

Monty Hall is not Hard

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1 Traditional Framing

The Monty Hall problem is a classic of introductory probability theory, traditionally used to teach students that nothing makes sense and math is bullshit. Competent teachers may twist the message somewhat to teach the more useful lesson that math is frequently subtle, and it is important to be careful lest we arrive confidently at wrong answers.

In either case, the problem is as follows. You are a contestant on a gameshow, looking at three closed doors. You are told that behind two of the doors there is a goat, and behind one of the doors there is a ferrari. At the end of the game you will have opened one of the doors, and be allowed to keep what you found behind it. You start the game by picking a door. The host will then open one of the other doors to reveal a goat, and ask you if you would like to open the door you started on (**stay**) or if you would like to open the other as yet unopened door (**switch**). Assuming you prefer sports cars to goats, should you switch or stay? What are the probabilities of walking away with the ferrari for each decision?

For people unfamiliar with the problem the most common instinct is to say that it doesn't matter, both doors are the same, so the probability of scoring a ferrari is $\frac{1}{2}$ in either case. People familiar with the problem know that it is better to switch, which gives a $\frac{2}{3}$ win rate, rather than to stay with a $\frac{1}{3}$ win rate. They know this, because somebody took the time to enumerate all the possible outcomes for them and count which would have **switch** winning and which would have **stay** winning, and the ratios were $\frac{2}{3}$ and $\frac{1}{3}$. Such an explanation is correct, and also does nothing to explain why "they're the same" was wrong. And if your brain is anything like ours, it will continue scream "but they're the same" for quite some time after you've learned the solution.

1.1 Counting Possibilities

We sketch a counting based solution here, but only for completeness. Readers are encouraged to skip to the next section for a more human-friendly, but still rigorous, solution.

A game of monty hall can be described by the door the contestant first selected, and by which door the car happened to start behind. We chart below the outcomes for every possible game of Monty Hall.

Contestant Door	Car Door	Winning Strategy
1	1	stay
1	2	switch
1	3	switch
2	1	switch
2	2	stay
2	3	switch
3	1	switch
3	2	switch
3	3	stay

Noting that the car is equally likely to be placed in position 1, 2, or 3, we see by rote calculation that **switch** wins $\frac{2}{3}$ games while **stay** wins the remaining $\frac{1}{3}$.

2 Intuitive Solution

The Monty Hall problem is a parlor trick designed to make people give the wrong answer, and like most parlor tricks it works by hiding important details of what's happening from its audience. More specifically, the reason people want **switch** to equal **stay** is that the human brain, and especially the human brain trained in mathematics or physics, *loves* symmetry. And the Monty Hall game ends with a powerfully symmetric image – two closed doors with a goat behind one and a ferrari behind the other. All the asymmetry derives from one step in the middle, the one where the host opens a goat door, and is remarkably easy to overlook.

Since the problem statement is pathologically bad, all we have to do is restate it equivalently and then solve the new problem. We propose another gameshow, named Monty Cargo Crates. The game starts with you standing

in front of three cargo crates. As before, two of them contain goats, and one contains a ferrari. Like before, you start the game by picking one of the crates. Once you've selected a crate, the host signals some longshoremen to bundle the two remaining crates together into a big crate. Then he runs into the big crate and frees a goat from inside it. Last, you are given the choice between switching to the big crate and the little crate.

We've made progress; the addition of a physical partition between the selected and un-selected sets to match the informational partition is quieting the voice in our head demanding we solve by symmetry, but it's not solvable by inspection yet, so we continue.

Suppose now that Monty Cargo Crates has been running for some time, and that on one of the shows the host broke his leg in a literal sense. Now running into the big crate to free a goat is troublesome, but he realizes that he can offload the work on the contestant. He decides to skip the free-a-goat step and just tell contestants that pick the big bundle that if there are two goats inside to please only take one. The game is now as follows: A contestant selects a crate, the two other crates are bundled together, and the contestant can either choose to search the bundle for a ferrari or her single selected crate. It is obvious at this point that the big bundle will have the ferrari $\frac{2}{3}$ of the time and the lone crate will have it only $\frac{1}{3}$ of the time. Since the big-bundle in Monty Cargo Crates maps onto **switch** in Monty Hall, we expect the same solution there.