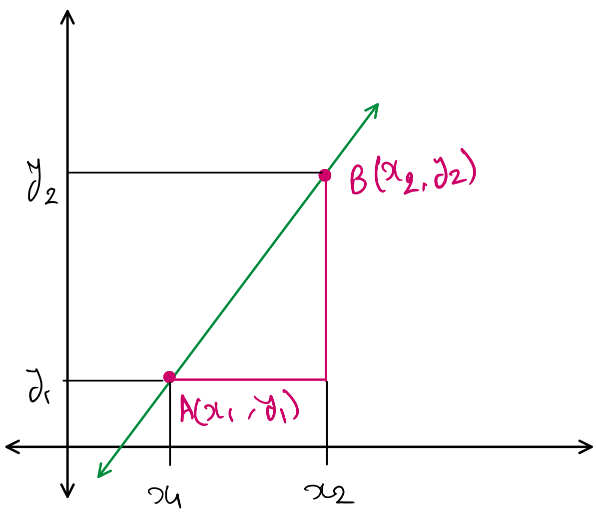
# Differentiation

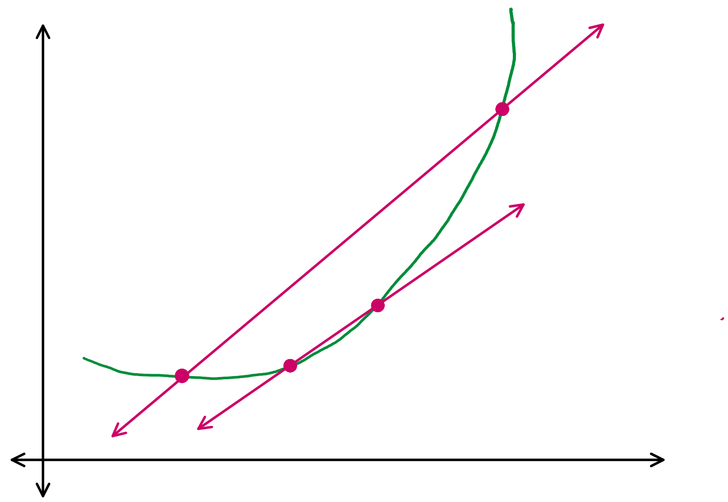
17 April 2025 09:19

Finding slope of a line:




$$m = \frac{y_2 - y_1}{x_2 - x_1} \Rightarrow \boxed{m = \frac{\Delta y}{\Delta x}}$$

Finding slope of a curve:



In case of a tangent $\Delta y$ & $\Delta x$ will be very small & to represent this fact, we call them $dy$ & $dx$.

$$\boxed{\therefore \text{ slope} = m = \frac{dy}{dx}}$$

In the above formula, dy and dx are very small changes in x & y respectively. Now suppose the function that represents our curve is:

$y = f(x)$

Therefore, for point $A(x_1, y_1)$ the value of $y_1$ can be given as:

$y_1 = f(x_1)$

And similarly for point $B(x_2, y_2)$:

$y_2 = f(x_2)$

As both the points are different, we can get $x_2$ by adding something (let's say "h") into $x_1$.

Therefore, by putting $x_2 = x_1 + h$ in the equation of $y_2$:

$y_2 = f(x_1 + h)$

Therefore, the slope will become:

$$m = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{f(x_1+h) - f(x_1)}{(x_1+h) - x_1} = \frac{f(x_1+h) - f(x_1)}{\cancel{x_1} + h - \cancel{x_1}}$$

$$\frac{\Delta y}{-} \quad f(x_1+h) - f(x_1)$$

$$\frac{\Delta y}{\Delta x} = \frac{f(x_1+h) - f(x_1)}{h}$$

Now if we consider difference in x & difference in y very small (ideally, both the points will fall on each other - that's how we will get tangent), the difference between x2 & x1 (that is "h") will tend towards 0. The same thing is also written mathematically as below:

$$\boxed{\frac{dy}{dx} = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}} \longrightarrow \text{core formula of differentiation}$$

**Example:**

Let $y = f(x) = x^2$ be our function $f(x)$.

$$\therefore \frac{dy}{dx} = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

if $\boxed{f(x) = x^2}$ then what will be $f(3) = 3^2 = 9$

$$f(5) = 5^2 = 25$$

$$\boxed{\therefore f(x+h) = (x+h)^2 = x^2 + 2hx + h^2}$$

$$\therefore \frac{dy}{dx} = \lim_{h \to 0} \frac{\cancel{x^2} + 2hx + h^2 - \cancel{x^2}}{h}$$

$$= \lim_{h \to 0} \frac{\cancel{h}(2x+h)}{\cancel{h}}$$

$$\frac{dy}{dx} = 2x + 0 \Rightarrow \frac{d}{dx}y = 2x$$

$$\boxed{\therefore \frac{d}{dx} x^2 = 2x}$$

| x | y ($y = x^2$) | dx (h) | new x (x + h) | new y $(\text{new } x)^2$ | dy (new y - old y) | dy/dx (By division) | dy/dx (By formula) |
|---|---|---|---|---|---|---|---|
| 2 | 4 | 0.00001 | 2.00001 | 4.00004 | 0.00004 | 4 | 4 |
| 3 | 9 | 0.00001 | 3.00001 | 9.00006 | 0.00006 | 6 | 6 |
| 4 | 16 | 0.00001 | 4.00001 | 16.00008 | 0.00008 | 8 | 8 |

| x | y ($y = x^3$) | dx (h) | new x (x + h) | new y $(\text{new } x)^3$ | dy (new y - old y) | dy/dx (By division) | dy/dx (By formula) |
|---|---|---|---|---|---|---|---|
| 2 | 8 | 0.00001 | 2.00001 | 8.00012 | 0.00012 | 12 | |
| 3 | 27 | 0.00001 | 3.00001 | 27.00027 | 0.00027 | 27 | |
| 4 | 64 | 0.00001 | 4.00001 | 64.00048 | 0.00048 | 48 | |

$$y = x^3 \Rightarrow \frac{dy}{dx} \neq 2x \Rightarrow \frac{d}{dx} x^3 = 3x^2$$

Therefore the differentiation is nothing but slope of the function at a particular point. This line/slope is also called Tangent or Gradient of the function at that point.

**But not all the functions are differentiable at all the points.**
Understanding differentiation of $y = x^2$ (using gif)
$y = x^2$ is differentiable at every point (even at 0 the slope is also 0)

But if we consider $y = |x|$, it is not differentiable at 0 as we get two tangents having different slopes.

**A function is said to be differentiable if & only if it is differentiable at every point**.

## Some commonly used differentiation formulae:

(1) $\frac{d}{dx} x^2 = 2x$, $\frac{d}{dx} x^3 = 3x^2$, $\frac{d}{dx} x^7 = 7x^6$

$$\frac{d}{dx} x^n = n \cdot x^{n-1}$$

② $\frac{d}{dx} x = 1$

③ $\frac{d}{dx} c = 0$ c = constant - A constant is anything (may be a function/variable) that is not dependent on $x$.

examples - $\frac{d}{dx} 5 = 0$ , $\frac{d}{dx} z = 0$ (if $z$ is not a function of $x$)

④ $\frac{d}{dx} \sin x = \cos x$ ⑤ $\frac{d}{dx} \cos x = -\sin x$

⑥ $\frac{d}{dx} \log x = \frac{1}{x}$ ⑦ $\frac{d}{dx} e^{x} = e^{x}$

**Some rules of differentiation:**

① Sum Rule - $$\boxed{\frac{d}{dx}\left(f(x) \pm g(x)\right) = \frac{d}{dx} f(x) \pm \frac{d}{dx} g(x)}$$

② Product Rule - $\frac{d}{dx}\left(f(x) \cdot g(x)\right) \neq \frac{d}{dx} f(x) \cdot \frac{d}{dx} g(x)$

$$\boxed{\frac{d}{dx}\left(f(x) \cdot g(x)\right) = g(x) \frac{d}{dx} f(x) + f(x) \cdot \frac{d}{dx} g(x)}$$

③ Division Rule - $\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) \neq \frac{d}{dx} f(x) \Big/ \frac{d}{dx} g(x)$

$$\boxed{\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = \frac{g(x)\cdot\frac{d}{dx}f(x) - f(x)\frac{d}{dx}g(x)}{(g(x))^2}}$$

$\frac{d}{dx}$ of any function $f(x)$ is also written as $f'(x)$

$$\therefore\ f'(x) = \frac{d}{dx}f(x)$$

$$\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = \frac{g(x)\cdot f'(x) - f(x)\cdot g'(x)}{(g(x))^2}$$

④ Chain Rule:

suppose $f(x) = 3x - 2$ & $g(x) = 2x^2$

$\therefore$ for $x=3$, $f(x) = 7$ & $g(x) = 18$

But $f(g(x))$ for $x = 3 \Rightarrow f(\underline{g(3)}) = f(2(3)^2)$

$$= f(18)$$

$$= 3(18) - 2$$

$$= 54 - 2$$

$$= 52$$

similarly, $g(f(x))$ for $x = 2$ : $g(f(2)) = g(3\times 2 - 2)$

$$= g(4)$$

$$= 2\times(4)^2$$

$$g(f(x)) = 32$$

$$\boxed{\frac{d}{dx} f(g(x)) = f'(g(x)) \cdot \frac{d}{dx} g(x)}$$

⑤ $$\frac{d}{dx} c \cdot f(x) = f(x) \cdot \frac{d}{dx} c + c \cdot \frac{d}{dx} f(x)$$

$$= 0 + c \cdot \frac{d}{dx} f(x)$$

$$\boxed{\frac{d}{dx} c \cdot f(x) = c \cdot \frac{d}{dx} f(x)}$$

**Now let's try out some examples:**

① $\frac{d}{dx} x^9 = 9x^8$

② $\frac{d}{dx} 7x = 7 \cdot \frac{d}{dx} x = 7$

③ $\frac{d}{dx} 5 \cdot x^3 = 5 \cdot \frac{d}{dx} x^3 = 5 \times 3 x^2 = 15x^2$

④ $\frac{d}{dx}(3x+4) = \frac{d}{dx} 3x + \frac{d}{dx} 4 = 3$

⑤ $\frac{d}{dx}(\sin x + \cos x) = \frac{d}{dx} \sin x + \frac{d}{dx} \cos x = \cos x - \sin x$

(6) $\dfrac{d}{dx}(3x^2 - 4x^5) = \dfrac{d}{dx}3x^2 - \dfrac{d}{dx}4x^5 = 6x - 20x^4$

(7) $\dfrac{d}{dx}(4x^3 - 5\cos x) = 12x^2 - 5\dfrac{d}{dx}\cos x = 12x^2 + 5\sin x$

(8) $\dfrac{d}{dx}\underbrace{2x^4}_{f(x)} \cdot \underbrace{\sin x}_{g(x)} = \sin x \cdot \dfrac{d}{dx}2x^4 + 2x^4\dfrac{d}{dx}\sin x$

$= 8x^3 \cdot \sin x + 2x^4 \cos x$

(9) $\dfrac{d}{dx}\left(\dfrac{\sin x}{\cos x}\right) = \dfrac{\cos x \cdot \dfrac{d}{dx}\sin x - \sin x \cdot \dfrac{d}{dx}\cos x}{\cos^2 x}$

$\tan x$

$= \dfrac{\cos^2 x - (-\sin^2 x)}{\cos^2 x}$

$= \dfrac{\cos^2 x + \sin^2 x}{\cos^2 x}$

$= \dfrac{1}{\cos^2 x} = \sec^2 x$

$\therefore \dfrac{d}{dx}\tan x = \sec^2 x$

(10) $\dfrac{d}{dx}3x^2 \cdot \log x = \log x \cdot \dfrac{d}{dx}3x^2 + 3x^2\dfrac{d}{dx}\log x$

$$= 6x \cdot \log x + \frac{3x^2}{x}$$

$$= 6x \cdot \log x + 3x$$

(11) $$\frac{d}{dx} \cos x \cdot \log x = \log x \cdot \frac{d}{dx} \cos x + \cos x \frac{d}{dx} \log x$$

$$= -\sin x \cdot \log x + \frac{\cos x}{x}$$

(12) $$\frac{d}{dx} \frac{\sin x}{\log x} = \frac{\log x \cdot \frac{d}{dx} \sin x - \sin x \cdot \frac{d}{dx} \log x}{(\log x)^2}$$

$$= \frac{\cos x \cdot \log x - \frac{\sin x}{x}}{(\log x)^2}$$

$$= \frac{x \cdot \cos x \cdot \log x - \sin x}{x (\log x)^2}$$

(13) $$\frac{d}{dx} \frac{e^x}{\sin x} = \frac{\sin x \cdot \frac{d}{dx} e^x - e^x \cdot \frac{d}{dx} \sin x}{\sin^2 x}$$

$$= \frac{\sin x \cdot e^x - e^x \cdot \cos x}{\sin^2 x}$$

(14) $\frac{d}{dx}(\sin x)^2 = \qquad \frac{d}{dx}(\_\_)^2 = 2(\_\_)'$

$$= 2(\sin x)' \cdot \frac{d}{dx}\sin x$$

$$= 2\sin x \cdot \cos x$$

(15) $\frac{d}{dx}(\cos x)^3 = 3(\cos x)^2 \cdot \frac{d}{dx}\cos x = -3\cos^2 x \cdot \sin x$

(16) $\frac{d}{dx}\log(3x+7) = \frac{1}{(3x+7)} \cdot \frac{d}{dx}(3x+7)$

$$= \frac{3}{3x+7}$$

(17) $\frac{d}{dx}\sin(3x^4) = \cos(3x^4) \cdot \frac{d}{dx}3x^4 = 12x^3 \cdot \cos(3x^4)$

(18) $\frac{d}{dx}\sin^2(2x^3) = \frac{d}{dx}(\sin(2x^3))^2 = 2\sin(2x^3) \cdot \frac{d}{dx}\sin(2x^3)$

$$= 2\sin 2x^3 \cdot \cos(2x^3) \cdot \frac{d}{dx}2x^3$$

$$= 12x^2 \cdot \sin 2x^3 \cdot \cos 2x^3$$

(19) $\frac{d}{dx}\log(\cos^3 4x^5) = \frac{1}{\cos^3 4x^5} \cdot \frac{d}{dx}(\cos 4x^5)^3$

$$= \frac{3\cancel{\cos^2 4x^5}}{\cos^{\cancel{3}} 4x^5} \cdot \frac{d}{dx}(\cos 4x^5)$$

$$= \frac{-3\sin 4x^5}{\cos 4x^5} \cdot \frac{d}{dx}(4x^5)$$

$$= -60x^4 \cdot \frac{\sin 4x^5}{\cos 4x^5} = -60x^4 \cdot \tan 4x^5$$

(20) $\frac{d}{dx} e^{7x^5} = e^{7x^5} \cdot \frac{d}{dx} 7x^5 = e^{7x^5} \cdot 35x^4$

(21) $\frac{d}{dx} e^{(5x^3-2x^2+9x+8)} = e^{(5x^3-2x^2+9x+8)} \cdot \frac{d}{dx}(5x^3-2x^2+9x+8)$

$$= e^{(5x^3-2x^2+9x+8)} \cdot (15x^2-4x+9)$$

(22) $\frac{d}{dx} e^{\log(\cos^2 4x)} = e^{\log(\cos^2 \cdot 4x)} \cdot \frac{d}{dx} \log(\cos^2 4x)$

$$= e^{\log(\cos^2 4x)} \cdot \frac{1}{\cos^2 4x} \cdot \frac{d}{dx} \cos^2 4x$$

$$= e^{\log(\cos^2 4x)} \cdot \frac{1}{\cos^2 4x} \cdot 2\cos 4x \cdot \frac{d}{dx} \cos 4x$$

$$= e^{\log(\cos^2 4x)} \cdot \frac{1}{\cos^2 4x} \cdot 2\cos 4x \, (-\sin 4x) \cdot \frac{d}{dx} 4x$$

$$= e^{\log(\cos^2 4x)} \cdot \frac{1}{\cos^2 4x} \cdot 2\cos 4x \, (-\sin 4x) \, \frac{d}{dx} 4x$$

$$= e^{\log(\cos^2 4x)} \cdot \frac{1}{\cos^{\cancel{2}} 4x} \cdot 2 \cancel{\cos 4x} \, (-\sin 4x) \cdot 4$$

$$= \frac{-8 \sin 4x \cdot e^{\log(\cos^2 4x)}}{\cos 4x} = -8 \tan 4x \cdot e^{\log(\cos^2 4x)}$$

(23) Find $\frac{dy}{dx}$ for $y = f(x) = \frac{1}{1+e^{-x}}$

$$\frac{d}{dx} e^{(-)} = e^{(-)} \cdot \frac{d}{dx}(-)$$

$$\frac{d}{dx}\left(\frac{1}{1+e^{-x}}\right) = \frac{(1+e^{-x}) \cdot \frac{d}{dx} 1 - 1 \frac{d}{dx}(1+e^{-x})}{(1+e^{-x})^2}$$

$$= \frac{0 - \frac{d}{dx}(1) - \frac{d}{dx}(e^{-x})}{(1+e^{-x})^2} = \frac{-e^{-x} \cdot \frac{d}{dx}(-x)}{(1+e^{-x})^2}$$

$$= \frac{e^{-x}}{(1+e^{-x})^2} = \boxed{\frac{e^{-x}}{(1+e^{-x})}} \cdot \frac{1}{(1+e^{-x})} \quad \text{— I}$$

$$f(x) = \frac{1}{1+e^{-x}} \Rightarrow 1 - f(x) = 1 - \frac{1}{1+e^{-x}}$$

$$= \frac{(1+e^{-x}) - 1}{1+e^{-x}}$$

$$\therefore \quad \boxed{1 - f(x) = \frac{e^{-x}}{1+e^{-x}}}$$

Putting this result into I :

$$\boxed{\frac{d}{dx} f(x) = (1 - f(x)) \cdot f(x)} \quad \text{for } f(x) = \frac{e^{-x}}{1-e^{-x}}$$

Although this is a very important result, it is **not a general rule of differentiation and cannot be applied to any function.** This result is only applicable for **sigmoid function.**