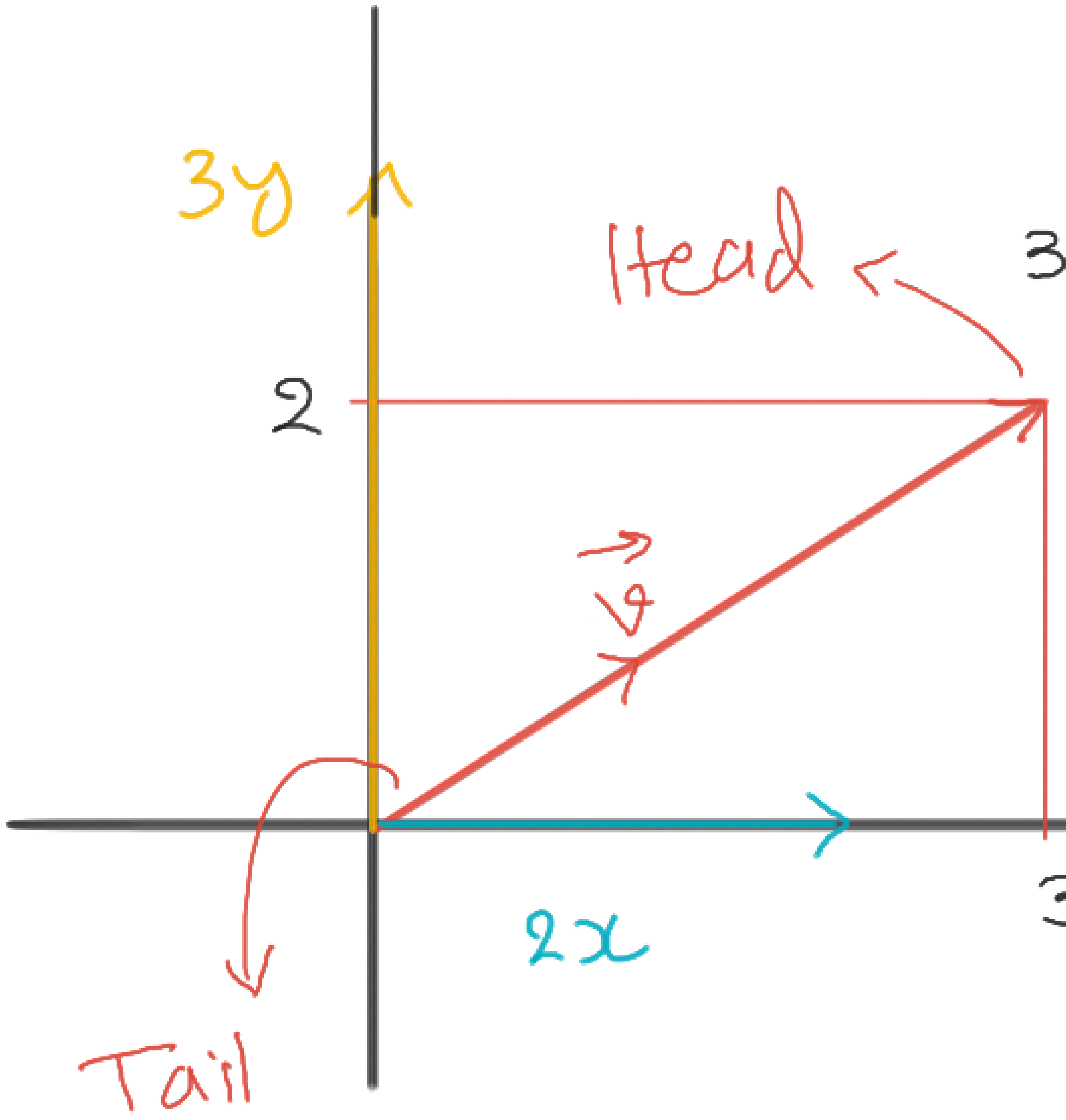
Linear Algebra - 2: Vectors

☆ Examples of vectors: Force, Velocity, Displacement



$3\hat{x} + 2\hat{y}$

(3, 2)

[3, 2]

$3x + 2y$

A vector is nothing but a 1D matrix.

or $\begin{bmatrix} 3 \\ 2 \end{bmatrix}$

☆ Unit vectors : A vector that has 1 (unit) magnitude is a unit vector.

→ Unit vector in the direction of +ve x-axis is usually represented as $\hat{x}$ or $\hat{i}$ and that in the direction of y-axis is called $\hat{y}$ or $\hat{j}$ and in z-axis $\hat{z}$ or $\hat{k}$.

→ Any vector

☆ Calculating magnitude of a vector:

$$\vec{v} = 3\hat{x} + 2\hat{y} \Rightarrow |\vec{v}| = \sqrt{(3)^2 + (2)^2} = \sqrt{13}$$

$$\vec{v} = a\hat{x} + b\hat{y} \Rightarrow |\vec{v}| = \sqrt{a^2 + b^2}$$

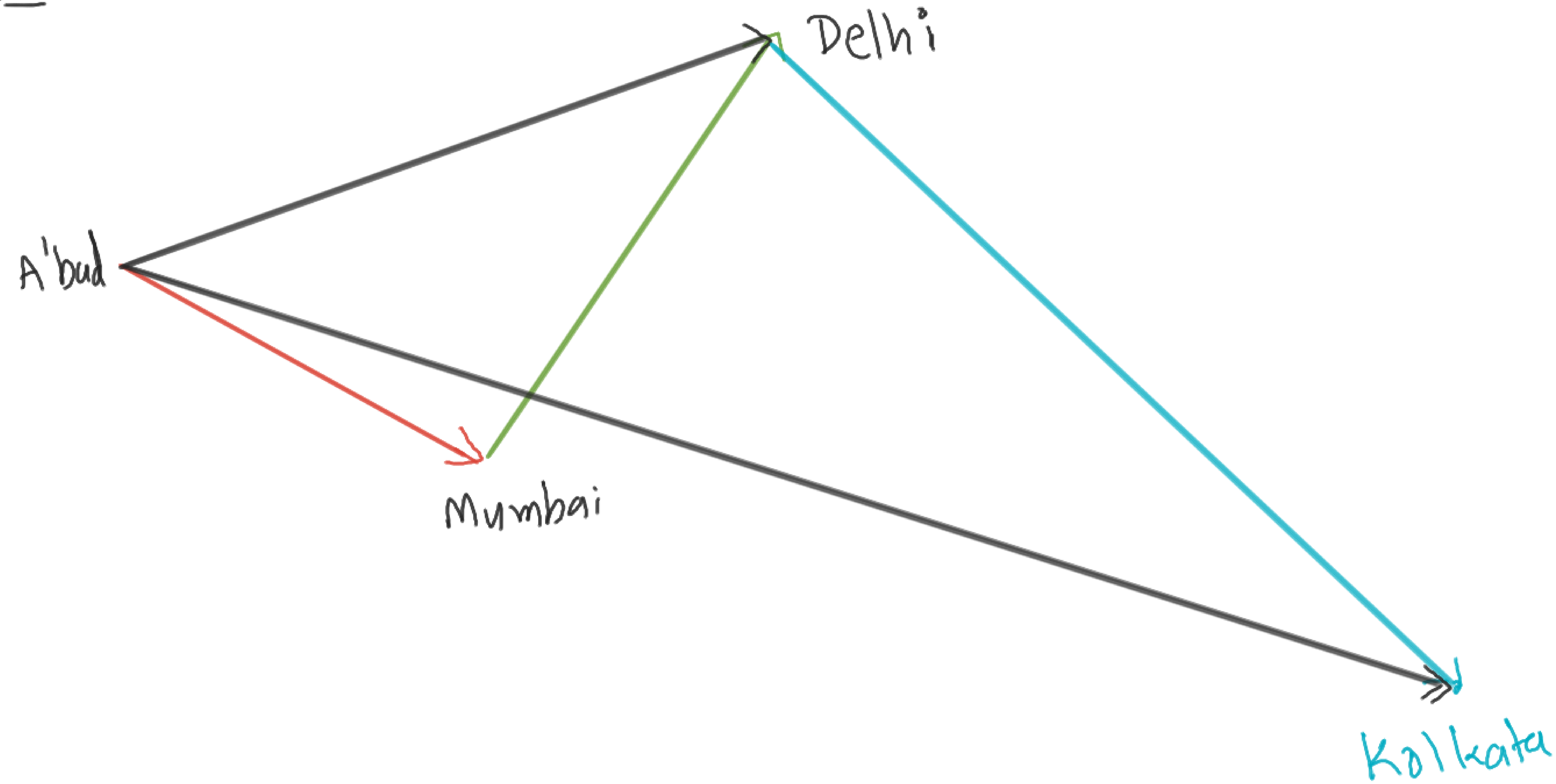
☆ Unit vector in the direction of $\vec{v}$

$$\hat{v} = 1\hat{x} + 1\hat{y} \Rightarrow |\hat{v}| = \sqrt{2} \quad \times$$

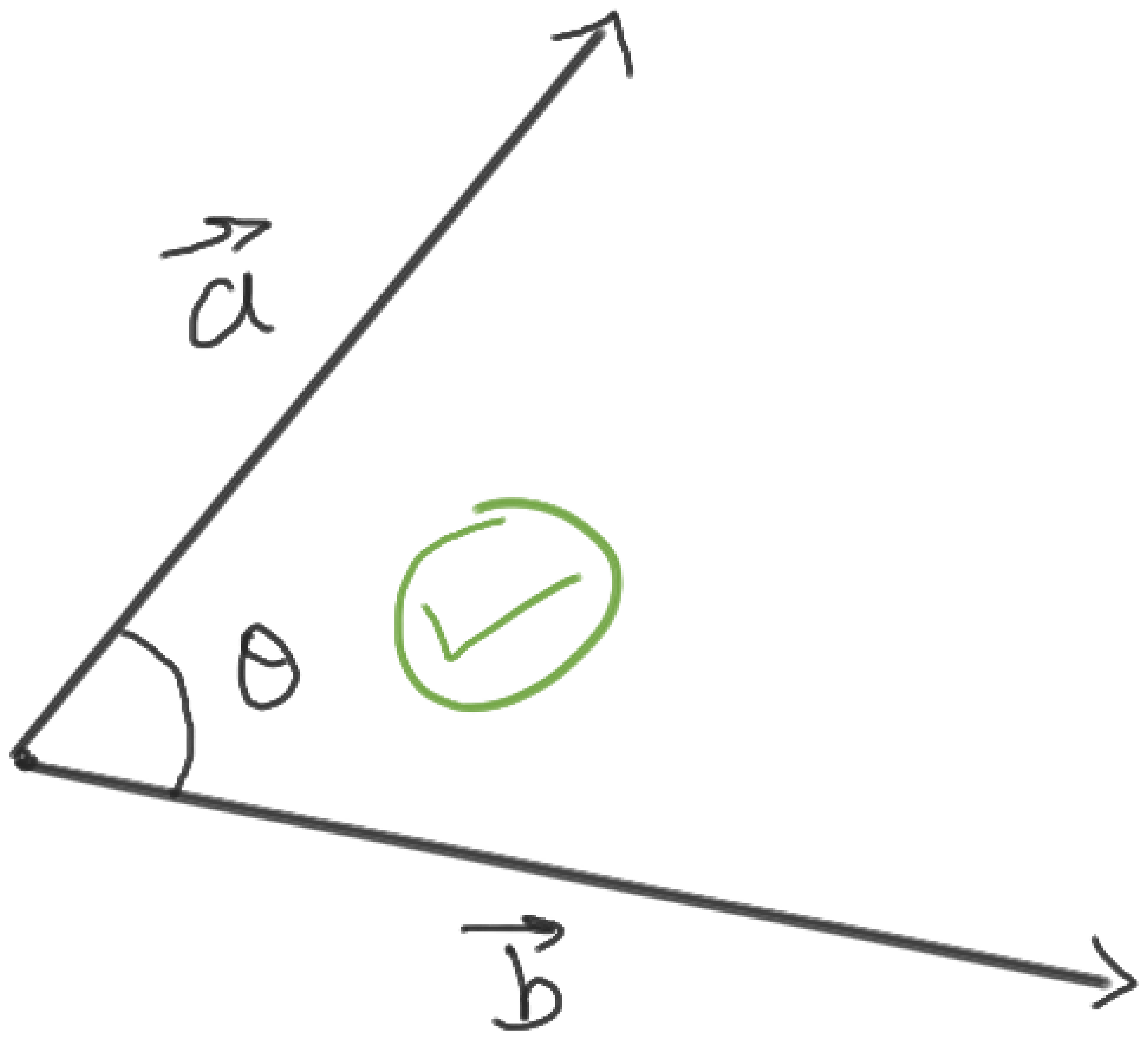
$$\Rightarrow \hat{v} = \begin{bmatrix} \frac{a}{\sqrt{a^2+b^2}} \\ \frac{b}{\sqrt{a^2+b^2}} \end{bmatrix}$$

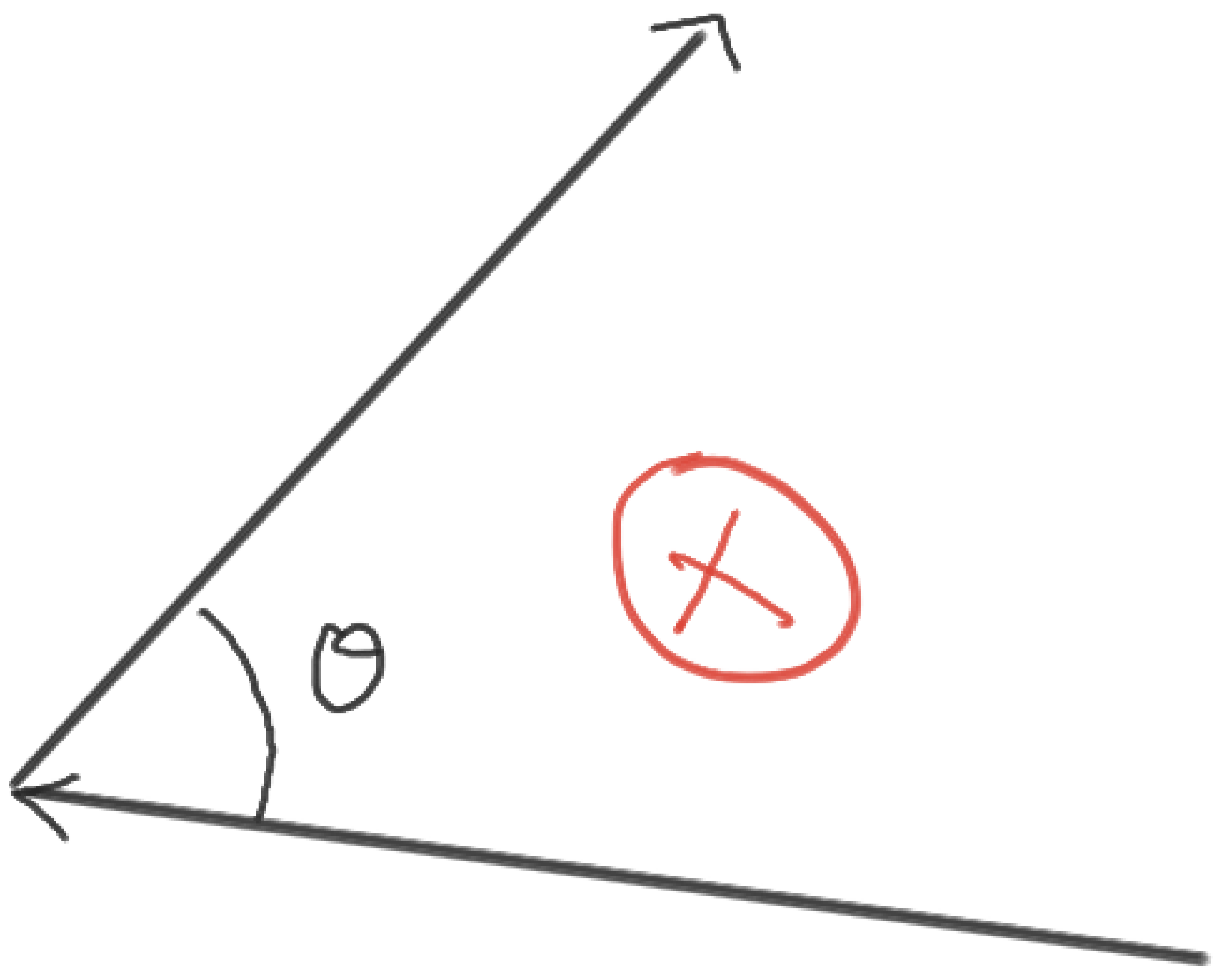
$$\hat{v} = \frac{\vec{v}}{|\vec{v}|} = \frac{a\hat{x} + b\hat{y}}{\sqrt{a^2+b^2}} = \frac{a}{\sqrt{a^2+b^2}} \cdot \hat{x} + \frac{b}{\sqrt{a^2+b^2}} \cdot \hat{y}$$

☆ Sum of two vectors:



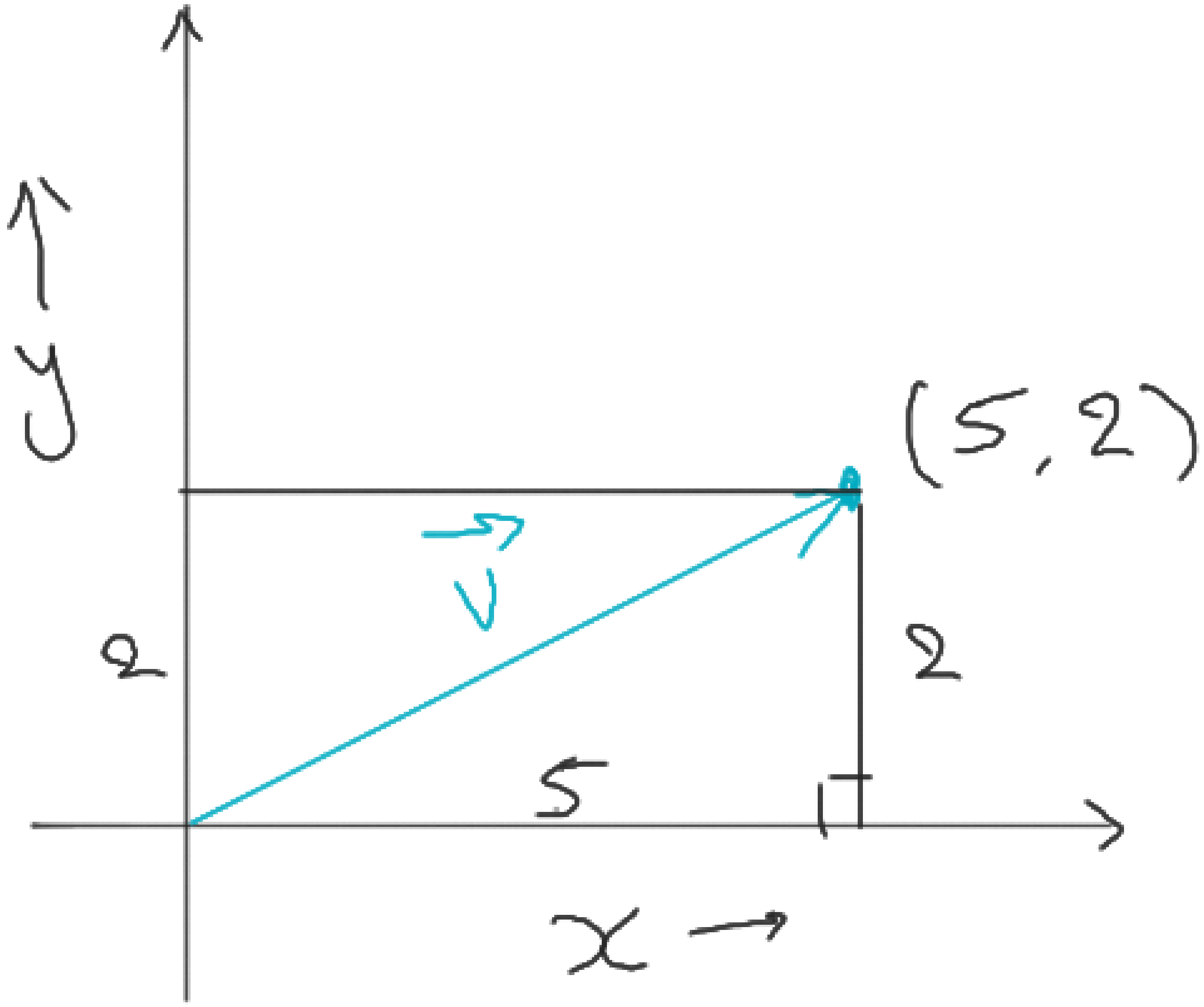
☆ Angle between two vectors:





Angle suspended betwee two vectors when their tails are connected.

☆ 'norm' of a vector:



$$|\vec{v}| = \sqrt{5^2 + 2^2}$$

For a vector $\vec{v}[a, b]$

$$|\vec{v}| = \sqrt{a^2 + b^2}$$

$$|\vec{v}| = \sqrt[3]{a^3 + b^3}$$

A'bad to Mumbai = 500 km
350 miles
300 vm

A'bad to Delhi = 1000 km
= 700
= 600

If $|\vec{v}| > |\vec{p}|$ using sqrt method, will $|\vec{v}| > |\vec{p}|$ using $\sqrt[3]{\ }$?

2-norm of $\vec{v} = \|\vec{v}\| = \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2}$

3-norm of $\vec{v} = \|\vec{v}\|_3 = \sqrt[3]{x_1^3 + x_2^3 + \cdots + x_n^3}$

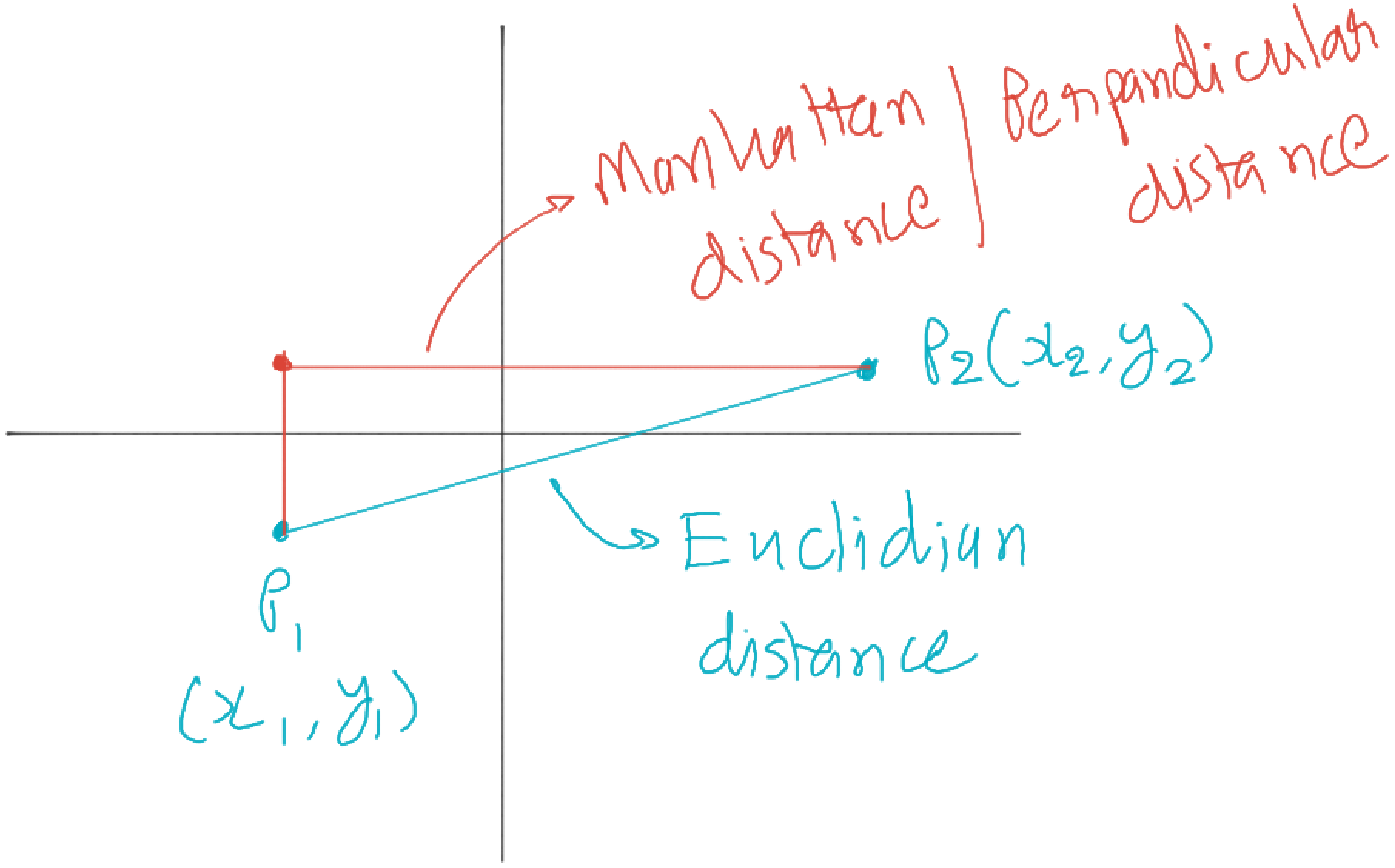
n-norm of $\vec{v} = \|\vec{v}\|_n = \sqrt[n]{x_1^n + x_2^n + \cdots + x_n^n}$

1-norm of $\vec{v} = \|\vec{v}\|_1 = |x_1| + |x_2| + \cdots + |x_n|$

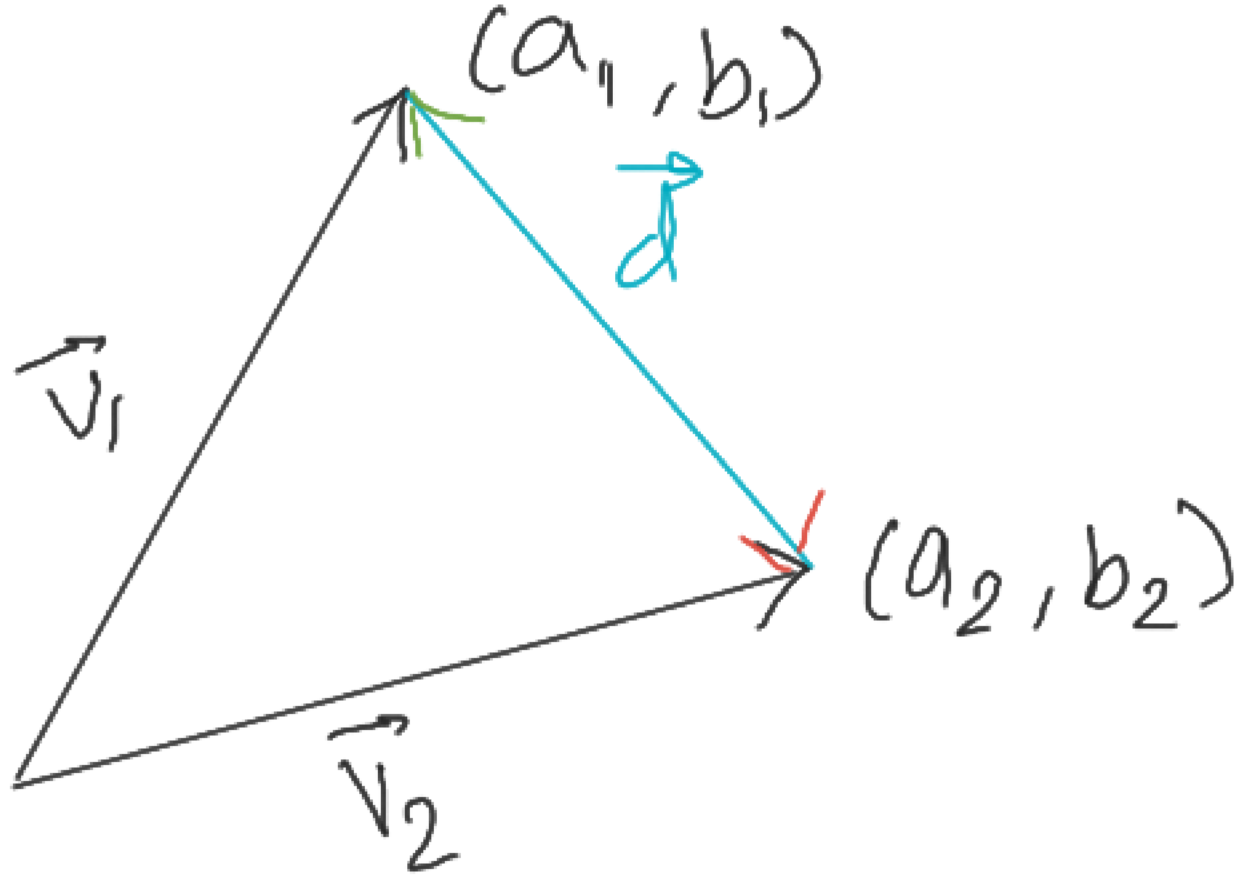
$\rightarrow$ Manhattan distance

2-norm gives us Euclidian distance

→ Euclidian distance is a straight line distance



☆ Distance between two vectors:



$$\vec{v}_1 = \vec{v}_2 + \vec{d} \qquad \vec{v}_2 = \vec{v}_1 + \vec{d}$$

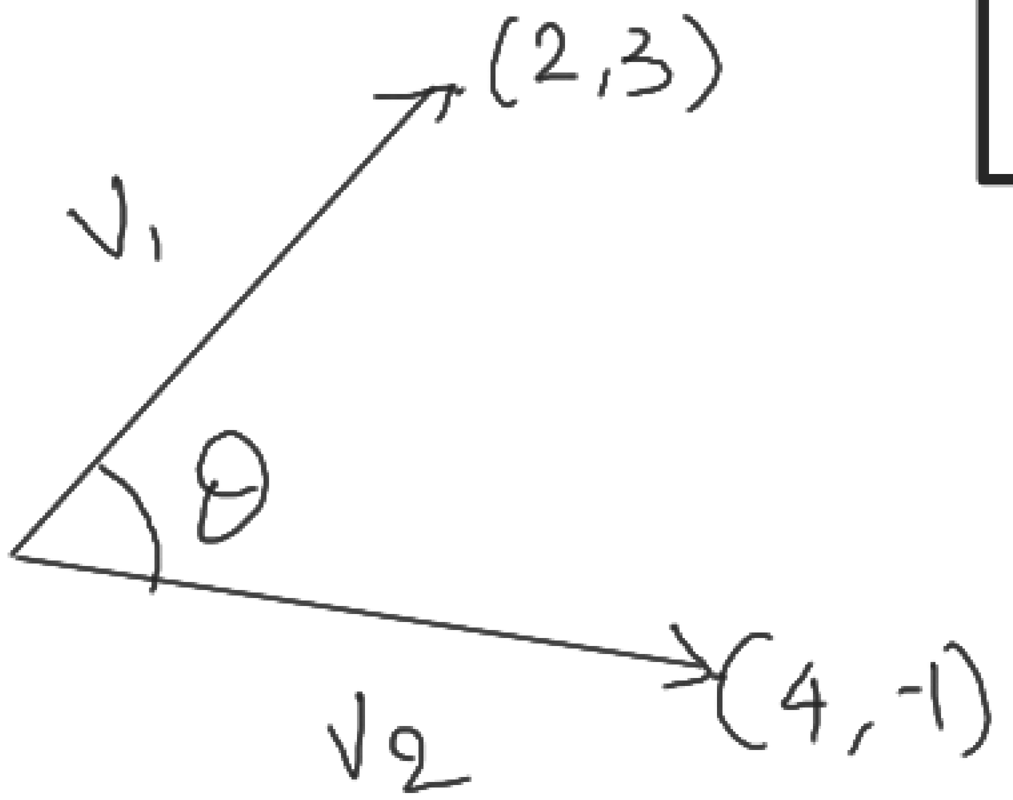
$$\therefore \vec{d} = \vec{v}_1 - \vec{v}_2 \qquad \vec{d} = \vec{v}_2 - \vec{v}_1$$

$$\therefore d = (a_2\hat{i} + b_2\hat{j}) - (a_1\hat{i} + b_1\hat{j})$$

$$\therefore d = (a_2 - a_1)\hat{i} + (b_2 - b_1)\hat{j}$$

$$\therefore \|\vec{d}\|_2 = \sqrt{(a_2 - a_1)^2 + (b_2 - b_1)^2}$$

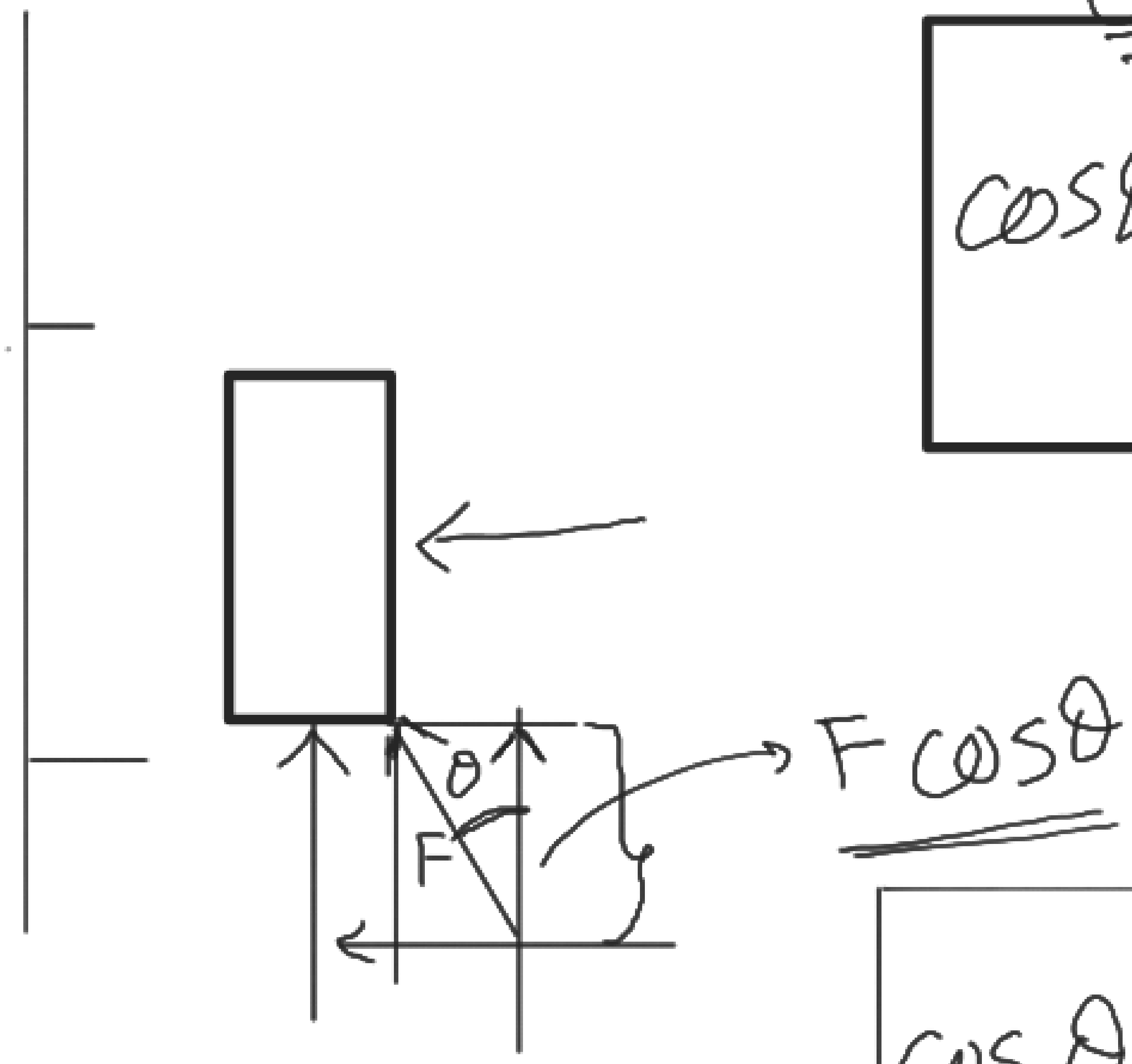
☆ Dot product : scalar multiplication



$$\boxed{\vec{v}_1 \cdot \vec{v}_2 = |\vec{v}_1| \cdot |\vec{v}_2| \cdot \cos\theta}$$

OR

$$\boxed{\cos\theta = \frac{\vec{v}_1 \cdot \vec{v}_2}{|\vec{v}_1| \cdot |\vec{v}_2|}}$$



$$\vec{v}_1 = \begin{bmatrix} 2 \\ 3 \end{bmatrix} \quad \vec{v}_2 = \begin{bmatrix} 4 \\ -1 \end{bmatrix}$$

$$\vec{v}_1^T = \begin{bmatrix} 2 & 3 \end{bmatrix} \quad \vec{v}_2 = \begin{bmatrix} 4 \\ -1 \end{bmatrix}$$

OR

$$\boxed{\cos\theta = \frac{\vec{v}_1^T \cdot \vec{v}_2}{\|\vec{v}_1\| \, \|\vec{v}_2\|}}$$

Let $v_1 \begin{bmatrix} 3 & 4 \end{bmatrix}_{1\times 2}$ & $v_2 \begin{bmatrix} 5 \\ -2 \end{bmatrix}_{2\times 1} \Rightarrow \vec{v_1}\vec{v_2} = 7$

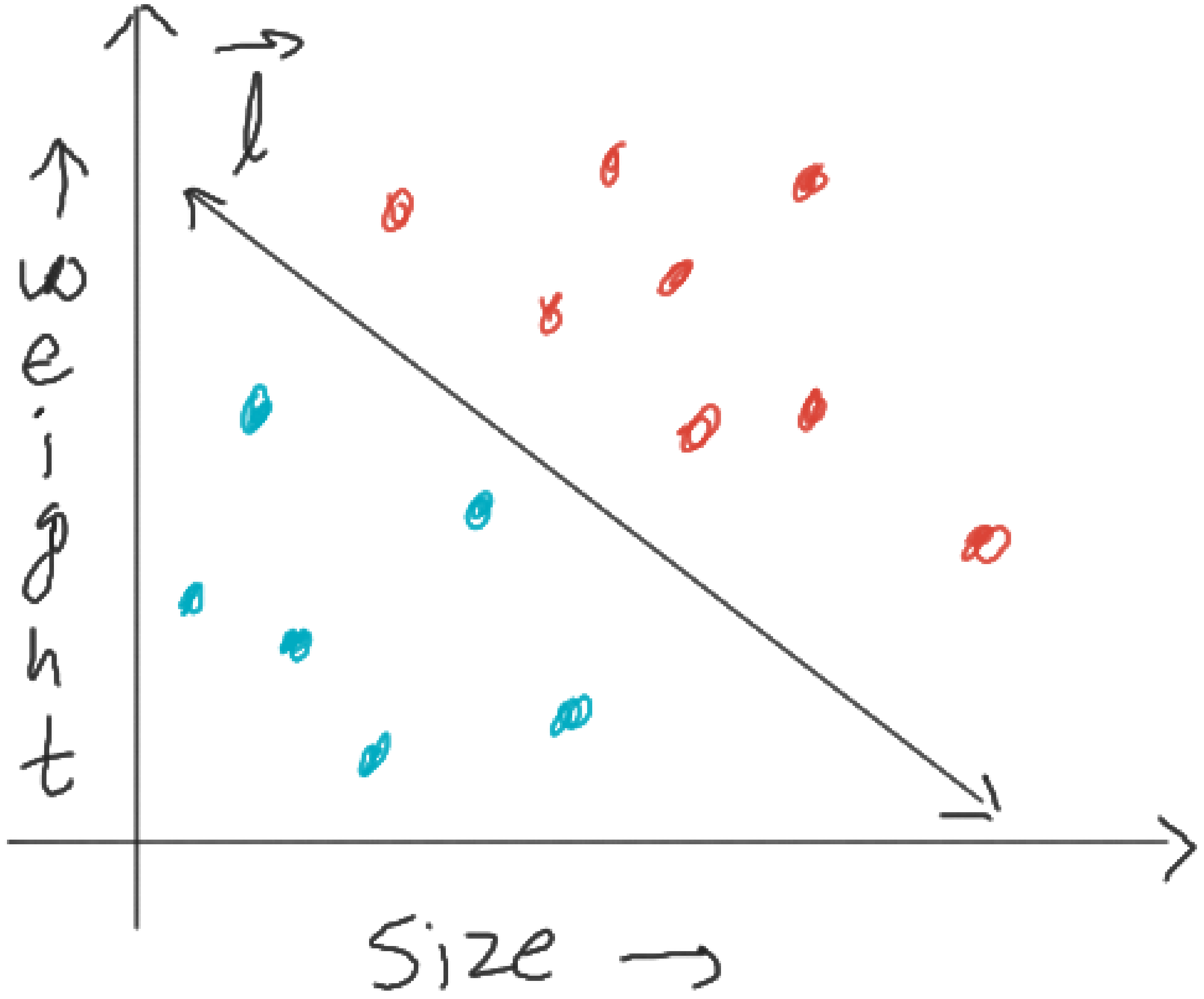
That means $\begin{bmatrix} \longrightarrow \\ \\ \\ \end{bmatrix}_{3\times 3} \times \begin{bmatrix} \big\downarrow & & \end{bmatrix}_{3\times 3}$

dot product

$= \begin{bmatrix} \bigcirc & & \\ & & \\ & & \end{bmatrix}_{3\times 3}$

Means matrix multiplication is nothing but a bunch of dot products.

☆ Coming back to fish-sorting problem....



$$\vec{l} : w_1 x_1 + w_2 x_2 + w_0 = 0$$

$$\begin{bmatrix} w_1 \\ w_2 \end{bmatrix}^T \cdot \begin{bmatrix} x_1 \\ x_2 \\ 1 \end{bmatrix} + w_0 = 0$$

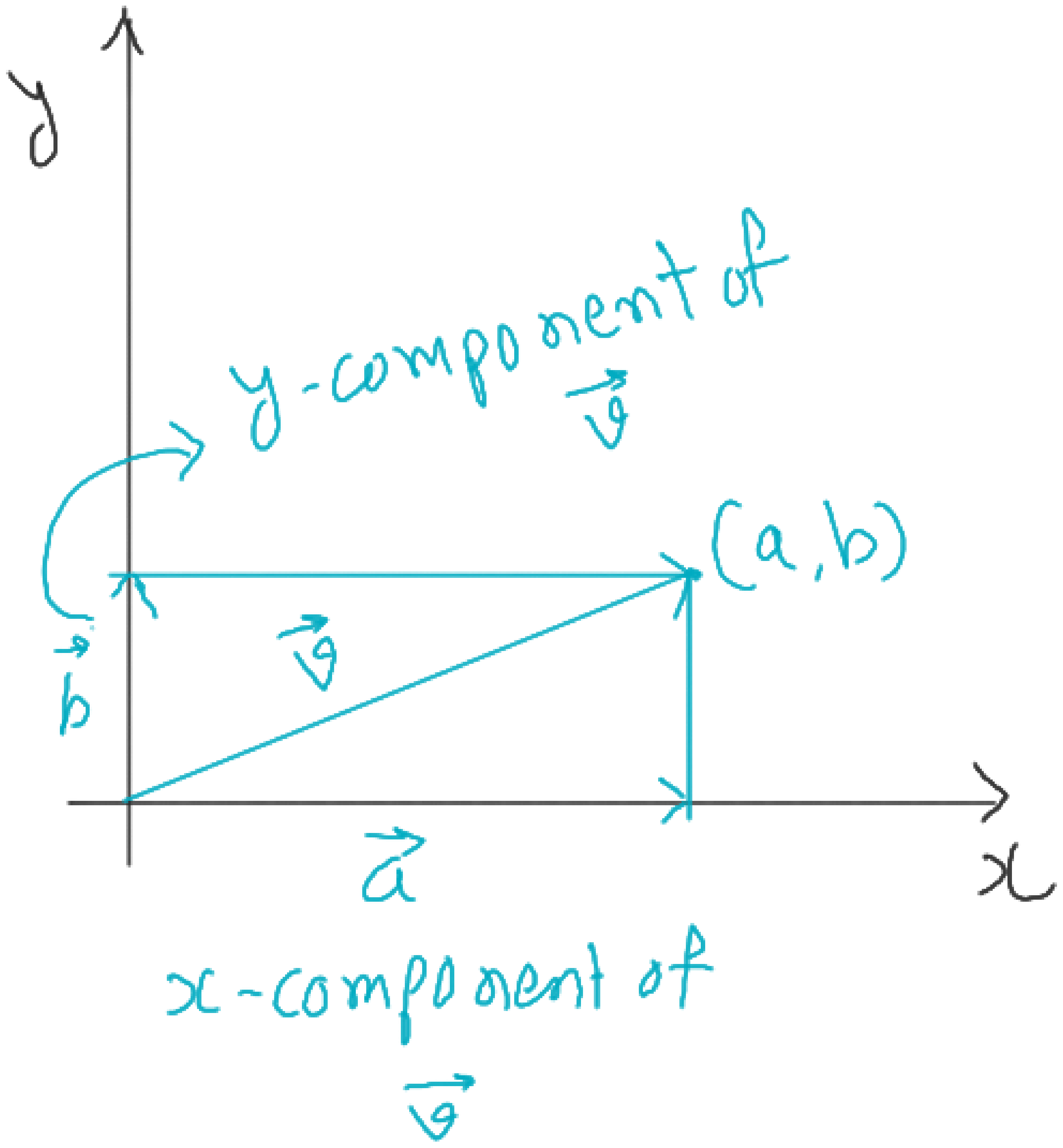
features 'x'

weight vector $\vec{w}$

$$\vec{l} : \vec{w^T} \cdot \vec{x} + w_0 = 0$$

<u>OR</u> $\vec{w^T} \cdot \vec{x} = -w_0$

☆ Components of a vector



$$\therefore \vec{v} = \vec{a} + \vec{b}$$

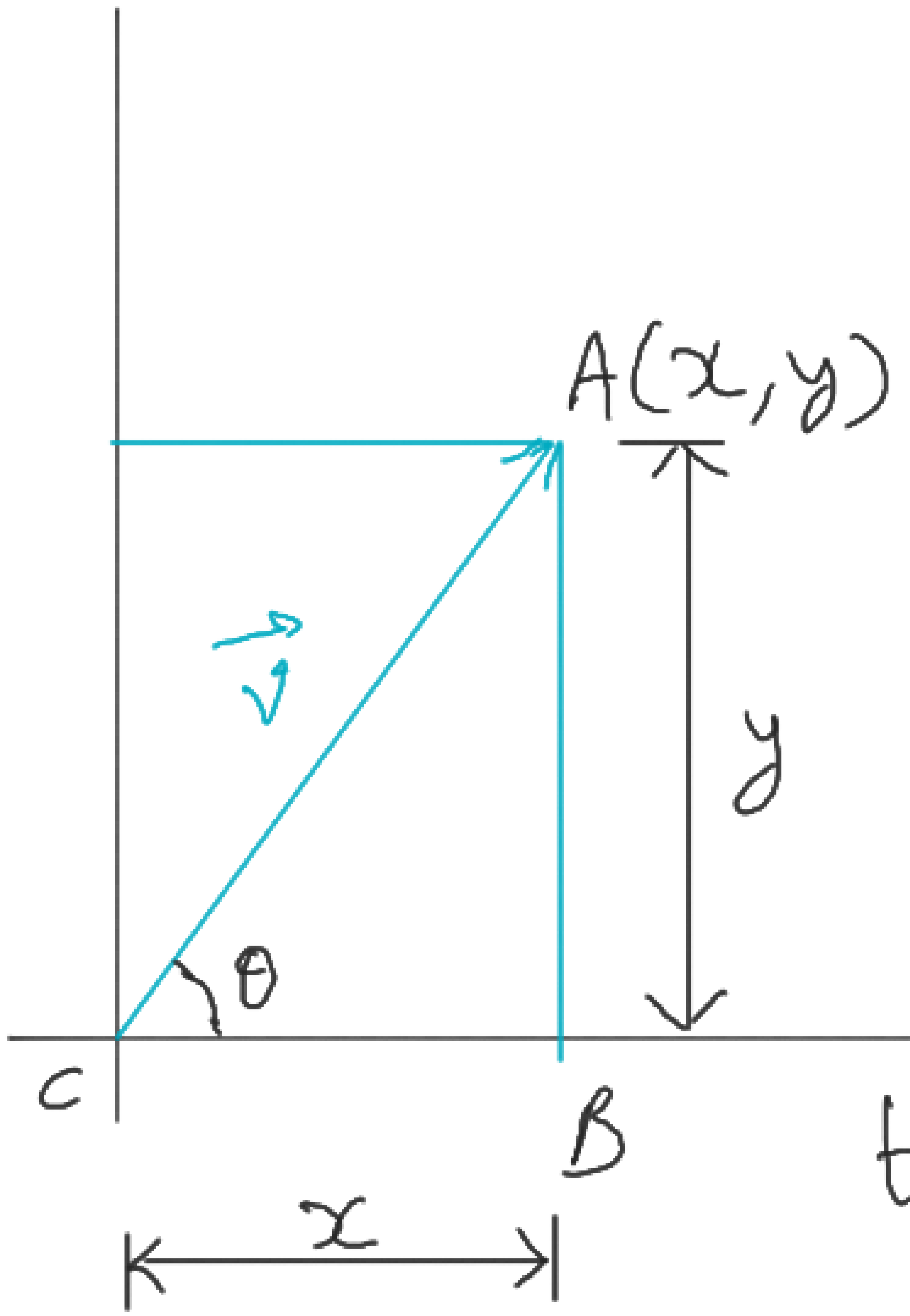
where $\vec{a} = a\,\hat{i}$ &

$\vec{b} = b \cdot \hat{j}$

$$\therefore \vec{v} = a\hat{i} + b\hat{j}$$

☆ Basic Trigonometry

$$|\vec{v}| = \sqrt{AB^2 + BC^2} = \sqrt{x^2 + y^2}$$



$$\sin\theta = \frac{\text{Opp. Side}}{\text{hyp}} = \frac{y}{|\vec{v}|}$$

$$\therefore \quad y = |\vec{v}| \sin\theta$$

$$\cos\theta = \frac{\text{Adj. side}}{\text{hyp}} = \frac{x}{|\vec{v}|}$$

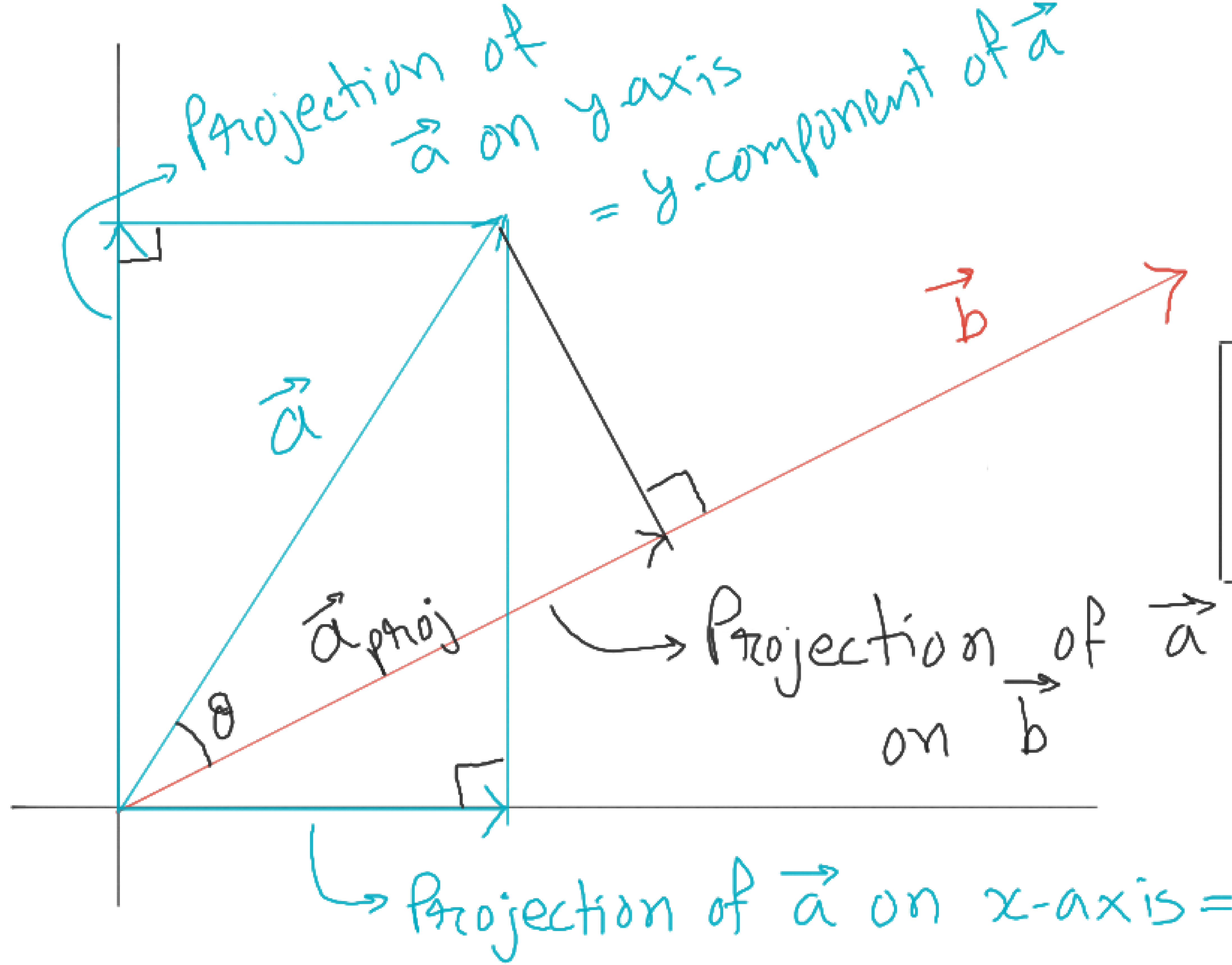
$$\therefore \quad x = |\vec{v}| \cos\theta$$

$$\tan\theta = \frac{y}{x} \Rightarrow y = x \cdot \tan\theta$$

$$\therefore \quad |\vec{v}| = \sqrt{x^2 + (x^2 \tan^2\theta)} \Rightarrow \boxed{|\vec{v}| = x\sqrt{(1 + \tan^2\theta)}}$$

☆ Projection Of A Vector :

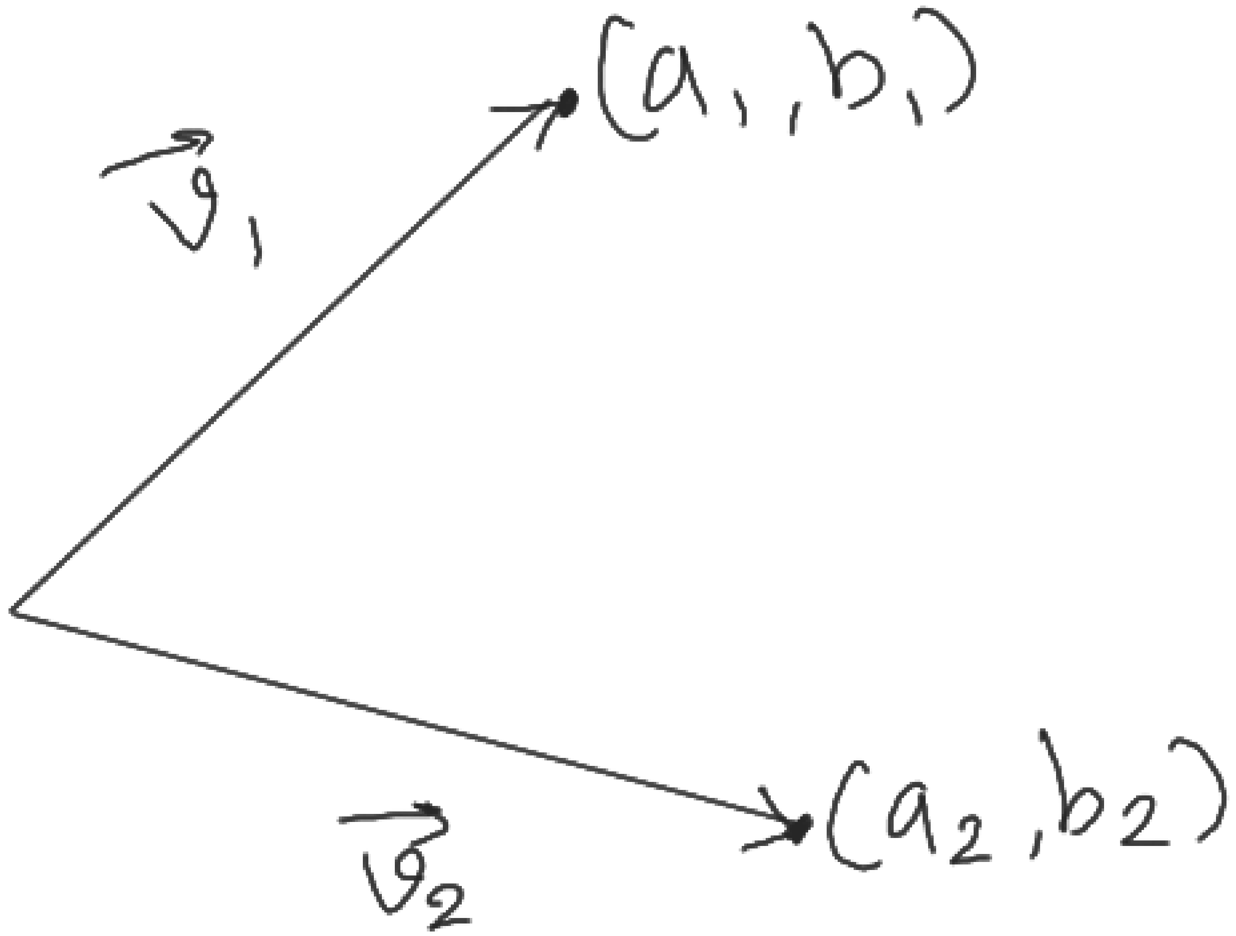
$$|\vec{a}_{proj}| = a \cdot \cos\theta$$



A vector is its magnitude * unit vector in its direction

$$\boxed{\therefore \vec{a}_{proj} = a \cdot \cos\theta \cdot \frac{\vec{b}}{|\vec{b}|}}$$

☆ An interesting fact about dot product of two vectors



$$\vec{v}_1 \cdot \vec{v}_2 = a_1 \cdot a_2 + b_1 \cdot b_2$$

☆ Normal Equation Formula of a straight line

General eqn of a line : $Ax + By + C = 0$

If I take a vector $\vec{n}(A, B)$ then $\vec{n} = A\hat{i} + B\hat{j}$

& another vector $\vec{p}(x, y)$ then $\vec{p} = x\hat{i} + y\hat{j}$

Therefore, $\vec{n} \cdot \vec{p} = Ax + By$. Substituting this in the general form:

$$\boxed{\vec{n}^T \cdot \vec{p} + C = 0}$$

OR

$$\boxed{\vec{n}^T \cdot \vec{p} = -C}$$ → Normal Equation Formula

same as in slide-12 given as:
$\vec{w}^T \cdot \vec{x} + w_0 = 0$

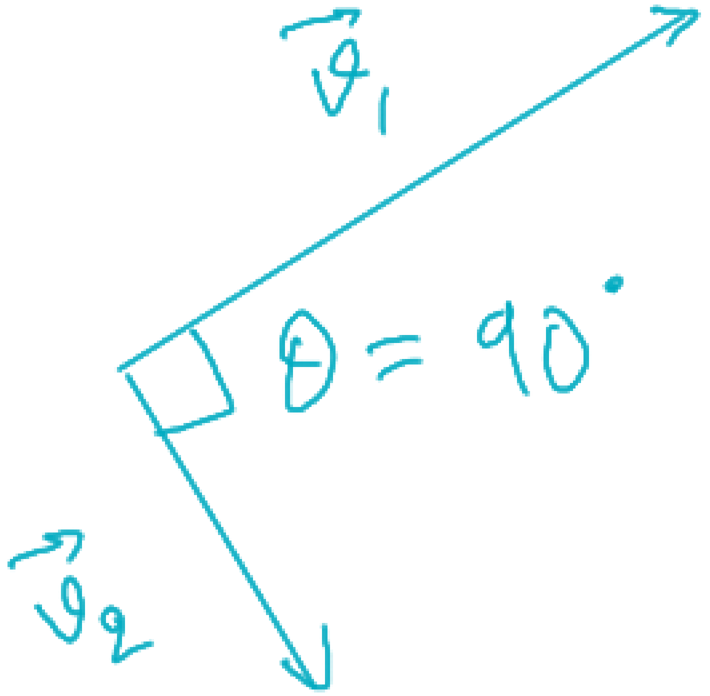
Why is it called 'Normal'?

↳ Because $\vec{n}$ is known as Normal Vector, Why?

↳ Because $\vec{n} = A\hat{i} + B\hat{j}$ is always perpandicular to line $Ax + By + C = 0$

$$\vec{v}_1 \cdot \vec{v}_2 = |\vec{v}_1| \cdot |\vec{v}_2| \cdot \cos\theta$$

$$= 0$$



$\theta = 90^\circ \Rightarrow \cos\theta = \cos 90^\circ = 0$

☆ One more interesting result:

If we have projection of $\vec{p}$ on $\vec{n}$ then,

$$\vec{p}_{proj} = |\vec{p}| \cdot \cos\theta \cdot \frac{\vec{n}}{|\vec{n}|} \quad \text{--------- (I)}$$

$\vec{a} = |\vec{a}| \cdot \hat{a}$

Now, $\vec{p} \cdot \vec{n} = |\vec{p}| \cdot |\vec{n}| \cdot \cos\theta \Rightarrow |\vec{p}| \cdot \cos\theta = \dfrac{\vec{p} \cdot \vec{n}}{|\vec{n}|}$

putting this value into (I)

$$\vec{p}_{proj} = \frac{\vec{p} \cdot \vec{n}}{|\vec{n}|} \cdot \frac{\vec{n}}{|\vec{n}|}$$

Unit vector in direction of $\vec{n}$

magnitude of $\vec{p}_{proj}$