

Auto-Differentiation

At the Intersection of Nifty and Obvious

A.C.

January 28, 2021

Outline

Derivatives Refresher

Autodiff

- Key Insight

- Forward Mode

- Backward Mode

A Practical Application

Univariate Derivatives

- ▶ The instantaneous rate of change of f in response to infinitesimal perturbations in x .
- ▶ The slope of the tangent line through $(x, f(x))$.

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Definition

Let $f : \mathbb{R} \rightarrow \mathbb{R}$. We say that f is differentiable wherever the limit

$$f' = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$$

exists and we call f' the derivative of f .

Derivatives of Transformations

Linearity:

$$(\alpha f + \beta g)' = \alpha f' + \beta g'$$

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Chain rule:

$$(f \circ g)' = g' \cdot (f' \circ g)$$

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Definition

Let $f : \mathbb{R}^n \rightarrow \mathbb{R}$, and u_i be the i th Cartesian unit vector in \mathbb{R}^n . We say that f is differentiable wherever the limit

$$\frac{\partial f}{\partial x_i} = \nabla f_i = \lim_{h \rightarrow 0} \frac{f(x + hu_i) - f(x)}{h}$$

exists for all $i \in [1, n]$. We call ∇f the gradient of f .

Jacobians

- ▶ Some functions have multiple outputs, too.
- ▶ Similar trick:
 - ▶ Compute the gradient for each output
 - ▶ Glue the gradients to one another
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Definition

Let $f : \mathbb{R}^n \rightarrow \mathbb{R}^m$, be differentiable in each of its outputs at x . We define the Jacobian of f to be the matrix such that

$$J_{ij} = \frac{\partial f_i}{\partial x_j}$$

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Autodiff

Key Insight

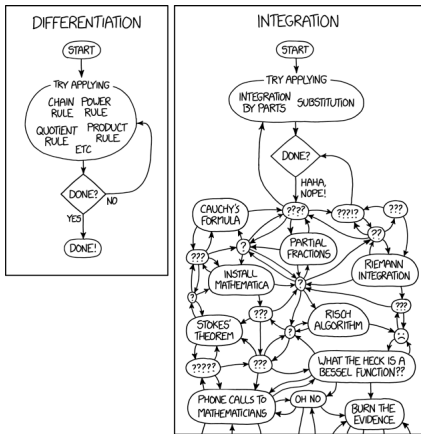
Forward Mode

Backward Mode

A Practical Application

First Piece

Differentiation is a flowchart. Flowcharts are programs.



Second Piece

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 - ▶ Not always easy to express in closed form

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- ▶ Numerical programs may include loops, branches, etc
 - ▶ Not always easy to express in closed form
- ▶ But when it executes it has to boil down to some finite composition of ALU-executable operations.
 - ▶ We know how to differentiate sums, products, differences, etc

Eureka

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- ▶ Apply the “differentiation flowchart” to the composite function defined by the *execution* of a numerical program.
- ▶ Compute $J_f(x)$ at the same time as $f(x)$!

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- ▶ Treat intermediate calculations as anonymous dependent variables.
- ▶ Keep track of every dependent variable's gradient WRT inputs.
 - ▶ Propagate forward with product rule, quotient rule, etc.
- ▶ Use independent variables to bootstrap the calculation
 - ▶ $\frac{\partial x_i}{\partial x_j} = I_{i=j}$

Dep Vars Example

Consider the function $f(x, y) = x^2y + 2y$, and let

$$g_1 = x \cdot x \implies \nabla g_1 = x \cdot (1, 0)^T + (1, 0)^T \cdot x$$

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$$f = g_2 + g_3 \implies \nabla f = \nabla g_2 + \nabla g_3$$

Exercise for the reader: confirm that $\nabla f = (2xy, x^2 + 2)^T$

Code I

```
class _FDepVar(object):
    def __init__(self, val, grad):
        self.val = val
        self.grad = grad

    def __add__(f, g):
        if not isinstance(g, _FDepVar):
            return _FDepVar(f.val + g, f.grad)
        return _FDepVar(f.val + g.val, f.grad + g.grad)

    def __mul__(f, g):
        if not isinstance(g, _FDepVar):
            return _FDepVar(f.val*g, f.grad*g)
        return _FDepVar(f.val*g.val,
                        f.grad*g.val + f.val*g.grad)
```

Code II

```
def forward(f):  
    def f_J(*args):  
        wrapped = []  
        for i, arg in enumerate(args):  
            grad = np.zeros(len(args))  
            grad[i] = 1  
            wrapped.append(_FDepVar(arg, grad))  
        out = f(*wrapped)  
        try:  
            return ([o.val for o in out],  
                    [o.grad for o in out])  
        except TypeError:  
            return out.val, out.grad  
    return f_J
```

Caveats

- ▶ Every scalar operation in our program now a vector op
- ▶ Calculation is n times more expensive, where n is our number of inputs.
 - ▶ No big deal for auto-diffing $f(x, y, z)$
 - ▶ But if f takes thousands of parameters ... hoo boy

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 - ▶ Derivatives of different functions with respect to x

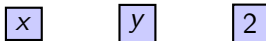
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 - ▶ Derivatives of different functions with respect to x
- ▶ Backward mode: take $\frac{df}{dg}, \frac{df}{dh}$, build $\frac{df}{dx}$
 - ▶ Derivatives of f with respect to different variables.
 - ▶ Multivariate chain rule is central:

$$\frac{df(g(x), h(x))}{dx} = \frac{\partial f}{\partial g} \cdot \frac{dg}{dx} + \frac{\partial f}{\partial h} \cdot \frac{dh}{dx}$$

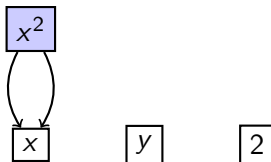
Calculation Graph

Calculations naturally form a DAG as they feed into one another:



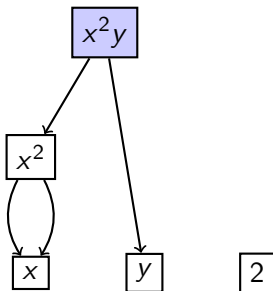
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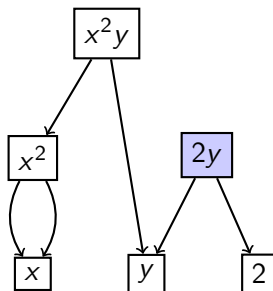
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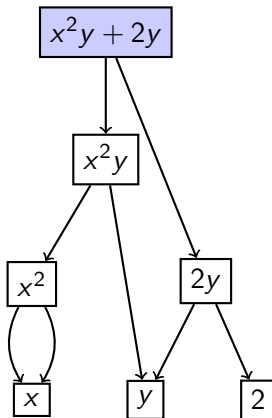
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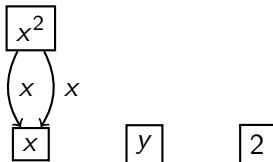
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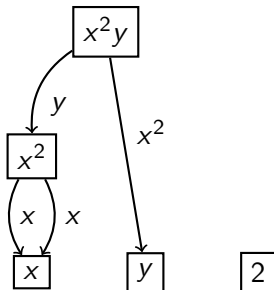
Edge Weights

Edges provide a convenient place to record our various $\frac{\partial g_i}{\partial g_j}$ s:



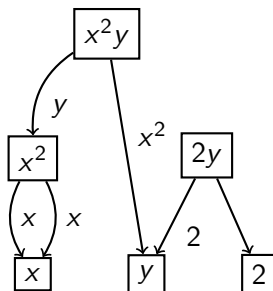
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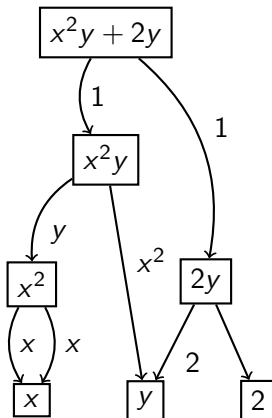
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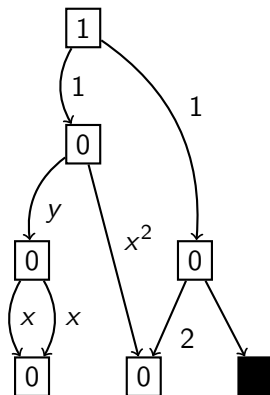
Propagation

- ▶ If we know all of $\frac{df}{d_{\text{parent}}}$, then by the chain rule $\frac{df}{d_{\text{self}}}$ is just the weighted sum of its parents.
 - ▶ Root node always knows its derivative: $\frac{df}{df} = 1$

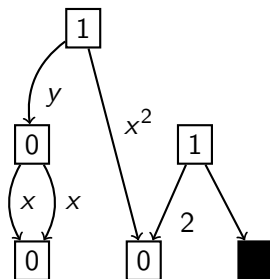
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 - ▶ Root node always knows its derivative: $\frac{df}{df} = 1$
- ▶ Note: DAG-order traversal, not just a DFS.
 - ▶ we can't visit a node until all its parents are done.
 - ▶ we have to traverse all edges, not visit every node.

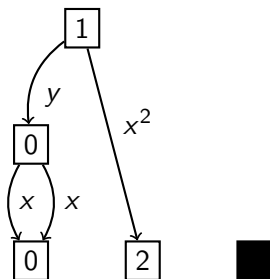
Propagation Example



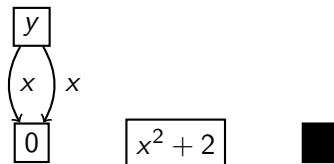
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Propagation Example



Propagation Example

$$2xy$$

$$x^2 + 2$$



Code

- ▶ The code for backward mode is a little less slide-deck friendly than for forward mode, so we'll skip it here.
- ▶ A (crude) implementation is available in this presentation's git repository for those interested in perusing it.

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- ▶ Have to propagate derivatives backward for every output.
 - ▶ Calculation is m times more expensive, where m is our number of outputs.
- ▶ Have to remember the calculation graph.
 - ▶ Requires additional space proportional to the length of the computation.

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A Practical Application

Disclaimer

ACHTUNG: Aggressive, true-from-10k-feet-but-not-below-that simplifications ahead.

ML \approx Optimization

- ▶ Most ML boils down to one of
 - ▶ Classification: Find me the best decision boundary for all this the training data I've marked as positive or negative.
 - ▶ Regression: Find me the function that best matches these data points I measured.

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 - ▶ Classification: Find me the best decision boundary for all this the training data I've marked as positive or negative.
 - ▶ Regression: Find me the function that best matches these data points I measured.
- ▶ If we define a loss function \mathcal{L} , and model parameters θ for tuning it, and call our training data X, Y , then we can formulate them as

$$\arg \min_{\theta} \mathcal{L}(X, Y; \theta)$$

- ▶ E.g. in linear regression, $\mathcal{L} = ||X^T \theta - y||_2^2$

Gradient Descent in a Nutshell

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- ▶ When you're on a hill and want to get to the bottom, *walk downhill*.
- ▶ The gradient of your loss function tells you which way is downhill.

$$\theta_n = \theta_{n-1} - h_n \nabla \mathcal{L}(\theta_{n-1})$$

Neural Networks & Backpropagation

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- ▶ Neural Networks are commonly optimized (trained) using gradient descent.
- ▶ But formulating the gradient is a PITA, sure would be nice if we could generate the gradient at a point programmatically and efficiently ...
- ▶ Which is exactly what autodiff does.
 - ▶ Neural networks have many, many parameters to train on, but we only have one loss function, so backward mode is a no-brainer.
 - ▶ Frequently called backpropagation or backprop for historical reasons.

References and Additional Materials

- ▶ The source for this presentation is hosted at <https://github.com/alan-christopher/autodiff-edu>.
- ▶ This deck borrows an image from XKCD: <https://xkcd.com/2117/>

Questions?