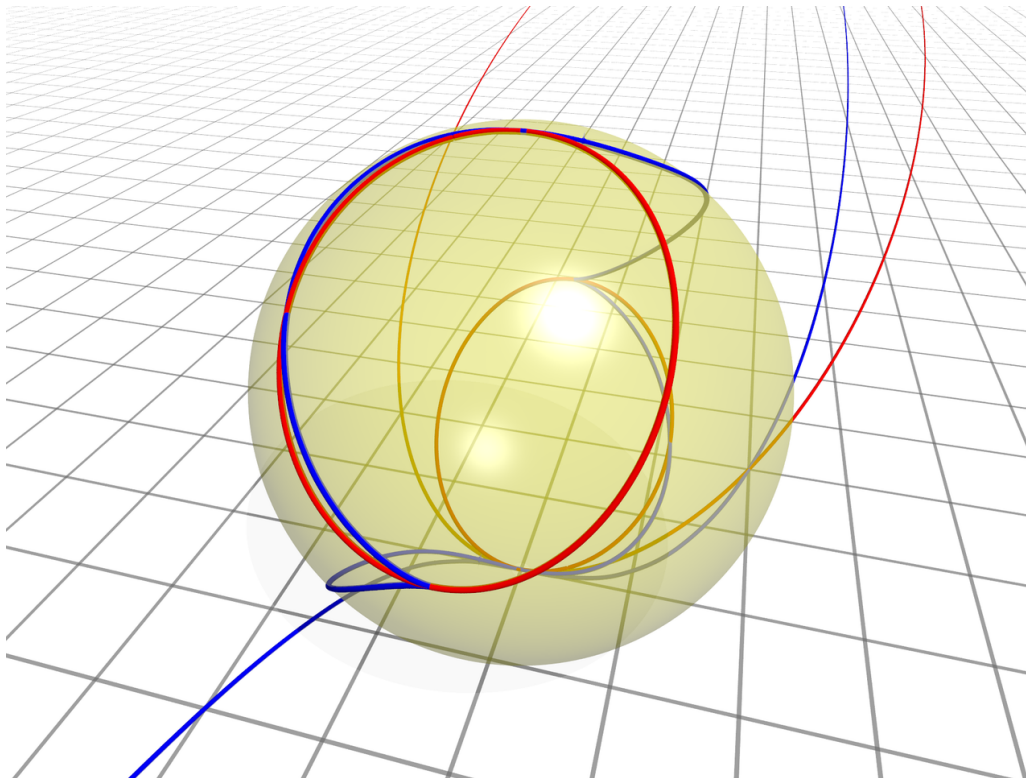


# Lecture Notes to Fundamental Concepts in Algebraic Geometry

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# Preface

## Technicalities

These aren't formal notes related to the course and henceforward there is *absolutely no guarantee* that the recorded material is in correspondence with the course expectations, or that these notes lack any mistakes.

In fact, there probably are mistakes in the notes! I would highly appreciate if any comments or corrections were sent to me via email at [tzorani.elad@gmail.com](mailto:tzorani.elad@gmail.com).

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# Goals of Algebraic Geometry

The main goals of algebraic geometry are the following.

- To classify algebraic varieties up to birational isomorphism. Either by enumeration of the varieties or classification by properties.
- Be able to tell whether two algebraic varieties are (birationally) isomorphic. This is done through different tools:
  - Cohomology (related to the Hodge Conjecture).
  - Vector bundles (related to K-theory).
  - Algebraic Cycles (subvarieties).
  - Coherent sheaves (derived category).
  - Topology.

# Chapter 1

## Sheaves

Let  $X \in \mathbf{Top}$ .

**Definition 1.0.1.** Let  $\mathbf{Top}_X$  be the category of open sets on  $X$  with inclusions as morphisms.

**Notation 1.0.2.** Denote by  $\mathbf{Ab}$  the category of abelian groups.

**Definition 1.0.3 (Presheaf).** A *presheaf (of groups)* on  $X$  is a contravariant functor

$$F: \mathbf{Top}_X \rightarrow \mathbf{Ab}.$$

Concretely, to give a presheaf we must give the following:

1. For each  $U \subseteq X$  open, an abelian group  $F(U)$ .
2. For inclusion  $V \hookrightarrow U$  of open sets, a group homomorphism  $F_{U,V}: F(U) \rightarrow F(V)$ .

Under the conditions:

1.  $F_{U,U} = \text{id}_{F(U)}$ .
2. If  $W \hookrightarrow V \hookrightarrow U$  then  $W \hookrightarrow U$  so the induced maps

$$\begin{array}{ccccc} F(U) & \longrightarrow & F(V) & \longrightarrow & F(W) \\ & \searrow & & \nearrow & \\ & & & & \end{array}$$

commute.

**Examples.** 1. Let

$$F(U) := \text{Hom}(U, \mathbb{R}) = \{f: U \rightarrow \mathbb{R}\}.$$

If  $V \subseteq U$  are open,

$$\begin{array}{ccc} F(U) & \xrightarrow{F_{U,V}} & F(V) \\ f & \mapsto & f|_V \end{array}.$$

This is actually also a presheaf of rings on  $X$ .

2. **The constant presheaf:** Let  $A \in \mathbf{Ab}$  and define

$$F(U) := A, \quad F_{U,V} = \text{id}_A.$$

3. Let  $k$  be an algebraically closed field and  $X$  an algebraic variety over  $k$ . We can think  $X \subseteq \mathbb{P}_k^n$ .

Define  $\mathcal{O}_X(U) := \{f: U \rightarrow k \mid f \text{ is regular}\}$ . By “regular” we mean that for all  $P \in U$  there’s a neighbourhood  $V \subseteq U$  of  $P$  such that

$$f|_V = \frac{g}{h}$$

with  $g, h \in k[x_0, \dots, x_n]$ .

4. Let  $X = \mathbb{P}^1$ . Then  $\mathcal{O}_{\mathbb{P}^1}(\mathbb{P}^1) = k$ . Also  $\mathcal{O}_{\mathbb{P}^1}(\mathbb{P}^1 \setminus \{0\}) = \left\{ \frac{g(x,y)}{h(x,y)} \mid \forall (x,y) \neq (0,1): h(x,y) \neq 0 \right\} = \left\{ \frac{g(x,y)}{x^{\deg(g)}} \right\}$ .

**Definition 1.0.4 (Section, Restriction).** For any  $F$  as above,  $s \in F(U)$  is called a **section**.  $F_{U,V}(s)$  is written as  $s|_V$  and called **restriction of  $s$  to  $V$** .

**Definition 1.0.5 (Global Section).**  $F(X) := \Gamma(X, F)$  is the **global section of  $F$** .

**Remark 1.0.6.** Why “section”? Think about  $\mathcal{O}_X(U)$  and view the diagram

$$\begin{array}{ccc} U \times k & \hookrightarrow & X \times k \\ \tilde{s} \uparrow \downarrow \pi_U & & \downarrow \pi \\ U & \hookrightarrow & X \end{array}$$

and call  $X \times k$  the **trivial line bundle**.

Take  $s \in \mathcal{O}_X(U)$  which induces a map

$$\begin{aligned} \tilde{s}: U &\rightarrow U \times k \\ x &\mapsto (X, s(x)). \end{aligned}$$

Then  $\tilde{s}$  is a section of  $\pi$  over  $U$  in the sense that  $\pi_U \circ \tilde{s} = \text{id}_U$ .

The idea of sheaves is that global sections are difficult to study topologically, but  $F(U)$  is easier to study for  $U$  small, and computation on smaller  $U$  can imply properties of  $F(U)$ .

**Definition 1.0.7 (Sheaf).** A **sheaf (over an abelian group)**  $F$  is a presheaf over an abelian group such that the following holds.

1. If  $U \subseteq X$  is open and  $\{V_i\}_{i \in I}$  is an open cover of  $U$ , and if  $s \in F(U)$  such that  $\forall i \in I: s|_{V_i} = 0$  then  $s = 0$ .
2. If  $U \subseteq X$  is open and  $\{V_i\}_{i \in I}$  is an open cover, and if

$$\forall i \in I \exists s_i \in F(V_i) \forall i, j \in I: s_i|_{V_i \cap V_j} = s_j|_{V_i \cap V_j}$$

then

$$\exists! s \in F(U) \forall i \in I: s|_{V_i} = s_i.$$

**Remark 1.0.8.** 1. We could equivalently remove the uniqueness requirement in the second condition as it's implied by the first condition.

2. With the uniqueness condition, the first condition is implied by the second.

**Remark 1.0.9.** There is a categorical definition of a sheaf over any (not necessarily abelian) category, which might be defined later in these notes.

**Examples.** 1.  $\mathcal{O}_x$  is a sheaf on  $X$ .

2.  $F(U) = A \in \mathbf{Ab}$  is **not** in general a sheaf unless  $A = 0$ .

If  $X$  has two connected components one can choose different elements of  $A$  on two disjoint open sets, which agree on the (empty) intersection, but which we can't lift to the union.

3. Give  $A \in \mathbf{Ab}$  the discrete topology. Define  $F_A(U) := \text{Hom}_{\text{cont}}(U, A)$  the set of continuous functions  $U \rightarrow A$ . This is a sheaf.

This is sometimes called the **constant sheaf**, although it isn't actually constant. If  $U \subseteq A$  is connected then  $F_A(U) \cong A$ .

**Definition 1.0.10 (Stalk of a Sheaf).** Let  $p \in X$  and  $F$  a presheaf on  $X$ . The **stalk of  $F$  at  $p$**  is

$$F_p := \varinjlim_{p \in U \subseteq X} F(U) = \coprod_{p \in U \subseteq X} F(U) / \sim \text{ if } s|_{U \cap V} = t|_{U \cap V}.$$

**Definition 1.0.11 (Morphism of Presheaves).** Let  $F, G$  be presheaves on  $X$ , a morphism  $f: F \rightarrow G$  is a natural transformation.

**Remark 1.0.12.** Concretely, a morphism  $f: F \rightarrow G$  of presheaves is a group homomorphism

$$f_u: F(U) \rightarrow G(U)$$

for every  $U \subseteq X$  open such that the diagram

$$\begin{array}{ccc} F(U) & \xrightarrow{f_U} & G(U) \\ \text{res} \downarrow & & \downarrow \text{res} \\ F(V) & \xrightarrow{f_V} & G(V) \end{array}$$

**Definition 1.0.13 (Isomorphism of Presheaves).** A morphism of presheaves is an isomorphism if it has a two-sided inverse.

**Notation 1.0.14.** If  $\varphi: F \rightarrow G$  is a morphism there's a induced map on stalks

$$\varphi_p: F_p \rightarrow G_p$$

for every  $p \in X$  where we remind

$$F_p = \varinjlim_{p \in U} F(U)$$

$$G_p = \varinjlim_{p \in U} G(U).$$

**Definition 1.0.15 (Morphism of Sheaves).** A morphism of sheaves is a morphism between the respective presheaves.

**Remark 1.0.16.** The above implies that presheaves over  $X$  are a full subcategory of sheaves over  $X$ .

**Proposition 1.0.17.** Let  $\varphi: F \rightarrow G$  be a morphism of sheaves. Then  $\varphi$  is an isomorphism iff  $\varphi_p$  is an isomorphism for every  $p \in X$ .

**Remark 1.0.18.** The above does *not* say that  $F \cong G$  iff  $F_p \cong G_p$  for every  $p \in X$ .

This isn't true.

*Proof.*  $\Rightarrow$ : This is straightforward.

$\Leftarrow$ : **Injectivity:** Let  $U \subseteq X$  open and  $s \in F(U)$  such that  $\varphi(s) = 0$ . WTS (want to show)  $s = 0$ .

Observe the following commutative diagram.

$$\begin{array}{ccc} F(U) & \xrightarrow{\varphi} & G(U) \\ \downarrow & & \downarrow \\ F_p & \xrightarrow{\varphi_p} & G_p \end{array}$$

$\varphi_p(s_p) = 0$  implies by injectivity  $s_p = 0$ . Then

$$\exists p \in W_p \subseteq U: s|_{W_p} = 0.$$

We can cover  $U$  by these  $\{W_p\}_{p \in U}$  is a cover over  $U$  and by the first sheaf condition we get  $s = 0$ .

**Surjectivity:** We want to show that  $\varphi_U: F(U) \rightarrow G(U)$  is surjective for every  $U \subseteq X$  open.

Let  $s \in G(U)$ , we consider the following commutative diagram for every  $p \in X$ .

$$\begin{array}{ccc} F(U) & \xrightarrow{\varphi} & G(U) \\ \downarrow & & \downarrow \\ F_p & \xrightarrow{\varphi_p} & G_p \end{array}$$



Take  $s \in G(U)$  from which we get  $s_p \in G_p$ . Then by surjectivity of  $\varphi_p$  there's  $\tilde{t}_p \in F_p$  such that  $\varphi_p(\tilde{t}_p) = s_p$ . Then there's  $p \in W_p \subseteq U$  and  $t_p \in F(W_p)$  such that  $\varphi_{W_p}(t_p) = s|_{W_p}$ . We want to glue the  $t_p$  to get  $t \in F(U)$ .

Check that the overlaps agree by showing that for  $P, Q \in U$ ,  $t_P, t_Q$  agree. Indeed,

$$\begin{aligned} \varphi(t_P|_{W_P \cap W_Q}) &= \varphi(t_P)|_{W_P \cap W_Q} \\ &= (s|_{W_P})|_{W_P \cap W_Q} \\ &= [\text{same computation}] \\ &= \varphi(t_Q|_{W_P \cap W_Q}) \end{aligned}$$

and by injectivity we get

$$\forall P, Q \in U: t_P|_{W_P \cap W_Q} = t_Q|_{W_P \cap W_Q}.$$

Since  $F$  is a sheaf, there's  $t \in F(U)$  such that  $t|_{W_p} = t_p$ .

We want to check that  $\varphi(t) = s \in G(U)$ . Because  $G$  is a sheaf we can check this locally.

$$\begin{aligned} \varphi(t)|_{W_p} &= \varphi(t|_{W_p}) \\ &= \varphi(t_p) \\ &= s|_{W_p} \end{aligned}$$

Then by uniqueness of the restrictions  $\varphi(t) = s$ , so  $\varphi$  is indeed surjective. ■

**Definition 1.0.19.** Let  $\varphi: F \rightarrow G$  a morphism of presheaves of abelian groups. Define the following presheaves.

$$\begin{aligned} (\ker \varphi)U &:= \ker(\varphi_U) \subseteq F(U) \\ (\widetilde{\text{coker}} \varphi)(U) &:= \text{coker}(\varphi_U) = G(U) / \varphi(F(U)) \\ (\widetilde{\text{Im}} \varphi)(U) &= \text{Im}(\varphi_U) \subseteq G(U) \end{aligned}$$

$\ker \varphi$  is a sheaf if  $F, G$  are. The other two aren't necessarily sheaves in this case.

In order to define cokernel and image which are sheaves, we produce a functor from presheaves to sheaves, which is adjoint to the forgetful functor.

**Proposition 1.0.20 (Sheafification).** Let  $F$  be a presheaf on  $X$ , there's a sheaf  $F^+$  and a morphism  $\theta: F \rightarrow F^+$  such that the following holds.

If  $G$  is a sheaf and  $F \rightarrow G$  is a morphism of presheaves, there's a unique morphism  $\psi: F^+ \rightarrow G$  such that the following diagram commutes.

$$\begin{array}{ccc} F & \xrightarrow{\quad} & G \\ & \searrow \theta & \nearrow \psi \\ & F^+ & \end{array}$$

Moreover,  $(F^+, \theta)$  is unique up to unique isomorphism. I.e if  $F \xrightarrow{\theta'} F'^+$  is another such object, there's a unique isomorphism such that the following commutes.

$$\begin{array}{ccc} F & \xrightarrow{\theta'} & (F')^+ \\ & \searrow \theta & \downarrow \wr \\ & & F^+ \end{array}$$

*Proof.* Define  $F^+$  as follows. For  $U \subseteq X$  open let

$$F^+(U) = \left\{ s: U \rightarrow \prod_{p \in U} F_p \mid \begin{array}{l} \forall p \in U: s(p) \in F_p \\ \forall p \in U \exists p \in V \subseteq U \exists t \in F(U) \forall Q \in V: t_Q = s(Q) \in F_Q \end{array} \right\}.$$

Check that this is a sheaf.

Define

$$\begin{aligned}\theta: F &\rightarrow F^+ \\ F(U) &\rightarrow F^+(U) \\ s \in F(U) &\mapsto \left[ \begin{smallmatrix} U \rightarrow \coprod_{p \rightarrow s_p} F_p \end{smallmatrix} \right].\end{aligned}$$

Check that  $(F^+, \theta)$  satisfies the universal property. ■

We have the following properties.

**Proposition 1.0.21.** 1. If  $F$  is a sheaf,  $\theta: F \rightarrow F^+$  is an isomorphism.

This follows from the universal property.

2. If  $p \in X$  then  $F_p \cong (F^+)_p$  via  $\theta$ .

**Definition 1.0.22 (Subsheaf).** Let  $F$  a sheaf, a **subsheaf**  $F'$  is a sheaf such that the diagram

$$\begin{array}{ccc} F'(U) & \hookrightarrow & F(U) \\ \downarrow & & \downarrow \\ F'(V) & \hookrightarrow & F(V) \end{array}$$

commutes for all  $V \subseteq U$ .

**Definition 1.0.23 (coker, Im).** Let  $\varphi: F \rightarrow G$  a morphism of sheaves then  $\ker \varphi$  is already a sheaf.

Define

$$\begin{aligned}\operatorname{Im} \varphi &:= \left( \widetilde{\operatorname{Im} \varphi} \right)^+ \\ \operatorname{coker} \varphi &:= \left( \widetilde{\operatorname{coker} \varphi} \right)^+.\end{aligned}$$

**Remark 1.0.24.** By the universal property  $\operatorname{Im} \varphi \hookrightarrow G$  is an injection.

**Definition 1.0.25 (Injective Morphism).** We say  $\varphi$  is **injective** if  $\ker \varphi = 0$ , i.e. if  $\varphi_U$  is injective for every  $U$ .

**Definition 1.0.26 (Surjective Morphism).** We say  $\varphi$  is **surjective** if  $\operatorname{Im} \varphi = G$ .

**Remark 1.0.27.** Surjectivity is **not** equivalent to  $\varphi_U$  being surjective for all  $U$ .

**Definition 1.0.28 (Exact Sequence).** Say a sequence of sheaves

$$\cdots \rightarrow F_{-2} \xrightarrow{\varphi_{-2}} F_{-1} \xrightarrow{\varphi_{-1}} F_0 \xrightarrow{\varphi_0} F_1 \xrightarrow{\varphi_1} \cdots$$

is **exact** if  $\ker \varphi_i = \operatorname{Im} \varphi_{i-1}$ .

If  $F' \subseteq F$ , define  $F' / F$  to be the sheafification of the presheaf

$$U \mapsto F'(U) / F(U).$$

**Definition 1.0.29 (Pushward and Inverse Image Sheaves).** Let  $f: X \rightarrow Y$  be a continuous map of topological spaces.

Let  $F$  be a sheaf on  $X$ , we define for  $V \subseteq Y$  open

$$(f_* F)(V) := F(f^{-1}(V)).$$

Then  $f_*F$  is a sheaf on  $Y$  and is called the **pushforward of  $F$** .

If  $f: X \rightarrow Y$  and  $G$  a sheaf on  $Y$ . We define for  $U \subseteq X$  open

$$(f^{-1}G)(U) = \left( \varinjlim_{V \supseteq f(U)} G(V) \right)^+.$$

Then  $f^{-1}G$  is a sheaf and called the **inverse image sheaf**.

**Remark 1.0.30.**  $f_*$  and  $f^{-1}$  are functors between sheaves on  $X$  and  $Y$ .

**Exercise 1.**  $f_*$  and  $f^{-1}$  are adjoint functors. I.e.

$$\mathrm{Hom}_X(f^{-1}G, F) \cong \mathrm{Hom}_Y(G, f_*F)$$

naturally.

**Notation 1.0.31.** Let  $Z \subseteq X$  and  $i: Z \hookrightarrow X$  the inclusion map. For a sheaf  $F$  on  $X$  we write  $F|_Z$  instead of  $i^{-1}F$ .

**Exercise 2.** Assume  $F(A) = A \in \mathbf{Ab}$  and that  $\theta U \subseteq X$ . Let

$$F_A(U) := \mathrm{Hom}_{\mathrm{cont}}(U, A),$$

we get  $F_A \cong F^+$ .

# Chapter 2

## Schemes

### 2.1 Motivation

We notice that in classical algebraic geometry the interesting object of an algebraic variety is its ring of functions  $R$ . It induces a map on the points which are  $\text{mSpec}(R)$ .

There is however the problem that this lacks functoriality. Images and preimages of maximal ideals aren't necessarily maximal. For this one looks at the space of prime ideals  $\text{Spec}(R)$ . If we look at  $R = \mathbb{C}[x]$  we get  $\text{Spec } R = \mathbb{A}^1 \cup \{(0)\}$ . We don't have a natural way to add  $(0)$  to the topology of  $\mathbb{A}^1$ . In higher dimension this is even more problematic. However, these points have important meaning which we want to understand through the notion of schemes.

### 2.2 Schemes - Basic Definitions

Let  $A \in \mathbf{Ring}$ , we define  $\text{Spec } A$  as the space of prime ideals of  $A$ .

Given  $\mathfrak{a} \trianglelefteq A$  we define

$$V(\mathfrak{a}) = \{\mathfrak{p} \in \text{Spec}(A) \mid \mathfrak{p} \supseteq \mathfrak{a}\}.$$

This is in bijection with  $\text{Spec}(A/\mathfrak{a})$ .

**Lemma 2.2.1.** 1. Let  $\mathfrak{a}_1, \mathfrak{a}_2 \trianglelefteq A$ . Then

$$V(\mathfrak{a}_1 \mathfrak{a}_2) = V(\mathfrak{a}_1) \cup V(\mathfrak{a}_2).$$

2. If  $\{\mathfrak{a}_i\}_{i \in I}$  are ideals of  $A$  then

$$V\left(\sum_{i \in I} \mathfrak{a}_i\right) = \bigcap_{i \in I} V(\mathfrak{a}_i).$$

3. If  $\mathfrak{a}_1, \mathfrak{a}_2 \trianglelefteq A$  then

$$V(\mathfrak{a}_1) \subseteq V(\mathfrak{a}_2) \iff \sqrt{\mathfrak{a}_1} \supseteq \sqrt{\mathfrak{a}_2}.$$

*Proof.* See Hartshorne. ■

**Remark 2.2.2.**  $V(A) = \emptyset$  and  $V((0)) = \text{Spec } A$ .

**Definition 2.2.3 (Zariski Topology).** We give  $\text{Spec } A$  the topology defined by the closed sets  $V(\mathfrak{a})$  for  $\mathfrak{a} \trianglelefteq A$ . This is called the *Zariski topology*.

**Definition 2.2.4.** Define a sheaf of rings  $\mathcal{O} = \mathcal{O}_A$  on  $\text{Spec } A$  as follows. For  $U \subseteq \text{Spec } A$  open we define

$$\mathcal{O}(U) := \left\{ s: U \rightarrow \prod_{\mathfrak{p} \in U} A_{\mathfrak{p}} \mid \begin{array}{l} \forall \mathfrak{p} \in U: s(\mathfrak{p}) \in A_{\mathfrak{p}} \\ \forall \mathfrak{p} \in U \exists \mathfrak{p}' \in V \subseteq U \exists a, f \in A (f \notin \mathfrak{p}' \wedge \forall \mathfrak{q} \in V: s(\mathfrak{q}) = \frac{a}{f} \in A_{\mathfrak{q}}) \end{array} \right\}.$$

**Proposition 2.2.5.** •  $\mathcal{O}(U)$  is a commutative unital ring.

- If  $V \subseteq U$ , the map

$$\begin{aligned} \mathcal{O}(U) &\rightarrow \mathcal{O}(V) \\ s &\mapsto s|_V \end{aligned}$$

is a homomorphism of rings.

- $\mathcal{O}$  is a sheaf on  $\operatorname{Spec} A$ .

**Definition 2.2.6.** If  $f \in A$ , set

$$D(f) := \operatorname{Spec} A \setminus V(f).$$