Counting Numerals

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1 Introduction

It occurred to me recently that in all my math and computer science education I have never come across a proper demonstration of how and why numerals are an adequate representation of counting numbers. The appropriateness of carry-based arithmetic on Arabic numerals is a fairly obvious consequence of commutativity, at least for students of new math, but it is not immediately clear[1] that every counting number can be represented by an Arabic numeral, and that its representation is unique. Let's take a few minutes here to address that gap.

We'll start by defining Arabic numerals, and their more general counterpart, counting numerals.

Definition 1. A counting numeral in radix $r \in \{2, 3, \ldots\}$ is a finite but arbitrarily long ordered list of digits $(d_0, d_1, \ldots, d_{n-1})$ s.t. $d_k \in \{0, 1, \ldots, r-1\}$. In its standard usage, the numerical value represented by a digit $(d_0, d_1, \ldots, d_{n-1})$ is $d_0 + d_1 r^1 + \cdots + d_{n-1} r^{n-1}$, or in sigma notation $\sum_{k=0}^{n-1} d_k r^k$.

Definition 2. An Arabic numeral is a counting numeral in radix 10.

Of course, we do not commonly write a numeral out as $(d_0, d_1, \ldots, d_{n-1})$, but rather as $d_{n-1} \ldots d_1 d_0$, and sometimes we write the radix as a subscript to make it explicit. E.g. 1337_8 would be the numeral $(7, 3, 3, 1)$ in radix 8.

2 Intuition

The core trick to counting numerals is in a well-known property about them: more significant digits dominate less significant digits. If we wish to know

[1]If the reader believes that it *is* obvious, then we postulate that he or she has simply gotten used to it.

which of a pair of numerals is larger, we only need to compare the most significant digit. We know that 238 is larger than 199 even without comparing the ones and tens places, because $2 > 1$. More than that, we know that the domination is minimal: 100 is precisely one more than 99, 200 is precisely one more than 199, etc. This first fact ensures that different places will not intrude on one another's "turf", and will eventually yield uniqueness, while the second ensures that no numbers can slip through the cracks, and will eventually yield completeness.

3 Rigor

Lemma 3.1. The largest n-digit numeral in radix r has value $r^n - 1$.

Proof. The largest n-digit numeral in radix r is clearly $(r-1, r-1, \ldots, r-1)$, and so its value must be

$$\begin{aligned} &(r-1) + (r-1)r + (r-1)r^2 + \cdots + (r-1)r^{n-1} \\ = &(r-1) + (r^2 - r) + (r^3 - r^2) + \cdots + (r^n - r^{n-1}) \end{aligned}$$

Observe the $+r$ in the first term, the $-r$ in the second term, the $+r^2$ in the second term and the $-r^2$ in the third, and so on. Regrouping, we see that the only two terms that do not cancel out are the -1 from the first term and the r^n from the last, giving us $r^n - 1$, as expected.

Now, math with ellipses should make us at least somewhat uncomfortable, as it can hide mistakes. We therefore repeat the same proof in sigma notation:

$$\begin{aligned} \sum_{k=0}^{n-1} (r-1)r^k &= \sum_{k=0}^{n-1} (r^{k+1} - r^k) \\ &= \sum_{k=0}^{n-1} r^{k+1} - \sum_{k=0}^{n-1} r^k \\ &= \sum_{k=1}^{n} r^k - \sum_{k=0}^{n-1} r^k \\ &= \left(r^n + \sum_{k=1}^{n-1} r^k \right) - \left(\sum_{k=1}^{n-1} r^k + 1 \right) \\ &= r^n - 1 \end{aligned}$$

$\square$

Theorem 3.2. For any arbitrary counting number $0 \leq x < r^n$, there exists an n-digit numeral in radix r whose value is x.

Proof. We proceed inductively on the number of digits in the numeral.

Base case: The value of a one-digit numeral is simply the digit, so the base case of $n = 1$ is obviously satisfied.

Induction step: Find the largest counting number m such that $mr^{n-1} < x$. Observe that $m < r$, or we would have $x \geq r^n$, contrary to the conditions of our construction. Observe also that $x - mr^{n-1} < r^{n-1}$, or we could have chosen a larger m. Write $d_{n-1} := m$, and choose $(d_0, \ldots, d_{n-2})$ to represent $x - mr^{n-1}$, as per the induction hypothesis, and we have

$$\begin{aligned}
x &= x - mr^{n-1} + mr^{n-1} \\
&= \left(\sum_{k=0}^{n-2} d_k r^k \right) + mr^{n-1} \\
&= \left(\sum_{k=0}^{n-2} d_k r^k \right) + d_{n-1} r^{n-1} \\
&= \sum_{k=0}^{n-1} d_k r^k
\end{aligned}$$

$\square$

Theorem 3.3. Given any two numerals in radix r $(a_0, \ldots, a_{n-1})$ and $(b_0, \ldots, b_{n-1})$ with values a and b, if $a = b$, then $a_k = b_k \forall k$.

Proof. We prove the contrapositive: if there exists an index i s.t. $a_i \neq b_i$, then $a \neq b$.

If there exists an i s.t. $a_i \neq b_i$, then there must exist a greatest index s.t. the same holds – our numerals are of finite length. Call this index k.

Without loss of generality, assume $a \geq b$ and consider the difference $a - b$. Because every digit more significant than k is equal, we may write this difference as

$$
\begin{aligned}
a - b &= r^k(a_k - b_k) + \sum_{i=0}^{k-1} \left(a_i r^i - b_i r^i\right) \\
&\geq r^k + \sum_{i=0}^{k-1} \left(a_i r^i - b_i r^i\right) \\
&\geq r^k - (r^k - 1) \\
&= 1
\end{aligned}
$$

The difference between a and b is nonzero, and so $a \neq b$. $\square$