CHAPTER 2

TWO-DIMENSIONAL TRANSFORMATION

2.1 Introduction

As stated earlier, Computer Aided Design consists of three components, namely, Design (Geometric Modeling), Analysis (FEA, etc), and Visualization (Computer Graphics). Geometric Modeling provides a mathematical description of a geometric object - point, line, conic section, surface, or a solid. Visualization deals with creation of visual effects, e.g., creation of pie charts, contour plots, shading, animation, etc. Computer graphics provides visual displays and manipulations of objects, e.g., transformation, editing, printing, etc. Fortran and visual C languages are used to effect these operations. Transformation is the backbone of computer graphics, enabling us to manipulate the shape, size, and location of the object. It can be used to effect the following changes in a geometric object:

- Change the location
- Change the Shape
- Change the size
- Rotate
- Copy
- Generate a surface from a line
- Generate a solid from a surface
- Animate the object

2.2 Two-Dimensional Transformation

Geometric transformations have numerous applications in geometric modeling, e.g., manipulation of size, shape, and location of an object. In CAD, transformation is also used to generate surfaces and solids by sweeping curves and surfaces, respectively. The term 'sweeping' refers to parametric transformations, which are utilized to generate surfaces and solids. When we sweep a curve, it is transformed through several positions along or around an axis, generating a surface. The appearance of the generated surface depends on the number of instances of the transformation. A parameter t or s is varied from 0 to 1, with the interval value equal to the fraction of the parameter. For example, to generate 10 instances, the parameter will have a value t/10 or s/10. To develop an easier understanding of transformations, we will first study the two-dimensional transformations and then extend it to the study of three-dimensional transformations. Until we get to the discussion of surfaces and solids, we will limit our discussion of transformation to only the simple cases of scaling, translation, rotation, and the combinations of these. Applications of transformations will become apparent when we discuss the surface and solid modeling.

There are two types of transformations:

Modeling Transformation: this transformation alters the coordinate values of the object. Basic operations are scaling, translation, rotation and, combination of one or more of these basic transformations. Examples of these transformations can be easily found in any commercial CAD software. For instance, AutoCAD uses SCALE, MOVE, and ROTATE commands for scaling, translation, and rotation transformations, respectively.

Visual Transformation: In this transformation there is no change in either the geometry or the coordinates of the object. A copy of the object is placed at the desired sight, without changing the coordinate values of the object. In AutoCAD, the ZOOM and PAN commands are good examples of visual transformation.

2.3 Basic Modeling Transformations

There are three basic modeling transformations: Scaling, Translation, and Rotation. Other transformations, which are modification or combination of any of the basic transformations, are Shearing, Mirroring, copy, etc.

Let us look at the procedure for carrying out basic transformations, which are based on matrix operation. A transformation can be expressed as

$$[P^*] = [P]\,[T]$$

where, $[P^*]$ is the new coordinates matrix
$[P]$ is the original coordinates matrix, or points matrix
$[T]$ is the transformation matrix

With the z-terms set to zero, the P matrix can be written as,

$$[P] = \begin{pmatrix} x_1 & y_1 & 0 \\ x_2 & y_2 & 0 \\ x_3 & y_3 & 0 \\ \vdots & \vdots & \vdots \\ x_n & y_n & 0 \end{pmatrix} \qquad (2.1)$$

The size of this matrix depends on the geometry of the object, e.g., a point is defined by a single set of coordinates (x_1, y_1, z_1), a line is defined by two sets of coordinates (x_1, y_1, z_1) and (x_2, y_2, z_2), etc. Thus a point matrix will have the size 1x3, line will be 2x3, etc.

A transformation matrix is always written as a 4x4 matrix, with a basic shape shown below,

$$[T] = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad (2.2)$$

Values of the elements in the matrix will change according to the type of transformation being used, as we will see shortly. The transformation matrix changes the size, position, and orientation of an object, by mathematically adding, or multiplying its coordinate values. We will now discuss the mathematical procedure for scaling, translation, and rotation transformations.

2.4 Scaling

In scaling transformation, the original coordinates of an object are multiplied by the given scale factor. There are two types of scaling transformations: uniform and non-uniform. In the uniform scaling, the coordinate values change uniformly along the x, y, and z coordinates, where as, in non-uniform scaling, the change is not necessarily the same in all the coordinate directions.

2.4.1 Uniform Scaling

For uniform scaling, the scaling transformation matrix is given as

$$[T] = \begin{pmatrix} s & 0 & 0 & 0 \\ 0 & s & 0 & 0 \\ 0 & 0 & s & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad (2.3)$$

Here, s is the scale factor.

2.4.2 Non-Uniform Scaling

Matrix equation of a non-uniform scaling has the form:

$$[T] = \begin{pmatrix} s_x & 0 & 0 & 0 \\ 0 & s_y & 0 & 0 \\ 0 & 0 & s_z & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad (2.4)$$

where, s_x, s_x, s_x are the scale factors for the x, y, and z coordinates of the object.

2.5 Homogeneous Coordinates

Before proceeding further, we should review the concept of homogeneous coordinate system. Since the points matrix has three columns for the x, y, and z values, and a transformation matrix is always 4x4 matrix, the two matrices are incompatible for multiplication. A matrix multiplication is compatible only if the number of columns in the first matrix equals the number of row in the second matrix. For this reason, a points matrix is written as,

$$[P] = \begin{pmatrix} x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \\ \vdots & \vdots & \vdots & \vdots \\ x_n & y_n & z_n & 1 \end{pmatrix} \tag{2.5}$$

Here, we have converted the Cartesian coordinates into homogeneous coordinates by adding a 4th column, with unit value in all rows. When a fourth column, with values of 1 in each row, is added in the points matrix, the matrix multiplication between the [P] and [T] becomes compatible. The values (x_1, y_1, z_1, 1) represent the coordinates of the point (x_1, y_1, z_1), and the coordinates are called as homogeneous coordinates. In homogeneous coordinates, the points (2,3,1), (4,6,2), (6,9,3), (8,12,4), represent the same point (2,3,1), along the plane z = 1, z = 2, z = 3, and z = 4, respectively. In our subsequent discussion on transformation, we will use homogeneous coordinates.

Example 1: If the triangle A(1,1), B(2,1), C(1,3) is scaled by a factor 2, find the new coordinates of the triangle**.**

Solution: Writing the points matrix in homogeneous coordinates, we have

$$[P] = \begin{pmatrix} 1 & 1 & 0 & 1 \\ 2 & 1 & 0 & 1 \\ 1 & 3 & 0 & 1 \end{pmatrix}$$

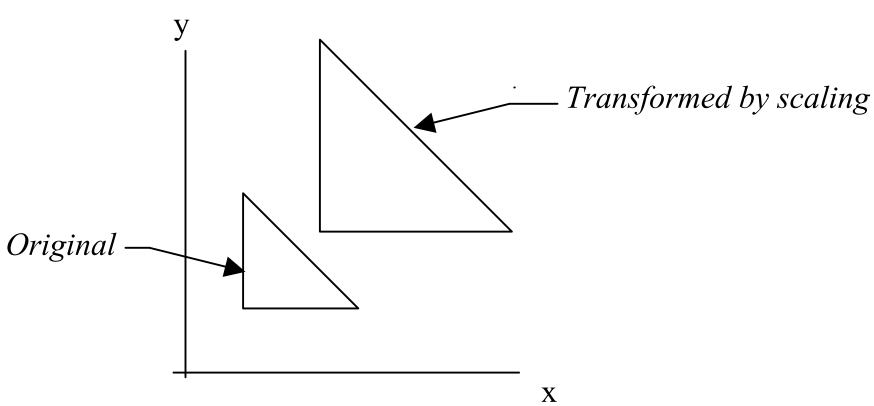
and the scaling transformation matrix is,

$$[T_s] = \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

The new points matrix can be evaluated by the equation

$[P^*] = [P]\,[T]$, and by substitution of the P and T values, we get

$$P^* = \begin{pmatrix} 1 & 1 & 0 & 1 \\ 2 & 1 & 0 & 1 \\ 1 & 3 & 0 & 1 \end{pmatrix} \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 2 & 0 & 1 \\ 4 & 2 & 0 & 1 \\ 2 & 6 & 0 & 1 \end{pmatrix}$$



Note that the new coordinates represent the original value times the scale factor. The old and the new positions of the triangle are shown in the figure.

2.6 Translation Transformation

In translation, every point on an object translates exactly the same distance. The effect of a translation transformation is that the original coordinate values increase or decrease by the amount of the translation along the x, y, and z-axes. For example, if line A(2,4), B(5,6) is translated 2 units along the positive x axis and 3 units along the positive y axis, then the new coordinates of the line would be

A'(2+2, 4+3), B'(5+2, 6+3) or

A'(4,7), B'(7,9).

The transformation matrix has the form:

$$[T_t] = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ x & y & 0 & 1 \end{pmatrix} \qquad (2.6)$$

where, x and y are the values of translation in the x and y direction, respectively. For translation transformation, the matrix equation is

$$[P^*] = [P]\,[T_t] \qquad (2.7)$$

where, $[T_t]$ is the translation transformation matrix.

***Example 2*:** Translate the rectangle (2,2), (2,8), (10,8), (10,2) 2 units along x-axis and 3 units along y-axis.

***Solution*:** Using the matrix equation for translation, we have

$[P^*] = [P]\,[T_t]$, substituting the numbers, we get

$$[P^*] = \begin{pmatrix} 2 & 2 & 0 & 1 \\ 2 & 8 & 0 & 1 \\ 10 & 8 & 0 & 1 \\ 10 & 2 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 2 & 3 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 4 & 5 & 0 & 1 \\ 4 & 11 & 0 & 1 \\ 12 & 11 & 0 & 1 \\ 12 & 5 & 0 & 1 \end{pmatrix}$$

Note that the resultant coordinates are equal to the original x and y values plus the 2 and 3 units added to these values, respectively.

2.7 Rotation

We will first consider rotation about the z-axis, which passes through the origin (0,0,0), since it is the simplest transformation for understanding the rotation transformation. Rotation about an arbitrary axis, other than an axis passing through the origin, requires a combination of three or more transformations, as we will see later.

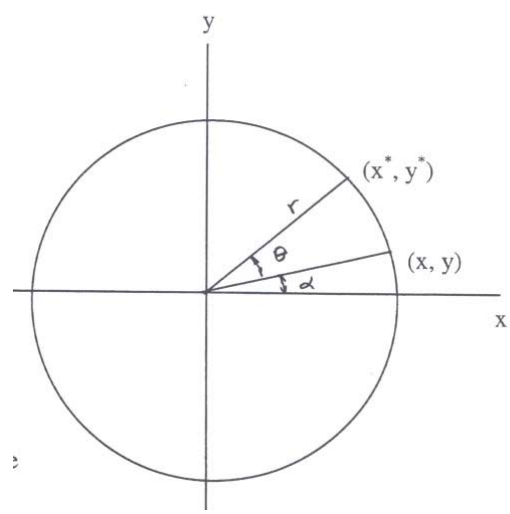
When an object is rotated about the z-axis, all the points on the object rotate in a circular arc, and the center of the arc lies at the origin. Similarly, rotation of an object about an arbitrary axis has the same relationship with the axis, i.e., all the points on the object rotate in a circular arc, and the center of rotation lies at the given point through which the axis is passing.

2.7.1 Derivation of the Rotation Transformation Matrix

Using trigonometric relations, as given below, we can derive the rotation transformation matrix. Let the point P(x, y) be on the circle, located at an angle α, as shown. If the point P is rotated an additional angle θ, the new point will have the coordinates (x^*, y^*). The angle and the original coordinate relationship is found as follows.

$$\left.\begin{aligned} x &= r\cos\alpha \\ y &= r\sin\alpha \end{aligned}\right\} \textit{Original coordinates of point P.}$$

$$\left.\begin{aligned} x^* &= r\cos(\alpha+\theta) \\ y^* &= r\sin(\alpha+\theta) \end{aligned}\right\} \textit{The new coordinates.}$$



where, α is the angle between the line joining the initial position of the point and the x-axis, and θ is the angle between the original and the new position of the point.

Using the trigonometric relations,
we get,

$$x^* = r\,(\cos\alpha\,\cos\theta - \sin\alpha\,\sin\theta) = x\cos\theta - y\sin\theta$$
$$y^* = r\,(\cos\alpha\,\sin\theta + \sin\alpha\,\cos\theta) = x\sin\theta + y\cos\theta$$

In matrix form we can write these equations as

$$[x^* \;\; y^*] = [\,x \quad y\,] \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \qquad (2.8)$$

In general, the points matrix and the transformation matrix given in equation (2.8) are re-written as

$$[x^* \quad y^* \quad 0 \quad 1] = [\,x \quad y \quad 0 \quad 1] \begin{pmatrix} \cos\theta & \sin\theta & 0 & 0 \\ -\sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad (2.9)$$

Thus, a point or any object can be rotated about the z-axis (in 2-D) and the new coordinates of the object found by the product of the points matrix and the rotation matrix, derived here.

2.7.2 Rotation of an Object about an Arbitrary Axis

Rotation of a geometric model about an arbitrary axis, other than any of the coordinate axes, involves several rotational and translation transformations. When we rotate an object about the origin (in 2-D), we in fact rotate it about the z-axis. Every point on the object rotates along a circular path, with the center of rotation at the origin. If we wish to rotate an object about an arbitrary axis, which is perpendicular to the xy-plane, we will have to first translate the axis to the origin and then rotate the model, and finally, translate so that the axis of rotation is restored to its initial position. If we erroneously use the equation (2.9) directly, to rotate the object about a fixed axis, and skip the translation of this point to the origin, we will in fact end up rotating the object about the z-axis, and not about the fixed axis.

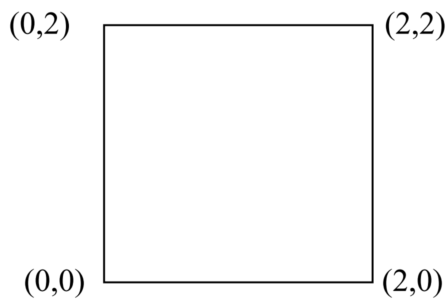
Thus, the rotation of an object about an arbitrary axis, involves three steps:

Step 1: Translate the fixed axis so that it coincides with the z-axis
Step 2: Rotate the object about the axis
Step 3: Translate the fixed axis back to the original position.

Note: When the fixed axis is translated, the object is also translated. The axis and the object go through all the transformations simultaneously.

We will now illustrate the above procedure by the following example.

Example 3: Rotate the rectangle (0,0), (2,0), (2, 2), (0, 2) shown below, 30^0 ccw about its centroid and find the new coordinates of the rectangle.



Solution: Centroid of the rectangle is at point (1, 1). We will first translate the centroid to the origin, then rotate the rectangle, and finally, translate the rectangle so that the centroid is restored to its original position.

1. ***Translate the centroid to the origin***: The matrix equation for this step is

$$[P^*]_1 = [P]\,[T_t], \text{ where } \quad [P] = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 2 & 0 & 0 & 1 \\ 2 & 2 & 0 & 1 \\ 0 & 2 & 0 & 1 \end{pmatrix}$$

and

$$[T_t] = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -1 & -1 & 0 & 1 \end{pmatrix}$$

2. ***Rotate the Rectangle 30^0 ccw About the z-axis*:** The matrix equation for this step is given as

$[P^*]_2 = [P^*]_1 [T_r]$, where, $[P^*]_1$ is the resultant points matrix obtained in step 1, and $[T_r]$ is the rotation transformation, where $\theta = 30^0$ ccw. The transformation matrix is,

$$[T_r]_\theta = \begin{pmatrix} \cos\theta & \sin\theta & 0 & 0 \\ -\sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} .866 & .5 & 0 & 0 \\ -.5 & .866 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

3. ***Translate the Rectangle so that the Centroid Lies at its Original Position*:** The matrix equation for this step is

$[P^*]_3 = [P^*]_2 [T_{-t}]$, where $[T_{-t}]$ is the reverse translation matrix, given as

$$[T_{-t}] = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{pmatrix}$$

Now we can write the entire matrix equation that combines all the three steps outlined above. The equation is,

$$[P^*] = [P] [T_t] [T_r] [T_{-t}]$$

Substituting the values given earlier, we get,

$$[P^*] = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 2 & 0 & 0 & 1 \\ 2 & 2 & 0 & 1 \\ 0 & 2 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -1 & -1 & 0 & 1 \end{pmatrix} \begin{pmatrix} \cos 30^0 & \sin 30^0 & 0 & 0 \\ -\sin 30^0 & \cos 30^0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{pmatrix} \text{x}$$

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 0.6340 & -0.3660 & 0 & 1 \\ -0.3660 & 1.3660 & 0 & 1 \\ 1.3660 & 2.3660 & 0 & 1 \\ -0.3660 & 1.3660 & 0 & 1 \end{pmatrix}$$

The first two columns represent the new coordinates of the rotated rectangle.

2.8 Combined Transformations

Most applications require the use of more than one basic transformation to achieve desired results. As stated earlier, scaling with an arbitrarily fixed point involves both scaling and translation. And rotation around a given point, other than the origin, involves rotation and translation. We will now consider these combined transformations.

2.8.1 Scaling With an Arbitrary Point

In uniform scaling, all points and their coordinates are scaled by a factor s. Therefore, unless the fixed point is located at (0, 0), it will be moved to a new location with coordinates s-times x and s-times y. To scale an object about a fixed point, the fixed point is first moved to the origin and then the object is scaled. Finally, the object is translated or moved so that the fixed point is restored to its original position. The transformation sequence is,

$$[P^*] = [P]\ [T_t]\ [T_s]\ [T_{-t}]$$

Where, $[T_t]$ is the translation transformation matrix, for translation of the fixed point to the origin,

$[T_s]$ is the scaling transformation matrix, and

$[T_{-t}]$ is the reverse translation matrix, to restore the fixed point to its original position.

Note: The order of matrix multiplication progresses from left to right and the order should not be changed.

The three transformation matrices $[T_t]\ [T_s]\ [T_{-t}]$ can be concatenated to produce a single transformation matrix, which uniformly scales an object while keeping the pivot point fixed. Thus, the resultant, concatenated transformation matrix for scaling is,

$$[T_s]_R = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -x & -y & 0 & 1 \end{pmatrix} \begin{pmatrix} s & 0 & 0 & 0 \\ 0 & s & 0 & 0 \\ 0 & 0 & s & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ x & y & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} s & 0 & 0 & 0 \\ 0 & s & 0 & 0 \\ 0 & 0 & s & 0 \\ x\text{-}sx & y\text{-}sy & 0 & 1 \end{pmatrix} \tag{2.10}$$

The concatenated equation can be used directly instead of the step-by-step matrix solution. This form is preferable when writing a CAD program.

Example 4: Given the triangle, described by the homogeneous points matrix below, scale it by a factor 3/4, keeping the centroid in the same location. Use (a) separate matrix operation and (b) condensed matrix for transformation.

$$[P] = \begin{pmatrix} 2 & 2 & 0 & 1 \\ 2 & 5 & 0 & 1 \\ 5 & 5 & 0 & 1 \end{pmatrix}$$

Solution

(a) The centroid of the triangle is at,

$x = (2+2+5)/3 = 3$, and $\quad$ $y = (2+5+5)/3 = 4$ or the centroid is C(3,4).

We will first translate the centroid to the origin, then scale the triangle, and finally translate it back to the centroid. Translation of triangle to the origin will give,

$$[P^*]_1 = [P]\,[T_t] = \begin{pmatrix} 2 & 2 & 0 & 1 \\ 2 & 5 & 0 & 1 \\ 5 & 5 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -3 & -4 & 0 & 1 \end{pmatrix} = \begin{pmatrix} -1 & -2 & 0 & 1 \\ -1 & 1 & 0 & 1 \\ 2 & 1 & 0 & 1 \end{pmatrix}$$

Scaling the triangle, we get,

$$[P^*]_2 = [P^*]_1\,[T_s] = \begin{pmatrix} -1 & -2 & 0 & 1 \\ -1 & 1 & 0 & 1 \\ 2 & 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} .75 & 0 & 0 & 0 \\ 0 & .75 & 0 & 0 \\ 0 & 0 & .75 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} -0.75 & -1.5 & 0 & 1 \\ -0.75 & 0.75 & 0 & 1 \\ 1.5 & 0.75 & 0 & 1 \end{pmatrix}$$

Translating the triangle so that the centroid is positioned at (3, 4), we get

$$[P^*] = [P^*]_2\,[T_{-t}] = \begin{pmatrix} -.75 & -1.5 & 0 & 1 \\ -.75 & .75 & 0 & 1 \\ 1.5 & .75 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 3 & 4 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 2.25 & 2.5 & 0 & 1 \\ 2.25 & 4.75 & 0 & 1 \\ 4.5 & 4.75 & 0 & 1 \end{pmatrix}$$

(b) The foregoing set of three operations can be reduced to a single operation using the condensed matrix with x = 3, and y = 4. See equation (2.10) on page 16.

$$[P^*] = [P]\,[T_{cond}] = \begin{pmatrix} 2 & 2 & 0 & 1 \\ 2 & 5 & 0 & 1 \\ 5 & 5 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0.75 & 0 & 0 & 0 \\ 0 & 0.75 & 0 & 0 \\ 0 & 0 & 0.75 & 0 \\ 3-0.75(3) & 4-0.75(4) & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 2.25 & 2.5 & 0 & 1 \\ 2.25 & 4.75 & 0 & 1 \\ 4.5 & 4.75 & 0 & 1 \end{pmatrix}$$

2.8.2 Rotation About an Arbitrary Point (in xy-plane)

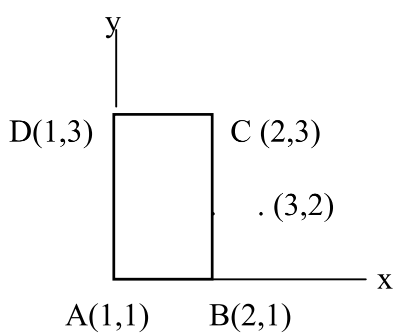
In order to rotate an object about a fixed point, the point is first moved (translated) to the origin. Then, the object is rotated around the origin. Finally, it is translated back so that the fixed point is restored to its original position. For rotation of an object about an arbitrary point, the sequence of the required transformation matrices and the condensed matrix is given as,

$[T_{cond}] = [T_t]\,[T_r]\,[T_{-t}]$ or

$$[T_{cond}] = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -x & -y & 0 & 1 \end{pmatrix} \begin{pmatrix} \cos\theta & \sin\theta & 0 & 0 \\ -\sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ x & y & 0 & 1 \end{pmatrix} \qquad (2.11)$$

where, θ is the angle of rotation and the point (x, y) lies in the xy plane.

Example 5: Rotate the rectangle formed by points A(1,1), B(2,1), C(2,3), and D(1,3) 30^0 ccw about the point (3,2).



Solution: We will first translate the point (3,2) to the origin, then rotate the rectangle about the origin, and finally, translate the rectangle back so that the original point is restores to its original position (3,2). The new coordinates of the rectangle are found as follows.

$$[P^*] = [P]\,[T_t]\,[T_r]\,[T_{-t}]$$

$$= \begin{pmatrix} 1 & 1 & 0 & 1 \\ 2 & 1 & 0 & 1 \\ 2 & 3 & 0 & 1 \\ 1 & 3 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -3 & -2 & 0 & 1 \end{pmatrix} \begin{pmatrix} .866 & .5 & 0 & 0 \\ -.5 & .866 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 3 & 2 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1.77 & .13 & 0 & 1 \\ 0.77 & 1.87 & 0 & 1 \\ 1.63 & 2.37 & 0 & 1 \\ 2.63 & 0.63 & 0 & 1 \end{pmatrix}$$

These are the new coordinates of the rectangle after the rotation.

2.9 Mirroring

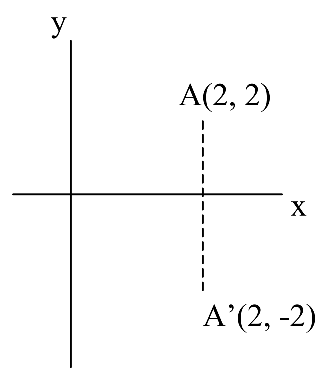
In modeling operations, one frequently used operation is mirroring an object. Mirroring is a convenient method used for copying an object while preserving its features. The mirror transformation is a special case of a negative scaling, as will be explained below.

Let us say, we want to mirror the point A(2,2) about the x-axis(i.e., xz-plane), as shown in the figure.

The new location of the point, when reflected about the x-axis, will be at (2, -2). The point matrix [P*] = [2 -2] can be obtained with the matrix transformation given below.

$$[P^*] = [2 \quad 2 \quad 0 \quad 1] \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$= [2 \quad -2 \quad 0 \quad 1]$$



The transformation matrix above is a special case of a non-uniform scaling with $s_x = 1$ and $s_y = -1$. We can extend this concept to mirroring around the y, z, and any arbitrary axis, as will be explained in the following discussion.

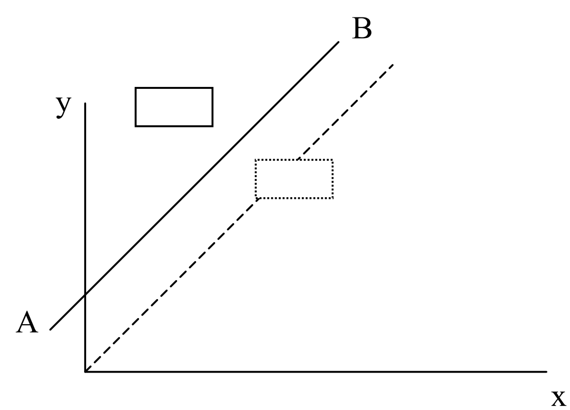
2.9.1 Mirroring About an Arbitrary Plane

If mirroring is required about an arbitrary plane, other than one defined by the coordinate axes, translation and/or rotation can be used to align the given plane with one of the coordinate planes. After mirroring, translation or rotation must be done in reverse order to restore the original geometry of the axis.

We will use the figure shown below, to illustrate the procedure for mirroring an object about an arbitrary plane. We will mirror the given rectangle about a plane passing through the line AB and perpendicular to the xy-plane. It should be noted that in each of the transformations, the plane and the rectangle have a fixed relationship, i.e., when we move the plane (or line AB, the rectangle also moves with it. A step-by-step procedure for mirroring the rectangle about the plane follows.

Note: We are using line AB to represent the plane, which passes through it. Mirroring can be done only about a plane, and not about a line.

Step 1: Translate the line AB (i.e., the plane) such that it passes through the origin, as shown by the dashed line.



Step 2: Next, rotate the line about the origin (or the z-axis) such that it coincides with x or y axes (we will use the x-axis).

Step 3: Mirror the rectangle about the x-axis.

Step 4: Rotate the line back to its original orientation.

Step 5: Translate the line back to its original position.

The new points matrix, in terms of the original points matrix and the five transformation matrices is given as,

$[P^*] = [P]\ [T_t]\ [T_r]\ [T_m]\ [T_{-r}]\ [T_{-t}]$ *(Note: A negative sign is used in the subscripts to indicate a reverse transformation).*

Where, the subscripts t, r, and m represent the translation, rotation, and mirror operations, respectively.