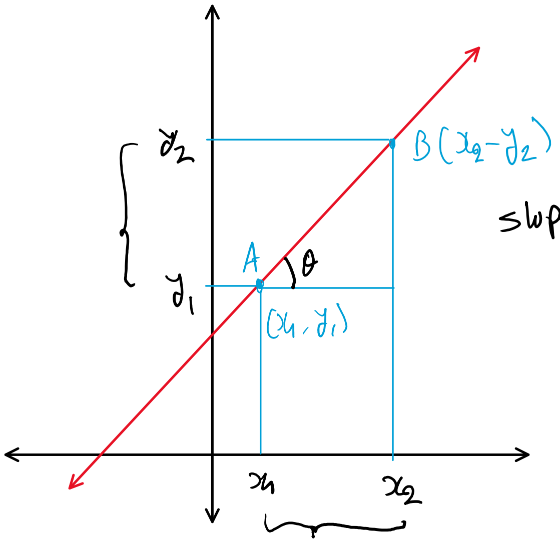
Calculus - 2

23 March 2025 13:56

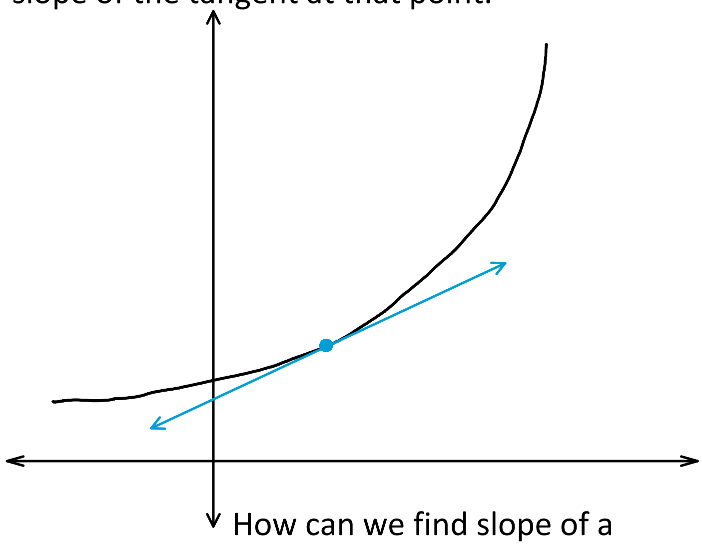
Differentiation



$$\text{slope} = \tan\theta = \frac{y_2 - y_1}{x_2 - x_1}$$

$$\text{slope} = \frac{\Delta y}{\Delta x}$$

Finding slope of the entire curve is not possible because it is changing continuously but, we can find its slope at a particular point by finding slope of the tangent at that point.



How can we find slope of a tangent?

$$\text{slope} = \frac{dy}{dx}$$

where dy & dx indicate very small changes in x & y

Let the equation of our curve is $y_1 = f(x_1)$ then $y_2 = f(x+h)$

where h is the difference between x_2 & x_1

$$\therefore \text{slope} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{f(x+h) - f(x)}{\cancel{x} + h - \cancel{x}} = \frac{f(x+h) - f(x)}{h}$$

$\therefore$ Slope of a curve at a point both these points x & $x+h$ should very near to each other and hence $h \to 0$

Mathematically, $\boxed{\frac{dy}{dx} = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}}$ "limit h tends to 0"

A practical example: Let **$y = x^2$** then, what is dy/dx? $y = f(x) = x^2$

$$\frac{dy}{dx} = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

$$\therefore f(x+h) = (x+h)^2$$

$$= \lim_{h\to 0} \frac{(x+h)^2 - x^2}{h} = \lim_{h\to 0} \frac{\cancel{x^2} + 2hx + h^2 - \cancel{x^2}}{h}$$

$$\therefore \frac{dy}{dx} = \lim_{h\to 0} \frac{\cancel{h}(2x+h)}{\cancel{h}} = \lim_{h\to 0} 2x+h = 2x+0$$

(by putting $h=0$)

$$\boxed{\frac{dy}{dx} = 2x}$$

x	y (y = x^2)	dx (h)	new x (x + h)	new y (new x^2)	dy (new y - old y)	dy/dx (By division)	dy/dx (By formula)
2	4	0.00001	2.00001	4.00004	0.00004	4	4
3	9	0.00001	3.00001	9.00006	0.00006	6	6
4	16	0.00001	4.00001	16.00008	0.00008	8	8

Therefore the differentiation is nothing but **slope** of the function at a particular point. This line/slope is also called **Tangent** or **Gradient** of the function at that point.

But not all the functions are differentiable at all the points!
e.g. y = |x| is not differentiable at the origine.

Let's look at some commonly used differentiation formulae:

① Power Rule: $\frac{d}{dx} x^n = n x^{n-1}$

② $\frac{d}{dx} c = 0$ where c is constant (constant means any term that doesnot have 'x' in it/ independent of x / not function of x)

③ $\frac{d}{dx} \sin x = \cos x$ ④ $\frac{d}{dx} \cos x = -\sin x$

⑤ $\frac{d}{dx} \log x = \frac{1}{x}$ ⑥ $\frac{d}{dx} e^x = e^x$

⑦ sum rule: $\frac{d}{dx}(f(x) + g(x)) = \frac{d}{dx} f(x) + \frac{d}{dx} g(x)$

⑧ Product Rule: $\frac{d}{dx}(f(x) \cdot g(x)) = g(x) \cdot \frac{d}{dx} f(x) + f(x) \frac{d}{dx} g(x)$

⑨ Division Rule: $\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = \frac{g(x) \frac{d}{dx} f(x) - f(x) \frac{d}{dx} g(x)}{(g(x))^2}$

$\frac{d}{dx} f(x)$ is also written as $f'(x)$

⑩ Chain Rule:

$$\boxed{\frac{d}{dx} f(g(x)) = f'(g(x)) \cdot g'(x)}$$

Try these out:

① $3x^2 \Rightarrow \frac{d}{dx} 3 x^2 = 3 \cdot \frac{d}{dx} x^2 = 3(2x) = 6x$

② $4x^5 - 2x^4 \Rightarrow \frac{d}{dx}(4x^5 - 2x^4) = \frac{d}{dx} 4x^5 - \frac{d}{dx} 2x^4$

$\frac{dy}{dx} = 20x^4 - 8x^3$

③ $4x^2 + 3\sin x \Rightarrow \frac{d}{dx}(4x^2 + 3\sin x) = \frac{d}{dx} 4x^2 + \frac{d}{dx} 3\sin x$

$\frac{dy}{dx} = 8x + 3\cos x$

④ $4x^3 \cdot \sin x$

$\sin x \cdot \frac{d}{dx} 4x^3 + 4x^3 \cdot \frac{d}{dx} \sin x = 12x^2 \sin x + 4x^3 \cos x$

⑤ $\frac{\sin x}{\cos x} \Rightarrow \frac{d}{dx} \frac{\sin x}{\cos x} = \frac{\cos x \frac{d}{dx} \sin x - \sin x \frac{d}{dx} \cos x}{\cos^2 x}$

$= \frac{\cos^2 x + \sin^2 x}{\cos^2 x} = \frac{1}{\cos^2 x}$

$= \frac{\cancel{\cos^2 x}}{\cancel{\cos^2 x}} + \frac{\sin^2 x}{\cos^2 x} = 1 + \tan^2 x$

(6) $\dfrac{\sin x}{\log x} \Rightarrow \dfrac{d}{dx}\dfrac{\sin x}{\log x} = \dfrac{\log x \cdot \cos x - \dfrac{\sin x}{x}}{(\log x)^2}$

(7) $3x^5 - 7x^4 + 3x^3 + 87x - 145$

$= \dfrac{d}{dx}3x^5 - \dfrac{d}{dx}7x^4 + \dfrac{d}{dx}3x^3 + \dfrac{d}{dx}87x - \dfrac{d}{dx}145$

$= 15x^4 - 28x^3 + 9x^2 + 87 - 0$

(8) $\sin 3x^4 \rightarrow$ This is like $f(g(x))$

$\dfrac{d}{dx}f(g(x)) = f'(x) \cdot g'(x)$

$\dfrac{d}{dx}\sin 3x^4 = \cos(3x^4) \cdot \dfrac{d}{dx}3x^4$

$= (\cos 3x^4)(12x^3)$

$= 12x^3 \cdot \cos 3x^4$

(9) $\cos 5x^3 = -\sin 5x^3 \cdot \dfrac{d}{dx}5x^3 = -15x^2 \sin 5x^3$

(10) $\sin^2 2x^3 = \dfrac{d}{dx}(\sin 2x^3)^2 = 2\sin 2x^3 \cdot \dfrac{d}{dx}\sin 2x^3$

$= 2\sin 2x^3 \cdot \cos 2x^3 \cdot \dfrac{d}{dx}(2x^3)$

$= 2\sin 2x^3 \cdot \cos 2x^3 \, (6x^2)$

$= 12x^2 \cdot \sin 2x^3 \cdot \cos 2x^3$

(11) $\log(\cos^3 4x^5) = \dfrac{1}{\cos^3 4x^5} \cdot \dfrac{d}{dx}(\cos 4x^5)^3$

$$= \frac{1}{\cos^3 4x^5} \cdot 3(\cos 4x^5)^2 \cdot \frac{d}{dx}\cos 4x^5$$

$$= \frac{1}{\cos^3 4x^5} \cdot 3(\cos 4x^5)^2 \cdot (-\sin 4x^5) \cdot \frac{d}{dx} 4x^5$$

$$= \frac{1}{\cos^3 4x^5} \cdot 3(\cos 4x^5)^2 \cdot (-\sin 4x^5) \cdot 20x^4$$

$$= \frac{-60x^4 \sin 4x^5 \cdot \cancel{\cos^2 4x^5}}{(\cancel{\cos^2 4x^5}) \cdot \cos 4x^5} = -60x^4 \tan 4x^5$$

(12) $e^{7x^5} = \dfrac{d}{dx} e^{7x^5} = e^{7x^5} \cdot \dfrac{d}{dx} 7x^5 = e^{7x^5} \cdot 35x^4$

(13) $\dfrac{d}{dx} e^{(5x^3-2x^2+9x+8)} = e^{(5x^3-2x^2+9x+8)} \cdot \dfrac{d}{dx}(5x^3-2x^2+9x+8)$

$$= e^{(5x^3-2x^2+9x+8)} \cdot (15x^2-4x+9)$$

(14) $e^{\log(\cos^2 4x)} = \dfrac{e^{\log(\cos^2 4x)}}{\cos^{\cancel{2}} 4x} \cdot \cancel{2(\cos 4x)}.(-\sin 4x).4$

$$= \frac{-8\, e^{\log(\cos^2 4x)} \cdot \sin 4x}{\cos 4x}$$

$$= -8\, e^{\log(\cos^2 4x)} \cdot \tan 4x$$

(15) Find $\dfrac{dy}{dx}$ for $y = f(x) = \dfrac{1}{1+e^{-x}}$

$$\frac{dy}{dx} = f'(x) = \frac{(1+e^{-x}) \cdot \frac{d}{dx} 1 - 1 \cdot \frac{d}{dx}(1+e^{-x})}{(1+e^{-x})^2}$$

$$= \frac{0 - \left(\frac{d}{dx}(1) + \frac{d}{dx} e^{-x}\right)}{(1+e^{-x})^2} = \frac{-\left(0 + e^{-x} \cdot \frac{d}{dx}(-x)\right)}{(1+e^{-x})^2}$$

$$= \frac{-e^{-x}(-1)}{(1+e^{-x})^2} = \frac{e^{-x}}{(1+e^{-x})^2} = \frac{e^{-x}}{(1+e^{-x})} \cdot \frac{1}{(1+e^{-x})}$$

$(1-f(x))$ $\qquad$ $f(x)$

Now, $1 - f(x) = 1 - \dfrac{1}{1+e^{-x}} = \dfrac{(1+e^{-x}) - 1}{(1+e^{-x})} = \dfrac{e^{-x}}{(1+e^{-x})}$

$$\boxed{\therefore f'(x) = \frac{dy}{dx} = f(x) \cdot (1 - f(x))}$$

Although this is a very important result, it is not a general rule for differentiation that can be applied to any function. **This result is only applicable for**

$$f(x) = \frac{1}{1+e^{-x}}$$

☆ Meaning of argmax | argmin:

Suppose $y = (x-2)^2$ is our function. then, what is the minimum possible value of y? Ans = 0.

Q: Which value of x gives us minimum value of y?

Ans: The minimum value of $y = 0$ and y becomes 0 for $x = 2$. The same thing is also written as:

$$\boxed{\underset{x}{\text{argmin}} \; y : (x-2)^2 = 2}$$

similarly,

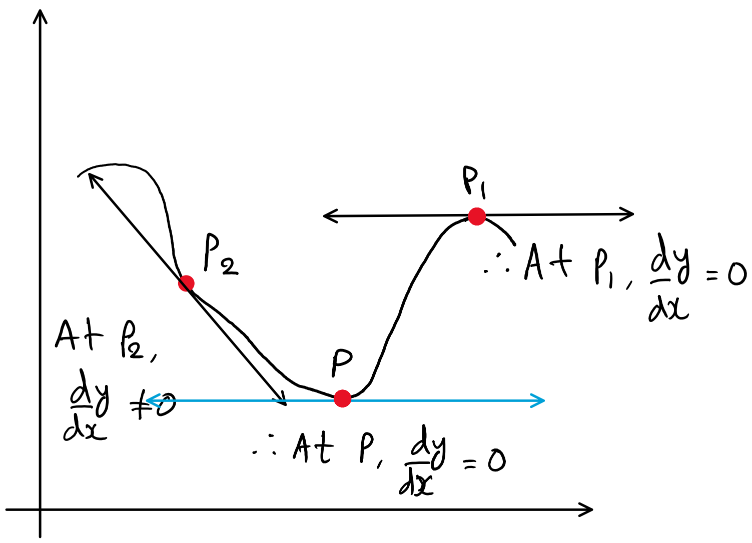
$$\boxed{\underset{x}{\text{argmax}} \; f(x) = p}$$

means at $x = p$, value of $f(x)$ is maximum.

$\therefore \underset{\vec{w}, w_0}{\text{argmin}} \; L(\vec{w}, w_0, \vec{x}, y)$ means the value of $\vec{w}$ & w_0 for which our loss function L is minimum.

Moving towards Gradient Descent

Suppose the graph of our Loss Function is as below & it is given by $y = f(x) = 41 - 72x - 18x^2$ and this function represents profit.



From the figure beside, we can see that the slope (or tangent or gradient or differentiation) at a point on the graph can be 0 if and only if that point is either maxima, minima or a saddle point. Otherwise at all other points, differentiation will be non-zero.

Hence we at first, are interested if finding the point where the differentiation becomes 0. Later we will separate the minima, maxima or the saddle point.

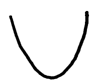
Therefore, let's first find out the differentiation of our f(x):

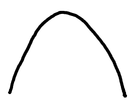
$$f'(x) = 0 - 72 - 36x \Rightarrow \boxed{f'(x) = -36x - 72}$$

$$f''(x) = \frac{d}{dx} f'(x) = -36$$

$$\boxed{f''(x) = -36}$$

An interesting observation:

If f''(x) > 0 then the function f(x) is concave in the upward direction 

And if f''(x) < 0 then the function f(x) is concave in the downward direction 

Steps to reach to the "Optima" (minima or maxima depending on our function):

1. Calculate f'(x)
2. Equate f'(x) = 0 to find the value of x at x = c
3. Calculate value of f''(x) for each value of x
4. Conclude:
   a. If f''(x) > 0 then x=c is minima
   b. If f''(x) < 0 then x=c is maxima and
   c. if f''(x) = 0 then x=c is a saddle point.

Examples:

(1) $f(x) = 41 - 72x - 18x^2$

step-1: $f'(x) = -72 - 36x$

step-2: $36x + 72 = 0$

$\therefore x = \frac{-72}{36}$

$\boxed{\therefore x = -2}$

step-3: $f''(x) = -36$

At $x = -2$,

$f''(-2) = -36$

$\boxed{f''(-2) < 0}$

Conclusion:

$x = -2$ is a maxima for $f(x) = 41 - 72x - 18x^2$

(2) $f(x) = 41 - 32x - 72x^2 - 18x^3$

$f'(x) = -32 - 144x - 54x^2$

$54x^2 + 144x + 32 = 0$

$27x^2 + 72x + 16 = 0 \Rightarrow \underline{a}x^2 + \underline{b}x + \underline{c} = 0$

$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-72 \pm 58.78}{54}$

$\boxed{x = -0.245}$ or $\boxed{x = -2.422}$

$f''(x) = -144 - 108x$

$\boxed{\therefore f''(-0.245) < 0}$ $\boxed{f''(-2.422) > 0}$

$\therefore x = -0.245$ is a maxima for $f(x) = 41 - 32x - 72x^2 - 18x^3$

And, $x = -2.422$ is a minima for $f(x) = 41 - 32x - 72x^2 - 18x^3$