Algorithmic Complexity:

Algorithmic complexity is concerned about how fast or slow particular algorithm performs. We define complexity as a numerical function *T(n)* - time versus the input size *n*. We want to define time taken by an algorithm without depending on the implementation details. But you agree that *T(n)* does depend on the implementation! A given algorithm will take different amounts of time on the same inputs depending on such factors as: processor speed; instruction set, disk speed, brand of compiler and etc. The way around is to estimate efficiency of each algorithm *asymptotically*. We will measure time *T(n)* as the number of elementary "steps" (defined in any way), provided each such step takes constant time.

Let us consider two classical examples: addition of two integers. We will add two integers digit by digit (or bit by bit), and this will define a "step" in our computational model. Therefore, we say that addition of two n-bit integers takes n steps. Consequently, the total computational time is $T(n) = c * n$, where c is time taken by addition of two bits. On different computers, additon of two bits might take different time, say c_1 and c_2, thus the additon of two n-bit integers takes $T(n) = c_1 * n$ and $T(n) = c_2 * n$ respectively. This shows that different machines result in different slopes, but time *T(n)* grows linearly as input size increases.

Asymptotic Notations

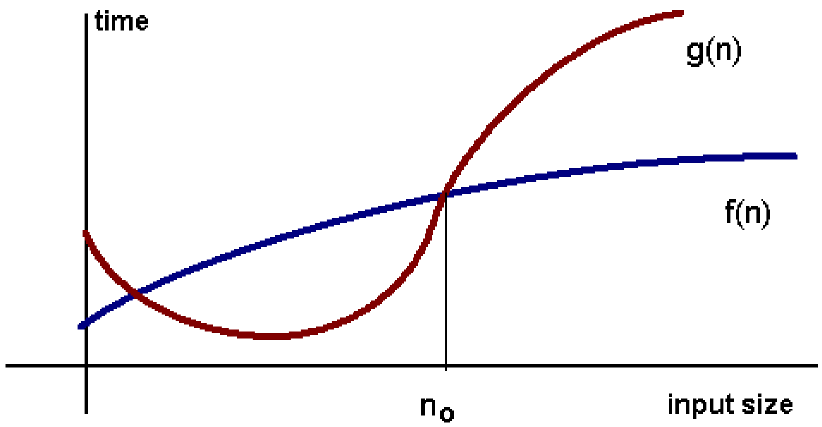
The goal of computational complexity is to classify algorithms according to their performances. We will represent the time function T(n) using the "big-O" notation to express an algorithm runtime complexity. For example, the following statement **$T(n) = O(n^2)$**
says that an algorithm has a quadratic time complexity.

Definition of "big Oh"

For any monotonic functions f(n) and g(n) from the positive integers to the positive integers, we say that $f(n) = O(g(n))$ when there exist constants $c > 0$ and $n_0 > 0$ such that $f(n) \leq c * g(n)$, for all $n \geq n_0$

Intuitively, this means that function f(n) does not grow faster than g(n), or that function g(n) is an **upper bound** for f(n), for all sufficiently large $n \to \infty$

Here is a graphic representation of $f(n) = O(g(n))$ relation:



Exercise. Let us prove $n^2 + 2n + 1 = O(n^2)$. We must find such c and n_0 that $n^2 + 2n + 1 \leq c*n^2$. Let $n_0=1$, then for $n \geq 1$

$$1 + 2n + n^2 \leq n + 2n + n^2 \leq n^2 + 2n^2 + n^2 = 4n^2$$

Therefore, c = 4.

Constant Time: O(1)
An algorithm is said to run in constant time if it requires the same amount of time regardless of the input size. Examples:

- array: accessing any element
- fixed-size stack: push and pop methods
- fixed-size queue: enqueue and dequeue methods

Linear Time: O(n)
An algorithm is said to run in linear time if its time execution is directly proportional to the input size, i.e. time grows linearly as input size increases. Examples:

- array: linear search, traversing, find minimum
- ArrayList: contains method
- queue: contains method

Logarithmic Time: O(log n)
An algorithm is said to run in logarithmic time if its time execution is proportional to the logarithm of the input size. Example:

- binary search

Recall the "twenty questions" game - the task is to guess the value of a hidden number in an interval. Each time you make a guess, you are told whether your guess iss too high or too low. Twenty questions game imploies a strategy that uses your guess number to halve the interval size. This is an example of the general problem-solving method known as **binary search**:

> locate the element a in a sorted (in ascending order) array by first comparing a with the middle element and then (if they are not equal) dividing the array into two subarrays; if a is less than the middle element you repeat the whole procedure in the left subarray, otherwise - in the right subarray. The procedure repeats until a is found or subarray is a zero dimension.

Note, $\log(n) < n$, when $n\to\infty$. Algorithms that run in O(log n) does not use the whole input.

Quadratic Time: $O(n^2)$
An algorithm is said to run in logarithmic time if its time execution is proportional to the square of the input size. Examples:

- bubble sort, selection sort, insertion sort

Definition of "big Omega"
We need the notation for the **lower bound**. A capital omega Ω notation is used in this case. We say that $f(n) = \Omega(g(n))$ when there exist constant c that $f(n) \geq c*g(n)$ for for all sufficiently large n. Examples

- $n = \Omega(1)$
- $n^2 = \Omega(n)$
- $n^2 = \Omega(n \log(n))$
- $2 n + 1 = O(n)$

Definition of "big Theta"
To measure the complexity of a particular algorithm, means to find the upper and lower bounds. A new notation is used in this case. We say that $f(n) = \Theta(g(n))$ if and only $f(n) = O(g(n))$ and $f(n) = \Omega(g(n))$. Examples

- $2 n = \Theta(n)$
- $n^2 + 2 n + 1 = \Theta(n^2)$

Analysis of Algorithms
The term analysis of algorithms is used to describe approaches to the study of the performance of algorithms. In this course we will perform the following types of analysis:

- the *worst-case runtime complexity* of the algorithm is the function defined by the maximum number of steps taken on any instance of size a.
- the *best-case runtime complexity* of the algorithm is the function defined by the minimum number of steps taken on any instance of size a.
- the *average case runtime complexity* of the algorithm is the function defined by an average number of steps taken on any instance of size a.
- the *amortized runtime complexity* of the algorithm is the function defined by a sequence of operations applied to the input of size a and averaged over time.

Example. Let us consider an algorithm of sequential searching in an array.of size n.
Its *worst-case runtime complexity* is O(n)
Its *best-case runtime complexity* is O(1)
Its *average case runtime complexity* is O(n/2)=O(n)