Beating the House

(You Will Not Beat the House)

A.C.

August 7, 2024

Outline

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2. The wheel spins.
3. Get paid . . . or don't.
   3.1 House advantage: $pr < 1$

$$X = \alpha\left(-1 + r\mathbb{I}(\mathrm{Win})\right)$$

Open Decisions

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- ~~What bet to place.~~
  - Black feels lucky.
  - $p \approx 0.47$
  - $r = 2$
- How much to bet.
- When to walk away.

The System

procedure $\textsc{ConstantBets}$
  $\alpha \leftarrow 1$
  while true **do**
    bet(α)

Playing it Out

$$W_n \sim \text{Bin(n, p)}$$

$$X_n = r \cdot W_n - n$$

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Expected losses grow as n, stddev as $\sqrt{n}$. Per Chebyshev, we are going broke, *almost certainly*.

The System

procedure MartingaleBetting
    $\alpha \leftarrow 1$
    while true **do**
        bet(α)
        if Win **then**
            return
        else
            $\alpha \leftarrow 2 \cdot \alpha$

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Let N be the round of betting where we finally win:

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Guaranteed profit, on *arbitrarily* bad bets!

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- Very hard to get unlimited betting rounds.
  - Credit limits
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Exercise for the reader: what happens if we can be cut off before the payout?

The System

procedure $\textsc{AntiMartingaleBetting}$

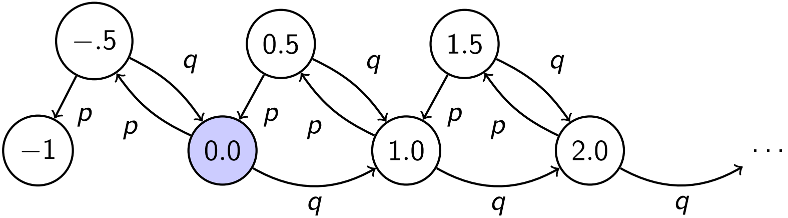
reserves $\leftarrow$ 1.0
target $\leftarrow$ 1.0
$\alpha \leftarrow$ 0.5
while reserves $\leq$ target **do**
reserves $\leftarrow$ reserves $+$ bet(α)
if $\alpha \geq$ reserves **then**
$\alpha \leftarrow 0.5 \cdot \alpha$
if $\alpha \leq 0.25 \cdot$ reserves **then**
$\alpha \leftarrow 2 \cdot \alpha$

Playing it Out

$$P(\text{Lose}) = 0$$

Can't lose, must win... Right?

Playing it Out II



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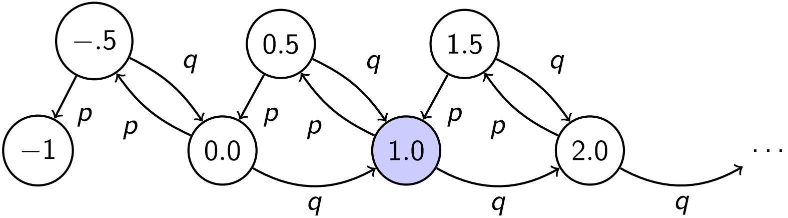

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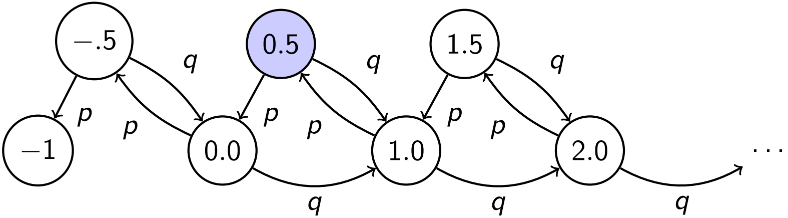

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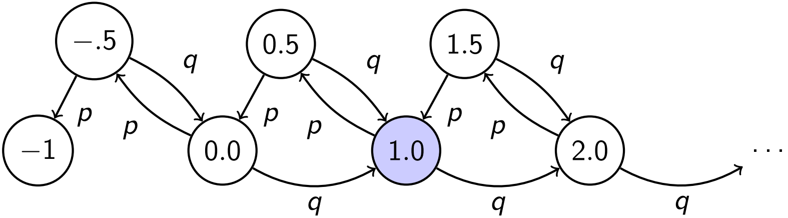

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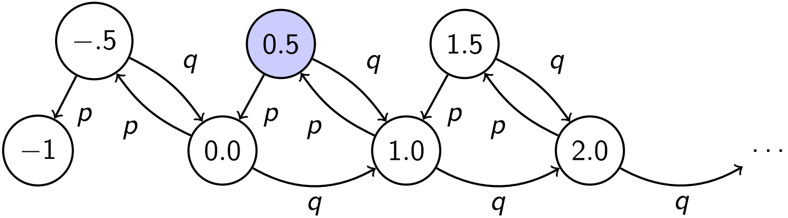

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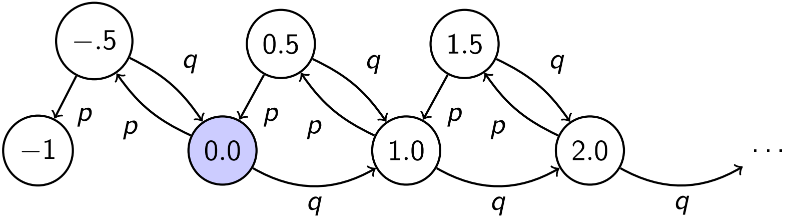

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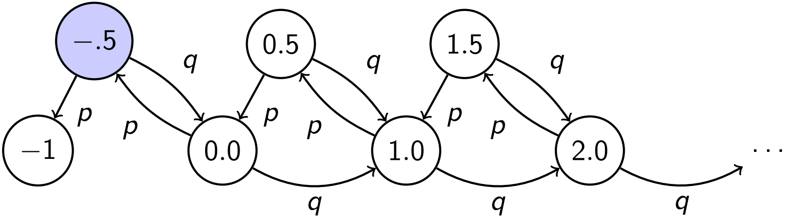

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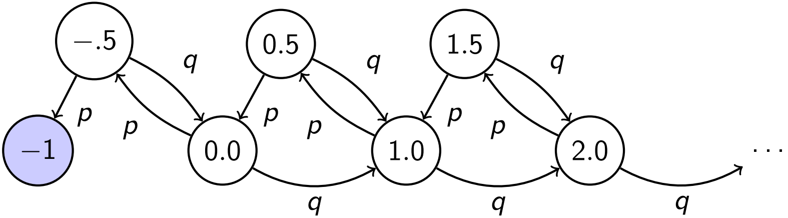

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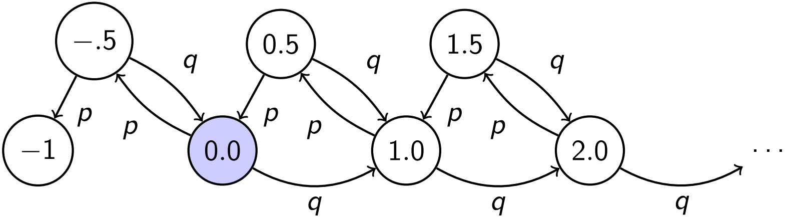

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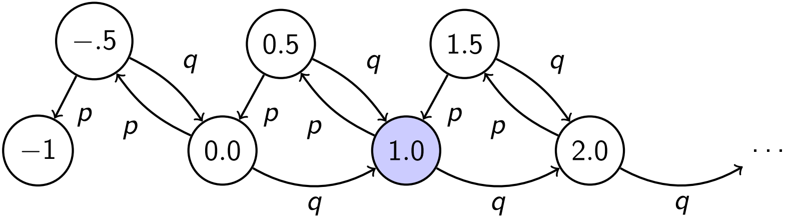

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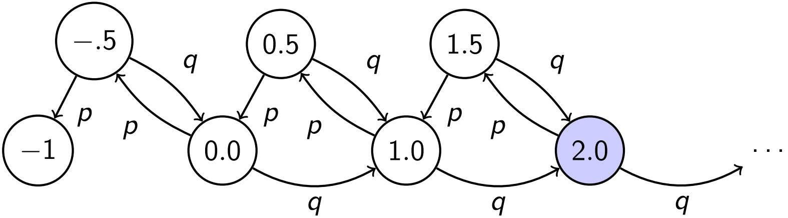

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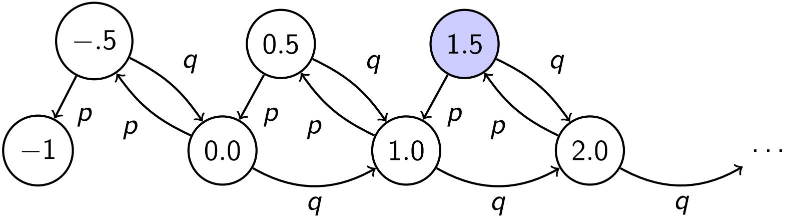

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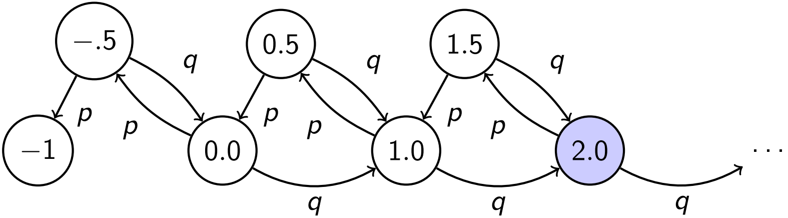

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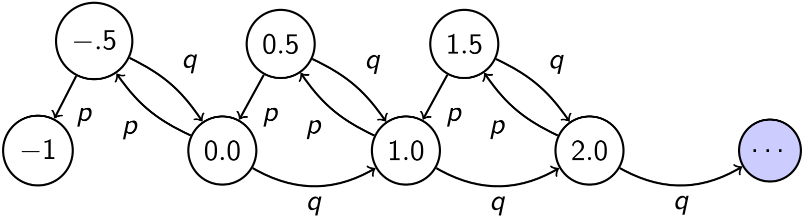

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Massage. series-term-cancellation.

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Massage. series-term-cancellation. limit as $n \to \infty$. Massage.

$$P_0 = \frac{\hat{p}}{\hat{q}}$$

Alas!

We've created a gambler's ruin problem, just flipped and in log scale.

- Nonzero probability of winning *at every step*.
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- Also, need to be able to place arbitrarily small bets.

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$$p = 0.48 \Rightarrow \frac{\hat{p}}{\hat{q}} \approx 0.44$$

Would've been better off laying it all on black.

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1. n horses running, each with probability of winning p_i and payout odds r_i
   1.1 House advantage: $\sum_i \frac{1}{r_i} > 1$

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2. Bet α_i units on each horse
3. The horses run.
4. Get paid according to your wager on the winning horse.

$$X = \sum_i \left[\alpha_i(-1 + r_i \mathbb{I}(h_i))\right]$$

Open Decisions

- Which horse(s) to back.

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- ~~When to walk away.~~
  - We're going to do single round betting here.

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- Still gambling though.

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- Rocinante is a long, long, *long* shot. Maybe 1 in 1000.
- Book-maker offers a special on Rocinante bets: 2000 to 1.
  - Does not update the rest of the book.
  - House advantage slips. Now $\sum_i \frac{1}{r_i} < 1$

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- Could just take the bookie up on the special.
- Bet is positive valued in expectation
- But I still need to make the rent this month, and this is a single round of gambling
- Too rich for my blood, and it's still gambling.

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A fixed, *positive* payout. This is what's known as a Dutch Book.

Alas!

Finding a bookie setting a line that is susceptible to a Dutch Book is. . . not easy.

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If $\sum_i r_i^* < 1$ we have a composite Dutch book! Let the arbitrage times roll.

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Also, if we pull this off we haven't so much beaten the house as become it.