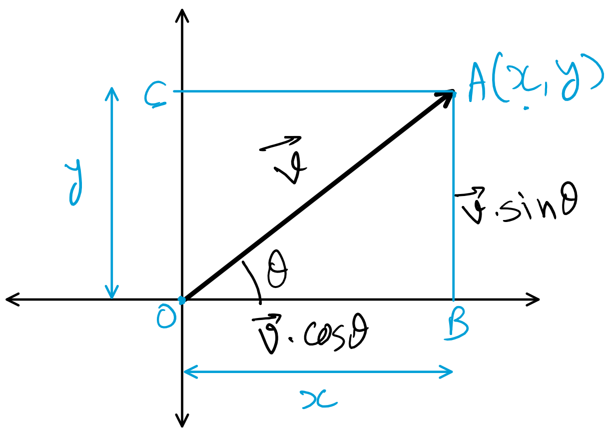
✯ Vectors (contd...)



$\cos\theta = \dfrac{OB}{OA}$ $\qquad$ $\sin\theta = \dfrac{AB}{OA}$

$\cos\theta = \dfrac{OB}{|\vec{v}|}$ $\qquad$ $\therefore \sin\theta = \dfrac{AB}{|\vec{v}|}$

$\therefore OB = |\vec{v}|\cos\theta$ $\qquad$ $\therefore AB = |\vec{v}|\sin\theta$

$\boxed{x = |\vec{v}|.\cos\theta}$ $\qquad$ $\boxed{\therefore y = |\vec{v}|.\sin\theta}$

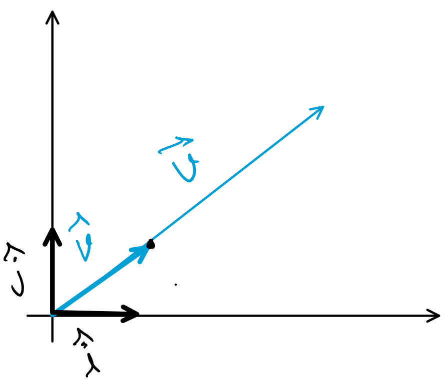
$$\tan\theta = \frac{\text{Opp. side}}{\text{Adj. side}} = \frac{y}{x} \Rightarrow y = x\tan\theta$$

$$\|\vec{v}\| = |\vec{v}| = \sqrt{x^2+y^2} \Rightarrow \|\vec{v}\| = \sqrt{x^2 + x^2\tan^2\theta} = \sqrt{x^2(1+\tan^2\theta)}$$

$$\boxed{\therefore \|\vec{v}\| = x\sqrt{1+\tan^2\theta}}$$

✯ Unit Vector:
Any vector with magnitude = 1 is called a unit vector

Suppose we want to find unit vector in the direction of $\vec{v}$



$\hat{v} \neq \hat{i} + \hat{j}$ why?

suppose $\hat{v} = \hat{i} + \hat{j} \Rightarrow \hat{v}(1, 1)$

And if it is so, $|\hat{v}| = \sqrt{x^2+y^2}$

$= \sqrt{1+1}$

$|\hat{v}| = \sqrt{2}$

But as $\hat{v}$ is a unit vector, its

magnitude must be 1.

$\therefore \hat{v} \neq \hat{i} + \hat{j}$

Then, how can we find unit vector in direction of any vector? Let $\vec{v}(x, y)$

$$\hat{v} = \frac{\vec{v}}{|\vec{v}|} = \frac{x \cdot \hat{i} + y \cdot \hat{j}}{\sqrt{x^2 + y^2}} = \boxed{\frac{x}{\sqrt{x^2+y^2}} \cdot \hat{i} + \frac{y}{\sqrt{x^2+y^2}} \cdot \hat{j}}$$

☆ Subtraction of two vectors:

Let $\vec{A}(a_1, b_1)$ & $\vec{B}(a_2, b_2)$ be two vectors

$\therefore \vec{A} = a_1 \cdot \hat{i} + b_1 \cdot \hat{j}$ & $\vec{B} = a_2 \cdot \hat{i} + b_2 \cdot \hat{j}$

$$\vec{d} = \vec{A} - \vec{B}$$

$$= \vec{A} + (-\vec{B}) = a_1 \cdot \hat{i} + b_1 \hat{j} - (a_2 \cdot \hat{i} + b_2 \hat{j})$$

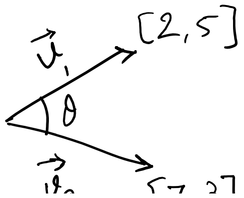
$$= a_1 \cdot \hat{i} - a_2 \cdot \hat{i} + b_1 \hat{j} - b_2 \hat{j}$$

$$\boxed{\vec{d} = (a_1 - a_2)\hat{i} + (b_1 - b_2)\hat{j}}$$

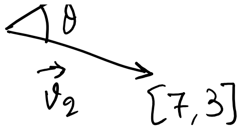
$\therefore$ coordinates of $\vec{d}$ are $(a_1 - a_2, b_1 - b_2)$

whereas $\boxed{\|\vec{d}\| = \sqrt{(a_1 - a_2)^2 + (b_1 - b_2)^2}}$

☆ Dot Product of two vectors (scalar multiplication)



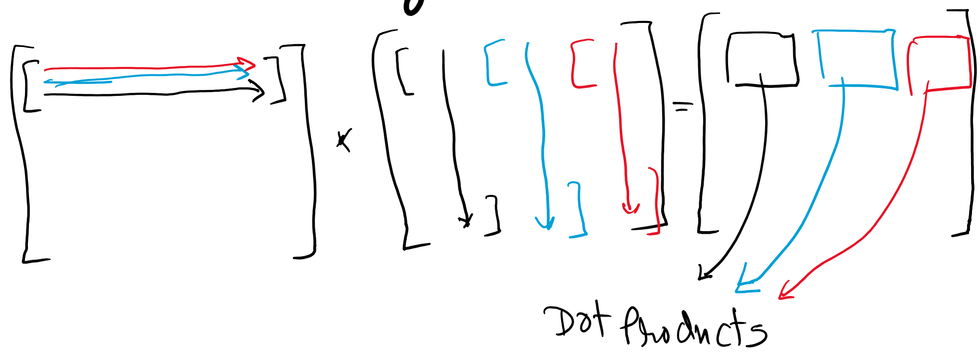
$\vec{v}_1 \cdot \vec{v}_2 = \|\vec{v}_1\| \cdot \|\vec{v}_2\| \cdot \cos\theta \longrightarrow$ same as matrix multiplication



$v_1 \cdot v_2 = \|v_1\| \, \|v_2\| \cos\theta$ ⟶ same as matrix multiplication of $[2,5]\begin{bmatrix}7\\3\end{bmatrix}$

$$\boxed{\therefore \cos\theta = \frac{\vec{v}_1 \cdot \vec{v}_2}{\|\vec{v}_1\| \cdot \|\vec{v}_2\|}}$$

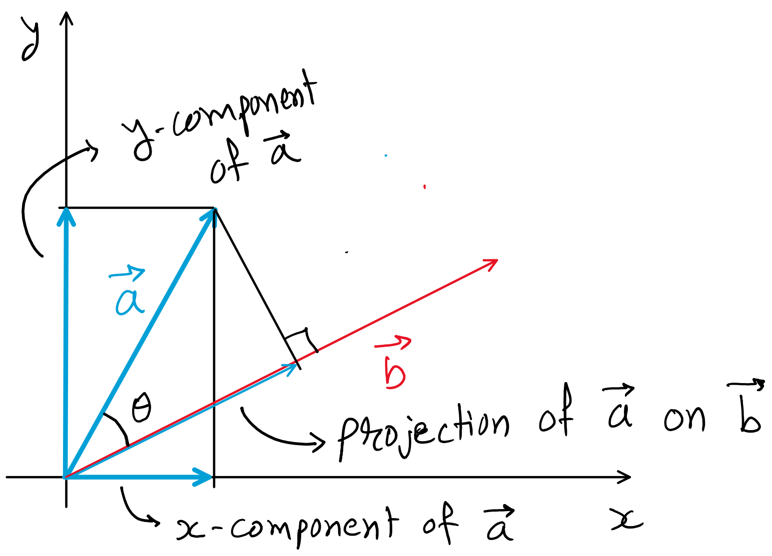
→ Matrix multiplication is nothing but a bunch of dot products



$\therefore \vec{v}_1(a_1, b_1)$ & $\vec{v}(a_2, b_2)$ then: $[a_1 \;\; b_1]\begin{bmatrix}a_2\\b_2\end{bmatrix}$

$$\boxed{\vec{v}_1 \cdot \vec{v}_2 = a_1 \cdot a_2 + b_1 \cdot b_2}$$

\# Projection of a vector



$\cos\theta$ = adj. side / hypotaneous

$$\cos\theta = \frac{\|\vec{a}_{proj}\|}{\|\vec{a}\|}$$

$$\boxed{\|\vec{a}_{proj}\| = \|\vec{a}\| \cdot \cos\theta} \quad \text{(I)}$$

any vector = magintude * unit vector in that dir^n

$$\vec{a}_{proj} = \|\vec{a}_{proj}\| \cdot \hat{b}$$

Hence the components of a vector are special cases of projection only.

$$\boxed{\therefore \vec{a}_{proj} = \|\vec{a}\| \cos\theta \cdot \hat{b}} \quad \text{(II)}$$

→ An interesting result:

$$\vec{a}\cdot\vec{b} = \|\vec{a}\|\cdot\|\vec{b}\|\cdot\cos\theta \Rightarrow \|\vec{a}\|\cos\theta = \frac{\vec{a}\cdot\vec{b}}{\|\vec{b}\|}$$

Substituting this result into (I):

$$\|\vec{a}_{proj}\| = \frac{\vec{a}\cdot\vec{b}}{\|\vec{b}\|}$$ Now, multiplying both sides by $\hat{b}$

$$\|\vec{a}_{proj}\|\cdot\hat{b} = \frac{\vec{a}\cdot\vec{b}}{\|\vec{b}\|}\cdot\hat{b} \Rightarrow \boxed{\vec{a}_{proj} = \frac{\vec{a}\cdot\vec{b}}{\|\vec{b}\|}\cdot\frac{\vec{b}}{\|\vec{b}\|}}$$

☆ Normal Form of Equation of a Straight Line:

As we know, general form of a line is:

$$ax + by + c = 0$$

Can we write $ax + by$ as: $\vec{\omega}\cdot\vec{x}$ where $\vec{\omega} = [a, b]$ & $\vec{x} = [x, y]$

Because for $v_1(a_1, b_1)$ & $v_2(a_2, b_2)$,

$$\vec{v}_1\cdot\vec{v}_2 = a_1\cdot a_2 + b_1\cdot b_2$$

∴ For $\vec{\omega}[a, b]$ & $\vec{x}[x, y]$, $\vec{\omega}\cdot\vec{x} = a\cdot x + by$

∴ The equation will become:

$$\vec{\omega}\cdot\vec{x} + c = 0$$

For higher dimensions (more features), we used to refer our boundary line as a "Hyperplane" to refer our boundary line as a hyperplane

eg, for a 2-D space the hyperplane will be 1-D. (line)

" " 3-D " " " " " 2-D

" " 4-D " " " " " 3-D

" " n-D " " " " " (n-1)-D

For 2D the line was $ax + by + c = 0$ | $w_1x_1 + w_2x_2 + w_0 = 0$

For 3D the line can be: $ax + by + cz + d = 0$ |

For 4D, 5D, — ., nD ? $w_1x_1 + w_2x_2 + w_3x_3 + w_0 = 0$

↳ $w_1x_1 + w_2x_2 + \dots + w_nx_n + w_0 = 0$

$$\vec{w} \cdot \vec{x} + w_0 = 0$$

where $\vec{w}\,[w_1, w_2, \dots w_n]$ & $\vec{x}\,[x_1, x_2, \dots x_n]$

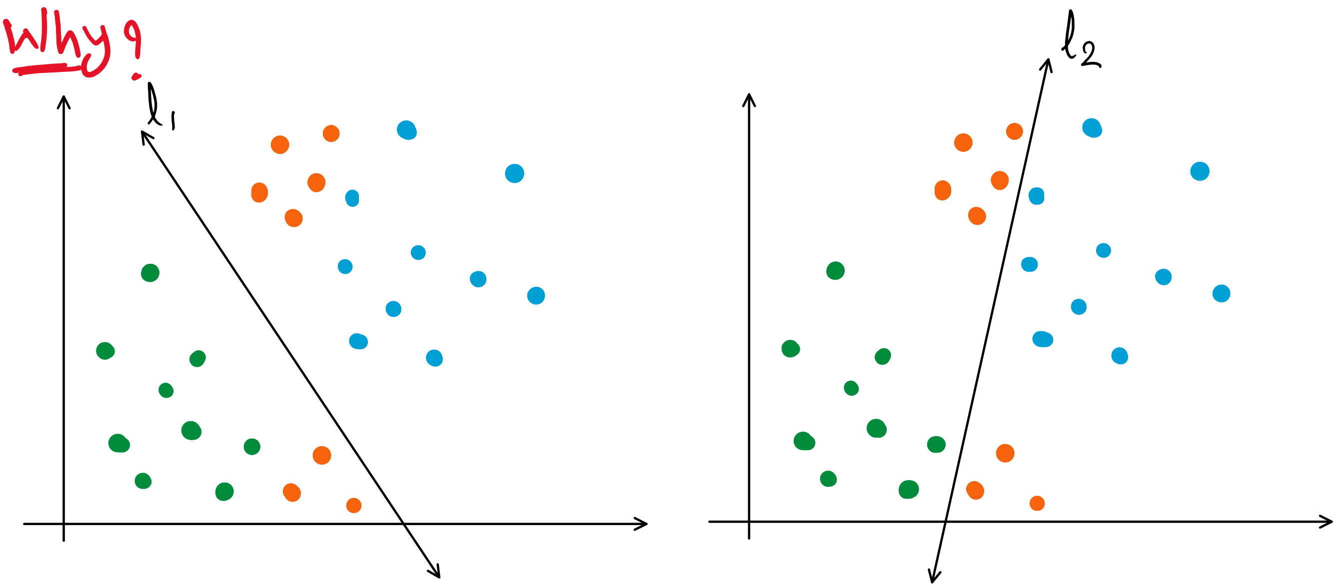
To make $\vec{w}$ & $\vec{x}$ compatible for matrix multiplication,

this eqn is converted to $\boxed{\vec{w}^T \cdot \vec{x} + w_0 = 0}$

This is called Normal Form of equation of straight line.

✱ Distance of a point from the line:

Why?



which seperator is better?

Which seperator is better?

l_1. Why? – intuitively

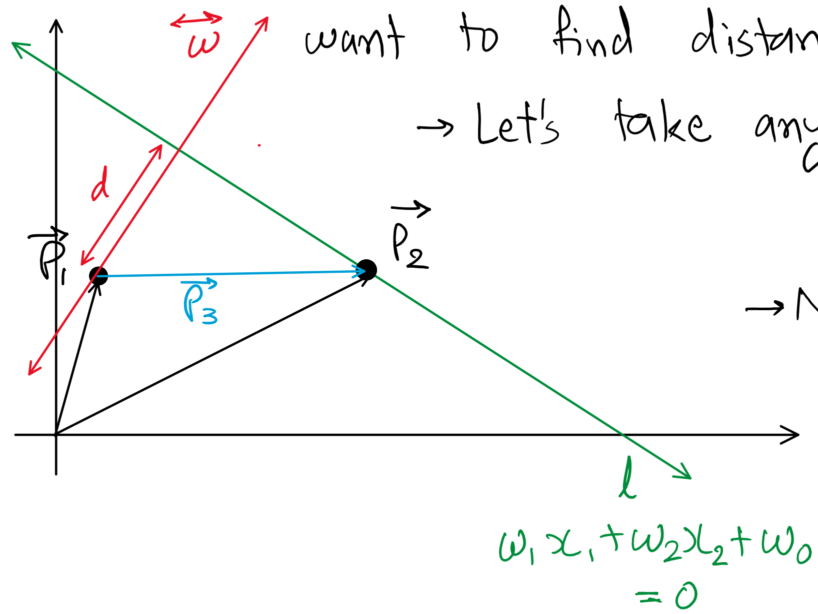
Logically – l_2 misclassifies newly inducted points (orange) while l_1 still classifies them correctly. Why did this happen?

<u>Ans</u>: The distances of points from l_1 are greater than distances of points from l_2.

$\therefore$ We want a boundary that is as far as possible from our datapoints.

$\boxed{\therefore \text{ We need to know how to find distance of a point from a line}}$

How?: Let P_1 be the point & l be the line. We want to find distance of P_1 from l.



→ Let's take any point on the line l. (Say P_2)

→ Now if we create a vector joining P_1 & P_2 (let's say $\vec{P_3}$), can we write this?

$$\vec{P_1} + \vec{P_3} = \vec{P_2}$$

$$\boxed{\therefore \vec{P_3} = \vec{P_2} - \vec{P_1}}$$

→ Now let's draw a perpandicular line $\vec{w}$ to $\overleftrightarrow{l}$ which is passing from point P_1. ∴ 'd' will be the distance of P_1 from l.

→ This 'd' is nothing but the projection of $\vec{P_3}$ on $\overleftrightarrow{w}$.

∴ From the formula of magnitude of projection vector,

$$\boxed{\|\vec{d}\| = \frac{\vec{P_3} \cdot \vec{w}}{\|\vec{w}\|}} \text{ —— } (A)$$

But here, we know nothing about either $\vec{P_3}$ or $\vec{w}$.

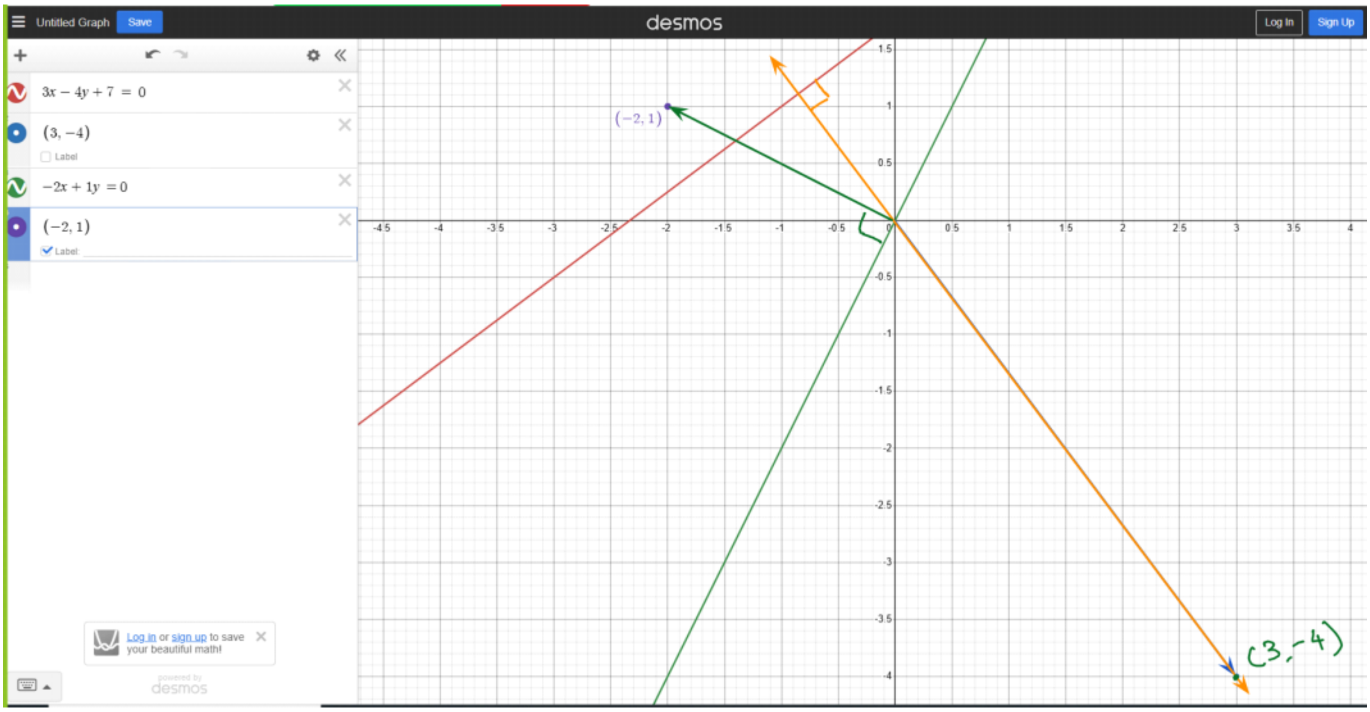
∴ Let $\vec{P_1}(x_1, y_1)$ & $P_2(x_2, y_2) \Rightarrow \vec{P_1} = x_1 \cdot \hat{i} + y_1 \cdot \hat{j}$ &

$$\vec{P_2} = x_2 \cdot \hat{i} + y_2 \cdot \hat{j}$$

But $\vec{P_3} = \vec{P_2} - \vec{P_1}$

$$\boxed{\therefore \vec{P_3} = (x_2 - x_1)\hat{i} + (y_2 - y_1)\cdot\hat{j}} \text{ —— } B$$

→ Why $\vec{w}^T \cdot \vec{x} + w_0 = 0$ with $\vec{w}[w_1, w_2]$ known as 'normal' equation? Ans: In maths, normal means perpandicular. & the $\vec{w}$ of a line is always perpandicular to that line.



$\therefore$ our $\vec{\omega}$ is nothing but $[\omega_1, \omega_2]$

$$\boxed{\therefore \vec{\omega} = \omega_1 \cdot \hat{i} + \omega_2 \cdot \hat{j}} \qquad \text{(C)}$$

$$\boxed{\|\vec{\omega}\| = \sqrt{\omega_1^2 + \omega_2^2}} \qquad \text{(D)}$$

Now substituting (B), (C) & (D) into (A):

$$\boxed{\|\vec{d}\| = \frac{[(x_2 - x_1)\cdot\hat{i} + (y_2 - y_1)\cdot\hat{j}]\cdot[\omega_1\cdot\hat{i} + \omega_2\cdot\hat{j}]}{\sqrt{\omega_1^2 + \omega_2^2}}}$$

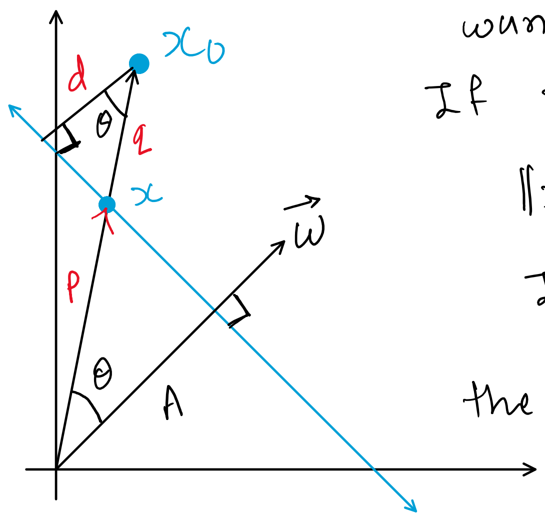
From the formula of dot product of two vectors $v_1(a_1, b_1)$ & $v_2(a_2, b_2) \Rightarrow v_1 \cdot v_2 = a_1 \cdot a_2 + b_1 \cdot b_2$

same way here in the numerator of $\|\vec{d}\|$, we have a dot product of two vectors.

$$\therefore \quad \boxed{\vec{d} = \frac{(x_2 - x_1)\cdot w_1 + (y_2 - y_1)\cdot w_2}{\sqrt{w_1^2 + w_2^2}}}$$

✱ Another formula of distance of a point from the line:

In the figure, x_0 is the point we want find distance from the line.



If x is a point on the line,

$$\|\vec{x_0}\| = p + q$$

In the diagram, if we consider the lower triangle, $\cos\theta = \frac{A}{p}$

$$\boxed{\therefore \; p = \frac{A}{\cos\theta}}$$

whereas $q = \|\vec{x_0}\| - p \quad \therefore \; q = \|\vec{x_0}\| - \frac{A}{\cos\theta}$

From the upper triangle, $\cos\theta = \frac{d}{q} \Rightarrow \boxed{d = q\cos\theta}$

$$\therefore \; d = \left(\|\vec{x_0}\| - \frac{A}{\cos\theta}\right)\cdot\cos\theta \Rightarrow \boxed{d = \|\vec{x_0}\|\cdot\cos\theta - A}$$

But from the dot product of $\vec{w}^T$ & $\vec{x_0}$

$$\vec{w}^T\cdot\vec{x_0} = \|\vec{w}^T\|\cdot\|\vec{x_0}\|\cdot\cos\theta$$

$$\therefore \; \cos\theta = \frac{\vec{w}^T\cdot\vec{x_0}}{\|\vec{w}^T\|\cdot\|\vec{x_0}\|}$$

$$\therefore\ d = \cancel{\|\vec{x_0}\|} \cdot \left(\frac{\vec{\omega}^T \cdot \vec{x_0}}{\|\vec{\omega}^T\| \cancel{\|\vec{x_0}\|}} \right) - A$$

Now we know $\vec{\omega}^T$, $\vec{x_0}$ & $\|\vec{\omega}^T\|$ but we still don't know anything about 'A'.

From the lower triangle, $A = p \cdot \cos\theta$ <u>OR</u>

$$A = \|\vec{x}\| \cdot \cos\theta$$

We can also have dot product of $\vec{\omega}^T$ & $\vec{x}$ to get $\cos\theta \Rightarrow \cos\theta = \dfrac{\vec{\omega}^T \cdot \vec{x}}{\|\vec{\omega}^T\| \cdot \|\vec{x}\|}$

Putting this in eqn of A: $A = \cancel{\|\vec{x}\|} \dfrac{\vec{\omega}^T \cdot \vec{x}}{\|\vec{\omega}\| \cdot \cancel{\|\vec{x}\|}}$

$$\therefore\ A = \frac{\vec{\omega}^T \cdot \vec{x}}{\|\vec{\omega}\|}$$

But, our eqn of line is: $\vec{\omega}^T \cdot \vec{x} + \omega_0 = 0$

$$\therefore\ \vec{\omega}^T \cdot \vec{x} = -\omega_0$$

$$\therefore\ A = \frac{-\omega_0}{\|\vec{\omega}\|}$$

substituting this 'A' into distance formula

$$d = \frac{\vec{\omega}^T \vec{x_0}}{\|\vec{\omega}\|} - \frac{(-\omega_0)}{\|\vec{\omega}\|}$$

$$\frac{\qquad}{\|\vec{w}\|} - \frac{\qquad}{\|\vec{w}\|}$$

$$\boxed{d = \frac{\vec{w}^T \cdot \vec{x_0} + w_0}{\|\vec{w}\|}}$$

Very important result