Optimization/Calculus - 1

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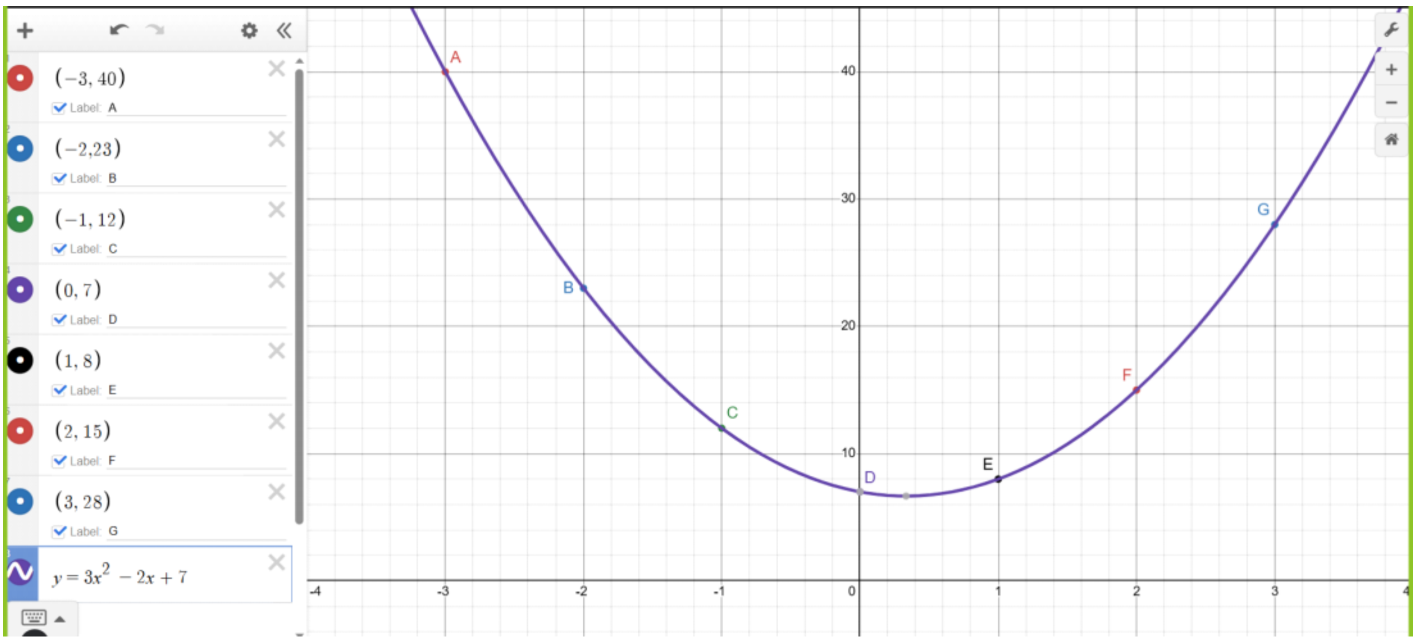
Why Calculus?

We can plot (draw graph) any function as below.
Example: suppose we have an equation as this: $y = 3x^2 - 2x + 7$, this equation can be used to compute the value of "y" for the given value of "x".
Therefore, it is said like this: "y is a function of x"
If we want to create a graph for this function, we simply need to put different values of "x" and find the corresponding values of "y" and join those points. Let's try to plot this equation/function:

x	-3	-2	-1	0	1	2	3
y	40	23	12	7	8	15	28

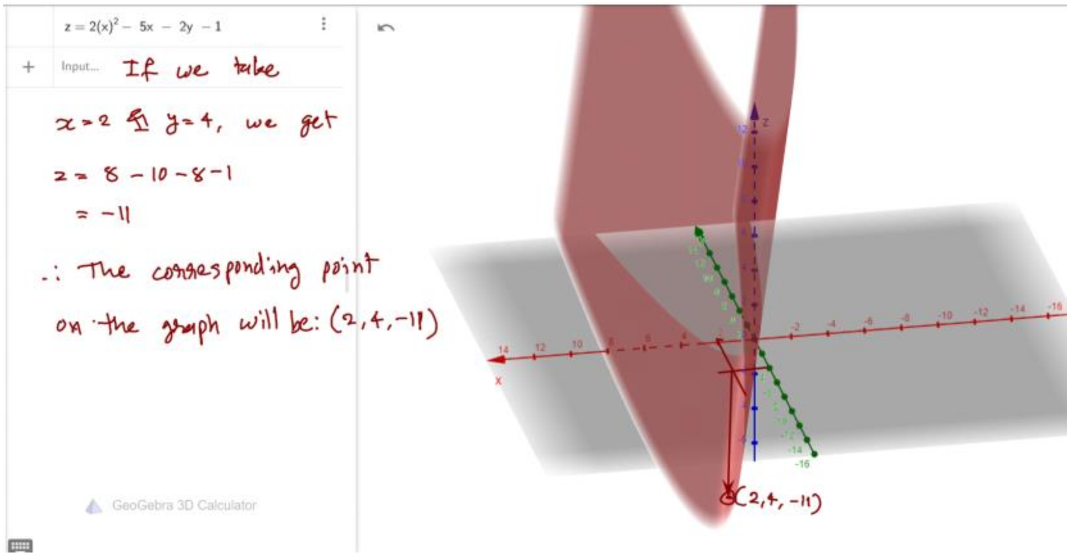
If connect all these points, we will get a graph like this:



Similarly, we can also draw the graph of Loss Function. Although, the loss function has multiple variables in it, the graph will be in higher dimensions. For example, instead of taking a function of only 1 variable "x", let's take an example of a function that has two variables "w1", "w2" as this:

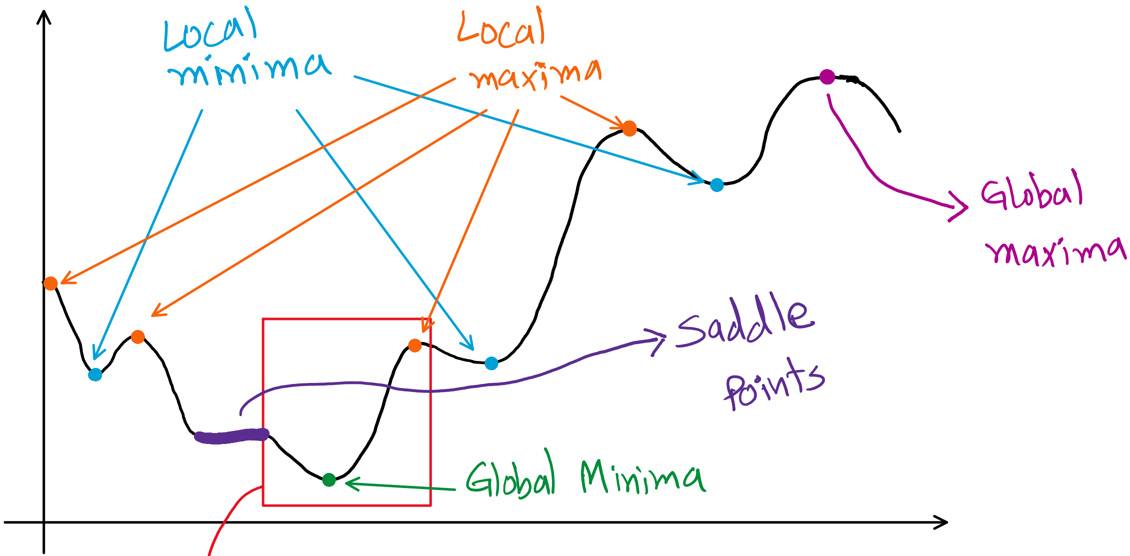
$L = 2(w1)^2 - 5(w1) - 3(w2)^2 + 4(w2) - 12$

Then the graph of this function will be in 3D as below:

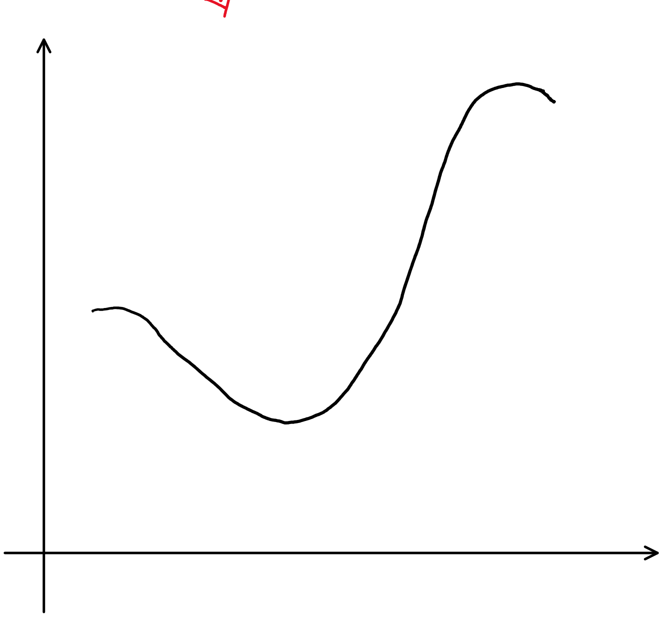


In fact, our loss function will have a lot of features (w_i) (typically minimum 10), it will be impossible to visualize it as we cannot visualize anything beyond 3D. So, to keep it simple to understand how calculus will help us, let's take a loss function only in 2D.

In order to understand how calculus is going to help us, let's first understand some technical jargons about a curve/graph of a function.



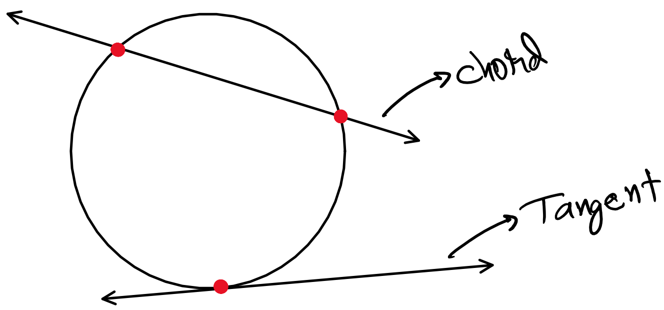
Let's zoomed into the selected portion of the graph of our loss function:



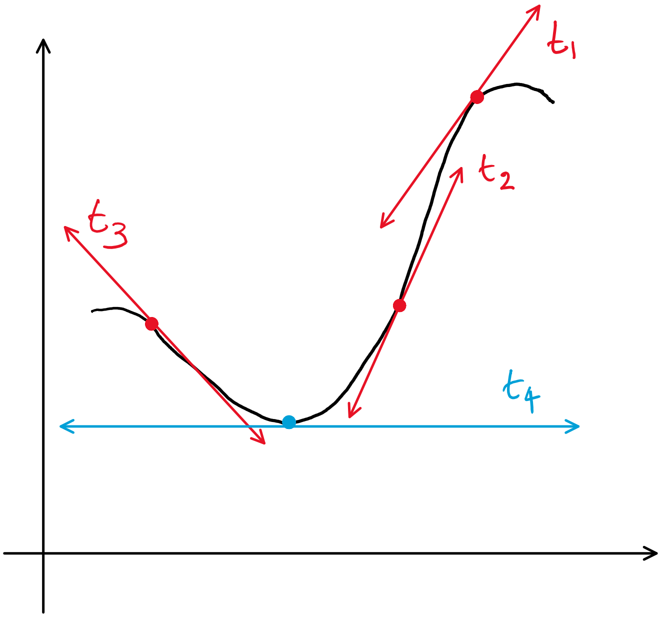
Recap of some concepts of a circle/curve (because a curve can also be considered as part of a circle):

Chord: Any line that intersects the circle in two different points

Tangent: A line that instead of intersecting the circle, just touches it **at one point**



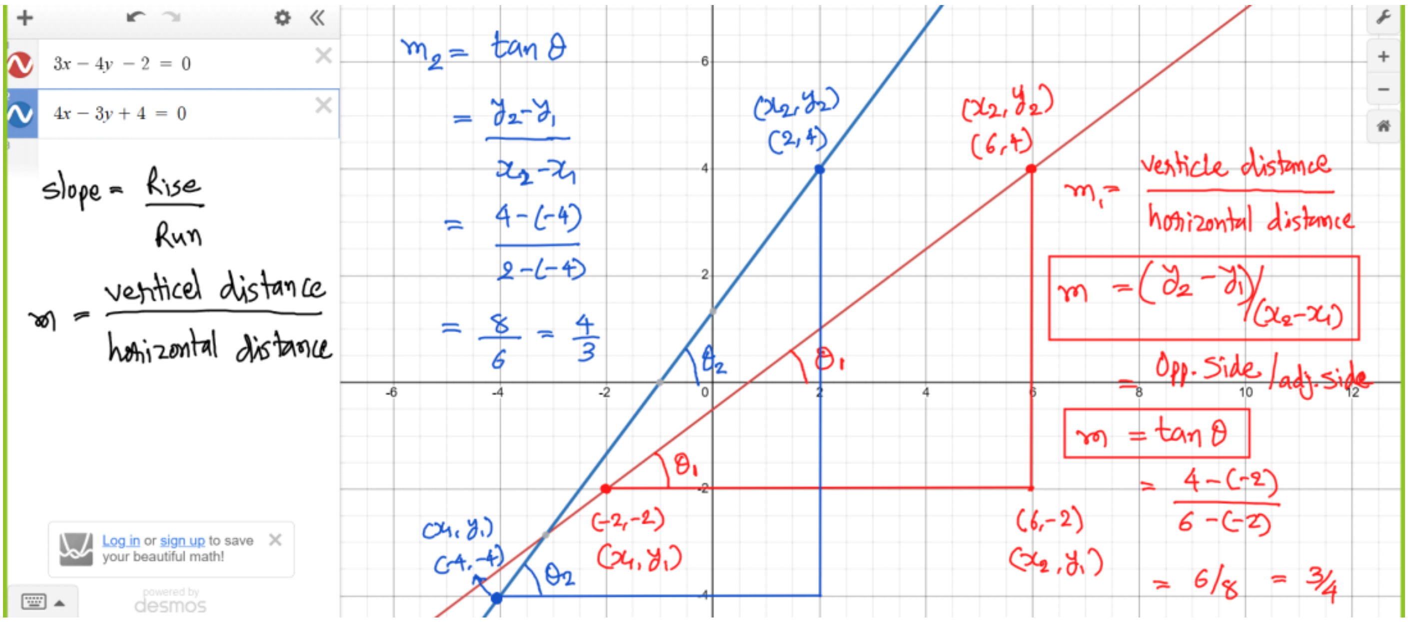
We can also draw tangents at different points of the graph of our loss function as below



As we can see, the slope of t_2 is "steeper"/more than the slope of t_1 and the slope of t_3 is altogether in a different direction!

So if we keep our discussion around this slope of the tangents, we are looking for the tangent whose slope is 0 that is t_4 (that is the minima).

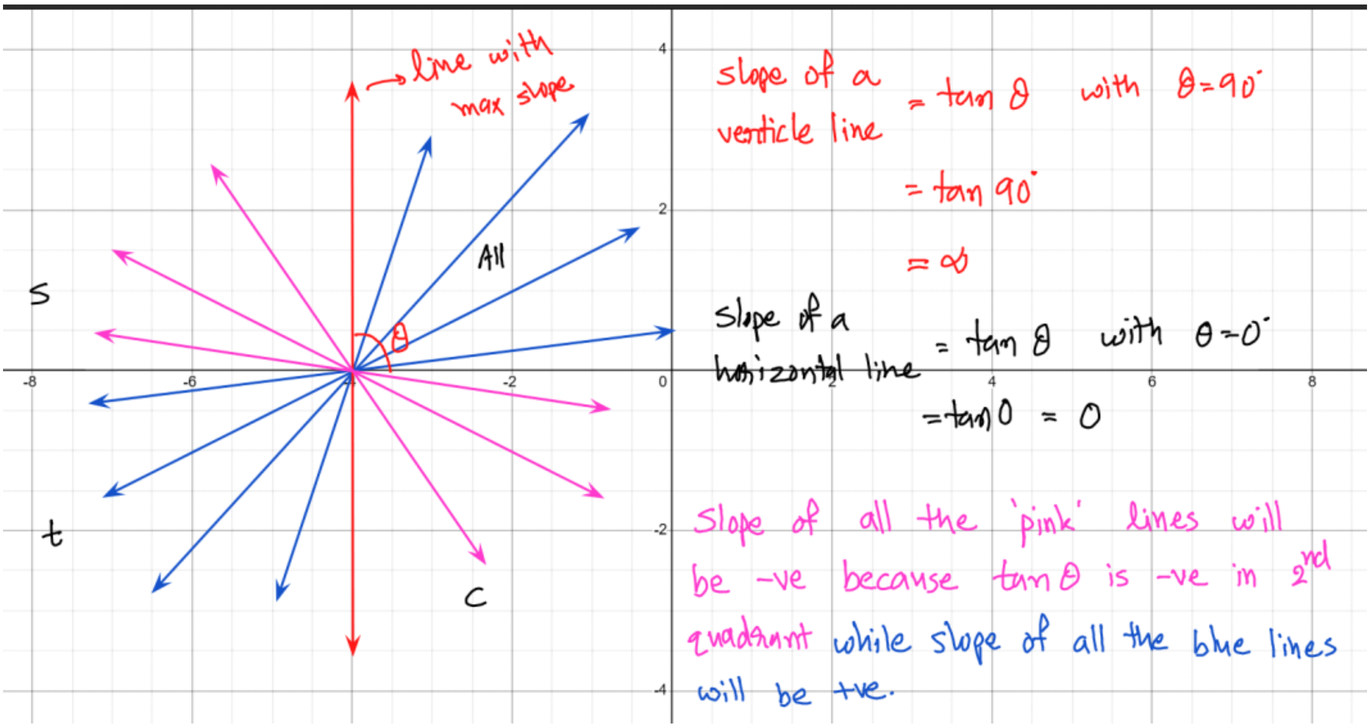
So, it will be a good idea to discuss about the slope of a line.



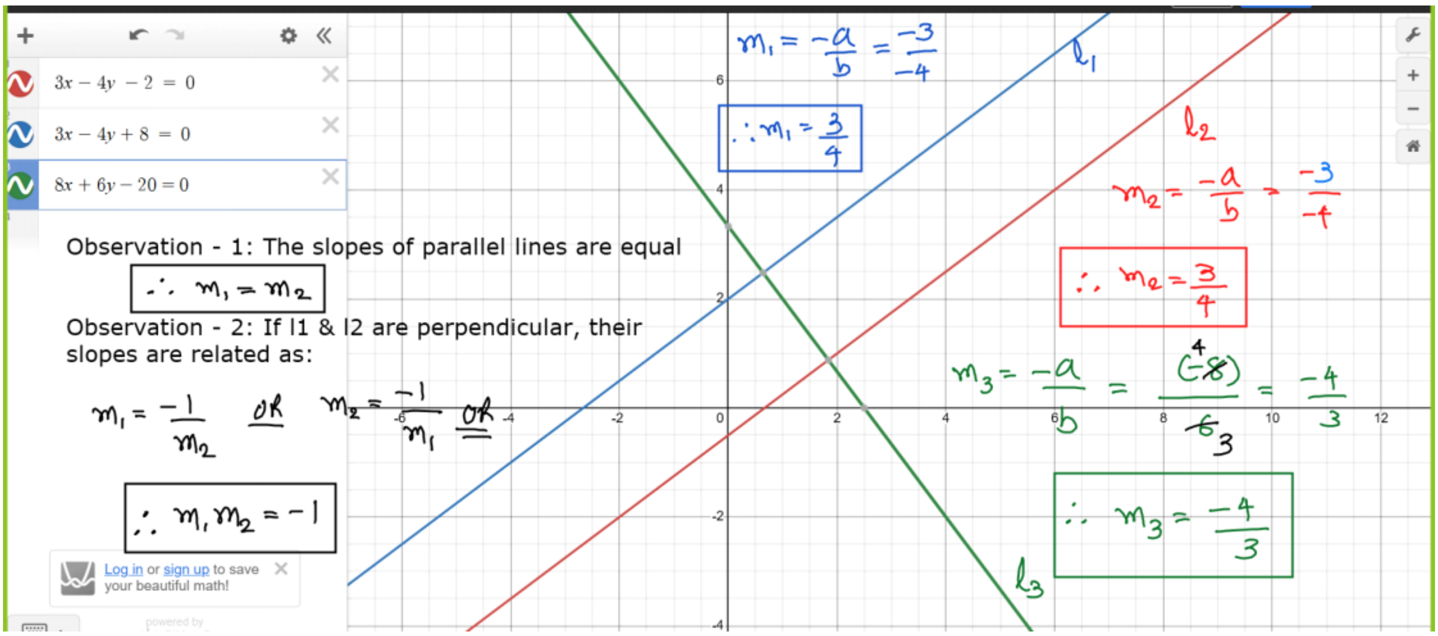
If we compare the equation of "Red" line with
ax + by + c = 0 then
a = 3, b = -4 & c = -2
Therefore, the slope is also given as:

$$\boxed{m = \frac{-a}{b}} = \frac{+3}{+4} = \frac{3}{4}$$

Lines with +ve, -ve, zero & infinite slope:



Relation between slopes of parallel & perpendicular lines:



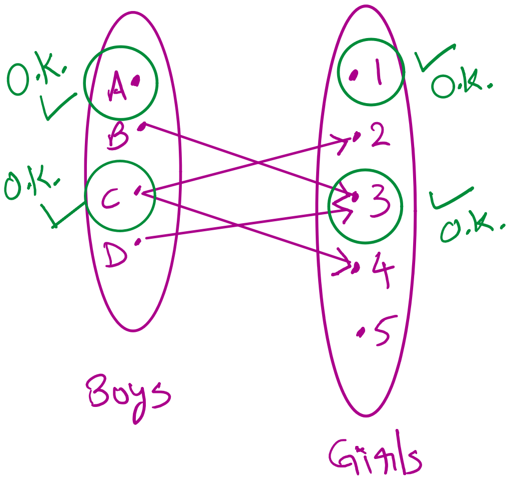
Now we can say that if we find slope of the graph of our loss function at different points, the point at which the slope becomes 0 is either a minima, maxima or a saddle point.

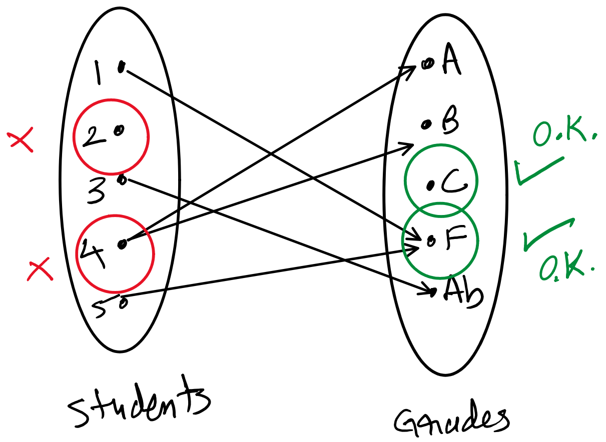
The slope (tangent) of a function at a particular point is also called **differentiation** of that **function** at that point.

Hence, we will first need to learn the concepts of **functions** & **differentiation**.

Functions: A function is a special type of relation

Always consider the following example to determine that a relation is function or not.





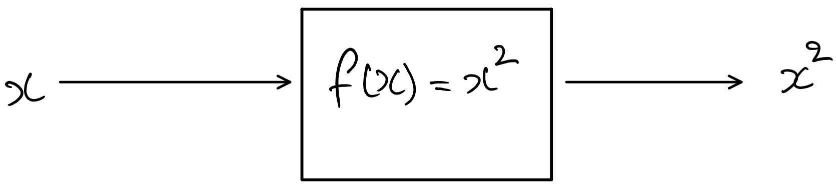
Example of a Relation

Example of a Relation that is Function

The "left" set in a function is called "domain" and the "right" set is called "co-domain". The set of all the elements of co-domain those are corresponded by some element of domain is known as "range" of the function.

If the above example was function then:
domain = {1,2,3,4,5}
codomain = {A, B, C, F, Ab}
range = {A, B, F, Ab}   - Why not "C"? Because it was not corresponded by anyone from the domain.

Example of a function: $y = x^2$ (or $f(x) = x^2$)



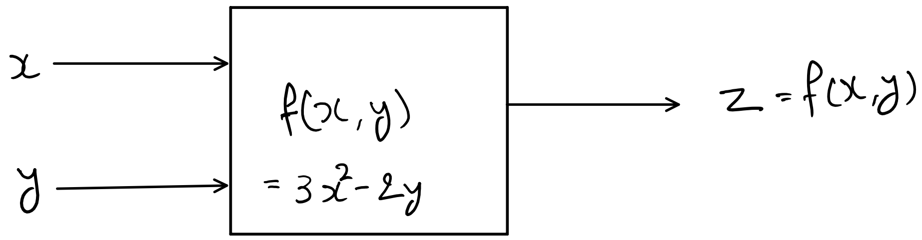
For this function, let's take Z (set of all integers) as the domain. What will be the range of this function?
Ans:

$f(-3) = (-3)^2 = 9$, $f(-2) = (-2)^2 = 4$, $f(0) = 0^2 = 0$, $f(5) = 25$,

Can we have $f(x) = 5$ for any x in Z? No.

Therefore, if we take Z as the domain, the range of this function will be the set of all the whole squares starting from 0. But, if we take R (set of all real numbers) as our domain then what will be the range of $f(x) = x^2$?
Ans: All non-negative real numbers

It is not necessary that a function can always have only one input. It may take multiple inputs as this:

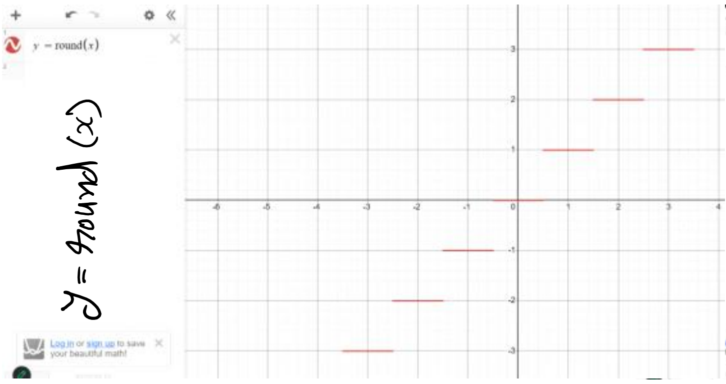


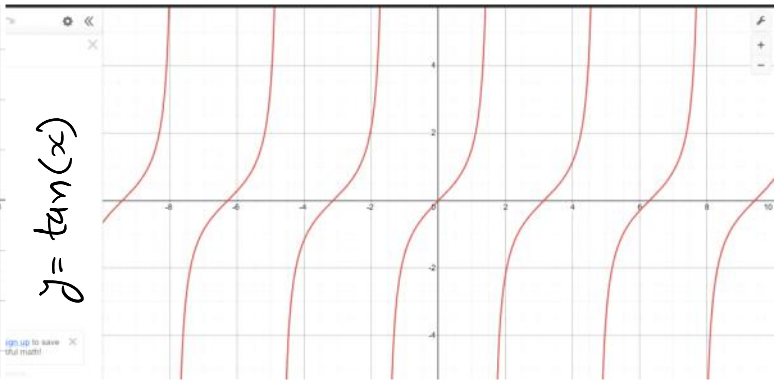
$$f(2,3) = 3(2)^2 - 2(3) = 12 - 6 = 6$$

$$\therefore z = 6$$

Continuous Vs. Non-continuous functions: A function is continuous if we can draw its graph without lifting the pen/pencil

Examples of discontinuous functions:





Understanding Logarithm (log): if $x^3 = 1000$ then $x = ?$

$x = \sqrt[3]{1000} \Rightarrow x = 10$. If $x^n = p \Rightarrow x = \sqrt[n]{p}$

$x^{10} = 2 \Rightarrow x = ? \Rightarrow x = \sqrt[10]{2}$

$10^{x} = 2 \Rightarrow x = ?$ power   base

$\log_{10} 2 = x$ (log of 2 with base 10)

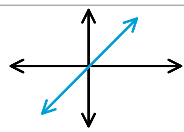
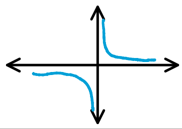
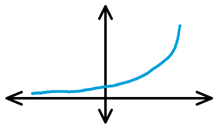
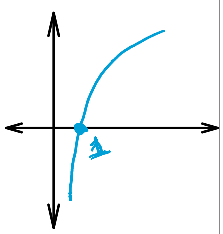
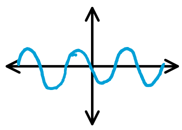
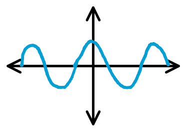
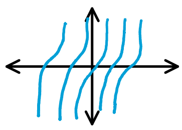
base of log

$\log_{10} 2 = 0.3010 \Rightarrow 10^{0.3010} = 2$

Where do we use log?

X	1	10	100	1000	10,000	1,00,000
Y	13	17	24	27	30	35
$\log_{10}X$	0	1	2	3	4	5

Some useful functions for ML:

function	Domain	Range	Continuous?	Graph of the function
$y = x$	R	R	Yes	
$y = 1/x$	R	R-{0}	No	
$y = e^x$	R	R^+	Yes	
$y = \log_{10}x$ or $y = \log_e x$ (Natural Log - $y = \ln x$)	R^+	R	Yes	
$y = \sin x$	R	[-1, 1]	Yes	
$y = \cos x$	R	[-1, 1]	Yes	
$y = \tan x$	R	$[-\alpha, +\alpha]$	No	
$y = \frac{1}{1+e^{-x}}$ (sigmoid function)	R	(0, 1)	Yes