Homework Set 3, CPSC 8420, Fall 2023

Last Name, First Name

Due 11/17/2023, Friday, 11:59PM EST

Problem 1

Considering soft margin SVM, where we have the objective and constraints as follows:

$$\begin{aligned} min \;\; & \frac{1}{2}||w||_2^2 + C\sum_{i=1}^{m}\xi_i \\ s.t. \;\; & y_i(w^T x_i + b) \geq 1 - \xi_i \;\; (i = 1, 2, ...m) \\ & \xi_i \geq 0 \;\; (i = 1, 2, ...m) \end{aligned} \tag{1}$$

Now we formulate another formulation as:

$$\begin{aligned} min \;\; & \frac{1}{2}||w||_2^2 + \frac{C}{2}\sum_{i=1}^{m}\xi_i^2 \\ s.t. \;\; & y_i(w^T x_i + b) \geq 1 - \xi_i \;\; (i = 1, 2, ...m) \end{aligned} \tag{2}$$

1. Different from Eq. (1), we now drop the non-negative constraint for ξ_i, please show that optimal value of the objective will be the same when ξ_i constraint is removed. The Lagrangian for the object is

$$L = \frac{1}{2}\omega^T\omega + \frac{C}{2}\sum_{i=1}^{m}\xi_i^2 - \sum_{i=1}^{m}\alpha_i[y_i(w^T x_i + b) - 1 + \xi_i]; \alpha_i \geq 0 \tag{3}$$

then the original problem can be converted to

$$\min_{x}\max_{\omega\;\xi\;b} L = \min_{x}\max_{\omega\;\xi\;b}\frac{1}{2}\omega^T\omega + \frac{C}{2}\sum_{i=1}^{m}\xi_i^2 - \sum_{i=1}^{m}\alpha_i[y_i(w^T x_i + b) - 1 + \xi_i] \tag{4}$$

$$= \min_{x}\left(\max_{\omega\;\xi\;b}\left(\frac{1}{2}\omega^T\omega + \frac{C}{2}\sum_{i=1}^{m}\xi_i^2 - \sum_{i=1}^{m}\alpha_i[y_i(w^T x_i + b) - 1]\right) + \max_{\xi}\sum_{i=1}^{m} -\alpha_i\xi_i\right) \tag{5}$$

$$\text{for } \xi_i < 0 \Rightarrow \max_{\xi}\sum_{i=1}^{m} -\alpha_i\xi_i = \infty \Rightarrow \max L = \infty \tag{6}$$

$$\text{for } \xi_i \geq 0 \Rightarrow \max L < \infty \tag{7}$$

Comnine (5), (6) and (7), we can greater

$$\min_{x} \max_{\omega\ \xi\ b} L = \min_{x} \left(\overbrace{\max L}^{\xi_i \geq 0}, \overbrace{+\infty}^{\xi_i < 0} \right) = \min_{x} \max L \tag{8}$$

then it proved, the optimization will drop the part for $\xi_i < 0$ automatically, so the optimal value of the objectives are the same

2. What's the generalized Lagrangian of the new soft margin SVM optimization problem?

$$L = \frac{1}{2}\omega^T\omega + \frac{C}{2}\sum_{i=1}^{m}\xi_i^2 - \sum_{i=1}^{m}\alpha_i[y_i(w^T x_i + b) - 1 + \xi_i]; \alpha_i \geq 0 \tag{9}$$

3. Now please minimize the Lagrangian with respect to w, b, and ξ.

$$\frac{\partial L}{\partial \omega} = 0 \Rightarrow \omega = \sum_{i=1}^{m} \alpha_i y_i x_i \tag{10}$$

$$\frac{\partial L}{\partial b} = 0 \Rightarrow \sum_{i=1}^{m} \alpha_i y_i = 0 \tag{11}$$

$$\frac{\partial L}{\partial \xi} = 0 \Rightarrow \xi_i = \frac{\alpha_i}{C} \tag{12}$$

4. What is the dual of this version soft margin SVM optimization problem? (should be similar to Eq. (10) in the slides)

$$\max_{\alpha} \min_{\omega\ \xi\ b} L(\omega\ \xi\ b\ \alpha) = \tag{13}$$

$$\min_{\alpha} \frac{1}{2}\sum_{i=1}^{m}\sum_{j=1}^{m}\alpha_i\alpha_j y_i y_j x_i^T x_i - \sum_{i=1}^{m}\alpha_i + \frac{1}{2}\sum_{i=1}^{m}\frac{\alpha_i^2}{C} \tag{14}$$

s.t

$$\sum_{i=1}^{m} \alpha_i y_i = 0 \quad , \alpha_i \geq 0$$

Problem 2

Recall vanilla SVM objective:

$$L(w, b, \alpha) = \frac{1}{2}||w||_2^2 - \sum_{i=1}^{m}\alpha_i[y_i(w^T x_i + b) - 1] \quad s.t. \quad \alpha_i \geq 0 \tag{15}$$

If we denote the margin as γ, and vector $\alpha = [\alpha_1, \alpha_2, \ldots, \alpha_m]$, now please show $\gamma^2 * \|\alpha\|_1 = 1$.

Since

$$\frac{\partial L}{\partial \omega} = 0 \Rightarrow \omega = \sum_{i=1}^{m} \alpha_i y_i x_i \tag{16}$$

$$\frac{\partial L}{\partial b} = 0 \Rightarrow \sum_{i=1}^{m} \alpha_i y_i = 0 \tag{17}$$

$$b = y_i - \sum_{j=1}^{m} \alpha_j y_j x_j^T x_i \tag{18}$$

then mutiply $\sum_{i=1}^{m} \alpha_i y_i$ on both sides of (18) then we got

$$\sum_{i=1}^{m} \alpha_i y_i b = \sum_{i=1}^{m} \alpha_i y_i^2 - \sum_{i=1}^{m} \sum_{j=1}^{m} \alpha_i y_i \alpha_j y_j x_j^T x_i \tag{19}$$

$$\text{Since } y_i^2 = 1 \ , \omega = \sum_{i=1}^{m} \alpha_i y_i x_i \ \sum_{i=1}^{m} \alpha_i y_i = 0 \Rightarrow \sum_{i=1}^{m} \alpha_i = ||\omega||^2 \tag{20}$$

$$\text{by definition } ||\alpha||_1 = \sum_{i=1}^{m} \alpha_i \ \ , \gamma = \frac{1}{||\omega||} \Rightarrow \gamma^2 \times ||\alpha||_1 = 1 \tag{21}$$