Beating the House

You will not beat the house

A.C.

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Outline

The Rules

1. Place a bet with probability p and payout ratio r.
2. The wheel spins.
3. Get paid ... or don't.
   3.1 House advantage: $pr < 1$

$$X = \alpha\left(-1 + r\mathbb{I}(\text{Win})\right)$$

Open Decisions

- ~~What bet to place.~~
  - Black feels lucky.
  - $p \approx 0.47$
  - $r = 2$
- How much to bet.
- When to walk away.

The System

procedure CONSTANTBETS
    $\alpha \leftarrow 1$
    while true **do**
        bet(α)

Playing it Out

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Expected losses grow linearly in n, stddev as $\sqrt{n}$. Per Chebyshev, we are going broke, *almost certainly*.

The System

```
procedure MartingaleBetting
    α ← 1
    while true do
        bet(α)
        if Win then
            return
        else
            α ← 2 · α
```

Playing it Out

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p nowhere to be found. Guaranteed profit!

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On *arbitrarily* bad bets!

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Exercise for the reader: what happens when we get cut off after N rounds?

The System

procedure $\textsc{AntiMartingaleBetting}$
  reserves $\leftarrow 1.0$
  target $\leftarrow 1.0$
  while reserves $\leq$ target **do**
    $\alpha \leftarrow$ reserves $\cdot\, 0.5$
    bet(α)
    if Win **then**
      reserves $\leftarrow$ reserves $+ \alpha$
    else
      reserves $\leftarrow$ reserves $- \alpha$

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$$P_0 = \frac{p}{q}$$

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We've created a gambler's ruin problem, just flipped and in log scale.

- Nonzero probability of winning *at every step*.
- Nonzero probability of never walking away from the table,
- Also, need to be able to place arbitrarily small bets.

Outline

Putting It All On Black

Go Razzmatazz, Go!

- Some Low-Level Accounting
- A Bigger, Dutcher Book

The Rules

1. n horses running, each with probability of winning p_i and payout odds r_i
   1.1 House advantage: $\sum_i \frac{1}{r_i} > 1$
2. α_i units on each horse
3. The horses run.
4. Get paid according to your wager on the winning horse.

$$X = \sum_i (\alpha_i(-1 + r_i \mathbb{I}(h_i)))$$

Open Decisions

- Which horse(s) to back.
- How much to bet.
- ~~When to walk away.~~
  - We're going to be doing single round betting here.

Pick the Right Horse

- We could play it straight, as we did in roulette.
- More (any) epistemic uncertainty
- Might even be able to turn a profit, if you're smarter than the market.
- Still gambling though. We're here to win, *guaranteed*.

The Booky's Favorite

- Suppose there is an old nag in the race, Rocinante.
- Rocinante is a long, long, *long* shot. Maybe 1 in 1000.
- Book-maker offers a special on Rocinante bets: 2000 to 1.
  - Does not update the rest of the book.
  - House advantage slips. Now $\sum_i \frac{1}{r_i} < 1$

The Obvious Approach

- Could just take the bookie up on the special.
- Bet is positive valued in expectation
- But I still need to make the rent this month, and this is a single round of gambling
- Too rich for my blood, can we do better?

Some Tulip Math

$$X = \sum_i \left(\alpha_i(-1 + r_i \mathbb{I}(h_i))\right)$$

What happens if we try to clear r_i out of our RV? Must have $\alpha_i = \frac{1}{r_i}$:

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That's a fixed payout.
A fixed, *positive* payout. This is what's known as a Dutch Book.

Alas!

Finding a bookie setting a line that is susceptible to a Dutch Book is. . . not easy.

N bookies, One Race

- Suppose multiple bookies are taking action on a single race.
- They have different clientelle, and are setting different lines
  - Bookie doesn't care about odds, bookie wants to take a vig off of evenly spread bets.
  - Just like we did with our Dutch book.
- Call the most favorable odds across all book makers for each horse r_i^*

If $\sum_i r_i^* < 1$ we have a composite Dutch book! Let the arbitrage times roll.

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With the right combination of speed and luck, however, arbitrage is possible. A frightening amount of effort goes into exactly these games, but played against stock markets.

Also, we haven't so much beaten the house as become it.