Lecture Notes to Fundamental Concepts in Algebraic Geometry

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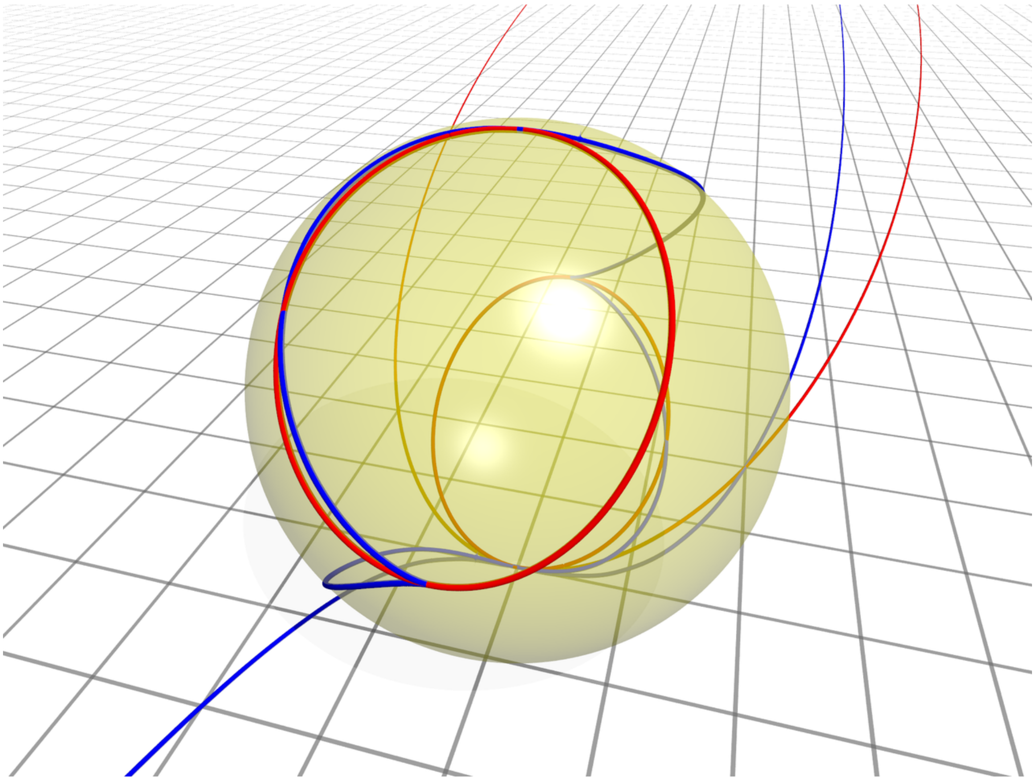

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Preface

Technicalities

These aren't formal notes related to the course and henceforward there is *absolutely no guarantee* that the recorded material is in correspondence with the course expectations, or that these notes lack any mistakes.
In fact, there probably are mistakes in the notes! I would highly appreciate if any comments or corrections were sent to me via email at tzorani.elad@gmail.com.
Elad Tzorani.

Goals of Algebraic Geometry

The main goals of algebraic geometry are the following.

- To classify algebraic varieties up to birational isomorphism. Either by enumeration of the varieties or classification by properties.
- Be able to tell whether two algebraic varieties are (birationally) isomorphic. This is done through different tools:
  - Cohomology (related to the Hodge Conjecture).
  - Vector bundles (related to K-theory).
  - Algebraic Cycles (subvarieties).
  - Coherent sheaves (derived category).
  - Topology.

Chapter 1

Sheaves

1.1 Basic Definitions

Let $X \in \mathbf{Top}$.

Definition 1.1.1. Let $\mathbf{Top}_X$ be the category of open sets on X with inclusions as morphisms.

Notation 1.1.2. Denote by $\mathbf{Ab}$ the category of abelian groups.

Definition 1.1.3 (Presheaf). A ***presheaf (of groups)*** on X is a contravariant functor

$$F\colon \mathbf{Top}_X \to \mathbf{Ab}.$$

Concretely, to give a presheaf we must give the following:

1. For each $U \subseteq X$ open, an abelian group $F(U)$.
2. For inclusion $V \hookrightarrow U$ of open sets, a group homomorphism $F_{U,V}\colon F(U) \to F(V)$.

Under the conditions:

1. $F_{U,U} = \mathrm{id}_{F(U)}$.
2. If $W \hookrightarrow V \hookrightarrow U$ then $W \hookrightarrow U$ so the induced maps

$$\begin{tikzcd} F(U) \arrow[r] \arrow[rr, bend right] & F(V) \arrow[r] & F(W) \end{tikzcd}$$

commute.

Examples. 1. Let

$$F(U) \coloneqq \mathrm{Hom}(U, \mathbb{R}) = \{f\colon U \to \mathbb{R}\}.$$

If $V \subseteq U$ are open,

$$\begin{aligned} F(U) &\xrightarrow{F_{U,V}} F(V) \\ f &\mapsto f|_V \end{aligned}.$$

This is actually also a presheaf of rings on X.

2. ***The constant presheaf:*** Let $A \in \mathbf{Ab}$ and define

$$F(U) \coloneqq A, \quad F_{U,V} = \mathrm{id}_A.$$

3. Let k be an algebraically closed field and X an algebraic variety over k. We can think $X \subseteq \mathbb{P}^n_k$.

   Define $\mathcal{O}_X(U) \coloneqq \{f \colon U \to k \mid f \text{ is regular}\}$. By "regular" we mean that for all $P \in U$ there's a neighbourhood $V \subseteq U$ of P such that
   $$f|_V = \frac{g}{h}$$
   with $g, h \in k[x_0, \ldots, x_n]$.

4. Let $X = \mathbb{P}^1$. Then $\mathcal{O}_{\mathbb{P}^1}(\mathbb{P}^1) = k$. Also $\mathcal{O}_{\mathbb{P}^1}(\mathbb{P}^1 \setminus \{0\}) = \left\{ \frac{g(x,y)}{h(x,y)} \;\middle|\; \substack{g,h \in k[x,y] \\ \forall (x,y) \neq (0,1) \colon h(x,y) \neq 0} \right\} = \left\{ \frac{g(x,y)}{x^{\deg(g)}} \right\}$.

Definition 1.1.4 (Section, Restriction). For any F as above, $s \in F(U)$ is called a ***section***. $F_{U,V}(s)$ is written as $s|_V$ and called ***restriction of*** s ***to*** V.

Definition 1.1.5 (Global Section). $F(X) \coloneqq \Gamma(X, F)$ is the ***global section of*** F.

Remark 1.1.6. Why "section"? Think about $\mathcal{O}_X(U)$ and view the diagram

$$\begin{tikzcd}
U \times k \arrow[r, hook] \arrow[d, "\pi_U"] & X \times k \arrow[d, "\pi"] \\
U \arrow[r, hook] \arrow[u, bend left, "\tilde{s}"] & X
\end{tikzcd}$$

and call $X \times k$ the ***trivial line bundle***.

Take $s \in \mathcal{O}_X(U)$ which induces a map

$$\begin{aligned}
\tilde{s} \colon U &\to U \times k \\
x &\mapsto (X, s(x)).
\end{aligned}$$

Then $\tilde{s}$ is a section of π over U in the sense that $\pi_U \circ \tilde{s} = \mathrm{id}_U$.

The idea of sheaves is that global sections are difficult to study topologically, but $F(U)$ is easier to study for U small, and computation on smaller U can imply properties of $F(U)$.