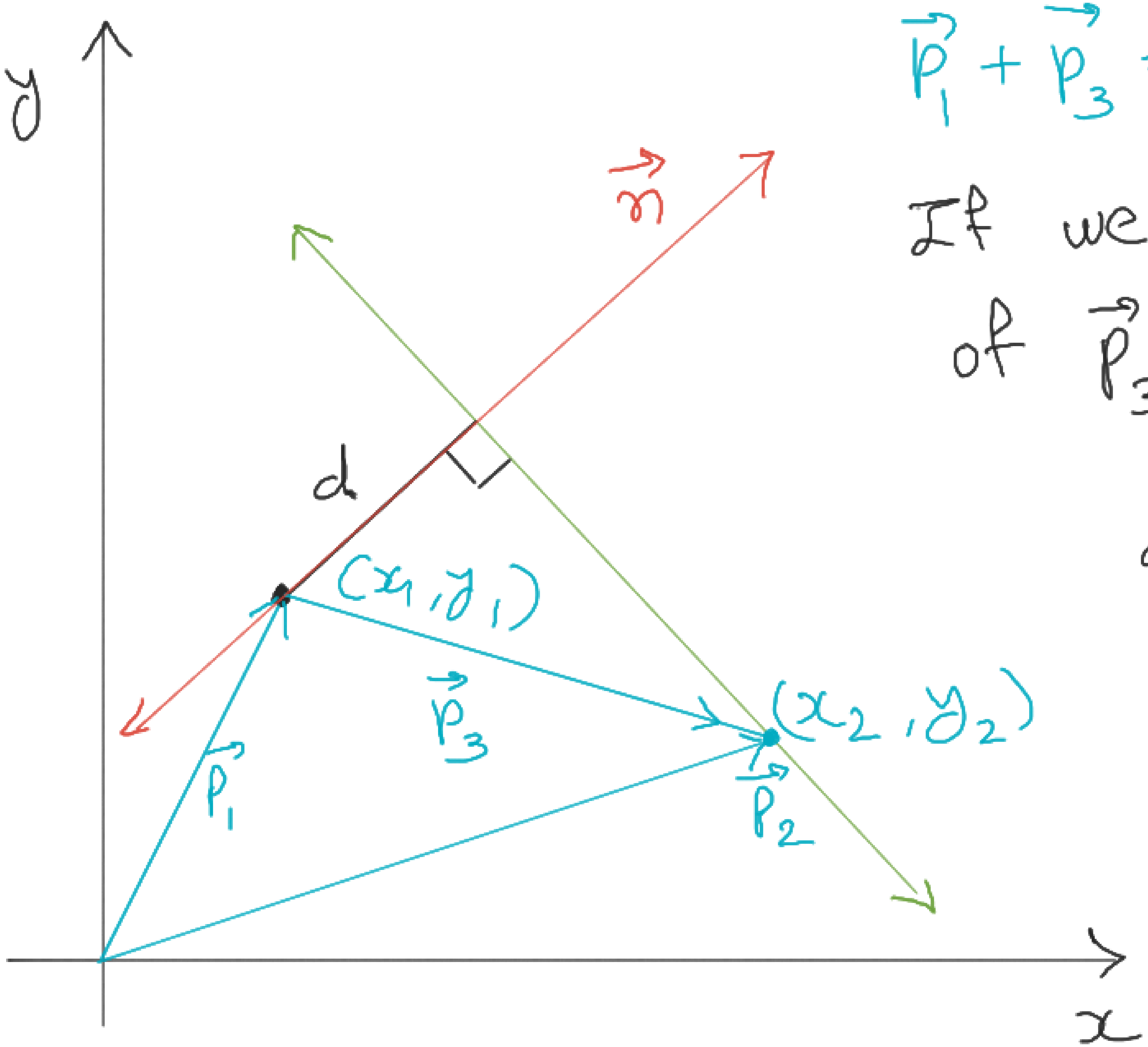
☆ Distance of a point from a line



$\vec{p}_1 + \vec{p}_3 = \vec{p}_2 \Rightarrow \vec{p}_3 = \vec{p}_2 - \vec{p}_1$

If we find magnitude of projection of $\vec{p}_3$ on $\vec{n}$ then it is d.

$$d = \frac{\vec{p}_3 \cdot \vec{n}}{|\vec{n}|}$$

Now, $\vec{p}_3 = \vec{p}_2 - \vec{p}_1$

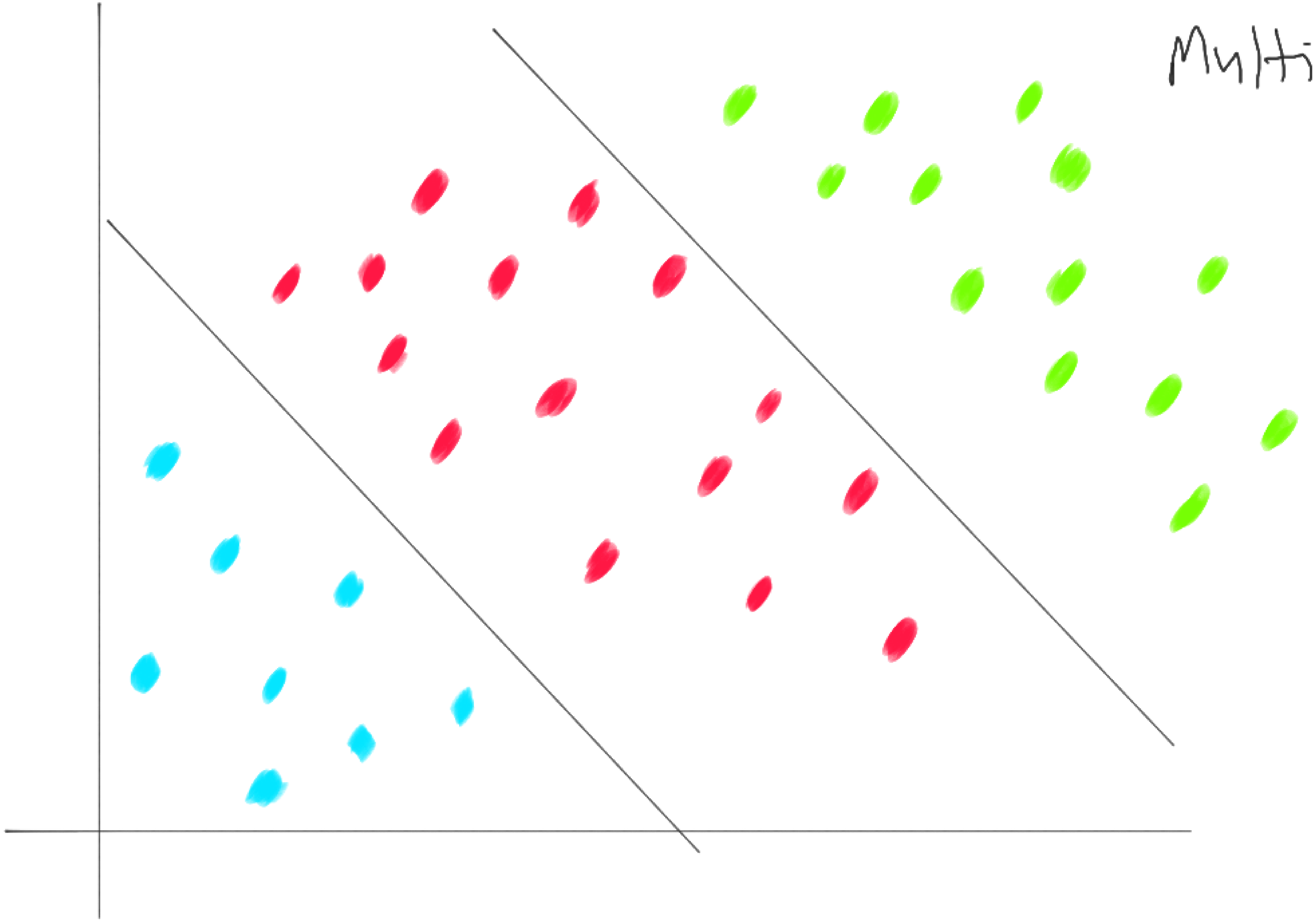
& $\vec{p}_1 = x_1 \cdot \hat{i} + y_1 \cdot \hat{j}$ &

$\vec{p}_2 = x_2 \cdot \hat{i} + y_2 \cdot \hat{j}$

$$\therefore \quad \vec{P_3} = (x_2 - x_1)\cdot\hat{i} + (y_2 - y_1)\cdot\hat{j}$$

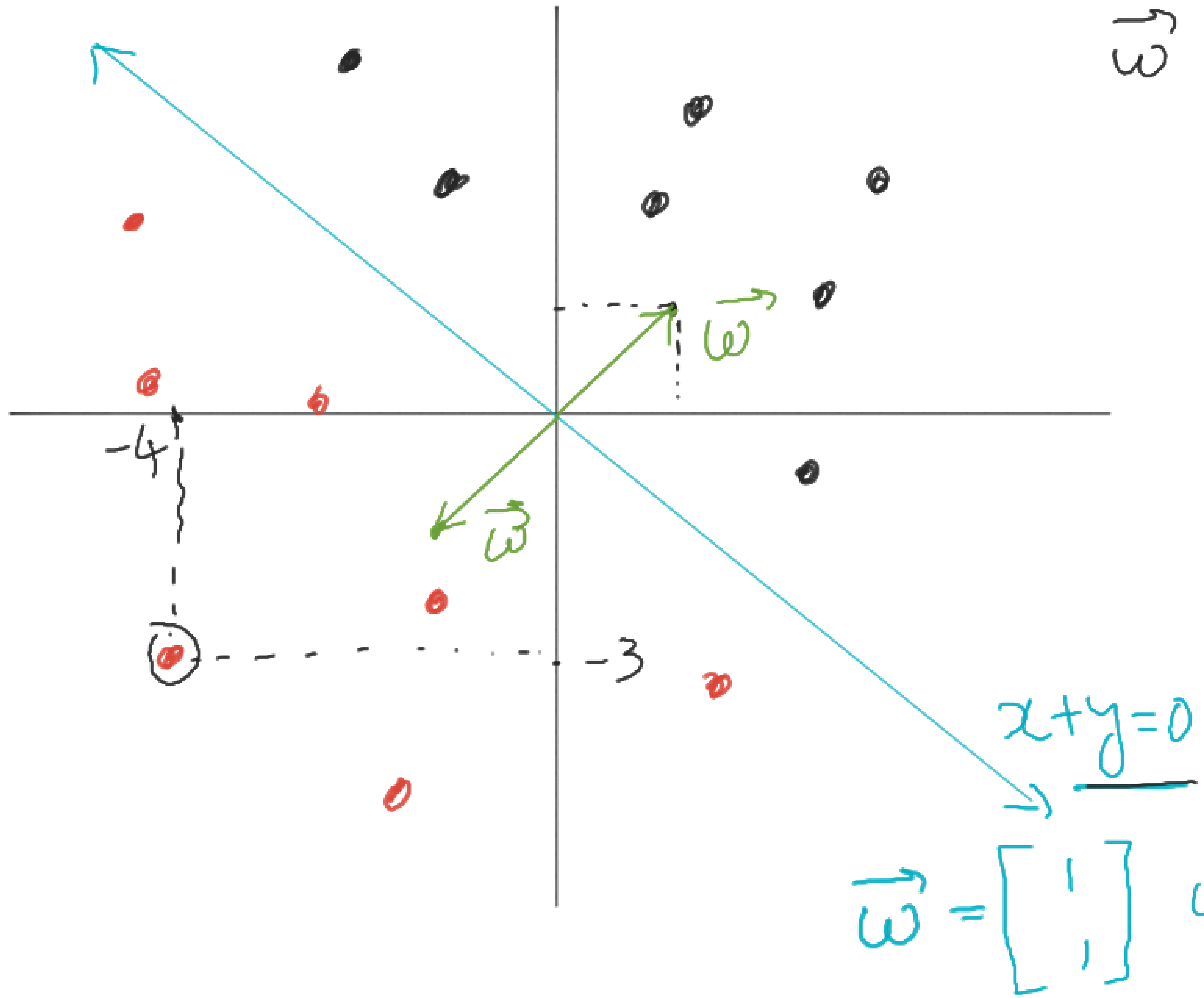
$$\vec{n} = A\cdot\hat{i} + B\cdot\hat{j}$$

$$d = \frac{\vec{P_3}\cdot\vec{n}}{|\vec{n}|} = \frac{A(x_2 - x_1) + B(y_2 - y_1)}{\sqrt{A^2 + B^2}}$$



Multi-class classification

☆ Significance of $\vec{w}$:



Let's consider another line

$-x-y=0$

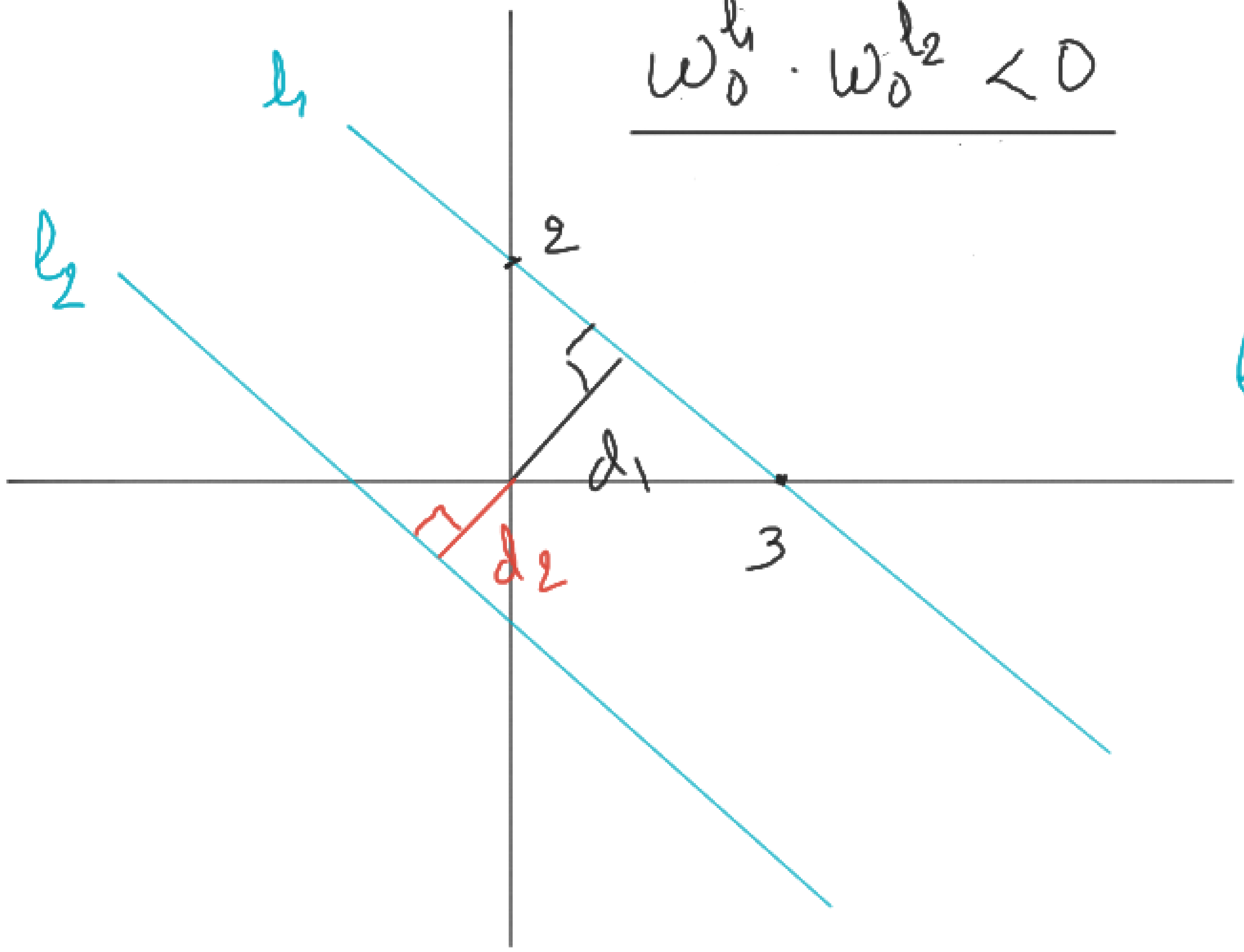
$$\vec{w} = \begin{bmatrix} -1 \\ -1 \end{bmatrix}$$

$$-(-4)-(-3) = 7 > 0$$

Hence $\vec{w}$ changes our perception.

☆ Distance between two lines:

$\underline{w_0^{l_1} \cdot w_0^{l_2} < 0}$

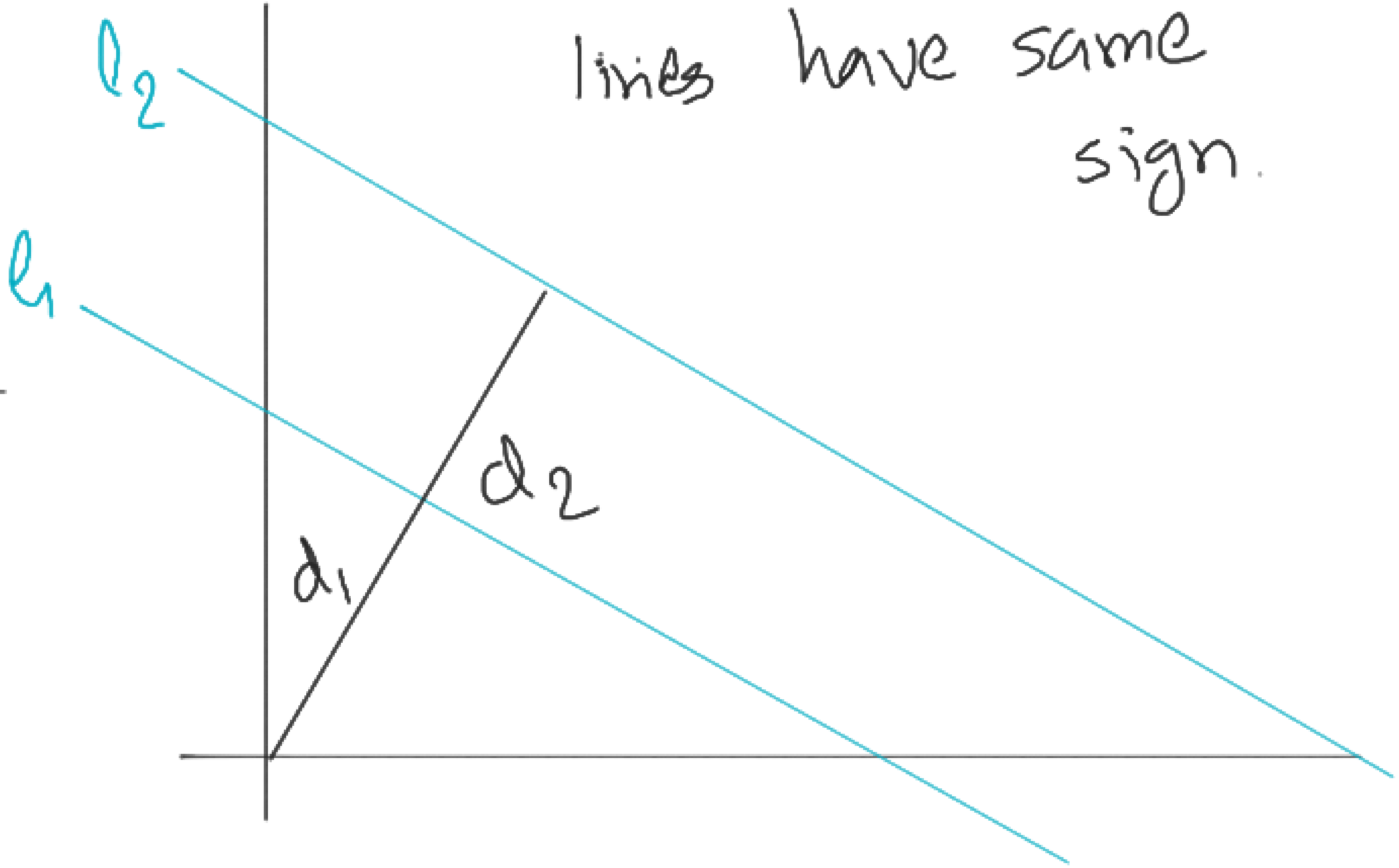


distance between l_1 & l_2 = $d_1 + d_2$

(when w_0 of both lines have diff sign)

$\underline{w_0^{l_1} \cdot w_0^{l_2} > 0}$

when w_0 of both the lines have same sign.

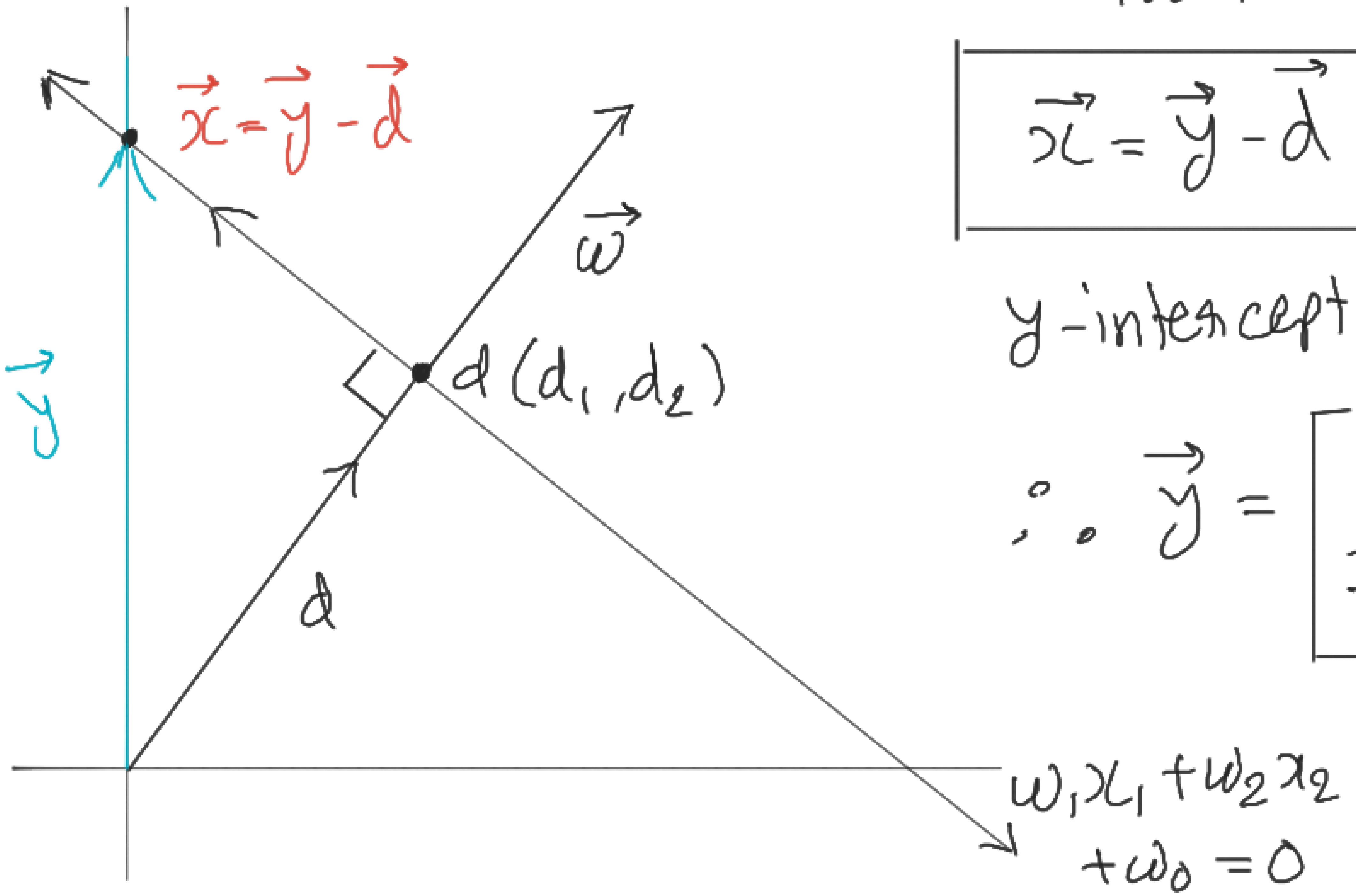


distance between l_1 & l_2

$= |d_1 - d_2|$

☆ Prove that $\vec{w} \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}$ is always perpendicular to the line $w_1 x_1 + w_2 x_2 + w_0 = 0$.

Proof: If we prove $\vec{w}^T \cdot \vec{x} = 0$ then we can solve this problem.



$$\boxed{\vec{x} = \vec{y} - \vec{d}} \quad \text{—— (I)}$$

y-intercept of the line $= \dfrac{-w_0}{w_2}$

$$\therefore \vec{y} = \begin{bmatrix} 0 \\ \dfrac{-w_0}{w_2} \end{bmatrix} \quad \text{—— (A)}$$

$\vec{d} = k\hat{\omega}$ But $\hat{\omega} = \dfrac{\vec{\omega}}{\|\vec{\omega}\|} = \dfrac{1}{\|\vec{\omega}\|}\begin{bmatrix} \omega_1 \\ \omega_2 \end{bmatrix} = \begin{bmatrix} \dfrac{\omega_1}{\|\vec{\omega}\|} \\ \dfrac{\omega_2}{\|\vec{\omega}\|} \end{bmatrix}$

$$\therefore \vec{d} = \begin{bmatrix} \dfrac{k\omega_1}{\|\vec{\omega}\|} \\ \dfrac{k\omega_2}{\|\vec{\omega}\|} \end{bmatrix} \text{ ——— } B$$

But we still donot know 'k'

As point $d(d_1, d_2)$ is on the line

$\omega_1 x_1 + \omega_2 x_2 + \omega_0 = 0$, d_1 & d_2 will satisfy it.

$$\omega_1 d_1 + \omega_2 d_2 + \omega_0 = 0$$

$$\therefore \omega_1\left(\frac{k\omega_1}{\|\vec{\omega}\|}\right) + \omega_2\left(\frac{k\omega_2}{\|\vec{\omega}\|}\right) = -\omega_0 \Rightarrow k\left(\frac{\omega_1^2}{\|\vec{\omega}\|} + \frac{\omega_2^2}{\|\vec{\omega}\|}\right) = -\omega_0$$

$$\therefore k = \frac{-\omega_0}{\left(\dfrac{\omega_1^2}{\|\vec{\omega}\|} + \dfrac{\omega_2^2}{\|\vec{\omega}\|}\right)}$$

$$\therefore k = \frac{-w_0 \, \|\vec{w}\|}{w_1^2 + w_2^2}$$ putting this in eqn (B)

$$d = \begin{bmatrix} \dfrac{-w_0 w_1 \cancel{\|\vec{w}\|}}{(w_1^2 + w_2^2)\cancel{\|\vec{w}\|}} \\ \\ \dfrac{w_2}{\cancel{\|\vec{w}\|}} \dfrac{(-w_0)\cancel{\|\vec{w}\|}}{(w_1^2 + w_2^2)} \end{bmatrix} = \begin{bmatrix} \dfrac{-w_0 w_1}{w_1^2 + w_2^2} \\ \\ \dfrac{-w_0 w_2}{w_1^2 + w_2^2} \end{bmatrix} \quad \text{——— (C)}$$

Substituting values of (A) & (C) into (I)

$$\vec{x} = \begin{bmatrix} \dfrac{\omega_0 \omega_1}{\omega_1^2 + \omega_2^2} \\[2em] \dfrac{\omega_0 \omega_2}{\omega_1^2 + \omega_2^2} - \dfrac{\omega_0}{\omega_2} \end{bmatrix} = \begin{bmatrix} \dfrac{\omega_0 \omega_1}{\omega_1^2 + \omega_2^2} \\[2em] \dfrac{\omega_0 \omega_2^2 - \omega_0(\omega_1^2 + \omega_2^2)}{\omega_2(\omega_1^2 + \omega_2^2)} \end{bmatrix} = \begin{bmatrix} \dfrac{\omega_0 \omega_1}{\omega_1^2 + \omega_2^2} \\[2em] \dfrac{-\omega_0 \omega_1^2}{\omega_2(\omega_1^2 + \omega_2^2)} \end{bmatrix}$$

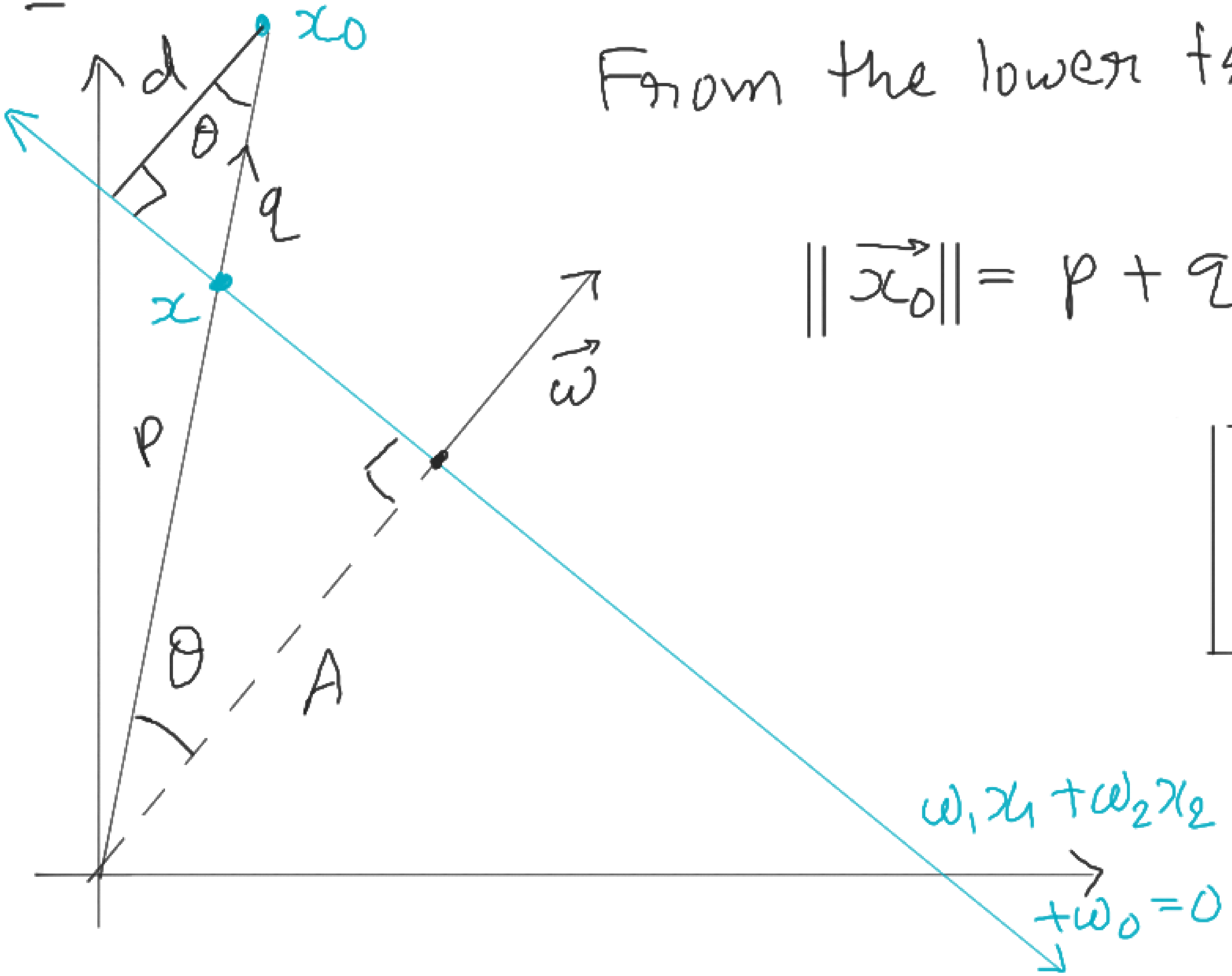
Now let's do $\vec{\omega}^T \cdot \vec{x}$

$$\vec{\omega}^T \cdot \vec{x} = \frac{\omega_0 \omega_1^2}{\omega_1^2 + \omega_2^2} - \frac{\omega_0 \omega_1^2 \cancel{\omega_2}}{\cancel{\omega_2}(\omega_1^2 + \omega_2^2)} = 0$$

$\therefore \cos\theta = 0$

$\therefore \theta = 90^\circ$

☆ Another formula of distance of a point from a line



From the lower triangle, $\cos\theta = \dfrac{A}{p} \Rightarrow p = \dfrac{A}{\cos\theta}$

$$\|\vec{x_0}\| = p + q \Rightarrow q = \|\vec{x_0}\| - p$$

$$\boxed{\therefore\ q = \|\vec{x_0}\| - \frac{A}{\cos\theta}} \text{ — (I)}$$

From the upper triangle,

$$\cos\theta = \frac{d}{q} \Rightarrow \boxed{d = q\cos\theta} \text{ — (II)}$$

Substituting value of q from I into II

$$d = \left(\|\vec{x_0}\| - \frac{A}{\cos\theta} \right) \cdot \cos\theta = \|\vec{x_0}\| \cdot \cos\theta - A$$

From the dot product of $\vec{\omega}^T$ & $\vec{x_0}$

$$\vec{\omega}^T \cdot \vec{x_0} = \|\vec{\omega}\| \cdot \|\vec{x_0}\| \cdot \cos\theta \Rightarrow \boxed{\cos\theta = \frac{\vec{\omega}^T \cdot \vec{x_0}}{\|\vec{\omega}\| \, \|\vec{x_0}\|}}$$

putting this value of $\cos\theta$ into the eqn of d:

$$d = \frac{\cancel{\|\vec{x_0}\|} \cdot \vec{\omega}^T \cdot \vec{x_0}}{\|\vec{\omega}\| \cancel{\|\vec{x_0}\|}} - A \qquad \text{III}$$

From the lower triangle, $p = \|\vec{x}\| \Rightarrow A = \|\vec{x}\| \cos\theta$

putting value of $\cos\theta$ (from the dot product formula) into this eq^n:

$$A = \frac{\cancel{\|\vec{x}\|} \cdot \vec{\omega}^T \cdot \vec{x}}{\|\vec{\omega}\| \cdot \cancel{\|\vec{x}\|}} = \frac{\vec{\omega}^T \cdot \vec{x}}{\|\vec{\omega}\|}$$

But the eq^n of line is $\omega_1 x_1 + \omega_2 x_2 + \omega_0 = 0$ OR

$$\vec{\omega}^T \cdot \vec{x} + \omega_0 = 0$$

$$\therefore \vec{\omega}^T \cdot \vec{x} = -\omega_0$$

$$\therefore A = \frac{-\omega_0}{\|\vec{\omega}\|}$$

Substituting this value of A into eqn III.

$$d = \frac{\vec{w}^T \cdot \vec{x}_0}{\|\vec{w}\|} + \frac{w_0}{\|\vec{w}\|}$$

$$\boxed{d = \frac{\vec{w}^T \cdot \vec{x}_0 + w_0}{\|\vec{w}\|}}$$ V.V. Imp Result