07_Optimization_Calculus

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argmax & argmin:

Let's take a function $f(x) = x^2$

What will be the minimum possible value of this function? Ans: 0

Let's take another function $f(x) = 3x^2 + 5$

What will be the minimum possible value of this function? Ans: 5

This is denoted by:

min $f(x) = 5$

But for what value of 'x', f(x) becomes minimum? Ans: x=0

This is denoted by:

$$\underset{x}{\operatorname{argmin}}\ f(x) = 5$$

similarly, $\underset{x}{\operatorname{argmax}}\ f(x)$ means the value of x for which $f(x)$ is minimum. <u>OK</u>

$\underset{x}{\operatorname{argmax}}\ f(x) = p$ mean at $x = p$ the function reaches to its maximum value (maxima)

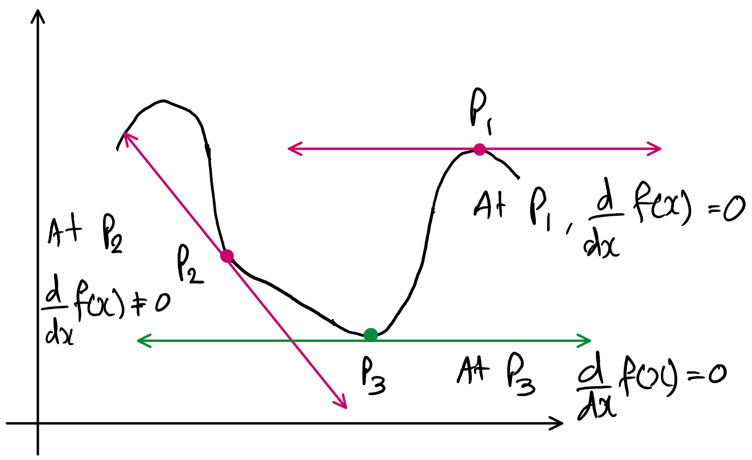
The function that we are interested in minimizing is the "Loss Function" and we want the line for which the value of this loss function is minimum. A "line" means values of $\vec{w}$ and w_0.

Hence we are interested to know:

$$\underset{\vec{w}, w_0}{\operatorname{argmin}}\ L(\vec{w}, w_0, \vec{X}, y)$$

Moving towards Gradient Descent:

Suppose the graph of our Loss function is as below and the function is: $f(x) = 41 - 72x - 18x^2$



We can see from this figure that the differentiation of the function (slope of the tangent) will be zero only at minima, maxima or saddle points. Nowhere else the differentiation can be zero so first, we want to find the points where d/dx of the f(x) is zero. Then we will try to separate the minima, maxima and the saddle points.

$$\therefore \frac{d}{dx} f(x) = -72 - 36x$$

The differentiation must be zero.

$$\therefore -72 - 36x = 0 \Rightarrow 36x = -72 \Rightarrow \boxed{x = -2}$$

This means, at x = -2 there is either minima, or maxima or saddle point. How will we identify it?
Solution: Take differentiation of f'(x) (double differentiation of f(x) - also known as f''(x)).

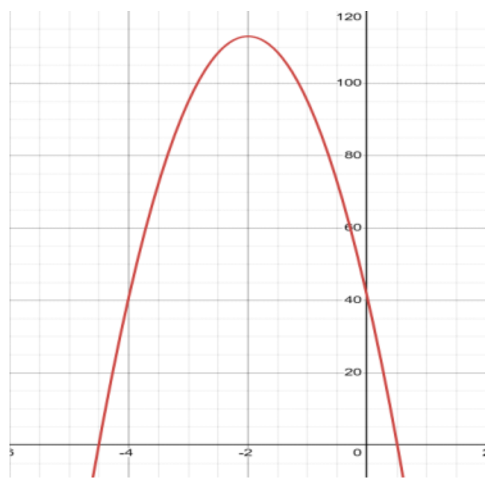
If f''(x) < 0 the point is maxima
If f''(x) > 0 then the point is minima &
If f''(x) = 0 then the point is a saddle point.

$$f'(x) = -72 - 36x$$

$$\Rightarrow f''(x) = \frac{d}{dx} f'(x) = -36$$

$$\therefore f''(x) < 0$$

$\therefore$ At $x = -2$, the function has maxima.



Now let's take an example of a different function:
$f(x) = 41 - 32x - 72x^2 - 18x^3$
Then the steps will look like:

Step - 1: Finding the differentiation of the f(x)
$f'(x) = -32 - 144x - 54x^2$

Step - 2: Equate f'(x) with 0 & find the value(s) of x where f'(x) = 0
$54x^2 + 144x + 32 = 0$ <u>or</u>

$27x^2 + 72x + 16 = 0$

Now this is not a linear equation, it's a quadratic equation hence we need to solve it using the formula:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \qquad \therefore \; b^2 - 4ac = (72)^2 - 4(27)(16)$$

$$= 3456$$

$$= \frac{-72 \pm 58.7878}{54}$$

$$\boxed{x_1 = -0.2447} \qquad \boxed{x_2 = -2.422}$$

Step - 3: Differentiating f'(x) one more time to find f''(x)

f''(x) = -144 - 108x

For x = -0.2447, f''(x) = -144 - 108*(-0.2447) = -117.5724

f''(x) < 0 for x = -0.2447 and hence at x= -0.2447, the function f(x) has a maxima.

For x = -2.422, f''(x) = -144 - 108*(-2.422) = 117.576

f''(x) > 0 for x = -2.422 and hence at x = -2.422, the function f(x) has a minima.