BB84

Quantum *Protected* Cryptography

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Outline

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- Maxwell's equations

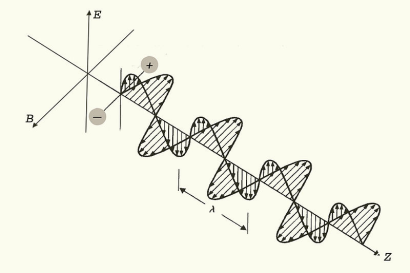
$$\oint_A E dA = \oint_A B dA = 0$$

$$\oint_C E dl = -\int_A \frac{\partial B}{\partial t} dA$$

$$\oint_C B dl = \mu_0 \epsilon \int_A \frac{\partial E}{\partial t} dA$$

Linearly Polarized Light

- Useful simplified model: assume harmonic, planar waves.
  - Between Fourier transforms and other superpositions this covers a surprising amount of ground.
  - Call the orientation of the E-wave the light's *polarization*



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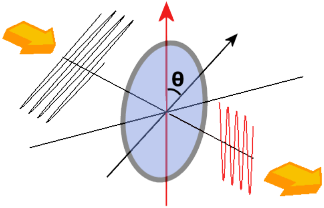
- Ideal polarizers follow Malus's law: $I = I_0 cos^2\theta$



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- Still have a polarization due to their wave aspect
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- Cannot be dimmed – quanta are indivisible.
- Instead polarizers absorb with probability $\sin^2 \theta$
  - or transmit with probability $\cos^2 \theta$

No Cloning Theorem

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- Observing a quantum entity, including photons, collapses it into a single state.
- Impossible to perfectly capture and reconstruct a photon of unknown polarization
  - Approximate cloning *is* possible, e.g. via stimulated emission
  - General upper bound on cloning fidelity of $\frac{5}{6}$
  - Fidelity bound for the case we'll care about of ≈ 0.8535

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- What happens if sender and receiver use different bases?
  - i.e., there's a phase offset of ϕ between their verticals
- Bit flip with probability $\sin^2 \phi$, by Malus' law.

The Protocol

Alice:

0	0	1	1	0	1	0	1

Alice selects random bits, bases

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Bob:	?	?	?	?	?	?	?	?

Bob selects random bases

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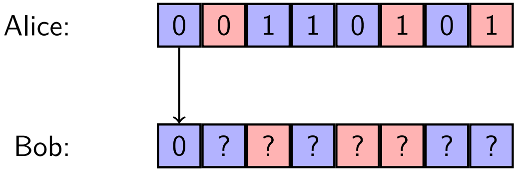

Bases agree, measurement succeeds

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Alice:	0	0	1	1	0	1	0	1
Bob:	0	0	?	?	?	?	?	?

Bases disagree, measurement *may* fail

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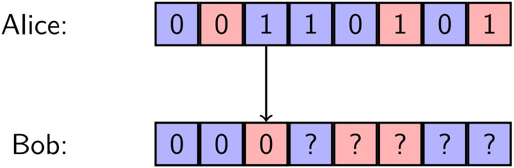

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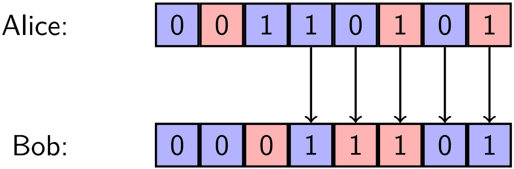


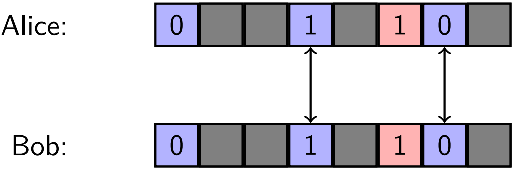
Etc...

The Protocol

Alice:	0			1		1	0	
Bob:	0			1		1	0	

Sift out mismatched bases

The Protocol



Sample bits to detect Eve

The Protocol

Alice:

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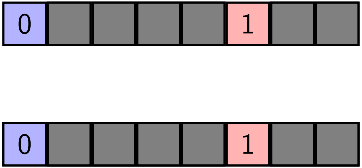

Redact sampled bits

The Protocol Summary

1. Alice and Bob publicly agree on two bases (rect, diag) with a $\frac{\pi}{4}$ offset between them
2. Alice creates a random sequence of bits
3. Alice randomly rect- or diag-encodes each bit, transmits to Bob
4. Bob randomly rect- or diag-decodes each photon
5. Alice and Bob publicly announce their basis choices, drop bits where their bases disagree.
6. Alice and Bob publicly compare a random subset of the undropped bits.
7. If they observe a high rate of corruption, they start over.
8. Otherwise, the undropped and unannounced bits are now a shared secret.

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  - 25% Bob gets a corrupted bit (assuming no intelligent cloning).

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  - 25% Bob gets a corrupted bit (assuming no intelligent cloning).
- Can't stop Eve, but can *detect* her.

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- Even if Eve isn't interfering, some of our bits will be scrambled.
- Secrets which are mostly the same are not nearly as useful as secrets which are exactly the same.
- We need to somehow rectify the differences between what Alice sent and Bob received.

Single Error Detection

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- Can write as a check matrix:

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  - Each parity bit will only cover some bits in the code word.
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- Hamming solved this for us in 1950 using powers of two
  - Code word: $(p_1, p_2, d_1, p_3, d_2, d_3, d_4)^T$
  - Check matrix:

$$H = \begin{pmatrix} 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 \end{pmatrix}$$

Full SECDED

- Glue a total parity bit onto a Hamming SEC scheme
  - No errors $\implies$ all parity checks report even
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- Commonly known as Hamming(8,4).
  - Code word: $(p_1, p_2, d_1, p_3, d_2, d_3, d_4, p_{\text{total}})^T$
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- We can reconcile errors by exchanging Hamming syndromes!

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7. Go back to (1) until convinced that the probability of surviving errors is negligible.

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- Paranoia requires that we treat all errors as due to eavesdropping
- *Eve is allowed to have partial information on our secret!*
  - Or we can just never succeed in negotiating a secret
- Need to do two things
  - Estimate how many bits of information Eve has
  - Somehow scrub those bits out of our secret

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  - Too easy to have $\hat{p} < p$
- Bayes rule to pick $\hat{p}$ s.t. $P(p > \hat{p}|\text{QBER}) < \epsilon$?
  - No sensible prior to put on p

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  - Sounds an awful lot like a confidence interval
- Exact CIs for binomials are computationally expensive
  - Almost always approximated using Poisson or Normal
  - Much of the literature focuses on methods that provide exact bounds, not estimates

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  - choose our hash function randomly, although it *can* be public
- This is a direct result of the leftover hash lemma
- Approach is obvious:
  1. Alice publicly announces a random seed
  2. Alice and Bob both use the seed to hash their secrets down

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  - But we can still use the scheme to stretch the bootstrap secret, e.g. to allow for one-time pads in the data plane.
- Our goal is unconditional security, so a standard HMAC won't cut it.

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3. Encrypt the tag with a one-time pad.
4. Discard the m bits used for the OTP.

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- Eve can't manufacture a hashes for messages, doesn't know the hash function.
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- The hash value is encrypted, so she can't manufacture a hash collision.
  - Universal hash function means she can't even make intelligent guesses about the hash value.
- Best she can do is guess the tag
  - Succeeds with probability $\frac{1}{2^m}$
  - Places a tradeoff between key expenditure and probability that Eve gets lucky.

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- Attack is commonly known as photon number splitting

Decoy States

- Alice uses two (or more) signal sources S and S'
  - S is the real photon source, averages μ photons per pulse
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- Impossible for Eve to tell whether a given pulse came from S or S'
- By blocking single photon pulses, Eve will affect the transmission rate for S more than for S'.
- Alice and Bob can compare empirical transmission loss for S and S' at the end.
  - If they differ, they conclude Eve did something naughty.

TL;DR

1. Send/receive a stream of quanta in random bases.
2. Throw away the bits where bases disagreed.
3. Reconcile keys using (adaptations of) standard error correction schemes.
4. Hash your keys down according to pessimistic estimates of how much information Eve could have.
5. Make sure to authenticate all your messages throughout.
6. Profit!

References and Additional Materials

- Presentation source: https://github.com/alan-christopher/bb84/tree/master/edu
- Quantum Cryptography Intro: https://arxiv.org/abs/1002.1237
- Quantum Cloning: https://arxiv.org/abs/quant-ph/0511088
- Privacy Amplification: https://link.springer.com/article/10.1007/BF00191318
- Winnow: https://arxiv.org/abs/quant-ph/0203096
- Decoy states: https://arxiv.org/abs/quant-ph/0211153
- A QKD implementation: https://arxiv.org/abs/1603.08387

Included Works

- Transverse Wave Image
- Polarizer Image

Questions?