BIFURCATIONAL ANALYSIS IN CHAOTIC SYSTEMS

Project report submitted
as part of summer internship

by

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DECLARATION

I hereby declare that this is an original project report done under Dr Chandrakala Meena from Indian Institute of Science Education and Research(IISER).

Signature of the Student

Date:

The project work reported in the project report entitled **"BIFURCATIONAL ANALYSIS IN CHAOTIC SYSTEMS"** was carried out under my supervision, in the school of physics at IISER, Thiruvananthapuram, India.

Chandrakala

Signature of the project supervisor

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Date:

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ABSTRACT

This project report begins with an overview of the basics of Nonlinear dynamics and finally concludes with bifurcation analysis of a few chaotic system. Nonlinear dynamics is a subset of dynamics which is involved in the study of systems evloving in time. Nonlinearity in these systems lead to interesting and complex behavior which has puzzled scientists for centuries. The birth of this branch of science primarily took place in the 1960's by the seminal work of meteorologist Ed Lorenz. Applications of Nonlinear dynamics and Chaos theory can be found almost everywhere including mechanics, fluid dynamics, meteorology, electronics, ecology, chemical reactions and cosmology. It can be used to analyse and study mechanical systems such as the double pendulum, astrophysical bodies, oscillating chemical reactions and ecological systems. In this report we build up from one dimensional systems and move onto two and three dimensions while analysing properties of these systems such as stability and bifurcation. We will end with the analysis of a few simple three dimensional chaotic systems using an algorithm called Parameter Continuation Method (PCM) to plot their bifurcation diagrams.

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Section 1
Introduction

1.1 Dynamical Systems

Dynamical systems refer to systems or processes whose state evolves in time following a particular differential equation. Here, state refers to a set of properties or configurations that is required to determine any future state of the system.

1.2 Differential equations

There are mainly two types of differential equations:

- ODEs(Ordinary Differential Equations): Dependent variables evolve as a function of a single independent variable, say time.
- PDEs(Partial Differential Equations): Dependent variables evolve as a function of multiple independent variables, say time and position.

1.3 ODE Framework

The general framework of a dynamical system is as follows:

$$\begin{aligned}\dot{x}_1 &= f_1(x_1, \ldots, x_n) \\ &\vdots \\ \dot{x}_n &= f_n(x_1, \ldots, x_n)\end{aligned} \tag{1.1}$$

OR in vector notation it can be represented as:

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, t) \tag{1.2}$$

where $\mathbf{x}$ is the state vector and $\mathbf{f}$ is a vector-valued function that describes the evolution of the system.

1.4 Phase Space

The phase space is a graphical visualization of the dynamical system where the axes represent the dynamic variables of the system. It illustrates the flow of the system over time, given its initial conditions, which are represented as a points in the phase space.

Section 2
First Order Systems

2.1 Fixed Points and Stability

For autonomous systems, fixed points can be thought of as points in the phase space which remain fixed in time in the absence of any perturbation. These are the points where the first time derivative of the dynamic variable is zero.
For a one-dimensional system $\dot{x} = f(x)$, a point x^* is a fixed point when $f(x^*) = 0$. Fixed points tell us how the system will behave long term given an initial condition or a perturbation from an initial condition.

2.1.1 Stability of a Fixed Point

Stable Fixed Point

A fixed point is considered stable if small perturbations from the initial conditions will result in the system approaching back to the same conditions. For a stable fixed point, the first time derivative is negative, indicating that the time derivative function locally decreases.

Unstable Fixed Point

A fixed point is considered unstable if small perturbations from the initial condition will result in the system drifting away from the same condition. For an unstable fixed point, the first time derivative is positive, indicating that the time derivative function locally increases.

Half-Stable Fixed Point

A fixed point is considered half-stable if small perturbations from the initial condition in one direction will result in the system drifting away from the same condition wheras perturbations in the opposite direction will result in system approaching back to the original condition. For a half-stable fixed point, the first time derivative is zero.

2.2 Linear Stability Analysis

The function $f(x)$ represents the time derivative of the variable x:

$$\dot{x} = f(x)$$

Let $\eta(t)$ be a small perturbation from x^*:

$$\eta(t) = x - x^*$$
$$\dot{\eta} = \dot{x} = f(x)$$

Taylor expanding f around x^* and neglecting higher-order terms, we get:

$$f(x) \approx f(x^*) + \eta f'(x^*)$$

Since $f(x^*) = 0$, we have:

$$\dot{\eta} = \eta f'(x^*)$$

Solving for η, we get:

$$\eta = e^{f'(x^*)t}$$

Therefore, a positive time derivative ensures that the perturbation grows exponentially, whereas a negative one leads to exponential decay.

2.3 Potential

The dynamics of a system can be visualized using a potential function $V(x)$, which is defined by the relationship:

$$f(x) = -\frac{dV}{dx}$$

This definition implies that the system's trajectories tend to move towards the valleys of the potential function over time, similar to how a ball would roll down a hill towards the lowest point.

2.4 Logistic Growth

Accounting for the limitation of resources, an approximate representation of population growth is given by the differential equation:

$$\frac{dx}{dt} = rx\left(1 - \frac{x}{K}\right)$$

where r is the growth rate and K is the carrying capacity.

We will perform the stability analysis graphically by plotting the function.

2.5 Population-Time Graph of the Logistic Model

From this graph, it is clear that the population size approaches K as time tends to infinity. Any initial population size close to K immediately decays to K, whereas values close to zero grow rapidly.

The reason it takes an infinite time to reach K is due to the feedback loop between the growth rate and the population size. As x tends to K, the derivative of x tends to zero.

2.6 More on First Order Systems

First order systems don't exhibit oscillations, which means they don't display periodic behavior. In contrast, second order systems can show oscillations, such as in the case of a simple pendulum.

The lack of oscillations in first order systems is due to the uniqueness of solutions, so a point on the real line can only move in one direction. Additionally, due to the properties of fixed points present on the real line, first order systems behave monotonically, either only increasing or only decreasing.

These properties can be observed in the damped simple pendulum. When the damping constant is increased beyond a certain limit, the system reaches an overdamped condition where the second-order term becomes negligible, resulting in first order behavior.

2.7 Numerical Methods of Integration

2.7.1 Euler's Method

Euler's method is used to find approximate solutions to indefinite integrals evaluated at a point. Given an initial condition (t_i, x_i), we take small time steps Δt in the x vs t graph.

Since we know the slope of the graph, we can use the slope-point format to find the next point on our approximate curve at $t_i + \Delta t$:

$$x_{n+1} = x_n + f(x_n)\Delta t$$

It is a first-order numerical method globally, which means after N iterations, the error, that is, the difference between the actual and approximate value, $|x\left(t_i + N\Delta t\right) - x_N|$, is a function of Δt:

$$E \approx O(\Delta t)$$

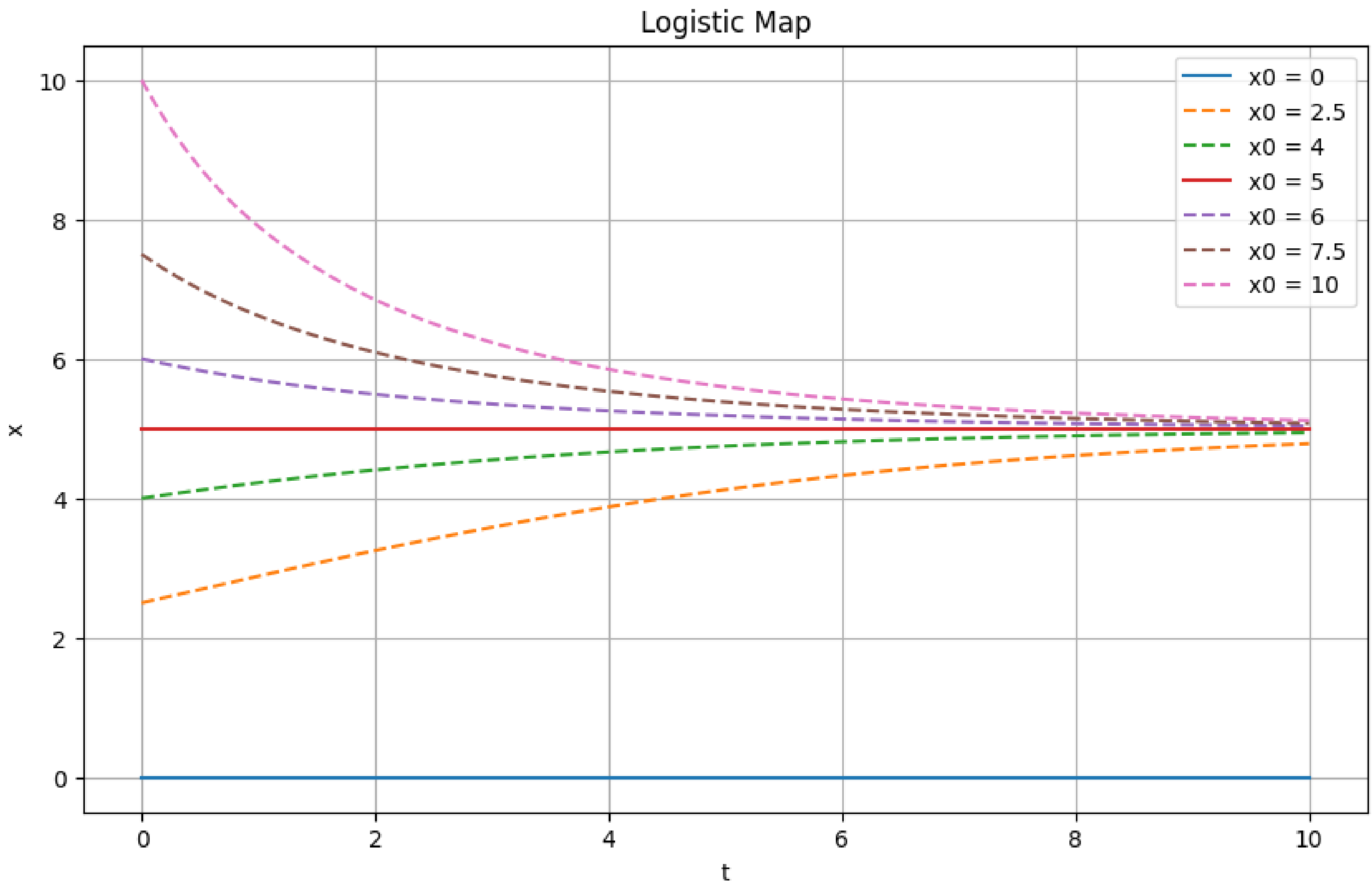


Figure 2.2: Population-Time graph of the logistic model.

2.7.2 Runge-Kutta Method (RK-4)

The Runge-Kutta method (RK-4) is superior to Euler's method because it is of fourth order, meaning the global error,

$$E \approx O(\Delta t^4)$$

This is due to the fact that each iteration takes a weighted average of four different slopes:

$$y_{n+1} = y_n + \frac{\Delta t}{6}\left(k_1 + 2k_2 + 2k_3 + k_4\right)$$

Section 3
Bifurcations

3.1 Introduction to Bifurcations

Bifurcations refer to the changes in the stability of dynamic variables as we vary a certain parameter. The parameter can refer to any constants in the differential equation.

3.1.1 Types of Bifurcations

Saddle-Node Bifurcation

Two fixed points move towards each other and annihilate.

Normal form:

$$\dot{x} = r + x^2$$

Transcritical Bifurcation

Fixed points come together and exchange stability.

Normal form:

$$\dot{x} = rx - x^2$$

Pitchfork Bifurcation

A fixed point trifurcates. This bifurcation demands symmetry.

Normal form:

$$\dot{x} = rx - x^3$$

3.2 Saddle-Node Bifurcation

Let's plot the normal form of such a system: $\dot{x} = f(x) = r + x^2$.

We can see that when $r > 0$, there are no stable fixed points. At $r = 0$, we have a half-stable fixed point. When $r < 0$, we have a stable and an unstable point.

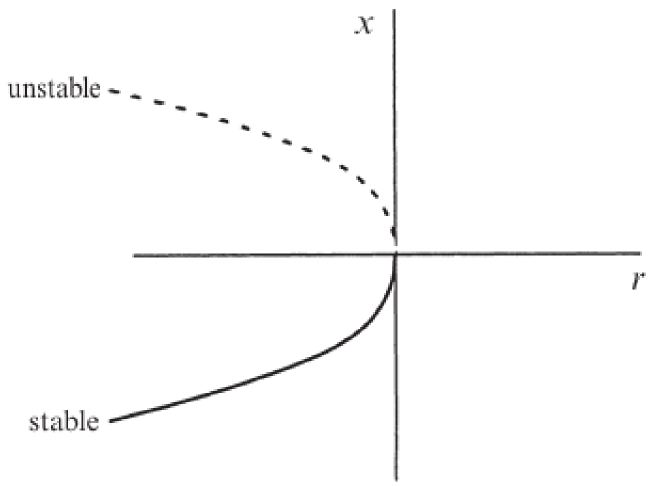
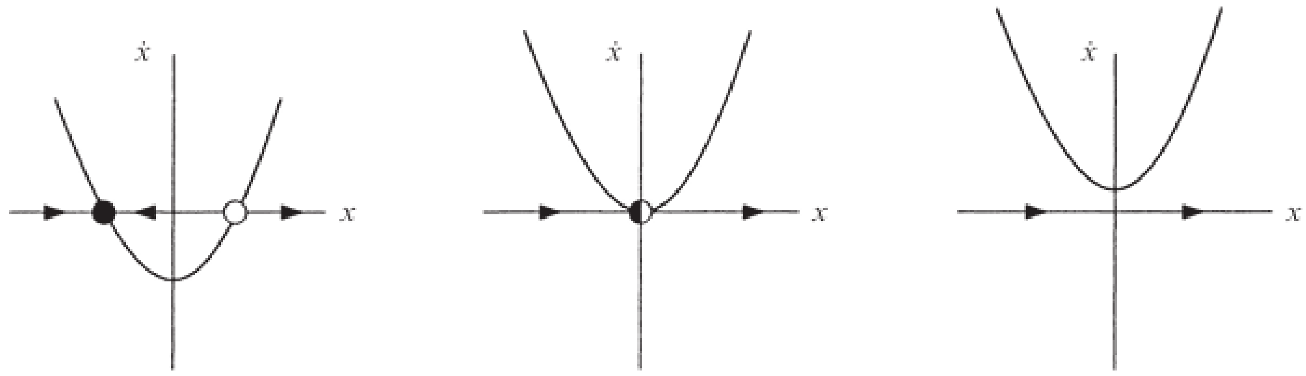


Figure 3.1: Bifurcation Diagram of the Saddle-node bifurcation.[1]

Here $r = 0$ is the bifurcation point and the dashed line represents the unstable fixed point, while the solid line shows the stable ones.



(a) $r<0$ (b) $r=0$ (c) $r>0$

Figure 3.2: Saddle node bifurcation.[1]

3.3 Transcritical Bifurcation

Let's plot the normal form of such a system: $\dot{x} = f(x) = rx - x^2$.

We can see that when $r > 0$, we have a stable point at the origin and another unstable point to its left . At $r = 0$, we have a half-stable fixed point. When $r < 0$, we have an unstable point at origin and a stable point to its right.

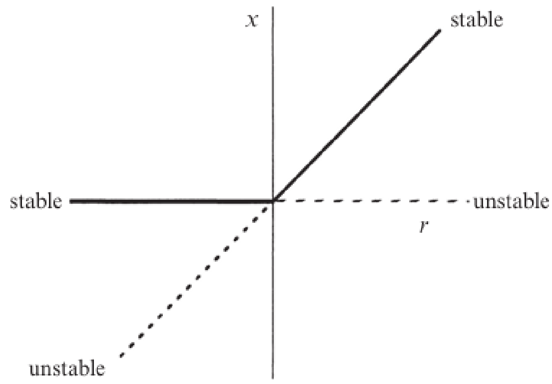
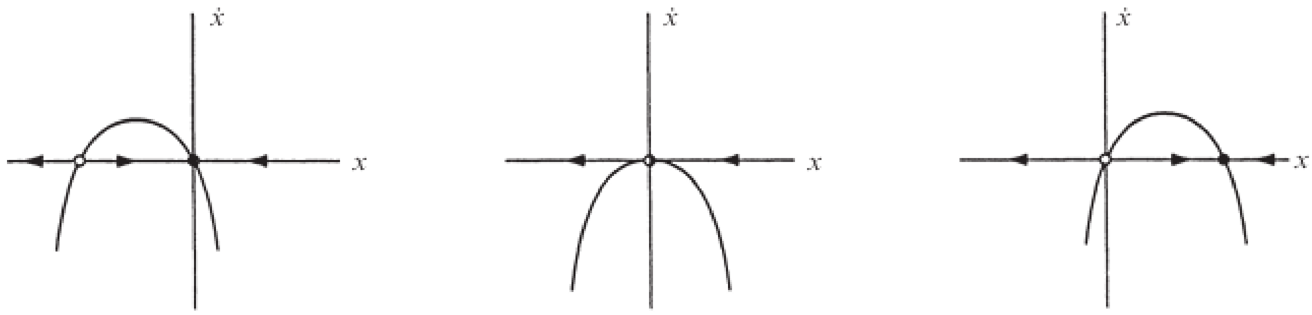


Figure 3.3: Bifurcation Diagram of the Transcritical Bifurcation.[1]

Here at the origin, fixed points exchange stabilities, which is typical of a Transcritical Bifurcation.



(a) $r<0$ (b) $r=0$ (c) $r>0$

Figure 3.4: Transcritical bifurcation.[1]

3.4 Pitchfork Bifurcation

Let's plot the normal form of such a system:

$$\dot{x} = f(x) = rx - x^3$$

As r crosses $x = 0$ we can see that the stable fixed point at origin changes to an unstable fixed point while 2 new stable fixed point emerges.

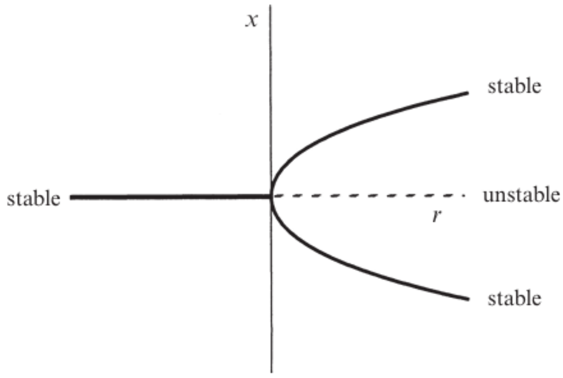


Figure 3.5: Bifurcation Diagram of the Pitchfork Bifurcation.[1]

The bifurcaiton diagram resembles a pitchfork hence the name.

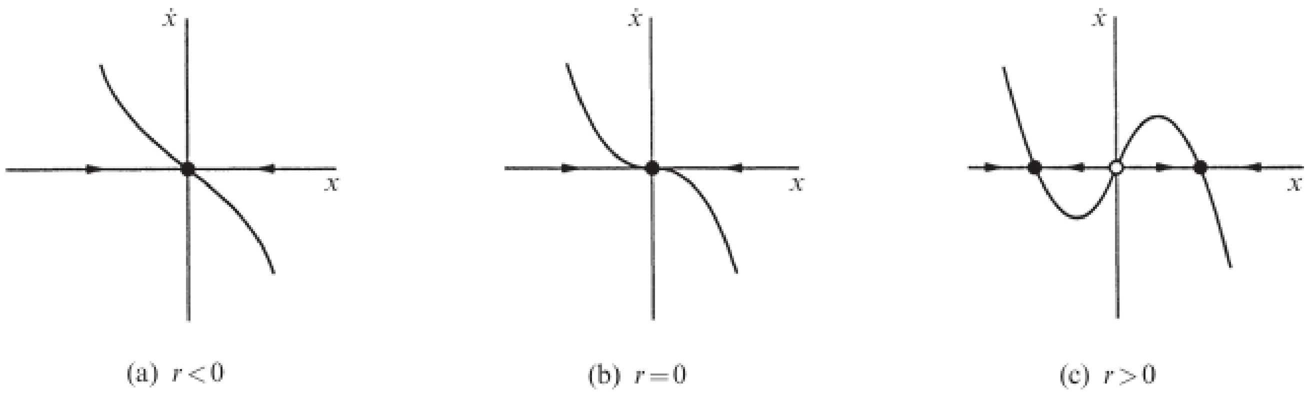


Figure 3.6: Pitchfork bifurcation.[1]

Section 4
Flows on the Circle

4.1 Introduction to Flows on the Circle

Functions which are periodic can be visualised using circular phase space. It provides a better visual representation of one dimensional first order systems which is periodic which can be described by some kind of oscillatory nature, like the overdamped pendulum.

- If the velocity of phase flow is a constant and is of the form $\dot{\theta} = \omega$ it is called a Uniform oscillator.
- If the velocity of phase flow is a function of θ such as $\dot{\theta} = \omega - a \sin \theta$ it is called a Nonuniform oscillator.

4.2 Phase Diagram Analysis

In the phase diagram of the system $\dot{\theta} = \sin \theta$, we observe the following key features:

- The fixed point at $\theta = 0$ is unstable.
- The fixed point at $\theta = \pi$ is stable.
- The phase flow attains maximum velocity at $\theta = \frac{\pi}{2}$.

4.3 Bottlenecks and Ghosts

In the study of phase space dynamics, a *bottleneck* refers to a region where the phase flow slows down significantly, approaching very small velocities before accelerating again. This can create a temporary "fixed point-like" behavior, even though the region is not an actual fixed point. These regions can be critical in understanding the system's behavior during transitions.

A *ghost* is a remnant of a fixed point observed at a bottleneck after a bifurcation has occurred. Despite not being an actual fixed point, the phase flow's near-zero velocity at the bottleneck gives the impression of a fixed point, hence the term "ghost."

4.3.1 Time Spent During Bottlenecks

The time spent by the system during a bottleneck can be approximated by the total time taken to move through phase space. This can be expressed as:

$$T = \int_{-\infty}^{+\infty} dt = \int_{-\infty}^{+\infty} \frac{1}{f(x)} \, dx$$

where $f(x)$ represents the velocity function of the phase flow. For a saddle-node bifurcation, the velocity function can be approximated by a parabolic form:

$$f(x) = r - x^2$$

Thus, the time spent during the bottleneck can be approximated as:

$$T = \int_{-\infty}^{+\infty} \frac{1}{r - x^2} \, dx = \frac{\pi}{\sqrt{r}}$$

Section 5
Two-Dimensional Linear Flows

5.1 General Framework

Consider a two-dimensional linear dynamical system with dependent variables x and y. The system of differential equations can be expressed as:

$$\dot{x} = f(x, y) = ax + by$$

$$\dot{y} = g(x, y) = cx + dy$$

This system can be compactly represented using linear algebra as:

$$\dot{\mathbf{x}} = A\mathbf{x}$$

where $\dot{\mathbf{x}} = \begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix}$ is the vector of time derivatives of the dependent variables, $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is the matrix of coefficients, and $\mathbf{x} = \begin{pmatrix} x \\ y \end{pmatrix}$ is the vector of dependent variables.

This matrix form simplifies the analysis of the system, enabling the use of linear algebra techniques to study the system's properties, such as eigenvalues and eigenvectors, and to understand the qualitative behavior of the system.

5.2 Analysis using Linear Algebra

Let's assume the function of the time-dependent variables such that straight-line solutions exist

$$\mathbf{x} = \mathbf{v}e^{\lambda t}$$

Differentiating with respect to time and substituting gives

$$\dot{\mathbf{x}} = \lambda \mathbf{v} e^{\lambda t}$$

$$A\mathbf{x} = \lambda \mathbf{x}$$

Where $\mathbf{v}$ and λ are the eigenvector and eigenvalue corresponding to A.

The solutions are given by the characteristic equation

$$\lambda_1, \lambda_2 = \frac{\tau \pm \sqrt{\tau^2 - 4\Delta}}{2}$$

The corresponding eigenvectors $\mathbf{v_1}, \mathbf{v_2}$ can be found from the previous equation.

5.3 Analysis of Analytical Solutions

If λ_1, λ_2 are distinct and real, we get $\mathbf{v_1}, \mathbf{v_2}$ linearly independent eigenvectors which span the entire space.

Given initial condition $\begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$, we can represent it using the new eigenvectors as $\begin{pmatrix} x_0 \\ y_0 \end{pmatrix} = c_1\mathbf{v_1} + c_2\mathbf{v_2}$.

Now the solution for our function according to the above initial condition can be written as the following using the superposition principle:

$$\mathbf{x} = c_1\mathbf{v_1}e^{\lambda_1 t} + c_2\mathbf{v_2}e^{\lambda_2 t}$$

From this, we can see that if the initial position is on one of the eigenvectors, the other vector's coefficient goes to zero. This implies the trajectory will be a straight line as it has a constant slope.

5.4 Classification of Linear Systems

We can classify systems based on just the trace and determinant of the transformation matrix A.

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

where trace $\tau = a + d$ and Δ is the determinant of A.

The curve plotted is the equation of the discriminant of the characteristic equation $\tau^2 - 4\Delta = 0$.

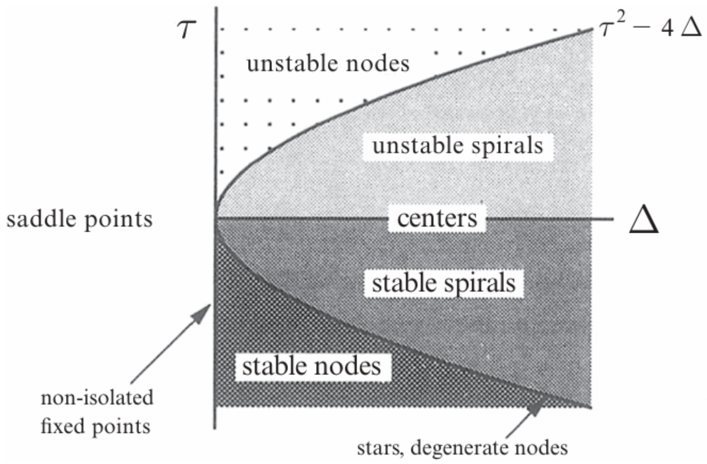


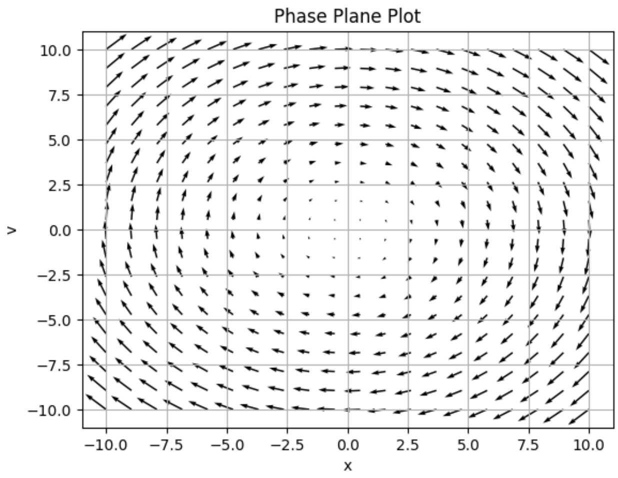
Figure 5.1: Classification of linear systems.[1]

5.5 Centers, Stable and Unstable Spirals

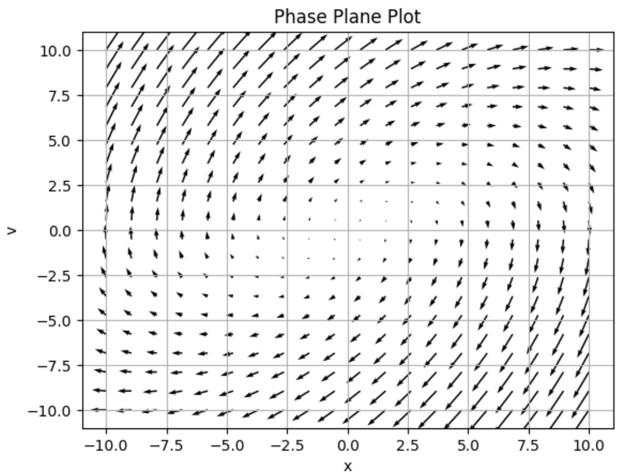
When $\tau^2 - 4\Delta < 0$ no real solutions exist for eigenvalues, hence there are no straight-line trajectories.

The complex solutions are of the form $\lambda_1, \lambda_2 = \alpha \pm \omega i$ where $\alpha = \frac{\tau}{2}$ and $\omega = \sqrt{4\Delta - \tau^2}$.

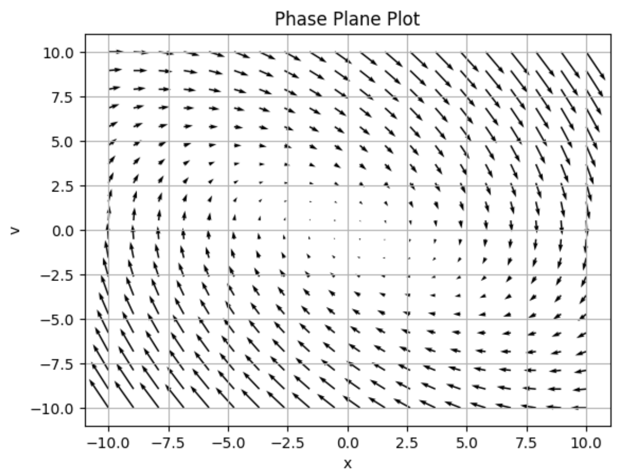
The solutions for x, y look like $e^{\alpha t}\cos(\omega t)$.



(a) Center



(b) Unstable spiral



(c) Stable spiral

Figure 5.2: Different types of behavior for $\tau^2 - 4\Delta < 0$.

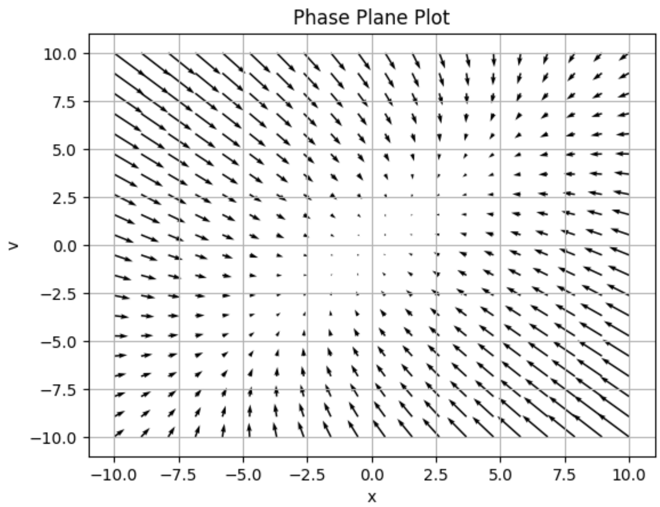
- If $\tau = 0$, $\alpha = 0$ and the function is truly periodic and looks like ellipses 5.2a. These are called centers. An example is the harmonic oscillator with $A = \begin{bmatrix} 0 & 1 \\ -\omega^2 & 0 \end{bmatrix}$.

- If $\tau > 0$, the function grows with an exponential envelope and these are called unstable spirals 5.2b. An example is $A = \begin{bmatrix} 0 & 1 \\ -\omega^2 & 1 \end{bmatrix}$.

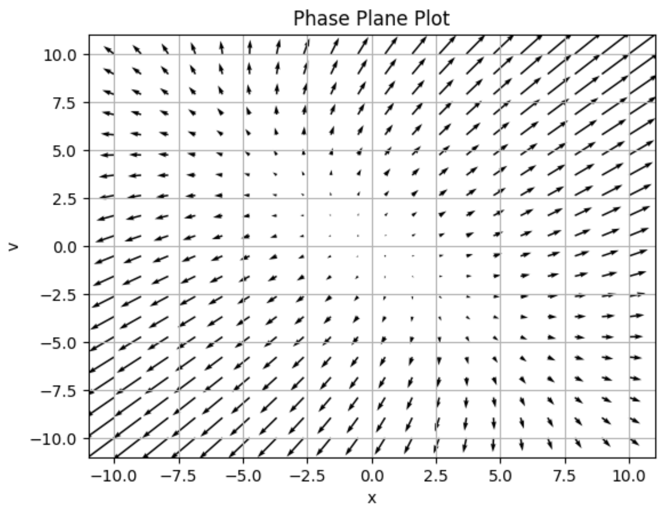
- If $\tau < 0$, the function decays with an exponential envelope and these are called stable spirals 5.2c. An example is $A = \begin{bmatrix} 0 & 1 \\ -\omega^2 & -1 \end{bmatrix}$ which represents a damped oscillator with unit damping constant.

5.6 Stable and Unstable Nodes

When $\tau^2 - 4\Delta > 0$ and $\Delta > 0$, we get two real solutions for eigenvalues; hence, there are two straight-line trajectories.



(a) Stable node



(b) Unstable node

Figure 5.3: Stable and Unstable Nodes

- When $\tau > 0$, both eigenvalues are positive, forming unstable node 5.3b

- When $\tau < 0$, both eigenvalues are negative, forming stable nodes 5.3a .

5.7 Saddle Nodes

When $\tau^2 - 4\Delta > 0$ and $\Delta < 0$, we get one positive and one negative eigenvalue; hence, there are two straight-line trajectories. The one with the negative eigenvalue is called the stable manifold, and the one with the positive eigenvalue is called the unstable manifold.

For example:

$$A = \begin{bmatrix} -1 & 0 \\ 0 & 2 \end{bmatrix}$$

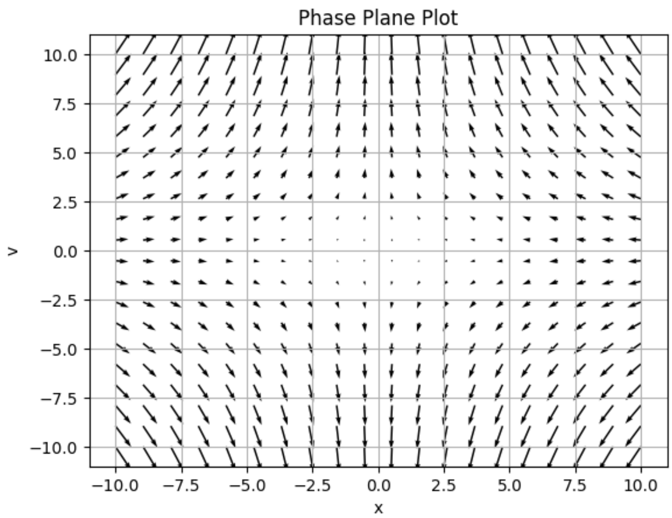


Figure 5.4: Saddle Node

5.8 Degenerate Nodes and Stars

If the discriminant is 0, then the only solutions are either $\frac{\tau}{2}$ or the entire space. One will correspond to a single straight-line trajectory, and the other corresponds to only straight-line trajectories.

5.8.1 Degenerate Node

An example is:

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

where the eigenvalue is 1 and the eigenvector $\mathbf{v} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

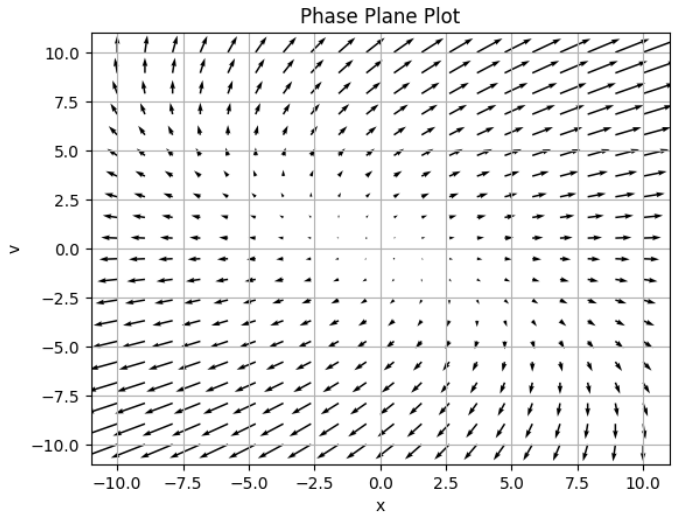


Figure 5.5: Degenerate Node

5.8.2 Star

When the transformation matrix is a scalar multiple of the identity matrix, all the vectors are eigenvectors with the scalar as the eigenvalue.

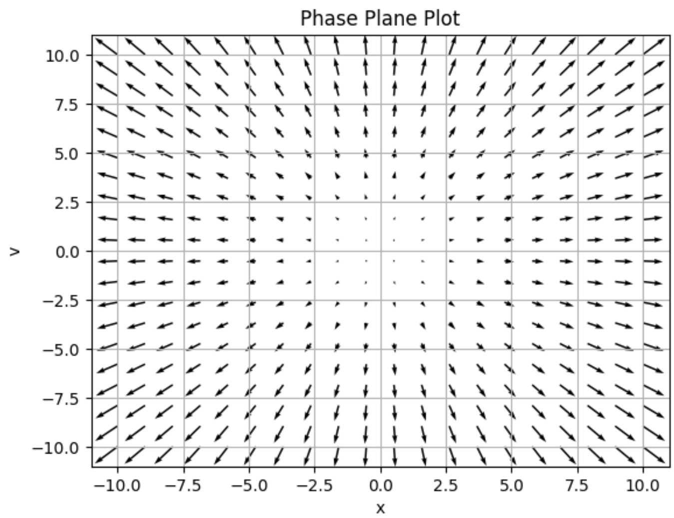


Figure 5.6: Star

5.9 Non-isolated Fixed Points

When $A \neq 0$ and $\Delta = 0$, the two eigenvalues are 0 and τ, which correspond to an entire line of fixed points and a straight-line trajectory. For example when $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ The eigenvector corresponding to 0 is $\mathbf{v} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$ and the one corresponding to $\tau = 2$ is $\mathbf{v} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

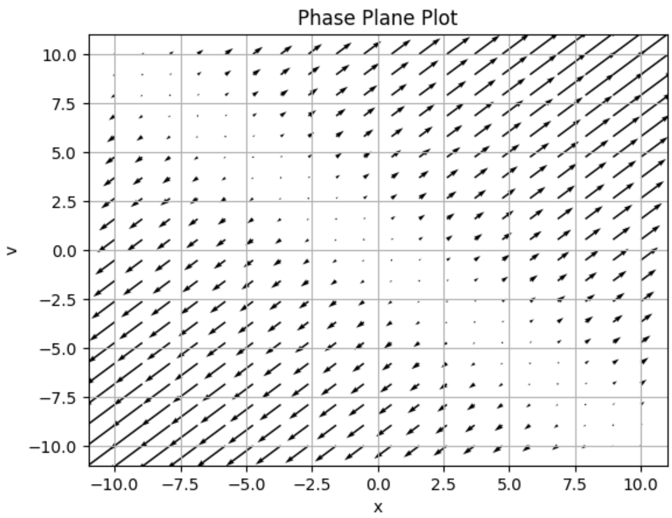


Figure 5.7: Unstable Line

When $A = 0$, all the points in the phase plane are fixed points.

Section 6
2D Nonlinear Flows

6.1 Introduction

Nonlinear 2-D systems are analytically hard to solve, so numerical methods come in handy.

According to our general framework, we have

$$\dot{x} = f(x, y)\dot{y} = g(x, y)$$

where at least one function contains nonlinear terms.

This prevents us from assigning a transformation matrix, A.

But we can get a linear approximation for A, which is called the Jacobian matrix of the transformation.

The Jacobian matrix is given by:

$$A = \begin{bmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} \end{bmatrix}$$

6.2 Derivation

Let $u(t)$ and $v(t)$ be perturbations from the stable fixed point $\begin{pmatrix} x^* \\ y^* \end{pmatrix}$ in the x and y directions, respectively. We express x and y in terms of these perturbations as follows:

$$x = u(t) + x^*$$

$$y = v(t) + y^*$$

Performing a Taylor series expansion of the functions $f(x, y)$ and $g(x, y)$ about the fixed point, and ignoring higher-order terms, we get:

$$\begin{aligned} f(x, y) &= f(x^* + u, y^* + v) \\ &= f(x^*, y^*) + u\frac{\partial f}{\partial x}(x^*, y^*) + v\frac{\partial f}{\partial y}(x^*, y^*) \end{aligned}$$

$$\begin{aligned} g(x, y) &= g(x^* + u, y^* + v) \\ &= g(x^*, y^*) + u\frac{\partial g}{\partial x}(x^*, y^*) + v\frac{\partial g}{\partial y}(x^*, y^*) \end{aligned}$$

Since the velocities are zero at the fixed points ($f(x^*, y^*) = 0$ and $g(x^*, y^*) = 0$), the equations simplify to:

$$f(x, y) = u\frac{\partial f}{\partial x}(x^*, y^*) + v\frac{\partial f}{\partial y}(x^*, y^*)$$

$$g(x, y) = u\frac{\partial g}{\partial x}(x^*, y^*) + v\frac{\partial g}{\partial y}(x^*, y^*)$$

In vector form, these can be written as:

$$\mathbf{f}(\mathbf{x}) = A\mathbf{x}$$

where

$$\mathbf{f}(\mathbf{x}) = \begin{pmatrix} f(x) \\ g(y) \end{pmatrix}$$

$$\mathbf{x} = \begin{pmatrix} x - x^* \\ y - y^* \end{pmatrix}$$

and

$$A = \begin{bmatrix} \frac{\partial f}{\partial x}(x^*, y^*) & \frac{\partial f}{\partial y}(x^*, y^*) \\ \frac{\partial g}{\partial x}(x^*, y^*) & \frac{\partial g}{\partial y}(x^*, y^*) \end{bmatrix}$$

is the Jacobian matrix evaluated at the fixed point. This linear approximation facilitates the analysis of the system's behavior near the fixed point.

6.3 Reliability of Linearization

Linear approximations of dynamical systems are generally reliable near fixed points that are not near borderline cases. Borderline cases include fixed points that lie on the boundaries of stability regions in phase space, such as centers, degenerate nodes, and stars. These cases are more complex and less amenable to straightforward linear analysis.

Fixed points can be classified based on their sensitivity to perturbations into two main categories:

- **Marginal Cases**: These include centers and non-isolated fixed points. Marginal cases are characterized by their susceptibility to changes in stability upon perturbation. The behavior of these fixed points can be significantly altered by small changes in system parameters, making the linear approximation less reliable near these points.

- **Robust Cases**: These fixed points are less sensitive to perturbations and maintain their stability characteristics under small changes in parameters. Stars are an example of robust but borderline cases. They are positioned between stable regions and are often referred to as hyperbolic points. For these points, the real parts of all eigenvalues are non-zero ($\text{Re}(\lambda) \neq 0$), indicating that they do not change stability upon small perturbations.

6.4 Conservative Systems

A dynamical system is considered conservative if there exists a conserved quantity $E(\mathbf{x})$ whose time derivative is zero along the trajectory of the system. This implies that $E(\mathbf{x})$ remains constant for a given trajectory, which means that $E(\mathbf{x})$ cannot be a trivial constant to avoid degenerate cases.

For systems described by equations of the form:

$$m\ddot{x} = F(x)$$

these are conservative systems. In such cases, we can define a potential function $V(x)$ such that:

$$F(x) = -\frac{dV}{dx}$$

The conserved quantity E in this context is given by:

$$E = \frac{1}{2}m\dot{x}^2 + V(x)$$

Here, E represents the total energy of the system, where $\frac{1}{2}m\dot{x}^2$ is the kinetic energy and $V(x)$ is the potential energy. For a given trajectory $x(t)$, the quantity E remains constant, reflecting the conservation of energy.

6.5 Reversibility

A dynamical system is called reversible if replacing y by $-y$ and t by $-t$ leaves the system invariant. This property indicates that the system exhibits symmetry under time reversal and reflection.

Reversibility often manifests as a symmetry in the system's phase space. In conservative systems, this symmetry is observed in phenomena such as closed loops around nonlinear centers. The ability of a system to return to its original state under time reversal is a hallmark of its reversible nature.

6.6 Pendulum

The pendulum is described by the set of equations

$$\dot{\theta} = v\dot{v} = -\sin\theta$$

The potential function for the system is given by

$$V = -\int \ddot{\theta}V = -\int -\sin\theta = -\cos\theta + C$$

Therefore, the energy of the system is given by

$$E = \frac{1}{2}v^2 - \cos\theta$$

For $(\pi, 0)$, $E = 1$, which corresponds to the unstable fixed point when the pendulum is at the top. Linear stability analysis tells us this is a saddle point.

For $(0, 0)$, $E = -1$, which corresponds to the stable fixed point when the pendulum is at the bottom. Linear stability analysis tells us this is a center.

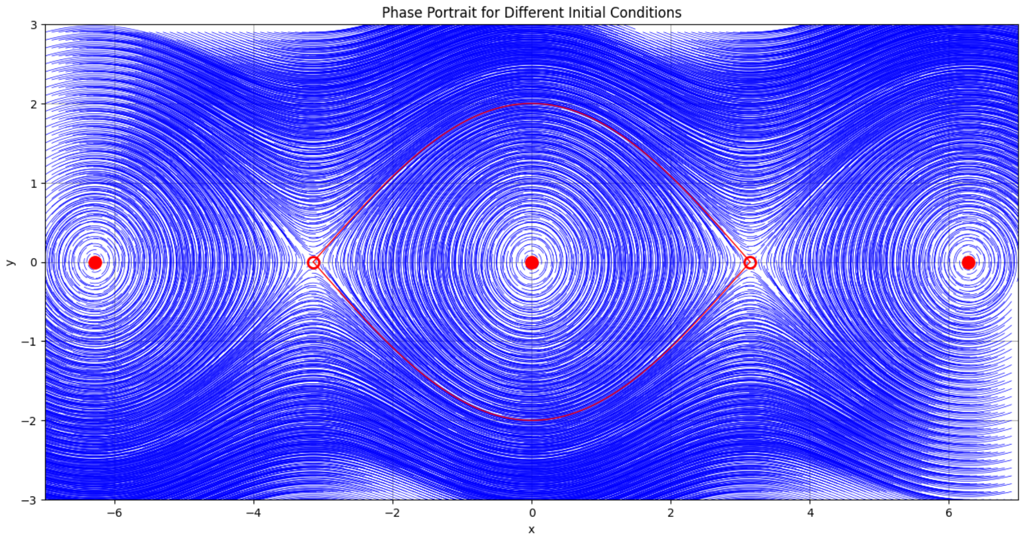


Figure 6.1: Phase portrait of the pendulum

The red lines are called as heteroclinic as they form a loop joining two unstable fixed points.

6.7 Index Theory

Index theory provides a global understanding of the phase plane by analyzing a closed curve C that does not enclose any fixed points. This curve does not need to be a trajectory within the phase plane.

We define:

$$\phi = \tan^{-1}\frac{\dot{x}}{\dot{y}}$$

where ϕ represents the direction of the net velocity vector at every point on the curve C.

The index of the curve C, denoted I_C, is defined as:

$$I_C = \frac{[\phi]}{2\pi}$$

where $[\phi]$ is the total change in ϕ around the closed curve C. This index represents the number of rotations made by the net velocity vector along the curve.

Key results from index theory include:

- All closed trajectories in the phase plane must have a net index of +1.
- Fixed points such as Spirals, Centers, Stars, and Nodes have an index of +1.
- Saddle nodes have an index of -1.
- Non-isolated fixed points cannot be assigned an index.
- Curves that enclose no fixed points have an index of 0, according to deformation theory.

Section 7
Bifurcations in 2D

7.1 Bifurcation of Fixed Points

Bifurcations of fixed points can be categorized into two types:

- The 2D analog of 1D bifurcations
- Hopf Bifurcation

7.1.1 2D Analog of 1D Bifurcations

Saddle-Node Bifurcations

The normal form for saddle-node bifurcations is given by:

$$\dot{x} = \mu - x^2$$

$$\dot{y} = -y$$

By setting both equations to zero, the fixed points are found to be:

$$(\pm\sqrt{\mu}, 0)$$

- **When** $\mu < 0$: The fixed point coordinates are $(\pm\sqrt{\mu}, 0)$, which is not real, indicating no real fixed points.
- **When** $\mu = 0$: The fixed point is $(0, 0)$.
- **When** $\mu > 0$: The fixed points are $(\pm\sqrt{\mu}, 0)$.

Transcritical Bifurcations

The normal form for transcritical bifurcations is:

$$\dot{x} = rx - x^2$$
$$\dot{y} = -y$$

The fixed points are given by:

$$(0, 0) \quad \text{and} \quad (r, 0)$$

Supercritical Pitchfork Bifurcations

The normal form for supercritical pitchfork bifurcations is:

$$\dot{x} = rx - x^3$$
$$\dot{y} = -y$$

The fixed points are given by:

$$(0, 0) \quad \text{and} \quad (\pm\sqrt{r}, 0)$$

Subcritical Pitchfork Bifurcations

For subcritical pitchfork bifurcations, the normal form is:

$$\dot{x} = rx + x^3$$
$$\dot{y} = -y$$

The fixed points are:

$$(0, 0) \quad \text{and} \quad (\pm\sqrt{-r}, 0)$$

In the subcritical case, the cubic term has a positive sign, which alters the stability characteristics compared to the supercritical case.

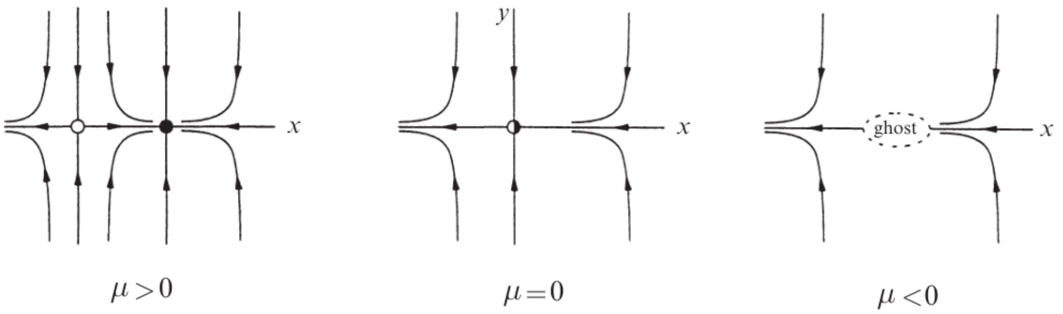


Figure 7.1: Saddle-Node Bifurcations 2D. [1]

7.1.2 Hopf Bifurcation

The fixed points here are spirals, and their bifurcation occurs as the eigenvalues move from negative $\mathrm{Re}(\lambda)$ to positive $\mathrm{Re}(\lambda)$. The bifurcation occurs when the real part is zero.

The normal form is given by:

$$\dot{r} = \mu r - r^3$$
$$\dot{\theta} = \omega + br^2$$

Let's set $\omega = 1$ and $b = 0$.

In the polar system, we can observe isolated closed orbits called limit cycles.

Setting $\dot{r} = 0$, we obtain a spiral fixed point at the origin and a stable isolated closed orbit at $r = \sqrt{\mu}$.

7.2 Bifurcation of Periodic Orbits

7.2.1 Saddle Node Coalescence

Let's consider the system:

$$\dot{r} = \mu r + r^3 - r^5 \dot{\theta} = 1$$

This bifurcation involves the creation or annihilation of a stable and unstable closed orbit to a half stable one.

Equating the first equation to zero gives:

- A fixed point at the origin.
- Closed orbits at $\sqrt{\frac{1 \pm \sqrt{1+4\mu}}{2}}$.

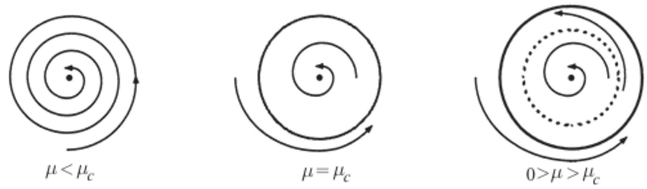


Figure 7.3: Saddle Node Coalescence.[1]

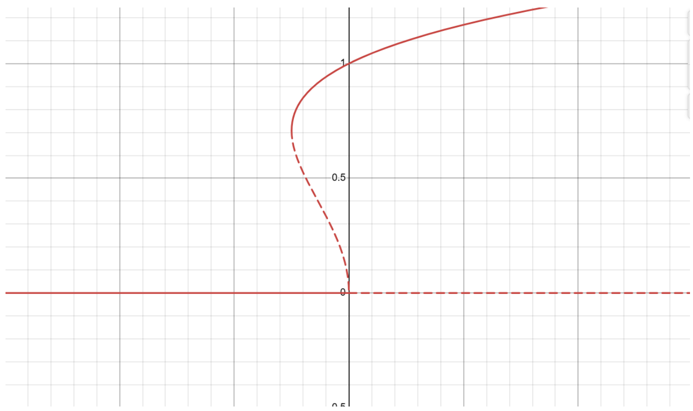


Figure 7.4: Bifurcation diagram

- For μ in the range $(-\infty, -\frac{1}{4})$, there exists only a stable spiral at the origin.
- For μ in the range $(-\frac{1}{4}, 0)$, there exists a stable spiral at the origin along with a pair of stable and unstable limit cycles which are born at $\mu = -\frac{1}{4}$.
- For $\mu > 0$, the stable spiral at the origin loses stability and we have only a stable limit cycle.

7.2.2 Infinite Period Bifurcation

This type of bifurcation can be described by the system:

$$\dot{r} = r(1 - r^2), \quad \dot{\theta} = \omega - \sin\theta$$

The time period on the closed orbit becomes infinite at bifurcation due to the formation of a fixed point on the orbit.

7.2.3 Homoclinic Bifurcation

This type of bifurcation can be described by the system:

$$\dot{x} = y, \quad \dot{y} = \mu y + x - x^2 + xy$$

A limit cycle bangs onto the saddle node upon bifurcation to form a homoclinic orbit which is destroyed after that.

Section 8
Chaos

8.1 Definition of Chaos

Qualitatively, a system is considered chaotic if it satisfies the following criteria:

- Aperiodic long-term behavior: Trajectories never settle down to fixed points, periodic orbits, or quasiperiodic orbits even as $t \to \infty$
- Sensitive Dependence on Initial Conditions: Small differences in initial conditions lead to exponentially diverging outcomes. This is often quantified using Lyapunov exponents.
- Deterministic: The system has no random or noisy inputs or parameters. The irregular behavior arises from the system's nonlinearity, rather than from noisy driving forces.

These properties ensure that chaotic systems exhibit complex, seemingly random behavior, even though deterministic rules govern them. We can use various tools to analyze and understand these properties, including bifurcation diagrams, Poincaré maps, and fractal dimensions.

8.2 Quantifying Chaos

Quantifying chaos involves measuring and analyzing the behavior of chaotic systems. This section covers key methods used to understand and characterize chaotic dynamics.

8.2.1 Lyapunov Exponents

Lyapunov exponents provide a quantitative measure of chaos by evaluating the rate at which nearby trajectories in the phase space diverge over time. They are defined as follows:

- Positive Lyapunov Exponent: Indicates chaotic behavior. A positive value implies that nearby trajectories will diverge exponentially, leading to sensitive dependence on initial conditions.
- Zero Lyapunov Exponent: Associated with periodic or quasi-periodic behavior. This indicates that the system does not exhibit exponential divergence of nearby trajectories.
- Negative Lyapunov Exponent: Suggests stable behavior, where trajectories converge or remain bounded.

The largest Lyapunov exponent, often referred to as the Lyapunov dimension, is particularly significant. If it is positive, the system exhibits chaotic dynamics. In practice, Lyapunov exponents are calculated using numerical methods and can provide insights into the stability and predictability of the system.

Section 9
Bifurcation Analysis by Parameter Continuation Method

9.1 Overview

In this chapter, we perform bifurcation analysis on several nonlinear dynamical systems to uncover "hidden behavior" using the parameter continuation method. Bifurcation analysis helps identify changes in the qualitative behavior of a system as parameters are varied. The parameter continuation method is a powerful algorithm that traces the bifurcation points and solutions as parameters change, revealing intricate structures and dynamics.

We will focus on the following systems for our analysis:

- Rössler System: A three-dimensional system with chaotic behavior, known for its simplicity and rich dynamics.
- Lorenz System: A well-known system in chaos theory, representing atmospheric convection and exhibiting complex chaotic dynamics.
- Chua's Circuit: An electronic circuit known for its chaotic behavior, used as a practical example of chaotic systems in engineering.

Each system will be analyzed to identify bifurcations and study the changes in their dynamics as parameters vary. The parameter continuation method will be employed to systematically track these changes and reveal the underlying structure of the bifurcations.

9.2 Bifurcation Analysis

Bifurcation diagrams provide insights into how the qualitative behavior of dynamical systems changes as parameters are varied. The meaning and representation of a bifurcation diagram can vary depending on the context. This section explores different contexts where bifurcation diagrams are used:

9.2.1 Fixed Points and Fixed Orbits

In the context of fixed points or fixed orbits, the bifurcation diagram illustrates how the number and stability of fixed points or fixed orbits change as a parameter varies.

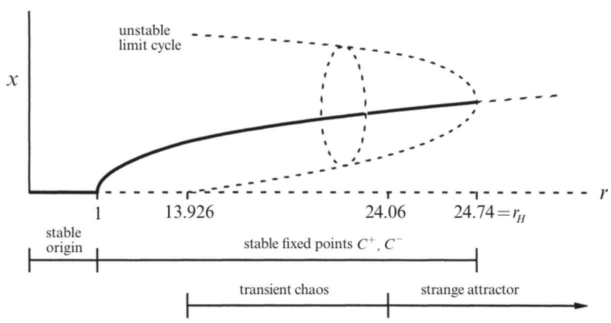


Figure 9.1: Bifurcation diagram illustrating fixed points in the lorenz system. [1]

The diagram helps in understanding how these fixed structures transition from one type to another, indicating critical changes in the system's behavior.

9.2.2 Discrete Time Systems

For discrete time systems, bifurcation diagrams often involve capturing all state values as a parameter varies. This approach helps in identifying how periodic or chaotic behavior emerges as the system's parameters are adjusted.

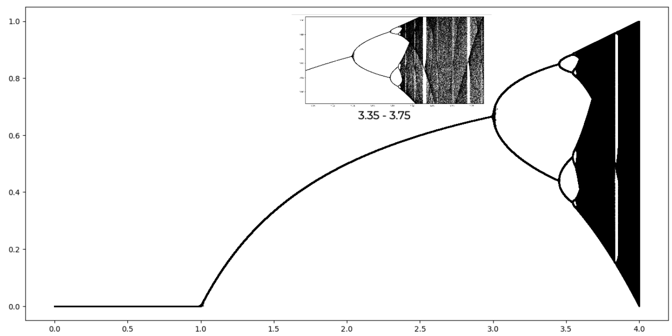


Figure 9.2: Bifurcation diagram for the logistic map.

9.2.3 Continuous Time Chaotic Systems

In continuous time chaotic systems, capturing periodic windows amidst chaos can be challenging. Two common approaches for bifurcation analysis in such systems are:

Poincaré Section

A Poincaré section involves using an $(n-1)$-dimensional section of an n-dimensional phase space to capture periodic orbits. The Poincaré map constructed from these sections provides valuable information about the dynamics of the system, revealing periodic structures within chaotic behavior.

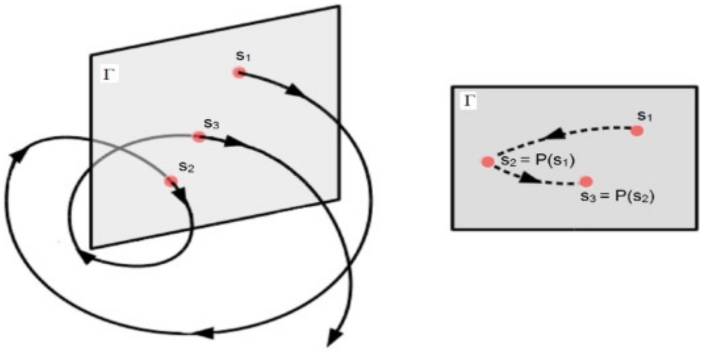


Figure 9.3: Poincaré section for a continuous time chaotic system. [2]

Peak Finding

Peak finding involves analyzing the time series of dynamic variables to identify periodicity. By locating peaks in the time series, one can discern periodic behavior within chaotic dynamics.

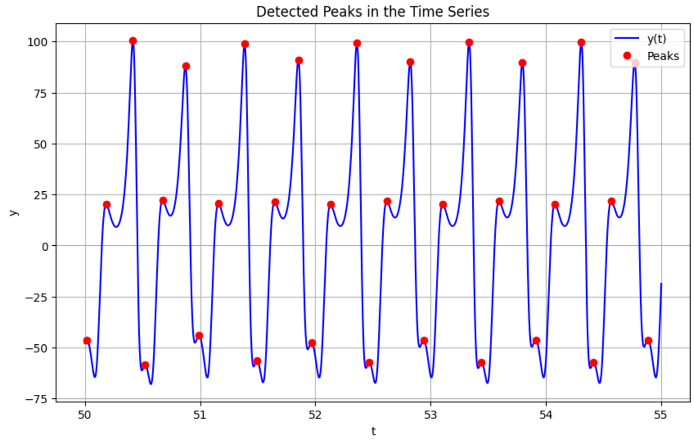


Figure 9.4: Peak finding in time series data of a chaotic system.

9.3 Parameter Continuation Method [2]

The Parameter Continuation Method is a numerical algorithm used to trace the evolution of steady-state solutions as a parameter changes. This method is particularly useful for bifurcation analysis and studying how qualitative features of dynamical systems shift with varying parameters.

9.3.1 Concept

The core idea behind the Parameter Continuation Method is to follow the steady-state solutions of a dynamical system as a parameter is varied. Given a steady-state solution at a parameter value α_0, the goal is to numerically determine the steady-state solution at a nearby parameter value $\alpha_0 + \delta$. This approach allows for tracking bifurcations and understanding how the system's behavior evolves.

9.3.2 Algorithm

The algorithm for the Parameter Continuation Method can be broken down into the following steps:

1. **Initialization:** Start with an initial parameter value α_0 and its corresponding steady-state solution $\mathbf{x}_0$. This solution is obtained by solving the equation $\mathbf{f}(\mathbf{x}, \alpha_0) = 0$, where $\mathbf{f}$ is the system of differential equations governing the dynamics.

2. **Perturbation:** Choose a small step size δ for the parameter change. Set the new parameter value to $\alpha_1 = \alpha_0 + \delta$.

3. **Predictor Step:** Use a numerical method to predict the new steady-state solution $\mathbf{x}_1$ at α_1. This prediction can be based on linear interpolation or an approximation that considers the behavior of $\mathbf{x}_0$ as the parameter changes. The prediction often uses the solution from α_0 as a starting point.

4. **Corrector Step:** Refine the predicted solution $\mathbf{x}_1$ by solving the system $\mathbf{f}(\mathbf{x}, \alpha_1) = 0$ numerically. This step involves iterating on $\mathbf{x}_1$ to find the accurate solution corresponding to α_1. Common iterative methods include Newton's method or other root-finding algorithms.

5. **Update:** Update the parameter value to α_1 and the steady-state solution to $\mathbf{x}_1$.

6. **Repeat:** Continue the process by incrementing the parameter value in steps and repeating the predictor-corrector steps. Track the steady-state solutions as the parameter varies to map out the bifurcation diagram.

Consider a system where we want to study the steady-state solutions as a parameter α changes. Suppose we start with α and its corresponding steady-state solution $\mathbf{x}_0$. We want to find the solution at $\alpha + \delta$.

The algorithm starts by predicting a new solution $\mathbf{x}_1$ based on $\mathbf{x}_0$. This prediction can be a simple linear approximation or use more sophisticated prediction techniques depending on the problem. After predicting $\mathbf{x}_1$, we solve the system $\mathbf{f}(\mathbf{x}, \alpha_0 + \delta) = 0$ to correct $\mathbf{x}_1$. The corrected $\mathbf{x}_1$ is then used as the starting point for the next step.

9.4 Rössler System

The Rössler system is described by the following set of differential equations:

$$\dot{x} = -y - z \tag{9.1}$$
$$\dot{y} = x + ay \tag{9.2}$$
$$\dot{z} = b + z(x - c) \tag{9.3}$$

For constructing the bifurcation diagram, we will chose the parameters $b = 2$ and $c = 4$ while varying a from 0.20 to 0.55.

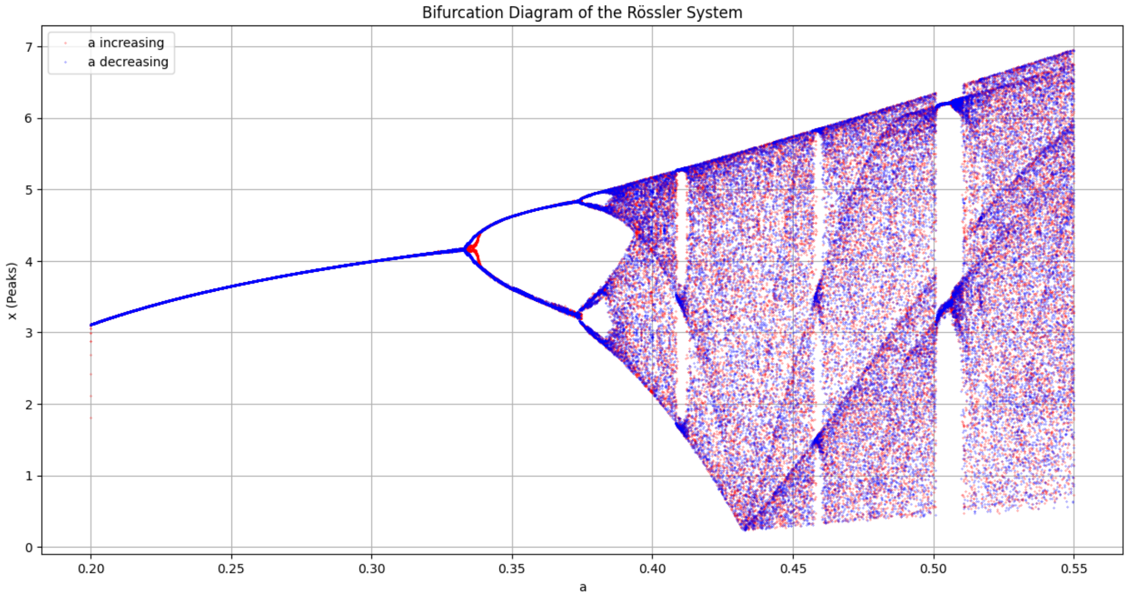


Figure 9.6: The bifurcation diagram for Rossler model for $0.2 < a < 0.55$.

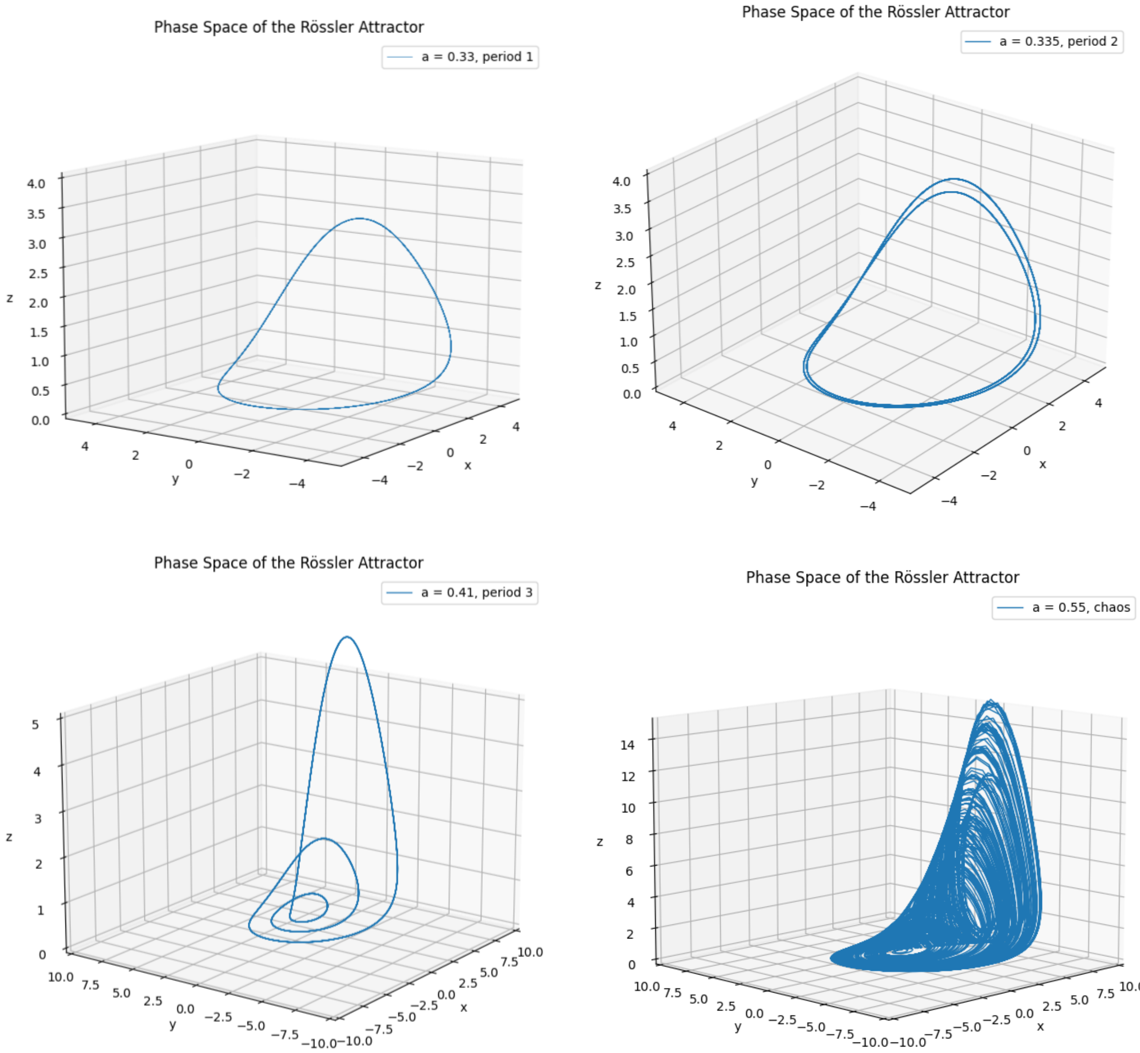


Figure 9.7: Validation of the bifurcation diagram for different values of the control parameter of Rossler model (in the zoom range)

9.5 Lorenz System

9.5.1 System Description

The Lorenz system is given by:

$$\dot{x} = \sigma(y - x) \tag{9.4}$$
$$\dot{y} = x(\rho - z) - y \tag{9.5}$$
$$\dot{z} = xy - \beta z \tag{9.6}$$

For constructing the bifurcation diagram, we will chose the parameters $p = 10$ and $b = \frac{8}{3}$ while varying r from 50 to 150.

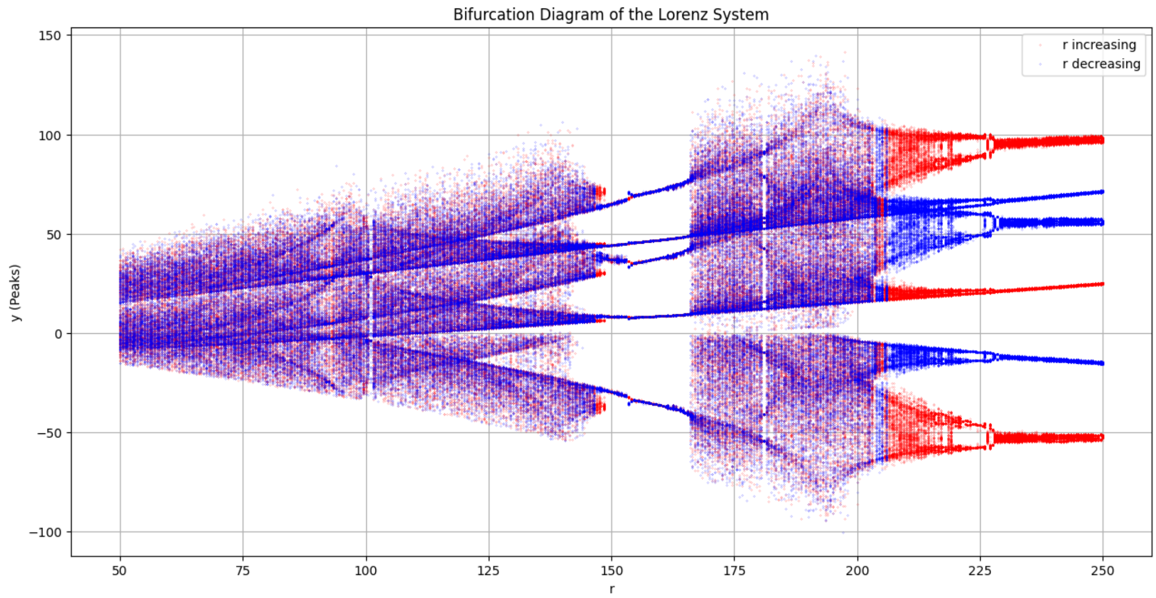


Figure 9.8: Bifurcation diagram for the state of y in the Lorenz system at $50 < r < 250$

Twin Attractor in the Lorenz System

The twin attractor in the Lorenz system can be justified because this system has the following properties:

1. Symmetry: $(x, y, z) \rightarrow (-x, -y, z)$ for all values of the parameters. This symmetry implies that if $(x(t), y(t), z(t))$ is a solution to the Lorenz equations, then $(-x(t), -y(t), z(t))$ is also a solution. This leads to the attractor in the phase space having a mirrored or "twin" structure, where the trajectories are symmetric about the z-axis.

2. The z-axis $(x = y = 0)$ is invariant, meaning all trajectories that start on it also end on it.

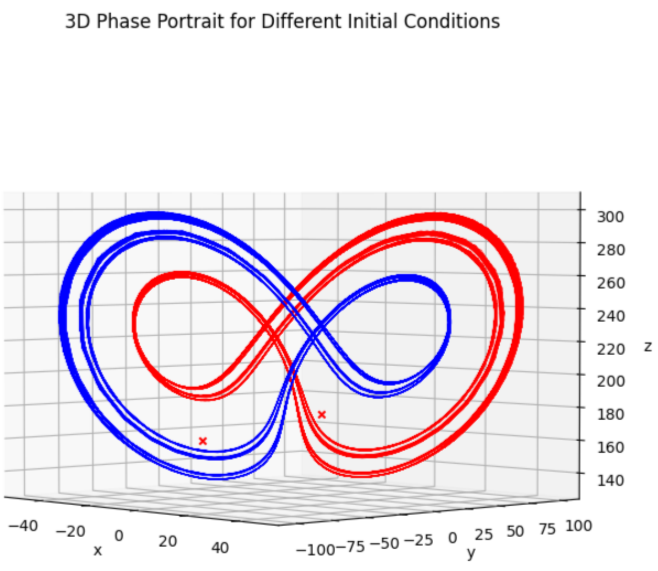


Figure 9.9: Twin attractors in the lorenz system for $r = 216.64$

9.6 Chua's Circuit

9.6.1 System Description

Chua's Circuit is defined by the equations:

$$\dot{x} = \alpha(y - x - f(x)) \tag{9.7}$$

$$\dot{y} = x - y + z \tag{9.8}$$

$$\dot{z} = -\beta y \tag{9.9}$$

where α, β, and the nonlinear function $f(x)$ characterize the circuit. Chua's Circuit is a classic example of a chaotic electronic system.

9.6.2 Hidden Attractors

- **Self-Excited Attractors:** Self-excited attractors are those whose basins of attraction are connected with the equilibrium points. These attractors can be easily localized numerically by standard computational procedures. For example, in systems with self-excited attractors, a small perturbation of the equilibrium state leads the system to oscillate and approach the attractor.

- **Hidden Attractors:** Hidden attractors, on the other hand, are those whose basins of attraction are not connected with any equilibrium points. That is Hidden attractors don't intersect with the neighborhood of any equilibrium point [3] and occur with a small basin of attraction in the space of initial conditions, within a limited range of parameter space. These attractors are not easily found by traditional numerical methods as they do not arise from perturbations of equilibria. Instead, hidden attractors require a different approach for detection and analysis, often involving sophisticated algorithms or numerical continuation methods.

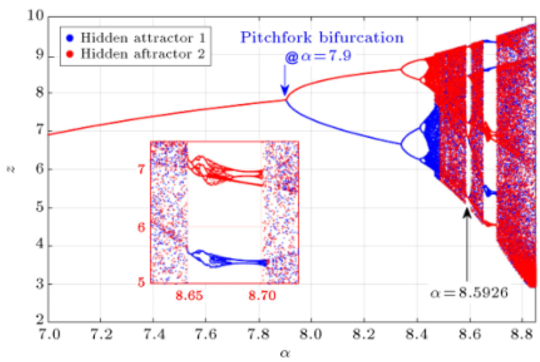


Figure 9.10: Bifurcation diagram of z for changes in the range [7;8.85]. [2]

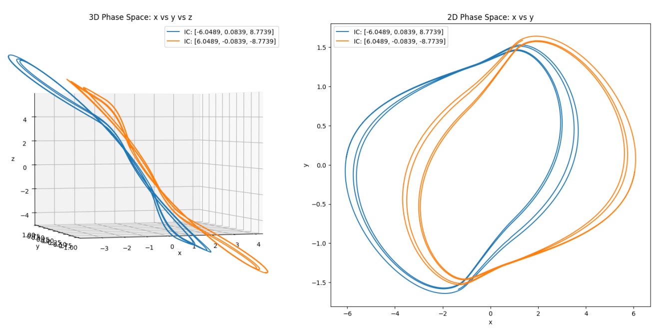


Figure 9.11: Twin Hidden attractor for $\alpha = 8.4543$

References

[1] Steven H. Strogatz. *Nonlinear Dynamics and Chaos*. Westview Press, 2015.

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