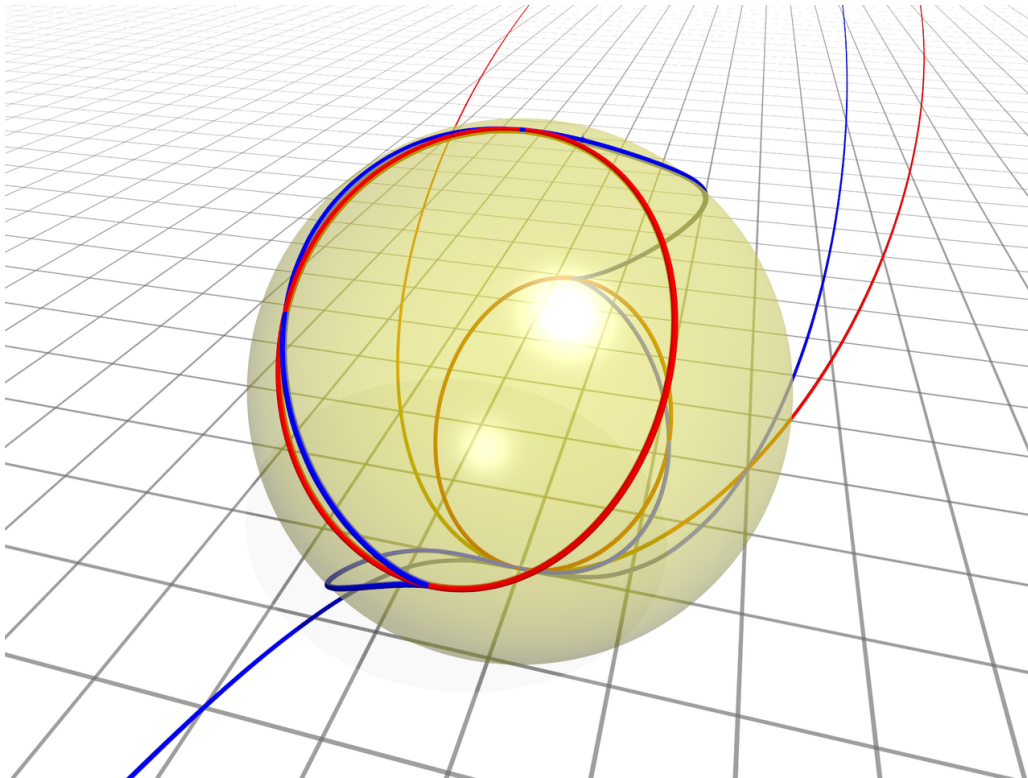


Lecture Notes to Fundamental Concepts in Algebraic Geometry

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Preface

Technicalities

These aren't formal notes related to the course and henceforward there is *absolutely no guarantee* that the recorded material is in correspondence with the course expectations, or that these notes lack any mistakes.

In fact, there probably are mistakes in the notes! I would highly appreciate if any comments or corrections were sent to me via email at tzorani.elad@gmail.com.

Elad Tzorani.

Goals of Algebraic Geometry

The main goals of algebraic geometry are the following.

- To classify algebraic varieties up to birational isomorphism. Either by enumeration of the varieties or classification by properties.
- Be able to tell whether two algebraic varieties are (birationally) isomorphic. This is done through different tools:
 - Cohomology (related to the Hodge Conjecture).
 - Vector bundles (related to K-theory).
 - Algebraic Cycles (subvarieties).
 - Coherent sheaves (derived category).
 - Topology.

Chapter 1

Sheaves

Let $X \in \mathbf{Top}$.

Definition 1.0.1. Let \mathbf{Top}_X be the category of open sets on X with inclusions as morphisms.

Notation 1.0.2. Denote by \mathbf{Ab} the category of abelian groups.

Definition 1.0.3 (Presheaf). A *presheaf (of groups)* on X is a contravariant functor

$$F: \mathbf{Top}_X \rightarrow \mathbf{Ab}.$$

Concretely, to give a presheaf we must give the following:

1. For each $U \subseteq X$ open, an abelian group $F(U)$.
2. For inclusion $V \hookrightarrow U$ of open sets, a group homomorphism $F_{U,V}: F(U) \rightarrow F(V)$.

Under the conditions:

1. $F_{U,U} = \text{id}_{F(U)}$.
2. If $W \hookrightarrow V \hookrightarrow U$ then $W \hookrightarrow U$ so the induced maps

$$\begin{array}{ccccc} F(U) & \longrightarrow & F(V) & \longrightarrow & F(W) \\ & \searrow & & \nearrow & \\ & & & & \end{array}$$

commute.

Examples. 1. Let

$$F(U) := \text{Hom}(U, \mathbb{R}) = \{f: U \rightarrow \mathbb{R}\}.$$

If $V \subseteq U$ are open,

$$\begin{array}{ccc} F(U) & \xrightarrow{F_{U,V}} & F(V) \\ f & \mapsto & f|_V \end{array}.$$

This is actually also a presheaf of rings on X .

2. **The constant presheaf:** Let $A \in \mathbf{Ab}$ and define

$$F(U) := A, \quad F_{U,V} = \text{id}_A.$$

3. Let k be an algebraically closed field and X an algebraic variety over k . We can think $X \subseteq \mathbb{P}_k^n$.

Define $\mathcal{O}_X(U) := \{f: U \rightarrow k \mid f \text{ is regular}\}$. By “regular” we mean that for all $P \in U$ there’s a neighbourhood $V \subseteq U$ of P such that

$$f|_V = \frac{g}{h}$$

with $g, h \in k[x_0, \dots, x_n]$.

4. Let $X = \mathbb{P}^1$. Then $\mathcal{O}_{\mathbb{P}^1}(\mathbb{P}^1) = k$. Also $\mathcal{O}_{\mathbb{P}^1}(\mathbb{P}^1 \setminus \{0\}) = \left\{ \frac{g(x,y)}{h(x,y)} \mid \forall (x,y) \neq (0,1): h(x,y) \neq 0 \right\} = \left\{ \frac{g(x,y)}{x^{\deg(g)}} \right\}.$

Definition 1.0.4 (Section, Restriction). For any F as above, $s \in F(U)$ is called a **section**. $F_{U,V}(s)$ is written as $s|_V$ and called **restriction of s to V** .

Definition 1.0.5 (Global Section). $F(X) := \Gamma(X, F)$ is the **global section of F** .

Remark 1.0.6. Why “section”? Think about $\mathcal{O}_X(U)$ and view the diagram

$$\begin{array}{ccc} U \times k & \hookrightarrow & X \times k \\ \tilde{s} \nearrow \downarrow \pi_U & & \downarrow \pi \\ U & \hookrightarrow & X \end{array}$$

and call $X \times k$ the **trivial line bundle**.

Take $s \in \mathcal{O}_X(U)$ which induces a map

$$\begin{aligned} \tilde{s}: U &\rightarrow U \times k \\ x &\mapsto (X, s(x)). \end{aligned}$$

Then \tilde{s} is a section of π over U in the sense that $\pi_U \circ \tilde{s} = \text{id}_U$.

The idea of sheaves is that global sections are difficult to study topologically, but $F(U)$ is easier to study for U small, and computation on smaller U can imply properties of $F(U)$.

Definition 1.0.7 (Sheaf). A **sheaf (over an abelian group)** F is a presheaf over an abelian group such that the following holds.

1. If $U \subseteq X$ is open and $\{V_i\}_{i \in I}$ is an open cover of U , and if $s \in F(U)$ such that $\forall i \in I: s|_{V_i} = 0$ then $s = 0$.
2. If $U \subseteq X$ is open and $\{V_i\}_{i \in I}$ is an open cover, and if

$$\forall i \in I \exists s_i \in F(V_i) \forall i, j \in I: s_i|_{V_i \cap V_j} = s_j|_{V_i \cap V_j}$$

then

$$\exists! s \in F(U) \forall i \in I: s|_{V_i} = s_i.$$

Remark 1.0.8. 1. We could equivalently remove the uniqueness requirement in the second condition as it's implied by the first condition.

2. With the uniqueness condition, the first condition is implied by the second.

Remark 1.0.9. There is a categorical definition of a sheaf over any (not necessarily abelian) category, which might be defined later in these notes.

Examples. 1. \mathcal{O}_x is a sheaf on X .

2. $F(U) = A \in \mathbf{Ab}$ is **not** in general a sheaf unless $A = 0$.

If X has two connected components one can choose different elements of A on two disjoint open sets, which agree on the (empty) intersection, but which we can't lift to the union.

3. Give $A \in \mathbf{Ab}$ the discrete topology. Define $F_A(U) := \text{Hom}_{\text{cont}}(U, A)$ the set of continuous functions $U \rightarrow A$. This is a sheaf.

This is sometimes called the **constant sheaf**, although it isn't actually constant. If $U \subseteq A$ is connected then $F_A(U) \cong A$.

Definition 1.0.10 (Stalk of a Sheaf). Let $p \in X$ and F a presheaf on X . The **stalk of F at p** is

$$F_p := \lim_{p \in U \subseteq X} F(U) = \coprod_{p \in U \subseteq X} F(U) / \sim \quad \begin{array}{l} (U, s \in F(U)) \sim (V, t \in F(V)) \\ \text{if } s|_{U \cap V} = t|_{U \cap V} \end{array}.$$

Definition 1.0.11 (Morphism of Presheaves). Let F, G be presheaves on X , a morphism $f: F \rightarrow G$ is a natural transformation.

Remark 1.0.12. Concretely, a morphism $f: F \rightarrow G$ of presheaves is a group homomorphism

$$f_U: F(U) \rightarrow G(U)$$

for every $U \subseteq X$ open such that the diagram

$$\begin{array}{ccc} F(U) & \xrightarrow{f_U} & G(U) \\ \text{res} \downarrow & & \downarrow \text{res} \\ F(V) & \xrightarrow{f_V} & G(V) \end{array}$$

commutes for every opens $V \subseteq U$.

Definition 1.0.13 (Isomorphism of Presheaves). A morphism of presheaves is an isomorphism if it has a two-sided inverse.

Notation 1.0.14. If $\varphi: F \rightarrow G$ is a morphism there's a induced map on stalks

$$\varphi_p: F_p \rightarrow G_p$$

for every $p \in X$ where we remind

$$F_p = \varinjlim_{p \in U} F(U)$$

$$G_p = \varinjlim_{p \in U} G(U).$$

Definition 1.0.15 (Morphism of Sheaves). A morphism of sheaves is a morphism between the respective presheaves.

Remark 1.0.16. The above implies that presheaves over X are a full subcategory of sheaves over X .

Proposition 1.0.17. Let $\varphi: F \rightarrow G$ be a morphism of sheaves. Then φ is an isomorphism iff φ_p is an isomorphism for every $p \in X$.

Remark 1.0.18. The above does *not* say that $F \cong G$ iff $F_p \cong G_p$ for every $p \in X$. This isn't true.

Proof. \Rightarrow : This is straightforward.

\Leftarrow : **Injectivity:** Let $U \subseteq X$ open and $s \in F(U)$ such that $\varphi(s) = 0$. WTS (want to show) $s = 0$. Observe the following commutative diagram.

$$\begin{array}{ccc} F(U) & \xrightarrow{\varphi} & G(U) \\ \downarrow & & \downarrow \\ F_p & \xrightarrow{\varphi_p} & G_p \end{array}$$

$\varphi_p(s_p) = 0$ implies by injectivity $s_p = 0$. Then

$$\exists p \in W_p \subseteq U: s|_{W_p} = 0.$$

We can cover U by these $\{W_p\}_{p \in U}$ is a cover over U and by the first sheaf condition we get $s = 0$.

Surjectivity: We want to show that $\varphi_U: F(U) \rightarrow G(U)$ is surjective for every $U \subseteq X$ open.

Let $s \in G(U)$, we consider the following commutative diagram for every $p \in X$.

$$\begin{array}{ccc} F(U) & \xrightarrow{\varphi} & G(U) \\ \downarrow & & \downarrow \\ F_p & \xrightarrow{\varphi_p} & G_p \end{array}$$

Take $s \in G(U)$ from which we get $s_p \in G_p$. Then by surjectivity of φ_p there's $\tilde{t}_p \in F_p$ such that $\varphi_p(\tilde{t}_p) = s_p$. Then there's $p \in W_p \subseteq U$ and $t_p \in F(W_p)$ such that $\varphi_{W_p}(t_p) = s|_{W_p}$. We want to glue the t_p to get $t \in F(U)$.

Check that the overlaps agree by showing that for $P, Q \in U$, t_P, t_Q agree. Indeed,

$$\begin{aligned}\varphi(t_P|_{W_P \cap W_Q}) &= \varphi(t_P)|_{W_P \cap W_Q} \\ &= (s|_{W_P})|_{W_P \cap W_Q} \\ &= [\text{same computation}] \\ &= \varphi(t_Q|_{W_P \cap W_Q})\end{aligned}$$

and by injectivity we get

$$\forall P, Q \in U: t_P|_{W_P \cap W_Q} = t_Q|_{W_P \cap W_Q}.$$

Since F is a sheaf, there's $t \in F(U)$ such that $t|_{W_P} = t_P$.

We want to check that $\varphi(t) = s \in G(U)$. Because G is a sheaf we can check this locally.

$$\begin{aligned}\varphi(t)|_{W_P} &= \varphi(t|_{W_P}) \\ &= \varphi(t_P) \\ &= s|_{W_P}\end{aligned}$$

Then by uniqueness of the restrictions $\varphi(t) = s$, so φ is indeed surjective. ■

Definition 1.0.19. Let $\varphi: F \rightarrow G$ a morphism of presheaves of abelian groups. Define the following presheaves.

$$\begin{aligned}(\ker \varphi)U &:= \ker(\varphi_U) \subseteq F(U) \\ (\widetilde{\text{coker}} \varphi)(U) &:= \text{coker}(\varphi_U) = G(U) / \varphi(F(U)) \\ (\widetilde{\text{Im}} \varphi)(U) &= \text{Im}(\varphi_U) \subseteq G(U)\end{aligned}$$

$\ker \varphi$ is a sheaf if F, G are. The other two aren't necessarily sheaves in this case.

In order to define cokernel and image which are sheaves, we produce a functor from presheaves to sheaves, which is adjoint to the forgetful functor.

Proposition 1.0.20 (Sheafification). Let F be a presheaf on X , there's a sheaf F^+ and a morphism $\theta: F \rightarrow F^+$ such that the following holds.

If G is a sheaf and $F \rightarrow G$ is a morphism of presheaves, there's a unique morphism $\psi: F^+ \rightarrow G$ such that the following diagram commutes.

$$\begin{array}{ccc} F & \xrightarrow{\quad} & G \\ \theta \searrow & & \nearrow \psi \\ & F^+ & \end{array}$$

Moreover, (F^+, θ) is unique up to unique isomorphism. I.e if $F \xrightarrow{\theta'} F'^+$ is another such object, there's a unique isomorphism such that the following commutes.

$$\begin{array}{ccc} F & \xrightarrow{\theta'} & (F')^+ \\ \theta \searrow & & \downarrow \wr \\ & & F^+ \end{array}$$

Proof. Define F^+ as follows. For $U \subseteq X$ open let

$$F^+(U) = \left\{ s: U \rightarrow \prod_{p \in U} F_p \mid \forall p \in U \exists p \in V \subseteq U \exists t \in F(V) \forall Q \in V: t_Q = s(Q) \in F_Q \right\}.$$

Check that this is a sheaf.

Define

$$\begin{aligned}\theta: F &\rightarrow F^+ \\ F(U) &\rightarrow F^+(U) \\ s \in F(U) &\mapsto [U \xrightarrow{\quad} \prod_{p \in U} F_p]_{p \mapsto s_p}.\end{aligned}$$

Check that (F^+, θ) satisfies the universal property. ■

We have the following properties.

Proposition 1.0.21. 1. If F is a sheaf, $\theta: F \rightarrow F^+$ is an isomorphism.

This follows from the universal property.

2. If $p \in X$ then $F_p \cong (F^+)_p$ via θ .

Definition 1.0.22 (Subsheaf). Let F a sheaf, a **subsheaf** F' is a sheaf such that the diagram

$$\begin{array}{ccc} F'(U) & \hookrightarrow & F(U) \\ \downarrow & & \downarrow \\ F'(V) & \hookrightarrow & F(V) \end{array}$$

commutes for all $V \subseteq U$.

Definition 1.0.23 (coker, Im). Let $\varphi: F \rightarrow G$ a morphism of sheaves then $\ker \varphi$ is already a sheaf. Define

$$\begin{aligned} \operatorname{Im} \varphi &:= \left(\widetilde{\operatorname{Im} \varphi} \right)^+ \\ \operatorname{coker} \varphi &:= \left(\widetilde{\operatorname{coker} \varphi} \right)^+. \end{aligned}$$

Remark 1.0.24. By the universal property $\operatorname{Im} \varphi \hookrightarrow G$ is an injection.

Definition 1.0.25 (Injective Morphism). We say φ is **injective** if $\ker \varphi = 0$, i.e. if φ_U is injective for every U .

Definition 1.0.26 (Surjective Morphism). We say φ is **surjective** if $\operatorname{Im} \varphi = G$.

Remark 1.0.27. Surjectivity is **not** equivalent to φ_U being surjective for all U .

Definition 1.0.28 (Exact Sequence). Say a sequence of sheaves

$$\cdots \rightarrow F_{-2} \xrightarrow{\varphi_{-2}} F_{-1} \xrightarrow{\varphi_{-1}} F_0 \xrightarrow{\varphi_0} F_1 \xrightarrow{\varphi_1} \cdots$$

is **exact** if $\ker \varphi_i = \operatorname{Im} \varphi_{i-1}$.

If $F' \subseteq F$, define F' / F to be the sheafification of the presheaf

$$U \mapsto F'(U) / F(U).$$

Definition 1.0.29 (Pushforward and Inverse Image Sheaves). Let $f: X \rightarrow Y$ be a continuous map of topological spaces.

Let F be a sheaf on X , we define for $V \subseteq Y$ open

$$(f_* F)(V) := F(f^{-1}(V)).$$

Then $f_* F$ is a sheaf on Y and is called the **pushforward of F** .

If $f: X \rightarrow Y$ and G a sheaf on Y . We define for $U \subseteq X$ open

$$(f^{-1} G)(U) = \left(\varinjlim_{V \supseteq f(U)} G(V) \right)^+.$$

Then $f^{-1} G$ is a sheaf and called the **inverse image sheaf**.

Remark 1.0.30. f_* and f^{-1} are functors between sheaves on X and Y .

Exercise 1. f_* and f^{-1} are adjoint functors. I.e.

$$\operatorname{Hom}_X(f^{-1} G, F) \cong \operatorname{Hom}_Y(G, f_* F)$$

naturally.

Notation 1.0.31. Let $Z \subseteq X$ and $i: Z \hookrightarrow X$ the inclusion map. For a sheaf F on X we write $F|_Z$ instead of $i^{-1} F$.

Exercise 2. Assume $F(A) = A \in \mathbf{Ab}$ and that $\theta U \subseteq X$. Let

$$F_A(U) := \operatorname{Hom}_{\text{cont}}(U, A),$$

we get $F_A \cong F^+$.

Chapter 2

Schemes

2.1 Motivation

We notice that in classical algebraic geometry the interesting object of an algebraic variety is its ring of functions R . It induces a map on the points which are $\text{mSpec}(R)$.

There is however the problem that this lacks functoriality. Images and preimages of maximal ideals aren't necessarily maximal. For this one looks at the space of prime ideals $\text{Spec}(R)$. If we look at $R = \mathbb{C}[x]$ we get $\text{Spec } R = \mathbb{A}^1 \cup \{(0)\}$. We don't have a natural way to add (0) to the topology of \mathbb{A}^1 . In higher dimension this is even more problematic. However, these points have important meaning which we want to understand through the notion of schemes.

2.2 Schemes - Basic Definitions

Let $A \in \mathbf{Ring}$, we define $\text{Spec } A$ as the space of prime ideals of A .

Given $\mathfrak{a} \subseteq A$ we define

$$V(\mathfrak{a}) = \{\mathfrak{p} \in \text{Spec}(A) \mid \mathfrak{p} \supseteq \mathfrak{a}\}.$$

This is in bijection with $\text{Spec}(A/\mathfrak{a})$.

Lemma 2.2.1. 1. Let $\mathfrak{a}_1, \mathfrak{a}_2 \subseteq A$. Then

$$V(\mathfrak{a}_1 \mathfrak{a}_2) = V(\mathfrak{a}_1) \cup V(\mathfrak{a}_2).$$

2. If $\{\mathfrak{a}_i\}_{i \in I}$ are ideals of A then

$$V\left(\sum_{i \in I} \mathfrak{a}_i\right) = \bigcap_{i \in I} V(\mathfrak{a}_i).$$

3. If $\mathfrak{a}_1, \mathfrak{a}_2 \subseteq A$ then

$$V(\mathfrak{a}_1) \subseteq V(\mathfrak{a}_2) \iff \sqrt{\mathfrak{a}_1} \supseteq \sqrt{\mathfrak{a}_2}.$$

Proof. See Hartshorne. ■

Remark 2.2.2. $V(A) = \emptyset$ and $V((0)) = \text{Spec } A$.

Definition 2.2.3 (Zariski Topology). We give $\text{Spec } A$ the topology defined by the closed sets $V(\mathfrak{a})$ for $\mathfrak{a} \subseteq A$. This is called the *Zariski topology*.

Definition 2.2.4. Define a sheaf of rings $\mathcal{O} = \mathcal{O}_A$ on $\text{Spec } A$ as follows. For $U \subseteq \text{Spec } A$ open we define

$$\mathcal{O}(U) := \left\{ s: U \rightarrow \prod_{\mathfrak{p} \in U} A_{\mathfrak{p}} \mid \begin{array}{l} \forall \mathfrak{p} \in U: s(\mathfrak{p}) \in A_{\mathfrak{p}} \\ \forall \mathfrak{p} \in U \exists \mathfrak{p}' \in V \subseteq U \exists a, f \in A (f \notin \mathfrak{p}' \wedge \forall \mathfrak{q} \in V: s(\mathfrak{q}) = \frac{a}{f} \in A_{\mathfrak{q}}) \end{array} \right\}.$$

Proposition 2.2.5. • $\mathcal{O}(U)$ is a commutative unital ring.

- If $V \subseteq U$, the map

$$\begin{aligned} \mathcal{O}(U) &\rightarrow \mathcal{O}(V) \\ s &\mapsto s|_V \end{aligned}$$

is a homomorphism of rings.

- \mathcal{O} is a sheaf on $\text{Spec } A$.

Definition 2.2.6. If $f \in A$, set

$$D(f) := \text{Spec } A \setminus V(f) = \{\mathfrak{p} \in \text{Spec } A \mid f \notin \mathfrak{p}\}.$$

Lemma 2.2.7. The sets $D(f)$ are a basis for the Zariski topology on $\text{Spec } A$.

Proof. In Hartshorne. ■

Proposition 2.2.8. 1. If $\mathfrak{p} \in \text{Spec } A$ then the stalk $\mathcal{O}_{\mathfrak{p}}$ of \mathcal{O} at \mathfrak{p} is isomorphic to $A_{\mathfrak{p}}$.

2. $\mathcal{O}(D(f)) \cong A_f$.

3.

$$\Gamma(\text{Spec } A, \mathcal{O}) = \mathcal{O}_A(\text{Spec } A) = \mathcal{O}(D(1)) \cong A_1 \cong A.$$

We want to give intuition for the sheaf \mathcal{O}_A on $\text{Spec } (A)$. Let $U \subseteq X$ open and $s \in \mathcal{O}_A(U)$. Then

$$s: U \rightarrow \prod_{\mathfrak{p} \in U} A_{\mathfrak{p}}.$$

$A_{\mathfrak{p}}$ is a local ring with maximal ideal $\mathfrak{p}A_{\mathfrak{p}}$. Then

$$k_{\mathfrak{p}} := A_{\mathfrak{p}} / \mathfrak{p}A_{\mathfrak{p}}$$

is a field called *the residue field of A at \mathfrak{p}* .

From $s \in \mathcal{O}(U)$ we get a map

$$\tilde{s}: U \rightarrow \prod_{\mathfrak{p} \in U} A_{\mathfrak{p}} \rightarrow \prod_{\mathfrak{p} \in U} k_{\mathfrak{p}}.$$

Example 1. Let $A = \mathbb{Z}$, we have

$$\text{Spec } A = \{(0), (2), (3), (5), (7), \dots\}.$$

Let

$$U = D(15) = \text{Spec } \mathbb{Z} \setminus V(15) = \text{Spec } (\mathbb{Z}) \setminus \{(3), (5)\}.$$

The above proposition says

$$\mathcal{O}_{\text{Spec } \mathbb{Z}}(U) = \mathbb{Z} \left[\frac{1}{15} \right].$$

Take $\frac{11}{5} \in \mathbb{Z} \left[\frac{1}{5} \right]$. Thinking of maps of the form \tilde{s} we have maps

$$(p) \mapsto \frac{11}{5} \in \mathbb{Z}_{\mathfrak{p}} \mapsto \frac{11}{5} \in \mathbb{F}_{\mathfrak{p}}$$

and

$$(0) \mapsto \frac{11}{5} \in \mathbb{Q} \mapsto \frac{11}{5} \in \mathbb{Q}.$$

Proof (2.2.8). 1. Define

$$\begin{aligned} \mathcal{O}_{\mathfrak{p}} &\xrightarrow{\varphi} A_{\mathfrak{p}} \\ (\mathfrak{p} \in U, s) &\mapsto s(\mathfrak{p}). \end{aligned}$$

Surjectivity: Let $\frac{a}{f} \in A_{\mathfrak{p}}$, so $a \in A$ and $f \notin \mathfrak{p}$. Take $U = D(f)$. Then

$$\begin{aligned} s: D(f) &\rightarrow \prod_{\mathfrak{q} \in D(f)} A_{\mathfrak{q}} \\ \mathfrak{q} &\mapsto \frac{a}{f} \in A_{\mathfrak{q}}. \end{aligned}$$

Then $\varphi(s) = \frac{a}{f}$.

Injectivity: Suppose $\mathfrak{p} \in U \subseteq \text{Spec } A$ and $s, t \in \mathcal{O}(U)$ such that $\varphi(s) = \varphi(t)$. So $s(\mathfrak{p}) = t(\mathfrak{p})$.

By definition of \mathcal{O} we can shrink U so that

$$s = \frac{a}{p}, \quad t = \frac{b}{g}$$

where $f, g \notin \mathfrak{p}$.

Then $\frac{a}{f} = \frac{b}{g} \in A_{\mathfrak{p}}$ so

$$\exists h \notin \mathfrak{p}: h(ga - fb) = 0 \in A.$$

Let

$$V = D(f) \cap D(g) \cap D(h)$$

then $\forall \mathfrak{q} \in V \subseteq U: S_{\mathfrak{q}} = t_{\mathfrak{q}}$. So

$$s|_V = t|_V$$

so $s = t \in \mathcal{O}_{\mathfrak{p}}$.

2. Let

$$\begin{aligned} \psi: A_f &\rightarrow \mathcal{O}(D(f)) \\ \frac{a}{f} &\mapsto \left\{ \begin{array}{l} D(f) \rightarrow \prod_{\mathfrak{q} \ni t} A_{\mathfrak{q}} \\ \mathfrak{q} \mapsto \frac{a}{f^n} \in A_{\mathfrak{q}} \end{array} \right. \end{aligned}$$

Then ψ is

Injective: $\psi\left(\frac{a}{f^n}\right) = \psi\left(\frac{b}{f^m}\right)$ hence $\frac{a}{f^n} = \frac{b}{f^m}$ in $A_{\mathfrak{q}}$. Hence

$$\exists h_{\mathfrak{p}} \notin \mathfrak{p}: h_{\mathfrak{p}}(f^m a - f^n b) = 0.$$

Set $\mathfrak{a} = \text{Ann}(f^m a - f^n b)$. So $h_{\mathfrak{p}} \in \mathfrak{a} \setminus \mathfrak{p}$ hence $\mathfrak{a} \not\subseteq \mathfrak{p}$. Hence $V(\mathfrak{a}) \cap D(f) = \emptyset$. Hence $V(\mathfrak{a}) \subseteq V(f)$ hence $f \in \sqrt{\mathfrak{a}}$ so $f^+ \in \mathfrak{a}$ so $f^+(f^m a - f^n b) = 0$ so $\frac{a}{f^n} = \frac{b}{f^m}$ in A_f so ψ is injective.

Surjective: We have

$$\begin{aligned} \psi: A_f &\rightarrow \mathcal{O}(D(f)) \\ \frac{a}{f^n} &\mapsto \left[\mathfrak{p} \mapsto \frac{a}{f^n} \right]. \end{aligned}$$

We notice two big facts.

Fact 2.2.9.

$$D(f) \subseteq D(g) \iff \exists n \in \mathbb{N}_+: f^n \in (g)$$

(Exercise / See Hartshorne or Atiyah McDonald).

Fact 2.2.10. If $D(f) \subseteq \bigcup_{i \in I} D(f_i)$ then $D(f) \subseteq \bigcup_{j \in J} D(f_j)$ for some $J \subseteq I$ finite.

Let $s \in \mathcal{O}(D(f))$, we cover $D(f)$ by $\{V_i\}_{i \in I}$ such that $s|_{V_i} = \frac{a_i}{g_i}$ and $\forall \mathfrak{p} \in V_i: g_i \notin \mathfrak{p}$. So $V_i \subseteq D(g_i)$. Check, using the first fact above, that we may assume that $V_i = D(g_i)$. Moreover, by the second fact we may assume I is finite.

Note that

$$\frac{a_i}{g_i} = \frac{a_j}{g_j}$$

on $D(g_i) \cap D(g_j) = D(g_i g_j)$, i.e. the equality holds in $A_{g_i g_j}$. There's $n_{i,j}$ such that

$$(g_i g_j)^{n_{i,j}} (g_j a_i - g_i a_j) = 0$$

and because the index set is finite we may assume $n_{i,j} = n$ for all i, j .

Then

$$g_j^{n+1} (g_i^n A_i) - g_i^{n+1} (g_j^n a_j) = 0.$$

Replace g_i by g_i^{n+1} and a_i by $g_1^n a_i$. We use $D(g_i) = D(g_i^{n+1})$. Then

$$\forall i, j: g_j a_i = g_i a_j.$$

Since $D(g_i)$ cover $D(f)$. So $f \in \sqrt{\sum_i (g_i)}$ so there's $m \in \mathbb{N}$ such that $f^m = \sum_i b_i g_i$ for some $b_i \in A$. Let $a = \sum_i b_i a_i$. Then

$$g_j a = \sum_i b_i a_i g_j = \sum_i b_i g_i a_j = f^n a_j.$$

Hence

$$\frac{a}{f^m} = \frac{a_j}{g_j}$$

on $D(g_j)$ hence

$$\psi\left(\frac{a}{f^m}\right) = s$$

so ψ is surjective. ■

Remark 2.2.11. There's some kind of analogy between $\text{Spec } \mathbb{Z}$ and $\text{Spec } \mathbb{C}[x] = \mathbb{A}^1$.

Definition 2.2.12 (Ringed Space). A *ringed space* is a pair (X, \mathcal{O}_X) where $X \in \mathbf{Top}$ and $\mathcal{O}_X \in \mathbf{Sh}(X, \mathbf{Ring})$ is a sheaf of rings on X .

Definition 2.2.13 (Morphism of Ringed Spaces). A morphism $(X, \mathcal{O}_X) \rightarrow (Y, \mathcal{O}_Y)$ of ringed spaces is a pair $(f, f^\#)$ where $f: X \rightarrow Y$ is a continuous map and $f^\#: \mathcal{O}_Y \rightarrow f_* \mathcal{O}_X$ is a morphism of sheaves on Y .

Definition 2.2.14 (Locally Ringed Space). A ringed space (X, \mathcal{O}_X) is a *locally ringed space* if for all $o \in X$ the stalk $\mathcal{O}_{X,p}$ of p is a local ring.

Definition 2.2.15 (Local Homomorphism). Let $R, S \in \mathbf{Ring}$ both local with respective maximal ideals $\mathfrak{m}_R, \mathfrak{m}_S$.

A ring homomorphism

$$\varphi: R \rightarrow S$$

is *local* if $\varphi^{-1}(\mathfrak{m}_S) = \mathfrak{m}_R$.

Definition 2.2.16 (Morphism of Locally Ringed Spaces). A *morphism of locally ringed spaces* is a morphism

$$(X, \mathcal{O}_X) \rightarrow (Y, \mathcal{O}_Y)$$

is a pair $(f, f^\#)$ such that for all $p \in X$,

$$f_p^\#: \mathcal{O}_{Y, f(p)} = \varinjlim_{V \ni f(p)} \mathcal{O}_Y(V) \rightarrow \varinjlim_{V \ni f(p)} \mathcal{O}_X(f^{-1}V) \rightarrow \varinjlim_{p \in U} \mathcal{O}_X(U) \rightarrow \mathcal{O}_{X,p}$$

is a local homomorphism.

Remark 2.2.17. $(f, f^\#)$ is an isomorphism iff f is a homeomorphism and $f^\#$ is an isomorphism of sheaves.

Example 2. To understand what $(f_* \mathcal{O}_X)_{f(p)}$ is, consider $f(x) = x^2$ where

$$f: \text{Spec } \mathbb{C}[t] = \mathbb{A}^1 \xrightarrow{f} \mathbb{A}^1.$$

Look at the stalk of $f_* \mathcal{O}_{\mathbb{A}^1}$ at the points $x = 0$ and $x = 1$. At $x = 1$ the map f isn't a local homeomorphism, and indeed the stalk isn't a local ring. It has two maximal ideals. At $x = 0$ the stalk is in fact a local ring.

Proposition 2.2.18. 1. Let $A \in \mathbf{Ring}$. Then $(\mathrm{Spec} A, \mathcal{O}_A)$ is a locally ringed space.

2. If $\varphi: A \rightarrow B$ is a homeomorphism of rings, then φ induces a morphism

$$(f, f^\#) : (\mathrm{Spec} B, \mathcal{O}_B) \rightarrow (\mathrm{Spec} A, \mathcal{O}_A)$$

of locally ringed spaces.

3. Any morphism

$$(\mathrm{Spec} B, \mathcal{O}_B) \rightarrow (\mathrm{Spec} A, \mathcal{O}_A)$$

of locally ringed spaces is induced from such a ring homomorphism.

Proof. 1. This is true since $\mathcal{O}_{A, \mathfrak{p}} \cong A_{\mathfrak{p}}$ which is a local ring.

2. Given $\varphi: A \rightarrow B$ we define

$$\begin{aligned} f: \mathrm{Spec} B &\rightarrow \mathrm{Spec} A \\ \mathfrak{p} &\mapsto \varphi^{-1}(\mathfrak{p}). \end{aligned}$$

For every $\mathfrak{a} \trianglelefteq A$ we have

$$f^{-1}(V(\mathfrak{a})) = V(\varphi(\mathfrak{a})).$$

Hence f is continuous.

For $V \subseteq \mathrm{Spec}(A)$ open define

$$\begin{aligned} f_V^\# : \mathcal{O}_A(V) &\rightarrow \mathcal{O}_B(f^{-1}(V)) \\ \left\{ s: V \rightarrow \coprod_{\mathfrak{q} \in V} A_{\mathfrak{q}} \right\} &\mapsto [\mathfrak{p} \mapsto \varphi_{\mathfrak{p}}(s(f(\mathfrak{p})))] \end{aligned}$$

where $\varphi_{\mathfrak{p}}: A_{\varphi^{-1}(\mathfrak{p})} \rightarrow B_{\mathfrak{p}}$ and by construction $\varphi_{\mathfrak{p}}$ is a local homeomorphism. The induced map $f_p^\#$ is φ_p , so $(f, f^\#)$ is a morphism of locally ringed spaces.

3. Given $(f, f^\#): \mathrm{Spec} B \rightarrow \mathrm{Spec} A$ where $f^\#: \mathcal{O}_A \rightarrow f_*\mathcal{O}_B$ we get

$$\varphi: \mathcal{O}_A(\mathrm{Spec} A) \rightarrow (f_*\mathcal{O}_B)(\mathrm{Spec} A).$$

We get a diagram

$$\begin{array}{ccc} \varphi: \mathcal{O}_A(\mathrm{Spec} A) & \xrightarrow{\varphi} & (f_*\mathcal{O}_B)(\mathrm{Spec} A) \\ \downarrow & & \downarrow \\ A_{f(\mathfrak{p})} & \xrightarrow{f_p^\#} & B_{\mathfrak{p}} \end{array}$$

where $f_p^\#$ is local. So $\varphi^{-1}(\mathfrak{p}) = f(\mathfrak{p})$.

Now the end of the proof follows from the following exercise.

Exercise 3. If $\psi_1, \psi_2: F \rightarrow G$ are morphisms of sheaves on X and $\psi_{1,p} = \psi_{2,p}$ for all $p \in X$ then $\psi_1 = \psi_2$.

Solution. Consider $\psi = \psi_1 - \psi_2$. Then the induced morphisms ψ_P on stalks are zero. Consider

$$0 \rightarrow \ker \psi \rightarrow F \xrightarrow{\psi} G.$$

We get for every P the diagram

$$0 \rightarrow (\ker \psi)_P \rightarrow F_P \xrightarrow{\psi_P=0} G_P.$$

Then the middle map is an isomorphism for every P , so the inclusion $\ker \psi \rightarrow F$ is an isomorphism, so $F = \ker \psi$ from which $\psi = 0$.

Warning: Let F, G be sheaves on X and consider

$$\underline{\mathrm{Hom}}(F, G)(U) = \mathrm{Hom}(F|_U, G|_U).$$

We have that ψ is a global section of this hom sheaf which is zero on every stalk. There's a map

$$\underline{\mathrm{Hom}}(F, G)_P \rightarrow \mathrm{Hom}(F_P, G_P).$$

This however isn't generally an isomorphism. ■

Definition 2.2.19 (Affine Scheme). An *affine scheme* is a locally ringed space X, \mathcal{O}_X which is isomorphic to $(\text{Spec } A, \mathcal{O}_A)$ for some $A \in \mathbf{Ring}$.

Definition 2.2.20 (Scheme). A *scheme* is a locally ringed space (X, \mathcal{O}_X) with an open cover $(U_i)_{i \in I}$ such that $(U_i, \mathcal{O}_X|_{U_i})$ is an affine scheme for all $i \in I$.

Definition 2.2.21 (Structure Sheaf). \mathcal{O}_X is the *structure sheaf* of the scheme X .

Remark 2.2.22. We usually write X instead of (X, \mathcal{O}_X) and remember there is an associated structure sheaf \mathcal{O}_X .

Examples. 1. If $k \in \mathbf{Field}$ then $\text{Spec } k = \{(0)\}$ is a scheme with $\mathcal{O}_{\text{Spec } k}(\text{Spec } k) = k$.

2. If $k \in \mathbf{Field}$ then $R := k[\varepsilon]/\varepsilon^2 = k + k\varepsilon$ is a local ring with maximal ideal $\mathfrak{m} := (\varepsilon) = k\varepsilon$. Then

$$X := \text{Spec } (k[\varepsilon]/\varepsilon^2) = \{\mathfrak{m}\}$$

is a scheme $\mathcal{O}_X(X) = R$.

The map

$$\begin{aligned} R &\rightarrow k \\ \varepsilon &\mapsto 0 \end{aligned}$$

induces a morphism of schemes

$$\text{Spec } k \hookrightarrow \text{Spec } k[\varepsilon]/\varepsilon^2.$$

This is an homeomorphism but not an isomorphism of schemes.

The intuition is that $\text{Spec } k$ is a point and X is a fattened point, as it's in some sense given by looking at both the point and its tangent space.

3. Let p a prime number and let

$$X := \text{Spec } \mathbb{Z}_{(p)} = \left\{ \frac{a}{b} \mid \begin{array}{l} a, b \in \mathbb{Z} \\ b \notin p\mathbb{Z} \end{array} \right\}.$$

Then $X = \{p\mathbb{Z}_{(p)}, (0)\}$. The point $p\mathbb{Z}_{(p)}$ is closed point (in the case of affine schemes this is equivalent to maximality). The point (0) is open and dense (it's not closed, and $|X| = 2$).

- We have

$$\mathcal{O}_X((0)) \cong \mathcal{O}_{X, p\mathbb{Z}_{(p)}} \cong \mathcal{O}_X(D(p)) \cong (\mathbb{Z}_{(p)})_p \cong \mathbb{Q}.$$

- The residue field at $p\mathbb{Z}_{(p)}$ is \mathbb{F}_p .
The residue field at (0) is \mathbb{Q} .

4. Define $\mathbb{A}_k^1 := \text{Spec } k[x]$. Then

$$\mathbb{A}_k^1 = \{(0)\} \cup \{(f(x)) \mid f \text{ is irreducible}\}.$$

The ideals coming from irreducible polynomials are maximal ideals, hence closed points. The point (0) is dense in \mathbb{A}_k^1 and is called *the generic point*.

If $k = \bar{k}$ then the closed points are in bijection with the field k as they come from polynomials of the form $x - \alpha$. Otherwise, the closed points are in bijection with monic irreducible polynomials, which we can think of as the $\text{Gal}(\bar{k}/k)$ orbits of elements in k .

5. Let $\mathbb{A}_k^n = \text{Spec } (k[x_1, \dots, x_n])$. If $k = \bar{k}$ we have

$$\mathbb{A}_k^2 = \{(0)\} \cup \{(x_1 - t_1, x_2 - t_2) \mid t_1, t_2 \in k\} \cup \{(f(x, y)) \mid f \text{ is irreducible}\} \cup \dots$$

The point (0) is the generic point which is here not dense or open. The points in the second set are closed and the points in the third set are with closure containing all closed points (t_1, t_2) such that $f(t_1, t_2) = 0$.

Remark 2.2.23. If F is a sheaf on X then $F(ns) = 0$.

It follows from covering the empty set by the empty open cover and using the sheaf axioms.

Definition 2.2.24 (Disjoint Union of Schemes). Let X, Y be schemes, and let $i_X: X \hookrightarrow X \amalg Y$ and $i_Y: Y \hookrightarrow X \amalg Y$ we define $X \amalg Y$ to be $(X \amalg Y, (i_X)_* \mathcal{O}_X \oplus (i_Y)_* \mathcal{O}_Y)$.

Definition 2.2.25 (Gluing of Schemes). Let X, Y be schemes and let $U \subseteq X, V \subseteq Y$ be open sets. Suppose

$$\varphi: (U, \mathcal{O}_X|_U) \xrightarrow{\sim} (V, \mathcal{O}_Y|_V)$$

is an isomorphism of locally ringed spaces. Suppose $i: X \hookrightarrow Z$ and $j: Y \hookrightarrow Z$ be topological inclusions. We define the **gluing** $Z = X \amalg Y / \varphi$ to be

$$\left(X \amalg Y / \varphi, \mathcal{O}_Z \right)$$

where for $W \subseteq Z$,

$$\mathcal{O}_Z(W) = \left\{ (s, s') \mid \begin{array}{l} s \in \mathcal{O}_X(i^{-1}W) \\ s' \in \mathcal{O}_Y(j^{-1}W) \end{array} \text{ such that } \varphi(s|_{U \cap i^{-1}W}) = s'|_{V \cap j^{-1}W} \right\}.$$

This is a scheme with scheme maps $X \hookrightarrow Z$ and $Y \hookrightarrow Z$.

Remark 2.2.26. One can interpret the gluing as a special case of a pushout in the category of schemes.

Example 3. We can glue \mathbb{A}^1 to itself along $\mathbb{A}^1 \setminus \{0\}$ using the identity on $\mathbb{A}^1 \setminus \{0\}$ as the gluing map φ . We get a line with a double point at 0.

Example 4. We can glue \mathbb{A}_k^1 to \mathbb{A}_k^1 by identifying $\mathbb{A}^1 \setminus \{0\}$ with

$$\mathbb{A}^1 \setminus \{0\} = \operatorname{Spec} k \left[x, \frac{1}{x} \right] \subseteq \mathbb{A}_k^1$$

under $x \mapsto \frac{1}{x}$.

We get \mathbb{P}_k^1 .

Let S be an \mathbb{N} -graded ring, so $S = \bigoplus_{d \geq 0} S_d$ with each S_d an abelian group such that $S_d \cdot S_e \subseteq S_{d+e}$.

Definition 2.2.27 (Homogeneous Element). An element $s \in S$ is **homogeneous** if $s \in S_d$ for some $d \in \mathbb{N}$.

Example 5. Let $k \in \mathbf{Field}$. Then $S = k[X]$ is graded with $S_d = kx^d$. $S' = k[x, y, z]$ is graded with S_d the set of monomials of total degree d .

Definition 2.2.28 (Homogeneous Ideal). An ideal $\mathfrak{a} \subseteq S$ is **homogeneous** if

$$\mathfrak{a} = \bigoplus_{d \geq 0} (\mathfrak{a} \cap S_d).$$

Fact 2.2.29. An homogeneous ideal is equivalently an ideal generated by homogeneous elements.

Notation 2.2.30. Denote

$$S_+ := \bigoplus_{d \geq 1} S_d.$$

Definition 2.2.31. Let S be an \mathbb{N} -graded ring, we define

$$\operatorname{Proj} S := \{ \mathfrak{p} \in S \mid \mathfrak{p} \text{ is a homogeneous prime ideal not containing } S_+ \}.$$

Definition 2.2.32. For any $\mathfrak{a} \subseteq S$ homogeneous let

$$V(\mathfrak{a}) = \{ \mathfrak{p} \in \operatorname{Proj} S \mid \mathfrak{p} \supseteq \mathfrak{a} \}.$$

Lemma 2.2.33.

$$V(\mathfrak{a}_1 \mathfrak{a}_2) = V(\mathfrak{a}_1) \cup V(\mathfrak{a}_2)$$

$$V\left(\sum_{i \in I} \mathfrak{a}_i\right) = \bigcap_{i \in I} V(\mathfrak{a}_i)$$

Definition 2.2.34. We put the Zariski topology on $\text{Proj } S$ such that $Z \subseteq \text{Proj } S$ is closed iff $Z = V(aa)$ for some homogeneous ideal $\mathfrak{a} \subseteq S$.

Definition 2.2.35 ($\mathcal{O}_{\text{Proj } S}$). For $\mathfrak{p} \in \text{Proj } S$ let

$$T_{\mathfrak{p}} := \left\{ s \in S \mid \begin{array}{l} s \text{ is homogeneous} \\ s \notin \mathfrak{p} \end{array} \right\}.$$

Let $T_{\mathfrak{p}}^{-1}S$ the localisation of S at $T_{\mathfrak{p}}$.

We define

$$S_{(p)} = \left\{ s \in T_{\mathfrak{p}}^{-1}S \mid s \text{ has degree } p \right\}.$$

Define a sheaf $\mathcal{O}_{\text{Proj } S}$ on $\text{Proj } S$ as follows. For $U \subseteq \text{Proj } S$ let

$$\mathcal{O}_{\text{Proj } S} := \left\{ s: U \rightarrow \prod_{\mathfrak{p} \in U} S_{(\mathfrak{p})} \mid \forall \mathfrak{p}: s(\mathfrak{p}) \in S_{(\mathfrak{p})}^{\star} \right\}$$

where condition \star means that for all $\mathfrak{p} \in U$ there's a neighbourhood $V \subseteq U$ of \mathfrak{p} and there are homogeneous $a, f \in S$ such that for all $\mathfrak{a} \in V$ it holds that $f \notin \mathfrak{a}$ and

$$s(\mathfrak{q}) = \frac{a}{f} \in S_{(\mathfrak{a})}.$$

Definition 2.2.36. For any $f \in S_+$ we define

$$D_+(f) = \{\mathfrak{p} \in \text{Proj } S \mid f \notin \mathfrak{p}\}.$$

Remark 2.2.37. The set $D_+(f)$ is open in $\text{Proj } S$.

Proposition 2.2.38. Denote $\mathcal{O} = \mathcal{O}_{\text{Proj } S}$.

1. For any $\mathfrak{p} \in \text{Proj } S$ the stalk $\mathcal{O}_{\mathfrak{p}}$ of \mathcal{O} at \mathfrak{p} is isomorphic to $S_{(\mathfrak{p})}$.
2. The sets $D_+(f)$ for $f \in S_+$ cover $\text{Proj } S$.
- 3.

$$(D_+(f), \mathcal{O}|_{D_+(f)}) \cong \text{Spec } S_{(f)}$$

as locally-ringed spaces where $S_{(f)}$ is defined to be the subring of S_f of degree 0 elements.

Proof. See Hartshorne. ■

Corollary 2.2.39 (Projective Space over A). $\text{Proj } S$ is a scheme.

Definition 2.2.40. For any $A \in \mathbf{Ring}$ we can define

$$\mathbb{P}_A^n := \text{Proj } A[x_0, \dots, x_n]$$

with the grading described in the proposition.

This is called *the projective space over A* .

Definition 2.2.41 (Scheme over a Scheme). Let S be a scheme. A *scheme over S* is a scheme X together with a map $X \rightarrow S$.

Definition 2.2.42 (). Let $X \xrightarrow{\pi_1} S, Y \xrightarrow{\pi_2} S$ be schemes over S . A morphism between π_1, π_2 is $\varphi: X \rightarrow Y$ such that the diagram

$$\begin{array}{ccc} X & \xrightarrow{\varphi} & Y \\ & \searrow \pi_1 & \swarrow \pi_2 \\ & S & \end{array}$$

commutes.

Notation 2.2.43. Denote by \mathbf{Sch}/S for the category of schemes over a scheme S .

Let $A \in \mathbf{Ring}$, we denote by \mathbf{Sch}/A for the category of schemes over $\mathrm{Spec} A$.

Proposition 2.2.44. Let $k \in \mathbf{Field}$ such that $k = \bar{k}$. Then there's a fully-faithful functor $t: \mathbf{Var}/k \rightarrow \mathbf{Sch}/k$ from the category of varieties over k to \mathbf{Sch}/k .

For $V \in \mathbf{Var}/k$, V is homeomorphic to the set of closed points of $t(V)$ and the sheaf \mathcal{O}_V of regular functions on V is isomorphic to $\mathcal{O}_{t(V)}|_V$.

Proof. Define

$$t(V) = \{\text{non-empty irreducible closed subsets of } V\}.$$

See Hartshorne for details. ■

Example 6. Let $f(x) = x^5 + 1$. Then the zero locus of $y^2 - f(x)$ in $\mathbb{A}_{\mathbb{C}}^2$ is not compact in some sense. We can homogenise this to

$$[0 : 1 : 0] \in \{z^3 y^2 - x^5 - z^5 = 0\} \subseteq \mathbb{P}_{\mathbb{C}}^2.$$

The first curve is smooth, but the second one isn't smooth at $[0 : 1 : 0]$. However, this can be embedded into a smooth curve.

We have

$$\mathrm{Spec} \left(\mathbb{C}[x, y] / y^2 - f(x) \right) \subseteq \mathrm{Spec} \mathbb{C}(x, y) = \mathbb{A}_{\mathbb{C}}^2.$$

Then

$$\mathrm{Proj} \left(\mathbb{C}[x, y, z] / y^2 - z^6 f\left(\frac{x}{z}\right) \right),$$

by considering $\deg(x) = \deg(z) = 1$ and $\deg(y) = 3$ can be seen as a graded algebra. We have

$$\mathrm{Spec} \left(\mathbb{C}[x, y] / y^2 - f(x) \right) = D(Z)$$

inside this, and that this projective variety is smooth.

2.2.1 Properties of Schemes:

Let $X \in \mathbf{Sch}$.

Definition 2.2.45 (Connected Scheme). X is *connected* if its underlying topological space is connected.

Definition 2.2.46 (Irreducible Scheme). X is *irreducible* if its underlying topological space is irreducible.

Definition 2.2.47 (Reduced Scheme). X is *reduced* if $\mathcal{O}_X(U)$ is reduced for every open $U \subseteq X$.

Exercise 4. X is reduced iff $\mathcal{O}_{X,P}$ is reduced for all $P \in X$.

Definition 2.2.48 (Integral Scheme). X is *integral* if $\mathcal{O}_X(U)$ is an integral domain for every open $U \subseteq X$.

Example 7. Let $X = \mathrm{Spec} A$ for some $A \in \mathbf{Ring}$.

- X is irreducible iff $\mathrm{nil}(A)$ is prime.
- X is reduced iff $\mathrm{nil} A = 0$.
- X is integral iff A is an integral domain.

Proposition 2.2.49. A scheme X is integral iff it's reduced and irreducible.

Proof. Assume X is integral. Then there are no nilpotent elements so X is reduced. Assume towards a contradiction X isn't irreducible, there are $U_1, U_2 \subseteq X$ open such that $U_1 \cap U_2 = \emptyset$. Now, $\mathcal{O}_X(U_1 \cap U_2) = \mathcal{O}_X(\emptyset) = 0$, so

$$\mathcal{O}_X(U_1 \cup U_2) = \mathcal{O}_X(U_1) \oplus \mathcal{O}_X(U_2)$$

which isn't an integral domain. Hence X is irreducible.

Conversely, assume X is reduced and irreducible. Let $U \subseteq X$ open and $f, g \in \mathcal{O}_X(U)$ nonzero such that $fg = 0$. Let

$$Y := \{x \in U \mid f_x \in \mathfrak{m}_x \trianglelefteq \mathcal{O}_{X,x}\}$$

and

$$Z := \{x \in U \mid g_x \in \mathfrak{m}_x \trianglelefteq \mathcal{O}_{X,x}\}.$$

These are closed in U , and $Y \cup Z = U$. Now, X is irreducible so U is irreducible so $Y = U$ or $Z = U$ and assume WLOG $Y = U$. If $Y = \text{Spec } A \subseteq U$ then $f|_Y \in \text{nil}_A = 0$ so $f = 0$. Otherwise, we vcover Y by affines and get the same result. Now $\mathcal{O}_X(U)$ is an integral domain. ■

Definition 2.2.50 (Locally Noetherian Scheme). X is *locally Noetherian* if it has an over cover $\{\text{Spec } A_i\}_{i \in I}$ with A_i noetherian for every $i \in I$.

Definition 2.2.51 (Quasi-Compact Scheme). X is *quasi-compact* if any open cover of X has a finite subcover.

Definition 2.2.52 (Quasi-Compact Morphism). A morphism $f: X \rightarrow Y$ is *quasi-compact* if there's an affine open cover $(V_i)_{i \in I}$ of Y such that $f^{-1}(V_i)$ is quasi-compact for all $i \in I$.

Definition 2.2.53 (Noetherian Scheme). X is *Noetherian* if it's locally Noetherian and quasi-compact.

Remark 2.2.54. X is Noetherian iff there's a finite cover of X by affine schemes $\text{Spec } A_i$ with A_i Noetherian.

Lemma 2.2.55. Let $\text{Spec } A$ and $\text{Spec } B$ be open affine schemes in X . Then $\text{Spec } A \cap \text{Spec } B$ is covered by open sets that are distinguished (i.e. of the form $D(f)$) in both $\text{Spec } A$ and $\text{Spec } B$.

Proof. Let $P \in \text{Spec } A \cap \text{Spec } B$. $\text{Spec } A \cap \text{Spec } B$ is open in A so there's $f \in A$ such that

$$P \in \text{Spec } A_f \subseteq \text{Spec } A \cap \text{Spec } B.$$

Now in $\text{Spec } A_f$ which is open in $\text{Spec } B$ we can get $\text{Spec } B_g$ similarly.

Denote by g' the map of g in $\mathcal{O}_X(\text{Spec } A_f)$ and write $g' = \frac{g}{f^k}$. Then

$$\text{Spec } B_g \subseteq \text{Spec } A_f \setminus \{Q \mid g' \in Q\} = \text{Spec } (A_f)_{g'} = \text{Spec } (A_f \bar{g}).$$

Lemma 2.2.56 (Affine Locality Test). Let \star be a property of affine subschemes of X such that the following holds.

1. If $\text{Spec } A \hookrightarrow X$ has \star then so does $\text{Spec } A_f$ for all $f \in A$.
2. If $(f_1, \dots, f_n) = A$ and $\text{Spec } A_{f_i}$ has \star , so does $\text{Spec } A$.

Assume $X = \bigcup_{i \in I} A_i$ and that every A_i has \star . Then every open affine $\text{Spec } B \hookrightarrow X$ has \star .

Proof. Let $\text{Spec } B \hookrightarrow X = \bigcup_{i \in I} \text{Spec } A_i$ when A_i has \star for every $i \in I$. Cover $\text{Spec } B$ with finitely many opens $\text{Spec } B_{g_j}$ which are simultaneously distinguished in $\text{Spec } B$ and $\text{Spec } A_i$ for some $i \in I$. This is possible by the previous lemma.

Then $\text{Spec } B_{g_j}$ has \star . Note $(g_1, \dots, g_n) = B$, so $\text{Spec } B$ has \star by the assumptions on \star . ■

Proposition 2.2.57. X is locally Noetherian iff for every open $\text{Spec } A \subseteq X$, A is Noetherian. In particular, $\text{Spec } A$ is a Noetherian scheme iff A is Noetherian.

Proof. We need to show the properties of the affine locality test.

For 1, if A is Noetherian so is A_f . For 2, we prove a lemma.

Lemma 2.2.58. If $\mathfrak{a} \subseteq A$ and $f_i \in A$ are such that $(f_1, \dots, f_n) = A$ with localisation maps $\varphi_i: A \rightarrow A_{f_i}$, then

$$\mathfrak{a} = \bigcap_{i \in [n]} \varphi_i^{-1}(\varphi_i(\mathfrak{a}) \cdot A_{f_i}).$$

Proof. \subseteq is clear, so we prove the other direction. Let b in the RHS and write $\varphi_i(b) = \frac{a_i}{f_i^k}$ with $a_i \in \mathfrak{a}$ and $k > 0$. Then

$$f_i^m (f_i^k b - a_i) = 0.$$

Then

$$f_i^{m+kb} \in \mathfrak{a}.$$

Note that $(f_1^{m+k}, \dots, f_n^{m+k}) = A$, because $I = A \iff \sqrt{I} = A$, or because $D(f^n) = D(f)$. So, $1 = \sum_{i \in [n]} c_i f_i^{k+m}$ for $c_i \in A$. Then

$$b = \sum_{i \in [n]} c_i f_i^{k+m} b \in \mathfrak{a}.$$

We want to show A is Noetherian assuming A_{f_i} are. If

$$\mathfrak{a}_1 \subseteq \mathfrak{a}_2 \subseteq \mathfrak{a}_3 \subseteq \dots$$

is an ascending chain of ideals in A then

$$\varphi_i(\mathfrak{a}_1) \subseteq \varphi_i(\mathfrak{a}_2) \subseteq \dots$$

is an ascending chain in A_{f_i} , so it stabilises. Since there are finitely many f_i and since $\mathfrak{a} = \bigcap_{i \in [n]} \varphi_i^{-1}(\varphi_i(\mathfrak{a}_i))$, the \mathfrak{a}_j must stabilise. \blacksquare

Definition 2.2.59 (Morphism Locally of Finite Type). A morphism of $f: X \rightarrow Y$ of schemes is **locally of finite type** if there's an open cover $(V_i)_{i \in I}$ of Y with $V_i \cong \text{Spec } B_i$ such that

$$f^{-1}(V_i) = \bigcup_{j \in J_i} \text{Spec } A_{i_j}$$

such that A_{i_j} is a finitely-generated B_i -algebra for every $i \in I$.

Definition 2.2.60 (Morphism of Finite Type). A morphism $f: X \rightarrow Y$ of schemes is **of finite type** if we can take the J_i above to be finite.

Definition 2.2.61 (Finite Morphism). A morphism $f: X \rightarrow Y$ of schemes is **finite** if there's an open cover $(V_i)_{i \in I}$ of Y with $V_i \cong \text{Spec } B_i$ and such that $f^{-1}(V_i) \cong \text{Spec } A_i$ with A_i a finite B_i -module.

Definition 2.2.62 (Open Immersion). Let X be a scheme and $U \subseteq X$ open. Then $(U, \mathcal{O}_X|_U)$ is a subscheme. Then the map $U \hookrightarrow X$ is an **open immersion**.

Definition 2.2.63 (Closed Immersion). A **closed immersion** is a morphism $f: Y \rightarrow X$ such that the following holds.

1. f gives a homeomorphism onto a closed subset of X .
2. $f^\#: \mathcal{O}_X \rightarrow f_* \mathcal{O}_Y$ is surjective.

Definition 2.2.64 (Closed Subscheme). A **closed subscheme** is an equivalence class of such morphisms.

Remark 2.2.65. Why can't we do what we did for an open immersion? Look at

$$\begin{aligned} \text{Spec } \mathbb{C} &\hookrightarrow \text{Spec } \mathbb{C}[t] \\ \mathbb{C}[x] &\rightarrow 0 \\ t &\mapsto 0. \end{aligned}$$

Let $i: \mathbb{C} \hookrightarrow \text{Spec } \mathbb{C}[t]$, and $X = \text{Spec } \mathbb{C}[t]$. Then $i^{-1} \mathcal{O}_X = \mathcal{O}_{X,x}$.

Remark 2.2.66. A map $A \rightarrow A/I$ induced a closed immersion $\mathrm{Spec}(A/I) \hookrightarrow \mathrm{Spec}(A)$.

Definition 2.2.67. Let $Y \subseteq X$ a closed subset of a scheme X . If $X = \mathrm{Spec} A$ set $I = \bigcap_{P \in Y} P \trianglelefteq A$. Then $V(I) = Y$. The induced closed subscheme structure is given by

$$Y = \mathrm{Spec}(A/I) \hookrightarrow \mathrm{Spec}(A).$$

Remark 2.2.68. $\mathrm{Spec}(A/I)$ is indeed reduced.

In general, to define the induced closed subscheme structure, we cover X with open affines U_i and set $Y_i = Y \cap U_i$. We get closed subschemes $Y_i \hookrightarrow U_i$. We glue these to obtain a sheaf on Y . To construct these we need to give isomorphisms

$$\mathcal{O}_{Y_i}|_{Y_i \cap Y_j} \xrightarrow{\varphi_{i,j}} \mathcal{O}_{Y_j}|_{Y_i \cap Y_j}$$

such that

$$\varphi_{i,k}|_{Y_{i,j,k}} = \varphi_{j,k}|_{Y_{i,j,k}} = \varphi_{i,j}|_{Y_{i,j,k}}$$

with the notation $Y_{i,j,k} = Y_i \cap Y_j \cap Y_k$.

Definition 2.2.69 (Dimension). Let $X \in \mathbf{Sch}$. The **dimension** of X is the supremum of the length of any chain

$$Z_0 \subsetneq Z_1 \subsetneq Z_2 \subsetneq \dots$$

of closed irreducible subsets of X .

Remark 2.2.70. If $X = \mathrm{Spec} A$, then $\dim X$ is the Krull dimension of A .

Proposition 2.2.71 (Fibre Products). Fibre products exist in \mathbf{Sch} .

Let $X \rightarrow S$ and $Y \rightarrow S$ be morphisms of schemes. There exists **the fibre product** $X \times_S Y$ together with maps $p_1: X \times_S Y \rightarrow X$ and $p_2: X \times_S Y \rightarrow Y$ such that for every S -scheme Z and any pair of morphisms $Z \rightarrow X$ and $Z \rightarrow Y$ there is a unique morphism $Z \rightarrow X \times_S Y$ such that the following diagram commutes.

$$\begin{array}{ccc} Z & \xrightarrow{\varphi} & X \times_S Y \\ & & \downarrow p_1 \quad \downarrow p_2 \\ & & X \longrightarrow S \end{array}$$

Remark 2.2.72. The fibre product when it exists is unique up to a unique isomorphism. We shall use this during the proof when moving from affine schemes to general ones.

Lemma 2.2.73. Let $X, Y \in \mathbf{Sch}(S)$ with respective maps f, g to S such that $X \times_S Y$ exists with maps p_1 to X and p_2 to Y . Let $U \subseteq X, V \subseteq Y, T \subseteq S$ open such that $f(U), g(V) \subseteq T$. Then

$$F := p_1^{-1}(U) \cap p_2^{-1}(V) \cong U \times_T V.$$

Proof. View the following diagram.

$$\begin{array}{ccc} Z & \longrightarrow & V \\ \downarrow & & \downarrow \\ U & \longrightarrow & T \end{array}$$

we extend it to a diagram

$$\begin{array}{ccccc} Z & \longrightarrow & V & \hookrightarrow & Y \\ \downarrow & & \downarrow & & \downarrow \\ U & \longrightarrow & T & & \\ \downarrow & & \searrow & & \downarrow \\ X & \longrightarrow & & & S \end{array}$$

then by diagram chase the maps from Z factor through $p_1^{-1}(U) \cap p_2^{-1}(V)$. ■

Proof. Affine Case: Assume first that $S = \operatorname{Spec} A, X = \operatorname{Spec} B, Y = \operatorname{Spec} C$. We show $\operatorname{Spec}(B \times_A C)$ is the tensor product $X \times_S Y$. This follows from exercise 2.4 in Hartshorne and the fact that the cofibre product of rings is the tensor product. We explain that explicitly.

Recall

$$\operatorname{Hom}(B \otimes_A C, R) \leftrightarrow \operatorname{Hom}(B, R) \times_{\operatorname{Hom}(A, R)} \operatorname{Hom}(C, R).$$

For any scheme Z and any ring A we have

$$\operatorname{Hom}(Z, \operatorname{Spec} A) \leftrightarrow \operatorname{Hom}(A, \Gamma(Z, \mathcal{O}_Z))$$

so

$$\begin{aligned} \operatorname{Hom}(Z, \operatorname{Spec}(B \otimes_A C)) &\rightarrow \operatorname{Hom}(B \otimes_A C, R) \leftrightarrow \operatorname{Hom}(B, R) \times_{\operatorname{Hom}(A, R)} \operatorname{Hom}(C, R) \\ &\leftrightarrow \operatorname{Hom}(Z, \operatorname{Spec} B) \times_{\operatorname{Hom}(Z, \operatorname{Spec} A)} \operatorname{Hom}(Z, \operatorname{Spec} C) \end{aligned}$$

where the arrows are natural bijections.

If S is affine: We apply 2.2.73. If S is affine, cover X, Y with open affines $(U_i)_{i \in I}, (V_j)_{j \in J}$ and then glue $U_i \times_S V_j$.

We need to give isomorphisms on the overlaps. By the lemma $(U_i \cap U_{i'}) \times_S (V_j \cap V_{j'})$ exists and is unique up to a unique isomorphism. Then uniqueness implies $\varphi_{i,j} = \varphi_{j,k} \circ \varphi_{i,j}$, as both sides satisfy the universal property.

Then we glue to get $X \times_S Y$.

General Case: Cover S with $(W_i)_{i \in I}$ affine. By the lemma we get the fibre products $f^{-1}(W_i) \times_{W_i} g^{-1}(W_i)$. Gluing these we get the result. ■

Example 8. Let $X \in \mathbf{Sch}$, then $X \times_X X \cong X$.

Specifically

$$\operatorname{Spec} k \times_{\operatorname{Spec} k} \operatorname{Spec} k \cong \operatorname{Spec} k.$$

Example 9. Examine $\operatorname{Spec} \mathbb{C} \times_{\operatorname{Spec} \mathbb{R}} \operatorname{Spec} \mathbb{C}$ under the inclusions $\mathbb{R} \hookrightarrow \mathbb{C}$ which induce $\operatorname{Spec} \mathbb{C} \rightarrow \operatorname{Spec} \mathbb{R}$. This is $\operatorname{Spec}(\mathbb{C} \otimes_{\mathbb{R}} \mathbb{C})$. This is $\operatorname{Spec}(\mathbb{C} \times \mathbb{C})$. There's an isomorphism

$$\begin{aligned} \mathbb{C} \otimes_{\mathbb{R}} \mathbb{C} &\xrightarrow{\sim} \mathbb{C} \times \mathbb{C} \\ z \otimes 1 &\mapsto (z, \bar{z}) \\ z \otimes t &\mapsto (zt, \bar{z}t). \end{aligned}$$

Now, $\operatorname{Spec}(\mathbb{C} \times \mathbb{C})$ is $\operatorname{Spec}(\mathbb{C}) \amalg \operatorname{Spec}(\mathbb{C})$ by exercise 2.4 and by the disjoint union being the coproduct of schemes.

Example 10. Examine the following diagram.

$$\begin{array}{ccc} & \overline{\mathbb{F}_p} & \\ \nearrow & & \nwarrow \\ \mathbb{F}_{p^2} & & \mathbb{F}_{p^2} \\ \nwarrow & & \nearrow \\ & \mathbb{F}_p & \end{array}$$

Now

$$\operatorname{Spec} \mathbb{F}_{p^3} \times_{\operatorname{Spec} \mathbb{F}_p} \operatorname{Spec} \mathbb{F}_{p^2} \cong \operatorname{Spec}(\mathbb{F}_{p^3} \otimes_{\mathbb{F}_p} \mathbb{F}_{p^2}) \cong \operatorname{Spec}(\mathbb{F}_{p^6}).$$

For the second isomorphism, we define

$$\begin{aligned} \mathbb{F}_{p^3} \otimes \mathbb{F}_{p^2} &\rightarrow \mathbb{F}_{p^6} \\ a \otimes b &\mapsto ab. \end{aligned}$$

This is clearly surjective and is a map between vector spaces of the same dimension and is therefore an isomorphism.

Definition 2.2.74 (Fibres). Let $f: X \rightarrow Y$ be a morphism of schemes and let $y \in Y$. Let $k_y = \mathcal{O}_{Y,y} / \mathfrak{m}_y$. Then there's an induced diagram

$$\begin{array}{ccc} X_y & \longrightarrow & X \\ \downarrow & & \downarrow f \\ \text{Spec } k_y & \hookrightarrow & Y \end{array}$$

where $X_y := \text{Spec } k_y \times_Y X$ is *the fibre of f above y* .

Exercise 5. X_y is homeomorphic to $f^{-1}(y)$.

Example 11. Let $R := \mathbb{Z}[x, y] / (y^2 - x^3 + 15)$, then $\text{Spec } R$ has a natural map to $\text{Spec } \mathbb{Z}$ given by the map $\mathbb{Z} \rightarrow \mathbb{Z}[x, y] \rightarrow R$. There is a closed immersion $\text{Spec } \mathbb{F}_p \hookrightarrow \text{Spec } \mathbb{Z}$ given by the morphism $\mathbb{Z} \rightarrow \mathbb{F}_p$. We get the fibre of f at (p) is $\text{Spec} \left(\mathbb{F}_p[x, y] / (y^2 - x^3 + 15) \right)$.

Looking at dimensions, we have $\dim \mathbb{Z} = 1$, $\dim R = 2$, $\dim \mathbb{F}_p = 0$ and $\dim \text{Spec} \left(\mathbb{F}_p[x, y] / (y^2 - x^3 + 15) \right) = 1$.

Example 12. Let $k = \mathbb{C}$ and examine $\text{Spec } R$ where $R := k[x, y, t] / (ty - x^2)$. We map $\text{Spec } R \rightarrow \text{Spec } k[t] \cong \mathbb{A}_{\mathbb{C}}^1$. The closed points of $\mathbb{A}_{\mathbb{C}}^1$ are in correspondence with elements of \mathbb{C} . So we get a map $\text{Spec } k \hookrightarrow \text{Spec } k[t]$ coming from the map

$$\begin{array}{ccc} k[t] & \rightarrow & k \\ t & \mapsto & a. \end{array}$$

We get that the fibre at a is $\text{Spec} \left(k[x, y] / (ay - x^2) \right)$, as in the following diagram.

$$\begin{array}{ccc} \text{Spec} \left(k[x, y] / (ay - x^2) \right) & \hookrightarrow & \text{Spec } R \\ \downarrow & & \downarrow f \\ \text{Spec } k & \hookrightarrow & \text{Spec } k[t] \end{array}$$

Denote $X = \text{Spec } R$ and X_a the fibre. Then at $a = 0$ we have $X_0 \cong \text{Spec} \left(k[x, y] / x^2 \right)$.

We say that the parabolas “degenerate” into the double line.

We can also take the fibre at the generic point η : $\text{Spec } k(t) \hookrightarrow \mathbb{A}_{\mathbb{C}}^1$. The fibre X_η is then $\text{Spec} \left(k(t)[x, y] / (ty - x^2) \right)$. This is a conic over the field $k(t)$.

Remark 2.2.75. Many questions in algebraic geometry amount to asking what we can learn about X_a from the *generic fibre* X_η , in the above notation.

Example 13. Take $k = \mathbb{C}$. We can take the same example over $\text{Spec } k[[x]]$. We have $\text{Spec } k[[t]] = \{(0), (t)\}$. Then there are two points, $t = 0$ and the generic point.

Example 14. We can examine the first example over $\text{Spec } \mathbb{Z}_p$ or $\text{Spec } \mathbb{Z}_{(p)} = \{(0), (p)\}$.

Definition 2.2.76 (Base Change). Let $S \in \mathbf{Sch}$ and $X, S' \in \mathbf{Sch}(S)$. Then $X \times_S S'$ is an S' -scheme. This is called *the base change of X to S'* .

Example 15. Let $X = \text{Spec } \mathbb{R}[x]$ which maps to $\text{Spec } \mathbb{R}$. We have also a morphism $\text{Spec } \mathbb{C} \rightarrow \text{Spec } \mathbb{R}$, and the fibre product of these maps is $\text{Spec } \mathbb{C}[x] \cong \mathbb{A}_{\mathbb{C}}^1$.

Remark 2.2.77. Base change is transitive. If $X, S', S'' \in \mathbf{Sch}(S)$, then

$$X \times_S S'' \cong (X \times_S S') \times_{S'} S''.$$

This can be checked by the universal property.

Definition 2.2.78 (Functor of Points). Let $X \in \mathbf{Sch}$. We get a functor

$$\mathbf{Sch}^{\text{op}} \rightarrow \mathbf{Set}$$

which sends S to $\text{Hom}(S, X)$ and a morphism $g: S \rightarrow S'$ to post-composition $f \rightarrow g \circ f$.

Remark 2.2.79. • By Yoneda's lemma, the functor $\text{Hom}(-, X)$ determines the scheme.

- It's enough to work with affine schemes to understand this.
- Most interesting schemes are best understood in terms of their functor of points.

Notation 2.2.80. Denote $X(S) := \text{Hom}(S, X)$.

Example 16. We define M_g the *moduli functor of (smooth, projective) curves of genus g* .

One can define genus generally such that the genus of a complex curve is its topological genus as a real manifold. We define

$$M_g(S) = \{\text{relative curves } C \rightarrow S \text{ of genus } g\} / \sim$$

where a *relative curve* $C \rightarrow S$ is a morphism such that for any $k = \bar{k}$ and any $x: \text{Spec } k \hookrightarrow S$, the fibre C_x of $C \rightarrow S$ over x is a *smooth connected projective*¹ curve of genus g over $\text{Spec } k$.

Fact 2.2.81 (Deligne-Mumford). *There is (almost) a scheme \widetilde{M}_g such that there's a bijection*

$$M_g(S) = \text{Hom}(S, M_g) \leftrightarrow M_g(S)$$

which is functorial in S .

Remark 2.2.82. We want to understand $\dim M_g$. We can compute it by computing

$$\dim_{\mathbb{C}} \text{Hom}_{C_0} \left(\text{Spec} \left(\mathbb{C}[x] / x^2 \right), M_g \times_{\mathbb{Z}} \mathbb{C} \right)$$

the dimension of the tangent space of M_g at $C_0 \in M_g(\mathbb{C})$. We have

$$\text{Hom}_{C_0} \left(\text{Spec} \left(\mathbb{C}[x] / x^2 \right), M_g \times_{\mathbb{Z}} \mathbb{C} \right) \subseteq M_g \left(\text{Spec} \left(\mathbb{C}[x] / x^2 \right) \right),$$

where the latter can be computed via deformation theory from commutative algebra.

Let $f: X \rightarrow Y$ be a morphism of schemes and let $\Delta: X \rightarrow X \times_Y X$ be the “diagonal” corresponding to f on each factor.

Definition 2.2.83. $f: X \rightarrow Y$ is *separated* if Δ is a closed immersion.

Definition 2.2.84. A scheme X is *separated* if $X \rightarrow \text{Spec } \mathbb{Z}$ is separated.

Notation 2.2.85. Let $k \in \mathbf{Field}$, we may write $X \times_k Y$ to mean $X \times_{\text{Spec } k} Y$.

Example 17. Let $k \in \mathbf{Field}$ and let X be the affine line with double origin over k . Then X is *not* separated over k .

$X \times_k X$ is geometrically the plane with doubled axes and four different origins. We claim $\Delta(X)$ isn't closed in $X \times_k X$. We can think of the fibre product as gluing of four planes so we get maps

$$\mathbb{A}_k^2 \hookrightarrow \coprod_{i \in [4]} \mathbb{A}_k^2 \rightarrow X \times_k X.$$

Now, pulling back $\Delta(X)$ to \mathbb{A}_k^2 we get the diagonal minus the origin, so X isn't separated.

Proposition 2.2.86. *Let $f: \text{Spec } A \rightarrow \text{Spec } B$. Then f is separated.*

Proof. Let $\varphi: B \rightarrow A$ induce f . Then $\Delta^\#: A \otimes_B A \rightarrow A$ send $a \otimes a'$ to aa' . This is surjective. Hence Δ is a closed immersion, hence f is separated. ■

Corollary 2.2.87. *If $f: X \rightarrow Y$ is a morphism of schemes, then f is separated iff $\Delta(X)$ is closed in $X \times_Y X$.*

Proof. Suppose $\Delta(X) \subseteq X \times_Y X$ is closed. We want to show the following.

1. $\Delta: X \rightarrow \Delta(X)$ is a homeomorphism onto the image. Sending X to $\Delta(X)$ and then projecting to X via one of the projections is a homeomorphism. This is clear. Hence $\Delta: X \rightarrow \Delta(X)$ is an injection, and is therefore an homeomorphism onto the image.

2. $\mathcal{O}_{X \times_Y X} \rightarrow \Delta_* \mathcal{O}_X$ is surjective. It's enough to show this on stalks. Check this at a point $\Delta(P) \in \Delta(X)$. Choose $U \subseteq X$ open affine such that $P \in U$ and $f(U) \subseteq V$ where $V \subseteq Y$ is open affine.

Then $U \times_Y U \subseteq X \times_Y X$ is an open affine subset of $\Delta(P)$. The corresponding map on stalks is surjective by the previous proposition. Hence $(\mathcal{O}_{X \times_Y X})_{\Delta(P)} \rightarrow (\Delta_*, \mathcal{O}_X)_{\Delta(P)}$ is surjective. ■

¹We haven't defined these formally, but they are analogous to the classical notions for varieties

2.3 Valuation Rings

Let $k \in \text{Field}$ and G a totally ordered abelian group (e.g. $\mathbb{Z}, \mathbb{R}, \mathbb{R}^n$).

Definition 2.3.1 (Valuation). A *valuation on k* is a map $v: k^\times \rightarrow G$ such that

1. $v(xy) = v(x) + v(y)$.
2. $v(x + y) \geq \min(v(x), v(y))$.

Remark 2.3.2. Clearly, $v(1) = 0$ and $v(r^{-1}) = -v(r)$.

Definition 2.3.3 (Discrete Valuation). A *discrete valuation* is a valuation into \mathbb{Z} .

Example 18. Let $k = \mathbb{C}((t))$, we can define the valuation

$$v\left(\sum_{i \geq i_0} a_i \cdot t^i\right) = i_0$$

is a discrete valuation.

Definition 2.3.4 (Valuation Ring). Let $v: k \rightarrow G$ be a valuation. The *valuation ring of v* is

$$R_v := \{v \in k^\times \mid v(r) \geq 0\} \cup \{0\}.$$

Remark 2.3.5. R_v is a local ring with maximal ideal

$$\mathfrak{m}_v = \{R \in R \mid v(r) > 0\} \cup \{0\}.$$

Definition 2.3.6 (Discrete Valuation Ring). A ring R is a *discrete valuation ring* if there's a valuation v such that $R \cong R_v$.

Lemma 2.3.7. A ring R is a discrete valuation ring iff R is a PID with a unique non-zero maximal ideal.

Proof. Use the fact that ideals of R are of the form \mathfrak{m}_v^n where $\mathfrak{m}_v = (t)$ and $v(t) > 1$. ■

Example 19. Let $k = \mathbb{Q}$ and $p \in \mathbb{Z}$ prime. Define $v: \mathbb{Q}^\times \rightarrow \mathbb{Z}$ by $p^{\frac{k}{n}} \mapsto k$ when $\gcd(p, mn) = 1$. This is the *p -adic valuation* and induces the DVR $R = \mathbb{Z}_{(p)}$ of the *p -adic integers*.

Remark 2.3.8. If R is a DVR then $\text{Spec } R = \{(0), \mathfrak{m}\}$ with \mathfrak{m} a maximal ideal.

Example 20. Let $R = \mathbb{C}[[t]] \subseteq \mathbb{C}((t)) \cong k$. We have an induced map $\text{Spec } k \hookrightarrow \text{Spec } R$.

Theorem 2.3.9 (Valuative Criterion for Separatedness). Let $f: X \rightarrow Y$ be a morphism of schemes and assume X is Noetherian. Then f is separated iff for all valuation rings $R \subseteq K$ (where $K = \text{Frac}(R)$) and all commutative diagrams

$$\begin{array}{ccc} \text{Spec } K & \longrightarrow & X \\ \downarrow & & \downarrow \\ \text{Spec } R & \longrightarrow & Y \end{array}$$

there's at most one morphism $\text{Spec } R \rightarrow X$ such that the resulting diagram commutes.

Proof. See Hartshorne. ■

Corollary 2.3.10. Under Noetherian hypothesis we have the following.

1. Open and closed immersions are separated.
2. A composition of separated morphisms is separated.
3. Any base change of a separated morphism is separated. I.e. if $f: X \rightarrow Y$ is separated and $g: Y' \rightarrow Y$ is a morphism then the induced map $f': X \times_Y Y' \rightarrow Y'$ is separated.

4. *Products of separated maps are separated. If $f: X \rightarrow Y$ and $f': X' \rightarrow Y'$ are separated maps of S -schemes then*

$$X \times_S X' \xrightarrow{f \times f'} Y \times_S Y'$$

is separated.

5. *If $X \xrightarrow{f} Y \xrightarrow{g} Z$ and gf is separated then f is separated.*

6. *$f: X \rightarrow Y$ is separated iff there's an open cover (equivalently, for any open cover) $\{Y_i\}_{i \in I}$ of Y such that $f^{-1}(Y_i) \rightarrow Y_i$ (the restriction of f) is separated for every $i \in I$.*

We prove some of the parts of the corollary.

Proof. 6. Assume f is separated, we want to show $f|_{f^{-1}(Y_i)} \rightarrow Y_i$ is separated. We want to use the valuative criterion. Examine a commutative diagram

$$\begin{array}{ccc} \text{Spec } k & \longrightarrow & f^{-1}(Y_i) \\ \downarrow & & \downarrow \\ \text{Spec } k & \longrightarrow & Y_i \end{array}$$

We add arrows and use the valuative criterion to say there's at most one dotted arrow

$$\begin{array}{ccccc} \text{Spec } k & \longrightarrow & f^{-1}(Y_i) & \hookrightarrow & X \\ \downarrow & & \downarrow & \nearrow \text{dotted} & \downarrow f \\ \text{Spec } k & \longrightarrow & Y_i & \hookrightarrow & Y \end{array}$$

from which there's at most one morphism $\text{Spec } k \rightarrow f^{-1}(Y_i)$ such that the following commutes.

$$\begin{array}{ccccc} \text{Spec } k & \longrightarrow & f^{-1}(Y_i) & \hookrightarrow & X \\ \downarrow & & \downarrow & \nearrow \text{dotted} & \downarrow f \\ \text{Spec } k & \longrightarrow & Y_i & \hookrightarrow & Y \end{array}$$

Then by the valuative criterion we're done.

1. Showing that a closed immersion is separated is also done via the valuative criterion by examining the following diagram.

$$\begin{array}{ccccc} \text{Spec } K & & X_0 & \hookrightarrow & X \\ \downarrow & \nearrow & \searrow & & \downarrow \\ \text{Spec } R & \longrightarrow & & & Y \end{array}$$

Definition 2.3.11. A morphism $f: X \rightarrow Y$ is **proper** if it satisfies the following.

1. f is separated.
2. f is of finite type.
3. f is **universally closed** in the sense that for every $Y' \rightarrow Y$, the base change f' is a closed map on the topological spaces.

Example 21. Let $k \in \mathbf{Field}$ and $X = \mathbb{A}_k^1$. Then X is not proper over k . We have $\mathbb{A}_k^1 \times_k \mathbb{A}_k^1 = \mathbb{A}_k^2$. Writing $f: \mathbb{A}_k^1 \rightarrow \text{Spec } k$ we get $f': (x, y) \mapsto y$. Let $Z = \text{Spec} \left(k[x, y] / xy - 1 \right)$. Then $f'(Z) \cong \mathbb{A}_k^1 \setminus \{0\}$ is not closed, so $\mathbb{A}_k^1 \rightarrow k$ is not proper.

Definition 2.3.12 (Specialisation). Let $X_0, x_1 \in X$. If $x_0 \in \overline{\{x_1\}}$ then x_1 **specialises to** x_0 .

Definition 2.3.13 (Dominating Ring). Say $A, B \subseteq K$ are local rings in a field. We say B **dominates** A if $A \subseteq B$ and $\mathfrak{m}_A = \mathfrak{m}_B \cap A$.

Theorem 2.3.14 (A-M Section 4). Let K be a field. A local ring $R \subseteq K$ is a valuation ring of K iff R is maximal with respect to domination.

Corollary 2.3.15. Every local ring in K is contained in some valuation ring.

Lemma 2.3.16. Let $f: X \rightarrow Y$ quasi-compact. Then $f(X)$ is closed in Y iff $f(X)$ is stable under specialisation.

Proof. See Hartshorne. ■

Theorem 2.3.17 (Valuative Criterion for Properness). Let $f: X \rightarrow Y$ be of finite type and assume X is Noetherian. Then f is proper iff for all valuation rings $R \subseteq K$ and all commuting diagrams

$$\begin{array}{ccc} \operatorname{Spec} K & \longrightarrow & X \\ \downarrow & & \downarrow \\ \operatorname{Spec} R & \longrightarrow & Y \end{array}$$

there's a unique $\psi: \operatorname{Spec} R \rightarrow X$ making the diagram

$$\begin{array}{ccc} \operatorname{Spec} K & \longrightarrow & X \\ \downarrow & \nearrow \psi & \downarrow \\ \operatorname{Spec} R & \longrightarrow & Y \end{array}$$

commute.

Proof. • Let $U := \operatorname{Spec} K$ and $T := \operatorname{Spec} R$.

If f is proper, it's separated so by the valuative criterion for separatedness it suffices to show that for every valuation ring R with $K := \operatorname{Frac}(R)$ and solid commutative diagram as follows there exists a dotted morphism ψ such that the diagram commutes.

$$\begin{array}{ccc} U & \longrightarrow & X \\ \downarrow & \nearrow \psi & \downarrow t \\ T & \longrightarrow & Y \end{array}$$

Let $X_T := X \times_Y T$, let $f': X_T \rightarrow T$ correspond to f and let $t_1 \in U$. We have a morphism $\varphi: U \rightarrow X_T$ by the universal property. Let $x_1 = \varphi(t_1)$. We have $Z := \overline{\{x_1\}} \xrightarrow{\text{closed}} X_T$ and that Z is integral. f is proper so f' is closed. Hence $f'(Z)$ is closed in T and contains the generic point, hence $f'(Z) = T$. Hence there's $x_0 \in X_T$ such that $f'(x_0) = t_0$ where t_0 is the unique closed point of T (corresponding to the maximal ideal).

Z is integral with function field $k(x_1) \hookrightarrow K$. We get that the morphism

$$\begin{array}{ccc} Z & \rightarrow & T \\ R & \xrightarrow{\text{local homo.}} & \mathcal{O}_{Z, x_0} \end{array}$$

can be post-composed to get a morphism to $k(x_1)$ and then to K . Hence \mathcal{O}_{Z, x_0} dominated R . Because R is a valuation ring, it's maximal with respect to domination, hence the map $R \rightarrow \mathcal{O}_{Z, x_0}$ is an isomorphism.

Define

$$\begin{array}{ccc} T & \rightarrow & Z \\ t_i & \mapsto & x_i \\ \mathcal{O}_{Z, x_0} & \rightarrow & X. \end{array}$$

We get a map $\operatorname{Spec} R = T \rightarrow \operatorname{Spec} \mathcal{O}_{Z, x_0} \rightarrow Z$. We post compose to get

$$\psi: T \rightarrow Z \rightarrow X_T \rightarrow X.$$

- Conversely, suppose there's a unique ψ in all cases. We need to show f is universally closed. Let $Y' \in \mathbf{Sch}(Y)$ and view the following commutative diagram.

$$\begin{array}{ccc} X' & \xrightarrow{f'} & Y' \\ \downarrow & & \downarrow \\ X & \xrightarrow{f} & Y \end{array}$$

Let $Z \subseteq X'$ closed. f is of finite type hence by an exercise in Hartshorne II.3, f' is of finite type. Then $f'|_Z$ is of finite type (since closed immersions are of finite type and compositions of finite type morphisms are of finite type). Hence $f'|_Z$ is quasi-compact.

By 2.3.16, $f'(Z)$ is then closed iff it's stable under specialisation. Let $z_1 \in Z$ and $y_1 = f'(z_1)$. Suppose $y_1 \rightsquigarrow y_0$ (y_1 specialises to y_0). Let \mathcal{O} be the local ring of y_0 in $\{y_1\}$. Hence

$$\mathrm{Frac}(\mathcal{O}) = k(y_1) \subseteq k(z_1) := K.$$

Let R be a valuation ring in K dominating \mathcal{O} . Let $U := \mathrm{Spec} K$ and $T := \mathrm{Spec} R$. We get the following commutative diagram.

$$\begin{array}{ccccccc} U & \longrightarrow & Z & \hookrightarrow & X' & \longrightarrow & X \\ \downarrow & & \downarrow & \swarrow f' & & & \downarrow f \\ T & \longrightarrow & Y' & \longrightarrow & & & Y \end{array}$$

We get an induced map $T \rightarrow X$ and we want to show this factors through Z . This is true because Z is closed and because closed points of T are in the closure of the generic point which is in K .

Set $z_0 = \psi(t_0)$. We get $f'(z_0) = y_0$. Hence $f(Z)$ is stable under specialisation. By 2.3.16 then $f'(Z)$ is closed so f' is closed so f is universally closed.

Hence f is proper. ■

Corollary 2.3.18. Assume $f: X \rightarrow Y$ under suitable Noetherian hypothesis on X, Y .

1. Closed immersions are proper.
2. Properness is stable under base-change.
3. Fibred products of proper morphisms are proper.
4. If $X \xrightarrow{f} Y \xrightarrow{g} Z$, gf is proper and g is separated, then f is also proper.
5. Properness is local on the base, as before.

Definition 2.3.19 (Projective Space of Affine Schemes). Let $A \in \mathbf{Ring}$, we define *the n^{th} projective space over A* to be

$$\mathbb{P}_A^n = \mathrm{Proj}(A[x_0, \dots, x_n]).$$

Remark 2.3.20. A map $A \rightarrow B$ induces

$$\mathbb{P}_B^n \xrightarrow{\sim} \mathbb{P}_A^n \times_{\mathrm{Spec} A} \mathrm{Spec} B.$$

Definition 2.3.21 (Projective Space of Schemes). Let X be a scheme, we define $\mathbb{P}_X^n = \mathbb{P}_{\mathbb{Z}}^n \times_{\mathrm{Spec} \mathbb{Z}} X$.

Definition 2.3.22 (Projective Morphism). A morphism $f: X \rightarrow Y$ is *projective* if it factors as $X \xrightarrow{i} \mathbb{P}_Y^n \rightarrow Y$ for some $n \in \mathbb{N}$ and where i is a closed immersion.

Definition 2.3.23 (Quasi-projective Morphism). $f: X \rightarrow Y$ is *quasi-projective* if it factors as $X \xrightarrow{i} X' \xrightarrow{f'} Y$ with i an open immersion and f' projective.

Claim 2.3.24. Let $A \in \mathbf{Ring}$ and $S = \bigoplus_{d \in \mathbb{N}} S_d$ an \mathbb{N} -graded ring with $S_0 = A$. Assume S is finitely-generated as an A -algebra by elements of S_1 . Then that the natural map $\mathrm{Proj}(S) \rightarrow \mathrm{Spec} A$ is projective.

Proof. Let $S' := A[x_0, \dots, x_n]$, there's a surjective map $S' \rightarrow S$ by taking degree 1 terms to degree 1 terms which include the generating set in S_1 . Then the map factors as

$$\mathrm{Proj} S \hookrightarrow \mathrm{Proj} S' = \mathbb{P}_A^n \rightarrow \mathrm{Spec} A.$$

Theorem 2.3.25. *A Projective morphism of Noetherian schemes is proper.*

Proof. Let $f: X \rightarrow Y$ a projection morphism of Noetherian schemes. We can factor $f: X \hookrightarrow \mathbb{P}_Y^n \rightarrow Y$ where $\mathbb{P}_Y^n = \mathbb{P}_{\mathbb{Z}}^n \times_{\mathrm{Spec} \mathbb{Z}} Y$. It's enough to show $\mathbb{P}_{\mathbb{Z}}^n \rightarrow \mathrm{Spec} \mathbb{Z}$ is proper, which then implies that the base change is proper. Because $X \hookrightarrow \mathbb{P}_Y^n$ is proper this gives the result.

Write

$$\mathbb{P}_{\mathbb{Z}}^n = \bigcup_{i=0}^n D^+(x_i) = \bigcup_{i=0}^n V_i$$

where $V_i \cong \mathrm{Spec} \mathbb{Z} \left[\frac{x_0}{x_i}, \dots, \frac{x_n}{x_i} \right]$. Hence $\mathbb{P}_{\mathbb{Z}}^n \rightarrow \mathrm{Spec} \mathbb{Z}$ is of finite type.

We have to show that for every diagram

$$\begin{array}{ccc} \mathrm{Spec} K = U & \longrightarrow & X = \mathbb{P}_{\mathbb{Z}}^n \\ \downarrow & \searrow \psi & \downarrow \\ \mathrm{Spec} R = T & \longrightarrow & \mathrm{Spec} \mathbb{Z} \end{array}$$

of solid arrows there's a unique ψ , where $R \subseteq K$ is a valuation ring.

Let $u_1 \in U$ and p_1 its image. We may assume by induction $p_1 \notin X \setminus V_i \cong \mathbb{P}_{\mathbb{Z}}^{n-1}$. Then $\frac{x_i}{x_j}$ are invertible in \mathcal{O}_{X, p_1} .

We have $k(p_1) \subseteq K$. Write $f_{i,j}$ to be the image of $\frac{x_i}{x_j}$ in K . Let $v: K^\times \rightarrow G$ be a valuations and let $g_i := v_i(f_{i,0})$ for $i \in \{0, \dots, n\}$. Let g_r be minimal. Then $v(f_{i,r}) = g_i - g_r$ because $f_{i,m} = f_{i,j}f_{j,m}$. Then $v(f_{i,r}) \geq 0$. So, define

$$\begin{aligned} \varphi: \mathbb{Z} \left[\frac{x_0}{x_r}, \dots, \frac{x_n}{x_r} \right] &\rightarrow R \\ \frac{x_i}{x_r} &\mapsto f_{i,r}. \end{aligned}$$

This gives a map $T \rightarrow V_r \hookrightarrow \mathbb{P}_{\mathbb{Z}}^n$ which we call ψ . ■

Definition 2.3.26 (Variety). A *variety* is an integral separated scheme X of finite type over a field $\mathrm{Spec} k$ where $k = \bar{k}$.

Remark 2.3.27. Usually, one omits the requirement $k = \bar{k}$ but requires that X is *geometrically integral* which we don't define now.

Proposition 2.3.28. Let $k = \bar{k}$. Recall the functor t from classical algebraic varieties to \mathbf{Sch}_k .

The image of t is the set of quasi-projective integral schemes over k (in our new terminology, quasi-projective varieties over k).

Moreover, if V is projective then so is $t(V) \rightarrow \mathrm{Spec} k$ is projective.

Chapter 3

Sheaves of Modules

Let (X, \mathcal{O}_X) be a ringed space.

Definition 3.0.1 (\mathcal{O}_X -Module). An \mathcal{O}_X -module is a sheaf \mathcal{F} on X of abelian groups such that for every open sets $U \subseteq X$ and $V \subseteq X$:

1. $\mathcal{F}(U)$ is an $\mathcal{O}_X(U)$ -module in the sense that we're given a map

$$\mathcal{O}_X(U) \times \mathcal{F}(U) \rightarrow \mathcal{F}(U)$$

for each U .

2. The morphism $\mathcal{F}(U) \rightarrow \mathcal{F}(V)$ is a morphism of $\mathcal{O}_X(U)$ -modules.

Definition 3.0.2 (Morphism of \mathcal{O}_X -Modules). A *morphism* $f: \mathcal{F} \rightarrow \mathcal{G}$ of \mathcal{O}_X -modules is a morphism of sheaves such that each $f_U: \mathcal{F}(U) \rightarrow \mathcal{G}(U)$ is $\mathcal{O}_X(U)$ -linear.

Lemma 3.0.3. If F is a presheaf of abelian groups satisfying conditions 1,2 above then F^+ naturally carries a structure of \mathcal{O}_X -modules.

Proof. We need to define a dotted map in the following diagram

$$\begin{array}{ccc} \mathcal{O}_X \times \mathcal{F} & \longrightarrow & \mathcal{F} \\ \downarrow & & \downarrow \\ \mathcal{O}_X \times \mathcal{F}^+ & \dashrightarrow & \mathcal{F}^+ \end{array}$$

where the sheaves are considered as sheaves of sets. This is given by the fact that sheafification commutes with direct products and by the universal property of sheafification. One can check that this map satisfies the desired axioms. ■

Fact 3.0.4. 1. If $f: \mathcal{F} \rightarrow \mathcal{G}$ is a morphism of \mathcal{O}_X -modules then so are $\ker(f)$, $\operatorname{coker}(f)$, $\operatorname{Im}(f)$.
2.

Definition 3.0.5 (Invertible Sheaf). A locally free sheaf of rank 1 is called *an invertible sheaf (alt. “line bundle”)*.

Remark 3.0.6. The reason that it's called an invertible sheaf is that $\mathcal{F} \otimes \underline{\operatorname{Hom}}(\mathcal{F}, \mathcal{O}_X) \cong \mathcal{O}_X$.

Definition 3.0.7 (\mathcal{O}_X -Submodule). An \mathcal{O}_X -submodule is an \mathcal{O}_X -module \mathcal{F} such that every $\mathcal{F}(U)$ is an $\mathcal{O}_X(U)$ -submodule with the given action.

This is also called a *sheaf of ideals on X* .

Proposition 3.0.8. Let $f: (X, \mathcal{O}_X) \rightarrow (Y, \mathcal{O}_Y)$ be a morphism of ringed spaces and let $\mathcal{F} \in \mathcal{O}_X\text{-Mod}$ (the category of \mathcal{O}_X -modules).

Then $f_*\mathcal{F}$ is an $f_*\mathcal{O}_X$ -module.

Proof. We have $(f_*\mathcal{F})(U) = \mathcal{F}(f^{-1}(U))$ which is a module over $\mathcal{O}_X(f^{-1}(U)) = (f_*\mathcal{O}_X)(U)$. But, $f_*\mathcal{O}_X$ is an \mathcal{O}_Y -module via

$$f^\# : \mathcal{O}_Y \rightarrow f_*\mathcal{O}_X.$$

So, $f_*\mathcal{F}$ is an \mathcal{O}_Y -module in a natural way. ■

Proposition 3.0.9. *Let $f: (X, \mathcal{O}_X) \rightarrow (Y, \mathcal{O}_Y)$ be a morphism of ringed spaces and let $\mathcal{G} \in \mathcal{O}_Y\text{-Mod}$. Then $f^{-1}\mathcal{G}$ is an $f^{-1}\mathcal{O}_Y$ -module.*

Proof. $(f^{-1}\mathcal{G})(U) \cong \lim_{V \supseteq f(U)} G(V)$ is a module over $\lim_{V \supseteq f(U)} \mathcal{O}_Y(V) \cong (f^{-1}\mathcal{O}_Y)(U)$. Adjunction applied to $f^\#$ gives $f^{-1}\mathcal{O}_Y \rightarrow \mathcal{O}_X$. Set $f^*\mathcal{G} := f^{-1}G \otimes_{f^{-1}\mathcal{O}_Y} \mathcal{O}_X$. ■

Remark 3.0.10. If f is an open immersion of schemes $f: U \hookrightarrow X$ then $f^*\mathcal{G} \cong \mathcal{G}|_U$.

Fact 3.0.11. *There's a natural isomorphism of groups*

$$\underline{\text{Hom}}_{\mathcal{O}_X}(f^*\mathcal{G}, \mathcal{F}) \cong \underline{\text{Hom}}_{\mathcal{O}_Y}(G, f_*\mathcal{F}).$$

Definition 3.0.12. Let $A \in \mathbf{Ring}$ and $M \in \mathbf{A-Mod}$. Let \tilde{M} be the sheaf on $\text{Spec } A$:

$$\tilde{M}(U) = \left\{ s: U \rightarrow \prod_{\mathfrak{p} \in \text{Spec } A} M_{\mathfrak{p}} \mid s(\mathfrak{p}) \in M_{\mathfrak{p}}^{\star} \right\}$$

where \star is the property that for all $\mathfrak{p} \in U$ there's a neighbourhood $V \subseteq U$ of \mathfrak{p} and there are $m \in M, f \in A$ such that for all $\mathfrak{q} \in V$ it holds that $f \notin \mathfrak{q}$ and $s(\mathfrak{q}) = \frac{m}{f} \in M_{\mathfrak{q}}$.

Proposition 3.0.13. *Let $X := \text{Spec } A$.*

1. \tilde{M} is an \mathcal{O}_X -module.
2. $(\tilde{M})_{\mathfrak{p}} \cong M_{\mathfrak{p}}$ as $A_{\mathfrak{p}}$ -modules for every $\mathfrak{p} \in X$.
3. $\tilde{M}(D(f)) \cong M_f$ as A_f -modules, for every $f \in A$, where $D(f)$ is the basic open in X .
4. $\tilde{M}(X) \cong M$.

Remark 3.0.14.

$$\begin{aligned} \mathbf{A-Mod} &\rightarrow \mathcal{O}_{\text{Spec } A}\text{-Mod} \\ M &\mapsto \tilde{M} \end{aligned}$$

is a functor.

Proposition 3.0.15. 1. *The functor $M \rightarrow \tilde{M}$ is exact and fully-faithful.*

2. $(M \otimes_A N)^\sim \cong \tilde{M} \otimes_{\mathcal{O}_X} \tilde{N}$.
3. $(\bigoplus_{i \in I} M_i)^\sim \cong \bigoplus_{i \in I} \tilde{M}_i$.
4. *Let $\varphi: A \rightarrow B$ a ring morphism and let $N \in \mathbf{B-Mod}$. Let $f: \text{Spec } B \rightarrow \text{Spec } A$ the induced map on spectra. Then $f_*\tilde{N} \cong \widetilde{{}_A N}$ where ${}_A N$ is N thought of as an A -module.*
5. *If $M \in \mathbf{A-Mod}$ then $f^*\tilde{M} \cong \widetilde{(M \otimes_A B)}$.*

Proof. 1. **Fully-faithfulness:** We want to show $\text{Hom}_A(M, N) \rightarrow \text{Hom}_{\mathcal{O}_X}(\tilde{M}, \tilde{N})$ is bijective. The inverse map is taking global sections.

Exactness: Let

$$0 \rightarrow M' \rightarrow M \rightarrow M'' \rightarrow 0$$

be exact. We want to show that the induced sequence

$$0 \rightarrow \tilde{M}' \rightarrow \tilde{M} \rightarrow \tilde{M}'' \rightarrow 0$$

is exact. We can check this on stalks which gives the result.

2. We have a natural map

$$(M \otimes_A N)^\sim \rightarrow \tilde{M} \otimes_{\mathcal{O}_X} \tilde{N}$$

by extending linearly. We have to show this is an isomorphism on the stalks. Because localisation commutes with tensor products we get

$$(M \otimes_A N)_\mathfrak{p} \cong M_\mathfrak{p} \otimes_{A_\mathfrak{p}} N_\mathfrak{p}$$

so the natural map is an isomorphism on stalks, and is therefore an isomorphism.

3. Exercise.

4. Exercise.

5. Exercise. ■

Definition 3.0.16 (Quasi-Coherent \mathcal{O}_X -Module). Let $X \in \mathbf{Sch}$ and $\mathcal{F} \in \mathcal{O}_X\text{-Mod}$. Then \mathcal{F} is called *quasi-coherent* if there's an open affine cover $(U_i)_{i \in I}$ with $U_i \cong \text{Spec } A_i$ of X such that $\mathcal{F}|_{U_i} \cong \tilde{M}_i$ for some A_i -module M_i , for all $i \in I$.

Definition 3.0.17 (Coherent \mathcal{O}_X -Module). A quasi-coherent \mathcal{O}_X -module \mathcal{F} is *coherent* if the M_i can be taken to be finite A_i -modules.

Remark 3.0.18. \mathcal{O}_X is coherent for every $X \in \mathbf{Sch}$.

Example 22. Let $X = \text{Spec } A$ and $Y \subseteq X$ a closed subscheme defined by $I \trianglelefteq A$. Then

$$i_* \mathcal{O}_Y \cong \widetilde{(A/I)}$$

where we think of A/I as an A -module.
This is coherent.

Definition 3.0.19 (Quasi-Coherent Sheaf). A quasi-coherent sheaf on X is a quasi-coherent \mathcal{O}_X -module.

Notation 3.0.20. Denote by $\mathbf{Qcoh}(X)$ the category of quasi-coherent sheaves on X .

Lemma 3.0.21. Let $X = \text{Spec } A$, $f \in A$, $D(f) \subseteq X$ and $\mathcal{F} \in \mathbf{Qcoh}(X)$.

1. If $s \in \Gamma(X, \mathcal{F})$ is such that $s|_{D(f)} = 0$ then $f^n s = 0$ for some $n > 0$.
2. If $t \in \mathcal{F}(D(f))$ then for some $n > 0$, $f^n t$ extends to a global section in $\Gamma(X, \mathcal{F})$.

Proof. Cover X with over affine $(V_i)_{i \in I}$ such that $V_i \cong \text{Spec } B_i$ and $\mathcal{F}|_{V_i} \cong \tilde{M}_i$ for M a B_i -module. For every $g \in A$ such that $D(g) \subseteq V_i$, and call $\iota: D(g) \hookrightarrow V_i$ the inclusion. We get a morphism $B_i \rightarrow A_g$. So

$$\mathcal{F}|_{D(g)} = (\mathcal{F}|_{V_i})|_{D(g)} = \tilde{M}_i|_{D(g)} = \iota^* \tilde{M} = (M \otimes_{B_i} A_g).$$

Hence there's an open affine cover $(D(g_i))_{i \in I'}$ of X which is and such that $\mathcal{F}|_{D(g_i)} \cong \tilde{M}_i$ and M_i is a A_{g_i} -module. We can take I' to be finite because X is affine and therefore quasi-compact.

1. If $s \in \Gamma(X, \mathcal{F})$ with $s|_{D(f)} = 0$, let $s_i := s|_{D(g_i)} \in \tilde{M}_i(D(g_i)) \cong M_i$. Then

$$D(f) \cap D(g_i) = D(fg_i)$$

so

$$\mathcal{F}|_{D(fg_i)} = \tilde{M}_i|_{D(fg_i)} \cong (\tilde{M}_i)_f.$$

Hence $s_i = 0 \in (M_i)_f$. Hence $f^n s_i = 0$ for some $n \in \mathbb{N}_+$. Because I' is finite we get the result by taking the maximal power.

2. Given $t \in \mathcal{F}(D(f))$ let $\tilde{t}_i \in \mathcal{F}(D(fg_i)) \cong (M_i)_f$ the restriction of t . Then there are $t_i \in M_i$ such that $t_i = f^n \tilde{t}_i$. Then the t_i agree on the double intersections $D(fg_i g_j)$. Then $f^m (t_i - t_j) = 0$ on $D(g_i g_j)$ by part 1. Hence $f^m t_i$ glue to a section $s \in \mathcal{F}(X)$ such that

$$s|_{D(f)} = f^m \{t_i\} = f^{m+n} t$$

where $\{t_i\}$ is the gluing of the t_i 's. ■

Proposition 3.0.22. *Let $X \in \mathbf{Sch}$ and let $\mathcal{F} \in \mathcal{O}_X\text{-Mod}$. Then:*

1. \mathcal{F} is quasi-coherent iff for all $U \subseteq X$ open affine there exists $M \in A\text{-Mod}$ such that $\mathcal{F}|_U \cong \tilde{M}$.
2. Assume X is Noetherian. \mathcal{F} is coherent iff for all $U \subseteq X$ open affine there exists $M \in A\text{-Mod}^{\text{fg}}$ such that $\mathcal{F}|_U \cong \tilde{M}$.

Proof. 1. Assume \mathcal{F} is quasi-coherent and let $U \subseteq X$ open and isomorphic to $\text{Spec } A$. Then $\mathcal{F}|_U$ is quasi-coherent because we can intersect a covering of X with U to get an appropriate cover. We can therefore assume by replacing X with U that X is affine with $\text{Spec } A$.

Let $M = \Gamma(X, \mathcal{F})$, we want to show $\mathcal{F} \cong \tilde{M}$. We have $\alpha: \tilde{M} \rightarrow \mathcal{F}$ from the universal property of the sheafification. We have that the sections of \tilde{M} at U are $\left\{ s: U \rightarrow \coprod_p M_p \mid s(p) = \frac{m}{f} \text{ locally} \right\}$. We can then send s to $\frac{m}{f}$ in $\mathcal{F}(U)$.

Cover X with $D(g_i)$ such that $\mathcal{F}|_{D(g_i)} \cong \tilde{M}_i$ where $M_i \in A_{g_i}\text{-Mod}$. By 3.0.21 we get $\mathcal{F}(D(g_i)) \cong M_{g_i}$. Then $M_{g_i} \cong M_i$ because we can recover both as global sections of the same sheaf.

Hence α is an isomorphism over $D(g_i)$ for all i , hence it's an isomorphism on every stalk and hence an isomorphism.

2. Assume \mathcal{F} is coherent and reduce similarly to the case where \mathcal{F} is affine. It's then also quasi-coherent so by the above $\mathcal{F} \cong \tilde{M}$ where $M \in A\text{-Mod}$. There are $(g_i)_{i \in [n]}$ which generate A such that M_{g_i} are finitely-generated A_{g_i} -modules. Then M_{g_i} are Noetherian hence so is M (via the same proof as for rings). ■

Proposition 3.0.23. *Let $X \in \mathbf{Sch}$ affine and let*

$$0 \rightarrow \mathcal{F}' \rightarrow \mathcal{F} \rightarrow \mathcal{F}'' \rightarrow 0$$

be an exact sequence of \mathcal{O}_X -modules. Assume $\mathcal{F}' \in \mathbf{Qcoh}(X)$. Then

$$0 \rightarrow \Gamma(X, \mathcal{F}') \rightarrow \Gamma(X, \mathcal{F}) \rightarrow \Gamma(X, \mathcal{F}'') \rightarrow 0$$

is exact.

Proof. By an exercise from Hartshorne, $\Gamma(X, -)$ is left exact. Hence it's enough to show that $\tilde{\beta}: \Gamma(X, \mathcal{F}) \rightarrow \Gamma(X, \mathcal{F}'')$ is surjective.

Let $s \in \Gamma(X, \mathcal{F}'')$. Let $x \in X$, there's $D(f)$ containing x such that $s|_{D(f)} = \beta(t_x)$ for some $t_x \in \mathcal{F}(D(g_i))$.

Claim 3.0.24. $f^n s$ lifts to $\mathcal{F}(X)$ for some $n > 0$.

Proof. Cover X with $D(g_i)$ such that $s|_{D(g_i)} = \beta(t_i)$ with $t_i \in \mathcal{F}(D(g_i))$.

On $D(f) \cap D(g_i) = D(fg_i)$ we have that t_x, t_i both map under β to s . Hence $t_x - t_i \in \ker \beta(D(fg_i))$ so $t_x - t_i \in \mathcal{F}'(D(fg_i))$. By 3.0.21 there's $n \in \mathbb{N}$ such that $f^n(t_x - t_i)$ extends to $u_i \in \mathcal{F}'(D(g_i))$. There are finitely many indices i so choose n big enough for all of them.

Let $t'_i := f^n t_i + u_i$. Then $\beta(t'_i) = \beta(f^n t_x) = f^n s$ on $D(g_i)$. Also $t'_i = f^n t_x$ on $D(fg_i)$.

So on $D(g_i g_j)$ we have that t'_i, t'_j both lift to $f^n s$. Then $t'_i - t'_j \in \mathcal{F}'(D(g_i g_j))$ so the difference is 0 on $D(fg_i g_j)$.

By 3.0.21, $f^m(t'_i - t'_j) = 0$ for m large enough, so $\{f^m t'_i\}_{i \in I}$ glue to $t'' \in \mathcal{F}(X)$ such that $\beta(t'') = f^{m+n} s \in \mathcal{F}''(X)$. ■

Now, cover X with $D(f_i)$. Then $s|_{D(f_i)}$ lift to $\mathcal{F}(D(f_i))$ and by the claim to $t_i \in \mathcal{F}(X)$ such that $\beta(t_i) = f_i^n s$. Now $(f_1, \dots, f_k) = A = (f_1^n, \dots, f_k^n)$ so $1 = \sum_i a_i f_i^n$ for some $a_i \in A$. Let $t = \sum a_i t_i \in \mathcal{F}(X)$. Then

$$\beta(t) = \sum a_i \beta(t_i) = \sum a_i f_i^n s = s \sum a_i f_i^n = s. \quad \blacksquare$$

Proposition 3.0.25. *Let $X \in \mathbf{Sch}$.*

1. *Kernels, cokernels and images exist in $\mathbf{Qcoh}(X)$.*
2. *Any extension of quasi-coherent sheaves is quasi-coherent. I.e. if the outer sheaves in a short exact sequence are quasi-coherent, so is the middle one.*
3. *If X is Noetherian, 1,2 hold in \mathbf{Coh} .*

Proof. It's enough to assume X is affine.

1. Take the following exact sequence $0 \rightarrow \mathcal{F}' \rightarrow \mathcal{F} \rightarrow \mathcal{F}''$ where $\mathcal{F}, \mathcal{F}''$ are quasi-coherent. Write $\mathcal{F} \cong \tilde{M}$ and $\mathcal{F}'' \cong \tilde{M}''$. The morphism $M \rightarrow M''$ has a kernel M' so that $0 \rightarrow M' \rightarrow M \rightarrow M''$ is exact. By exactness of $\tilde{}$ we get that $0 \rightarrow \tilde{M}' \rightarrow \tilde{M} \rightarrow \tilde{M}''$ is exact, so by fully-faithfulness $\mathcal{F}' \cong \tilde{M}'$.
2. Suppose we have a short exact sequence $0 \rightarrow \mathcal{F}' \rightarrow \mathcal{F} \rightarrow \mathcal{F}'' \rightarrow 0$ where $\mathcal{F}' \cong \tilde{M}'$ and $\mathcal{F}'' \cong \tilde{M}''$. Taking global sections we get

$$0 \rightarrow M' \rightarrow M \rightarrow M'' \rightarrow 0$$

which is exact by the above proposition. Then

$$0 \rightarrow \mathcal{F}' \rightarrow \tilde{M} \rightarrow \mathcal{F}'' \rightarrow 0$$

is exact. We have a functorial morphism $\alpha: \tilde{M} \rightarrow \mathcal{F}$. α on $\mathcal{F}', \mathcal{F}''$ is an isomorphism, so we have a diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{F}' & \longrightarrow & \tilde{M} & \longrightarrow & \mathcal{F}' & \longrightarrow & 0 \\ & & \downarrow & & \downarrow \alpha & & \downarrow & & \\ 0 & \longrightarrow & \mathcal{F}' & \longrightarrow & \mathcal{F} & \longrightarrow & \mathcal{F}'' & \longrightarrow & 0 \end{array}$$

where the vertical morphisms on the sides are isomorphisms. Then by the 5-lemma α is an isomorphism. Hence \mathcal{F} is quasi-coherent. \blacksquare

Proposition 3.0.26. *Let $f: X \rightarrow Y$ in **Sch**.*

1. If $\mathcal{G} \in \mathbf{Qcoh}(Y)$ then $f^*\mathcal{G} \in \mathbf{Qcoh}(X)$.
2. If X, Y are Noetherian and $\mathcal{G} \in \mathbf{Coh}(Y)$ then $f^*\mathcal{G} \in \mathbf{Coh}(X)$.
3. If either X is Noetherian or f is quasi-compact and separated then if $\mathcal{F} \in \mathbf{Qcoh}(X)$ then $f_*\mathcal{F} \in \mathbf{Qcoh}(Y)$.

Remark 3.0.27. Part 3 of the proposition fails badly for $\mathbf{Coh}(X)$. For example, take $\mathrm{Spec} k[x] \rightarrow \mathrm{Spec} k$.

Proof. For 1,2 we reduce to the case where X, Y are affine: Take $U \subseteq Y$ open affine and examine $f^{-1}(U)$. This can be covered by affines. Now $f^*\mathcal{G}|_U = (f|_U)^*\mathcal{G}|_U$.

1. We have $\mathcal{G} \cong \tilde{M}$ and $f^*\mathcal{G} \cong (M \otimes_A B)^\sim \in \mathbf{Qcoh}(X)$.
2. Similarly, where M is finitely-generated over A and hence over B by Noetherianity of both.
3. We have $(f_*\mathcal{F})|_U \cong f_*\left(\mathcal{F}|_{f^{-1}(U)}\right)$ for $U \subseteq Y$. Cover X with open affines $(U_i)_{i \in [n]}$.
 - If f is separated, $U_i \cap U_j$ is affine (exercise 4.3).
 - If X is Noetherian, we can cover $U_i \cap U_j$ with finitely many open affines $U_{i,j,k}$.

By the sheaf property we get the following exact sequence, where the last map is the identity if i appears in the first coordinate or minus the identity if it's on the second one.

$$0 \rightarrow f_*\mathcal{F} \rightarrow \bigoplus_{i \in [n]} f_*\left(\mathcal{F}|_{U_i}\right) \rightarrow \bigoplus_{i,j,k} f_*\left(\mathcal{F}|_{U_{i,j,k}}\right)$$

The direct sums are quasi-coherent because U_i are affine, so $f_*\mathcal{F}$ is quasi-coherent as the kernel of the morphism of quasi-coherent schemes. \blacksquare

3.0.1 Ideal Sheaves

Let $i: Y \hookrightarrow X$ be a closed subscheme. The **ideal sheaf** \mathcal{I}_Y of Y is $\ker(\mathcal{O}_X \rightarrow i_*\mathcal{O}_Y)$. We have an exact sequence

$$0 \rightarrow \mathcal{I}_Y \rightarrow \mathcal{O}_X \rightarrow i_*\mathcal{O}_Y \rightarrow 0.$$

Proposition 3.0.28. 1. \mathcal{I}_Y is quasi-coherent.

2. Any quasi-coherent sheaf of ideals on X arises from some Y in the above sense.

3. If X is Noetherian, \mathcal{I}_Y is coherent.

Proof. 1. We know \mathcal{O}_X is quasi-coherent, hence by the exact sequence we may prove $i_*\mathcal{O}_Y$ is quasi-coherent. Because i is a closed immersion, it's quasi-compact and separated (e.g. in the Noetherian case, this follows from the valuative criterion). Then by the previous proposition $i_*\mathcal{O}_Y$ is quasi-coherent from which we get the result as stated.

2. Let \mathcal{I} be a quasi-coherent sheaf of ideals of \mathcal{O}_X . Take

$$Y = \text{supp}(\mathcal{O}_X/\mathcal{I}) = \left\{ y \in X \mid (\mathcal{O}_X/\mathcal{I})_y \neq 0 \right\}.$$

Locally, \mathcal{I}_{U_i} is quasi-coherent sheaf of ideals on $\text{Spec } A_i \cong U_i$. So it's of the form $\tilde{\mathfrak{a}}_i$ for some $\mathfrak{a}_i \trianglelefteq A_i$. Hence $(V(\mathfrak{a}_i), A_i/\mathfrak{a}_i)$ is a closed subscheme of U_i and these glue to give $(Y, \mathcal{O}_Y/\mathcal{I}) \hookrightarrow X$. ■

Corollary 3.0.29. If $X = \text{Spec } A$ then there's a bijection between ideals $I \trianglelefteq A$ and closed subschemes $Y \hookrightarrow X$ which sends I to $\text{Spec } A/I$.

Let S be a graded ring and M a graded S -module (as in the previous course, $M = \bigoplus_{d \in \mathbb{Z}} M_d$ is such that $S_d \cdot M_e \subseteq M_{d+e}$ for all $d, e \in \mathbb{Z}$).

Definition 3.0.30 (Twisted Graded Module). Let M be an S -graded module and let $k \in \mathbb{Z}$. Define $M(k)$ by $M(k)_d = M_{d+k}$.

Definition 3.0.31. When viewing S as a graded S -module, we have $M(k) \cong M \otimes_S S(k)$.

Notation 3.0.32. Let $p \in \text{Proj}(S)$. We denote

$$T_p := \{t \in S \text{ homogeneous} \mid t \notin p\}$$

$$M_{(p)} := (T_p^{-1}M)_0$$

where the last subscript is to denote elements of degree 0.

Definition 3.0.33. Let M be a graded S -module, we define

$$\tilde{M}(U) = \left\{ s: U \rightarrow \prod_{p \in U} M_{(p)} \mid \star \right\}$$

where \star is the intersection of the following properties.

1. $\forall p \in U: s(p) \in M_{(p)}$.
2. For all $p \in U$ there's a neighbourhood $V \subseteq U$ and homogeneous elements $m \in M$ and $f \in S$ such that

$$\forall q \in V: f \notin q \wedge s(q) = \frac{m}{f} \in M_{(q)}.$$

Proposition 3.0.34. 1. $(\tilde{M})_p \cong M_{(p)}$.

2. For every homogeneous $f \in S_+$ we have $\tilde{M}|_{D_+(f)} \cong (M_{(f)})^\sim$.

3. \tilde{M} is quasi-coherent on $\text{Proj}(S)$.

4. If S is Noetherian and M is a finite (graded) S -module then \tilde{M} is coherent.

Definition 3.0.35 (Serre's Twistin Sheaf). Let $X = \text{Proj } S$. For $n \in \mathbb{Z}$ let $\mathcal{O}_X(n) := \widetilde{S(n)}$.

Definition 3.0.36 (Twisted \mathcal{O}_X -Module). Let $X = \text{Proj } S$ and $\mathcal{F} \in \mathcal{O}_X\text{-Mod}$. We define

$$\mathcal{F}(n) := \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{O}_X(n).$$

Proposition 3.0.37. Let $X = \text{Proj } S$ where S is generated by S_1 as an S_0 -algebra. Then

1. $\mathcal{O}_X(n)$ is invertible.
2. $\tilde{M}(n) \cong \widetilde{M(n)}$.
3. $\mathcal{O}_X(n) \otimes_{\mathcal{O}_X} \mathcal{O}_X(m) \cong \mathcal{O}_X(n+m)$.
4. Let $T = \bigoplus T_d$ be a graded ring, generated by T_1 as a T_0 -algebra. Let $\varphi: S \rightarrow T$ be a ring homomorphism preserving degrees.

Let

$$U := \{p \in \text{Proj}(T) \mid p \not\subseteq \varphi(S_+)\} \subseteq \text{Proj}(T) = Y.$$

There's an induced map $f: U \rightarrow \text{Proj}(S)$. Then

$$\begin{aligned} f^*(\mathcal{O}_X(n)) &\cong \mathcal{O}_Y(n)|_U \\ f_*(\mathcal{O}_Y(n)|_U) &\cong f_*(\mathcal{O}_U(n)). \end{aligned}$$

Proof. 1. Let $f \in S_1$. Then $\mathcal{O}_X(n)|_{D_+(f)} \cong \widetilde{S(n)}_{(f)}$ on $\text{Spec } S_{(f)}$ by looking at definitions.

Note that

$$S(n)_{(f)} = \{s \in S_p \text{ of degree } n\} \cong S(m)_{(f)}$$

by sending s to $f^{m-n}s$. In particular $S(n)_{(f)} \cong S(0)_{(f)} \cong S_{(f)}$ as $S_{(f)}$ -modules. Hence $S(n)_{(f)}$ is of rank 1.

X is covered by $D_+(f)$ for $f \in S_1$, hence by the above $\mathcal{O}_X(n)$ is locally free of rank 1, hence is invertible.

2. More generally,

$$(M \otimes_S N)^\sim \cong \tilde{M} \otimes_{\mathcal{O}_X} \tilde{N}.$$

This is true since for $f \in S_1$,

$$(M \otimes_S N)_{(f)} \cong M_{(f)} \otimes_{S_{(f)}} N_{(f)}.$$

Writing a global isomorphism is then easy.

3. This follows from the general formula above.

4. In general one can check that

$$\begin{aligned} f^*(\tilde{M}) &\cong \mathcal{O}_Y(n)|_U \\ f_*\left(\tilde{N}\Big|_U\right) &\cong ({}_S N)^\sim \end{aligned}$$

■

Example 23. Let $X = \mathbb{P}_k^1 = \text{Proj}(k[x, y])$.

Then $\mathcal{O}_X(1) \cong \tilde{M}$ is isomorphic to

$$k \oplus \bigoplus_{n \geq 1} \text{Sym}^n(k^2)^\vee$$

and to

$$k \oplus \bigoplus_{n \geq 1} \{\text{homogeneous binary } n\text{-ic forms over } k\}.$$

Note that $\Gamma(\mathbb{P}^1, \mathcal{O}_X(1)) \cong kx + ky$. Also,

$$\mathcal{O}_X(1)|_{D_+(y)} \cong \widetilde{k[x]} \cong \mathcal{O}_{D_+(y)}$$

where $D_+(y) \cong \text{Spec } k[x]$ by $y \mapsto 1$.

Definition 3.0.38. Let $X = \text{Proj}(S)$ and let $\mathcal{F} \in \mathcal{O}_X\text{-Mod}$. Define the graded S -module

$$\Gamma_*(\mathcal{F}) := \bigoplus_{n \in \mathbb{Z}} \Gamma(X, \mathcal{F}(n)).$$

Remark 3.0.39. $\Gamma_*(\mathcal{F})$ has an S -module structure. An element $s \in S_d$ gives a global section in $\Gamma(X, \mathcal{O}_X(d))$. An element $t \in \Gamma(X, \mathcal{F}(n))$ gives rise to

$$s \cdot t = s \otimes t \in \Gamma(X, \mathcal{O}_X(d) \otimes_{\mathcal{O}_X} \mathcal{F}(n)) \cong \Gamma(X, \mathcal{F}(n+d)).$$

Proposition 3.0.40. Let $A \in \mathbf{Ring}$ and $S = A[x_0, \dots, x_r]$ for $r \in \mathbb{N}_+$. Let $X = \text{Proj } S \cong \mathbb{P}_A^r$. Then $\Gamma_*(\mathcal{O}_X) \cong S$.

Proof. Write $x = \bigcup_i D_+(x_i)$. Giving $t \in \Gamma(X, \mathcal{O}_X(n))$ is the same as giving $t_i \in \mathcal{O}_X(n)(D_+(x_i))$ agreeing on the $D_+(x_i x_j)$.

We have

$$\Gamma_*(\mathcal{O}_X) = \{(t_0, \dots, t_r) \mid t_i = t_j \text{ in } S_{x_i x_j}\}.$$

We get a map

$$S \hookrightarrow S_{x_i} \hookrightarrow S_{x_i x_j} \hookrightarrow S_{x_1 \dots x_r} =: S'.$$

Hence

$$\Gamma_*(\mathcal{O}_X) = \bigcap_{i=0}^r S_{x_i} \subseteq S'.$$

If $s \in S'$ is homogeneous then

$$s = x_0^{i_0} \cdot x_1^{i_1} \cdot \dots \cdot x_r^{i_r} \cdot f(x_0, \dots, x_r)$$

where the i_j are in \mathbb{Z} and $f \in S$. We have $s \in S_{x_i}$ iff $i_j \geq 0$ for $j \neq i$, so $\bigcap_i S_{x_i} = S$. ■

Proposition 3.0.41. Let S be a graded ring finitely-generated by S_1 as an S_0 -algebra. Let $X = \text{Proj } S$ and let $\mathcal{F} \in \mathbf{Qcoh}(X)$. Then there's a natural isomorphism

$$\widetilde{\Gamma_*(\mathcal{F})} \xrightarrow{\sim} \mathcal{F}.$$

Proof. See Hartshorne. ■

Corollary 3.0.42. Let $A \in \mathbf{Ring}$.

1. If $Y \hookrightarrow \mathbb{P}_A^r$ is a closed immersion there's an homogeneous ideal $I \subseteq A[x_0, \dots, x_r]$ such that $Y \cong \text{Proj}(S/I) \hookrightarrow \text{Proj}(S)$.
2. $Y \in \mathbf{Sch}(\text{Spec } A)$ is projective iff $Y \cong \text{Proj}(S)$ for some graded ring S where $S_0 \cong A$ and S is finitely-generated by S_1 as an S_0 -algebra.

Proof. 1. Let \mathcal{I}_Y be the ideal sheaf on X so that the following is exact.

$$0 \rightarrow \mathcal{I}_Y \rightarrow \mathcal{O}_X \rightarrow i_* \mathcal{O}_Y \rightarrow 0$$

Note that $\Gamma(X, -)$ is left-exact. We have also that $(-) \otimes \mathcal{O}(n)$ is exact. In fact this is a specific example of the fact that $(-) \otimes \mathcal{I}$ is exact for any invertible sheaf \mathcal{I} . Here it's enough to work locally, and tensoring by \mathcal{O}_X is an isomorphism, so that's clear.

Hence $\Gamma_*(-)$ is left exact. Now, $0 \rightarrow \mathcal{I}_Y \rightarrow \mathcal{O}_X$ is exact hence so is

$$0 \rightarrow \Gamma_*(\mathcal{I}_Y) \rightarrow \Gamma(\mathcal{O}_X) \cong S.$$

Denote $\mathcal{I} := \Gamma_*(\mathcal{I}_Y)$. We get that $\text{Proj}(S/I) \hookrightarrow \text{Proj}(S)$ with sheaf of ideals being $\tilde{\mathcal{I}}$, which is quasi-coherent.

By 3.0.41, $\tilde{\mathcal{I}} = \Gamma_*(\mathcal{I}_Y) \cong \mathcal{I}_Y$. Then $Y \cong \text{Proj}(S/I)$.

2. See Hartshorne. ■

Definition 3.0.43. Let $Y \in \mathbf{Sch}$ and let $j: \mathbb{P}_Y^r \rightarrow \mathbb{P}_{\mathbb{Z}}^r$. Define $\mathcal{O}(1)$ a sheaf on \mathbb{P}_Y^r by $g^* \mathcal{O}_{\mathbb{P}_{\mathbb{Z}}^r}(1)$. This is invertible.

Definition 3.0.44 (Immersion). A morphism $i: X \rightarrow Z$ is an **immersion** if it's an isomorphism onto an open subscheme of a closed subscheme.

Definition 3.0.45 (Very Ample Invertible Sheaf). If $X \rightarrow Y$ is a morphism, an invertible sheaf \mathcal{L} on X is **very ample relative to Y** if there's an immersion $X \hookrightarrow \mathbb{P}_Y^r$ for some r such that $i^*\mathcal{O}(1) \cong \mathcal{L}$.

Lemma 3.0.46. Let $Y \in \mathbf{Sch}$ Noetherian and let $f: X \rightarrow Y$. Then f is projective iff it's proper and there exists a very ample invertible sheaf \mathcal{L} on X relative to Y .

Proof. \implies : This is clear by pulling back $\mathcal{O}(1)$ under $X \hookrightarrow \mathbb{P}_Y^r$.

\impliedby : There's an immersion $i: X \hookrightarrow \mathbb{P}_Y^n$. Since X is proper, this has to be a closed immersion. \blacksquare

Definition 3.0.47. Let $X \in \mathbf{Sch}$ and $\mathcal{F} \in \mathcal{O}_X\text{-Mod}$. Then \mathcal{F} is **generated by global sections** if there are $(s_i)_{i \in I} \subseteq \Gamma(X, \mathcal{F})$ such that the s_i generate \mathcal{F}_x as an $\mathcal{O}_{X,x}$ -module for every $x \in X$.

Remark 3.0.48. Equivalently, \mathcal{F} is generated by global sections if there's a surjection

$$\bigoplus_{i \in I} \mathcal{O}_X \twoheadrightarrow \mathcal{F}$$

(sending e_i to s_i).

Example 24. Let $X = \text{Proj } S$ where S is generated by S_1 as an S_0 -algebra. Then $\mathcal{O}_X(1)$ is globally generated (by S_1).

Theorem 3.0.49 (Serre). Let $X \in \mathbf{Sch}$ be projective over $\text{Spec } A$ for $A \in \mathbf{Ring}$ Noetherian. Let $\mathcal{O}(1)$ be pulled from $\mathcal{O}(1)$ of \mathbb{P}_A^r via some embedding $X \hookrightarrow \mathbb{P}_A^r$. Let $\mathcal{F} \in \mathcal{O}_X\text{-Mod}$. Then for all n large enough, $\mathcal{F}(n)$ is generated by global sections.

Exercise 6. Let $i: X \hookrightarrow Y$ be a closed immersion of Noetherian schemes, then $i_*\mathcal{F}$ is coherent on Y for all $\mathcal{F} \in \mathbf{Coh}(X)$.

Proof. By the exercise, $i_*\mathcal{F} \in \mathbf{Coh}(\mathbb{P}_A^r)$. Also $i_*(\mathcal{F}(n)) \cong (i_*\mathcal{F})(n)$ so $\mathcal{F}(n)$ is globally generated iff $i_*(\mathcal{F}(n))$ is. So, we may assume $X = \mathbb{P}_A^r$.

Cover X with $(D_+(X_i))_{i=0}^r$. \mathcal{F} is coherent hence $\mathcal{F}|_{D_+(x_i)} \cong \tilde{M}_i$ for finitely-generated B_i -modules M_i where $B_i = A \left[\frac{x_0}{x_i}, \dots, \frac{x_r}{x_i} \right]$.

Say $s_{i,j}$ generated M_i , then $x_i^n s_{i,j}$ extend to global sections $t_{i,j}$ of $\mathcal{F}(n)$ for large enough n . $\mathcal{F}(n)$ is coherent, so $\mathcal{F}|_{D_+(x_i)} \cong \tilde{M}'_i$ for B_i -modules M'_i .

Now the map $\mathcal{F} \rightarrow \mathcal{F}(n)$ defined by $\alpha \mapsto x_i^n \alpha$ is an isomorphism over $D_+(x_i)$. Hence $M'_i \cong M_i$. Hence $x_i^n s_{i,j}$ generate M'_i .

Hence $t_{i,j} \in \Gamma(X, \mathcal{F}(n))$ generate $\mathcal{F}(n)$ at all stalks. \blacksquare

Theorem 3.0.50. Let $k \in \mathbf{Field}$, $A \in k\text{-Alg}^{\text{fg}}$ and $f: X \rightarrow \text{Spec } A$ projective. Let $\mathcal{F} \in \mathbf{Coh}(X)$. Then $\Gamma(X, \mathcal{F})$ is a finitely-generated A -module.

Proof. • Assume first that $X = \text{Proj } S$ where S is a graded ring finitely-generated by S_1 over $A := S_0$.

Let

$$M := \Gamma_*(\mathcal{F}) \cong \bigoplus_n \Gamma(X, \mathcal{F}(n))$$

then $\tilde{M} \cong \mathcal{F}$. Now $\mathcal{F}(n)$ is globally-generated for large enough n , by finitely many sections in $\Gamma(X, \mathcal{F}(n))$. Let $M' \leq M$ the sub- S -module generated by these global sections.

\sim is exact, so $\tilde{M}' \subseteq \tilde{M} \cong \mathcal{F}$. Hence

$$\tilde{M}'(n) \hookrightarrow \mathcal{F}(n)$$

but this is a surjection because the sections generate locally all of the stalks of $\mathcal{F}(n)$. Hence we get an isomorphism $\tilde{M}'(n) \xrightarrow{\sim} \mathcal{F}(n)$. By untwisting we get $\tilde{M}' \cong \tilde{M} \cong \mathcal{F}$.

- Assume now that $\Gamma(X, \tilde{M})$ is a finitely-generated A -module if M is a finitely-generated S -module. Use the following fact.

Fact 3.0.51 (See *Hartshorne I.7.4*). Let M a finitely-generated S -module, there exists a finite filtration

$$0 = M_0 \subsetneq M_1 \subsetneq \dots \subsetneq M_n = M$$

such that $M^i / M^{i-1} \cong (S/\mathfrak{p}_i)(n_i)$ for some homogeneous $\mathfrak{p}_i \in \text{Spec}(S)$ and some $n_i \in \mathbb{Z}$.

We have the short exact sequences

$$0 \rightarrow \widetilde{M_{i-1}} \rightarrow \widetilde{M_i} \rightarrow \widetilde{M_i} / \widetilde{M_{i-1}} \rightarrow 0.$$

These induce exact sequences

$$0 \rightarrow \Gamma(X, \widetilde{M_{i-1}}) \rightarrow \Gamma(X, \widetilde{M_i}) \rightarrow \Gamma(X, (\widetilde{M_i} / \widetilde{M_{i-1}})) \rightarrow 0.$$

By induction on length it's enough to show that $\Gamma(X, \widetilde{S/\mathfrak{p}}(n)) = \Gamma(Y, \mathcal{O}_Y(n))$ for some prime $\mathfrak{p} \in \text{Spec}(S)$ and some $n \in \mathbb{Z}$ and where $Y = \text{Proj}(S/\mathfrak{p})$. Note that S/\mathfrak{p} is a graded domain.

We reduce to showing the statement where $X = \text{Proj } S$ and S is generated by S_1 as an $A := S_0$ -algebra. Then $\Gamma(X, \mathcal{O}_X(n))$ is a finitely-generated A -module. Take generators $(x_i)_{i \in [r]} \in S_1$, we get that

$$S \subseteq S' = \bigoplus_{n \geq 0} \Gamma(X, \mathcal{O}_X(n)) \subseteq \bigcap_i S_{x_i}.$$

Claim 3.0.52. S' is integral over S .

Proof. Let $s' \in S'$ homogeneous of degree $d \geq 0$. Then $s' \in S_{x_i}$ so $x_i^n s' \in S$ for all i , for some large enough n . For large enough m , $ys' \in S$ for all $y \in S_m$. Then $ys' \in S_{\geq m}$ for all $y \in S_{\geq m}$ so $y(s')^q \in S_{\geq m}$ for all $y \in S_{\geq m}$. Take $y = x_0^m$, we get that $(S')^q \subseteq \frac{1}{x_0^m} S$ is a finitely-generated S -module. Hence S' is integral over S . ■

Because S' is integral over S , it's contained in \bar{S} the integral closure of S in $\text{Frac}(S)$.

Fact 3.0.53. \bar{S} is finitely-generated as an S -module.

From the fact we see that S' is a finitely-generated S -module, hence $(S')_n = \Gamma(X, \mathcal{O}_X(n))$ is a finitely-generated A -module. ■

Chapter 4

Divisors

Definition 4.0.1 (Regular Ring). A local ring R is **regular** if $\dim_k \mathfrak{m}/\mathfrak{m}^2 = \dim R$ where \mathfrak{m} is the maximal ideal of R and $k = R/\mathfrak{m}$.

Theorem 4.0.2 (A-M, 9.2). Let R a Noetherian local domain of dimension 1 with a maximal ideal \mathfrak{m} . The following are equivalent.

1. R is a DVR.
2. R is integrally closed.
3. R is a regular local ring.
4. \mathfrak{m} is principal.

Definition 4.0.3 (Regular Scheme). A scheme X is **regular in codimension 1** if for every $x \in X$ if \mathcal{O}_x is of codimension 1, it's regular.

Notation 4.0.4. Let \star be the property such that X has \star iff it's a Noetherian, integral, separated scheme which is regular in codimension 1.

Remark 4.0.5. If X/k is non-singular then $\mathcal{O}_{X,x}$ is regular for all closed points $x \in X$.

Fact 4.0.6. A localisation at a prime of a regular local ring is itself regular.

Corollary 4.0.7. If X/k is non-singular, then X is regular of codimension 1.

Definition 4.0.8 (Prime Divisor). Say X has \star . A **prime divisor** Y on X is a closed integral subscheme of codimension 1.

Remark 4.0.9. If $Y \subseteq X$ is a prime divisor there are no closed irreducible $Z \subseteq X$ such that $Y \subsetneq Z \subsetneq X$.

Definition 4.0.10 (Weil Divisor). A **Weil divisor** D on X is an element of the free abelian group $\text{Div} X$ on the set of prime divisors of X .

Definition 4.0.11 (Effective (Weil) Divisor). If $D = \sum_{Y \in \text{Div} X} n_Y Y$ with $n_Y \geq 0$ we call D **effective**.

Observation 4.0.12. Let Y a prime divisor on X and let $\eta \in X$ be the generic point of Y . Then $\mathcal{O}_{X,\eta}$ is of Krull dimension 1, is a domain, regular and Noetherian. Hence $\mathcal{O}_{X,\eta}$ is a DVR.

Let $v_Y : K(X)^\times \rightarrow \mathbb{Z}$ be the corresponding valuation. Let $f \in K(X)^\times$. If $v_Y(f) > 0$ we say “ f has a zero at Y ” and if $v_Y(f) < 0$ then “ f has a pole at Y ”.

Example 25. Let $X = \mathbb{A}^3 = \text{Spec } k[x, y, z]$ and $Y = \{y = 0\} = \text{Spec } \left(k/[x, y, z](y) \right)$. Then

$$\mathcal{O}_{X,\eta} = \left\{ \frac{f(x, y, z)}{g(x, y, z)} \mid y \nmid g(x, y, z) \right\}$$

and

$$\mathfrak{m}_\eta = \left\{ \frac{f(x, y, z)}{g(x, y, z)} \mid y \mid g(x, y, z) \right\}.$$

Lemma 4.0.13. *Let X satisfy \star . Let $f \in K(X)^\times$. Then $v_Y(f) = 0$ for all but finitely many prime divisors Y on X .*

Proof. Say f is a germ (U, f) where $U \subseteq X$ is with $U \cong \text{Spec } A$ and $Z = X \setminus U$. Because X is Noetherian, there are finitely many prime divisors Y contained in Z .

So, let Y be such that $Y \cap U \neq \emptyset$. Then $Y \cap U$ is a prime divisor in U . We have $v_{Y \cap U}(f) = v_Y(f)$. Hence $K(X) \cong K(U)$.

Let $X \in \mathbf{Sch}$ satisfy \star and let $K := K(X)$.

Definition 4.0.14 (Principal Divisor). Let $f \in K^\times$, we define *the principal divisor*

$$\text{div}(f) = \sum_Y v_Y(f) Y \in \text{Div } X.$$

Remark 4.0.15. The valuation induces a homomorphism

$$\begin{aligned} K^\times &\rightarrow \text{Div}(X) \\ f &\mapsto \text{div}(f). \end{aligned}$$

Specifically, the set of principal divisors is a subgroup of $\text{Div}(X)$.

Definition 4.0.16 (Linear Equivalence). Let $D, D' \in \text{Div}(X)$. Write $D \sim D'$ if $D - D' = \text{div}(f)$ for some $f \in K^\times$. In this case say D, D' are *linearly equivalent*.

Definition 4.0.17 (Divisor Class Group). Define

$$\text{Cl}(X) := \text{Div}(X) / \{\text{div}(f) \mid f \in K^\times\}.$$

This is the *divisor class group* of X .

Definition 4.0.18 (Normal Scheme). A scheme X is *normal* if all local rings $\mathcal{O}_{X,x}$ are integrally-closed.

Remark 4.0.19. If A is a domain, A is integrally-closed iff $\text{Spec } A$ is normal. Indeed, because A is a domain we can write $A = \bigcap_{\mathfrak{p} \in \text{Spec } A} A_{\mathfrak{p}} \subseteq \text{Frac}(A)$ so if all $A_{\mathfrak{p}}$ are integrally closed then so are A .

Lemma 4.0.20 (Matsumura, Thm 38). *Let $A \in \mathbf{Ring}$ be an integrally-closed Noetherian domain. Then A is the intersection of the $A_{\mathfrak{p}}$ where $\text{height}(\mathfrak{p}) = 1$.*

Proposition 4.0.21. *Let A a Noetherian Domain. Then A is a UFD iff $X := \text{Spec } A$ is normal and $\text{Cl}(X) = 0$.*

Proof. If A is a UFD, it's integrally-closed and then $\text{Spec } A$ is normal. So it's enough to show that when A is an integrally closed Noetherian domain then all height 1 primes are principal iff $\text{Cl}(\text{Spec } A) = 0$ (since A is a UFD iff all height 1 primes are principal).

\Rightarrow : consider $Y \subseteq X$ a prime divisor. We get $Y \cong \text{Spec}(A/\mathfrak{p})$ for some $\mathfrak{p} \in \text{Spec}(X)$ of height 1. So $\mathfrak{p} = (f) = fA$ for some $f \in A$. Note $Y = \text{div}(f)$. Then $[Y] = 0$ in $\text{Cl}(X)$, so $\text{Cl}(X) = 0$.

\Leftarrow : Assume $\text{Cl}(X) = 0$ and let $\mathfrak{p} \in \text{Spec}(A)$ of height 1. Then $Y := V(\mathfrak{p})$ is a prime-divisor. Then $Y = \text{div}(f)$ for some $f \in K := \text{Frac}(A)$. Then $v_Y(f) = 1$ and $v_{Y'}(f) = 0$ for every prime divisor Y' different from Y . These imply $f \in A_{\mathfrak{p}}$ and $f \in A_{\mathfrak{p}'}$ respectively. Hence by 4.0.20 $f \in A$.

$v_Y(f) = 1$ implies also $f \in \mathfrak{p}A_{\mathfrak{p}}$, so $f \in A \cap \mathfrak{p}A_{\mathfrak{p}} = \mathfrak{p}$. We want to show $(f) = \mathfrak{p}$. Let $g \in \mathfrak{p}$. We have $v_Y(g) \geq 1$, $v_{Y'}(g) \geq 0$ from which $v_{Y''}\left(\frac{g}{f}\right) \geq 0$ for all Y'' including Y . Hence

$$\frac{g}{f} \in \bigcap_{\substack{\mathfrak{p} \in \text{Spec } A \\ \text{height}(\mathfrak{p})=1}} A_{\mathfrak{p}} = A$$

so $g \in fA$ so $\mathfrak{p} = fA = (f)$. ■

Example 26. Let $X := \mathbb{A}_k^n$. Then $\text{Cl}(X) = 0$ since $k[x_1, \dots, x_n]$ is a UFD.

Definition 4.0.22 (Degree of a Divisor in \mathbb{P}_k^n). Let $X := \mathbb{P}_k^n$, any prime divisor $Y \subseteq X$ is of the form $\text{Proj} \left(k[x_0, \dots, x_n] / f \right)$ for f homogeneous and irreducible. We define $\deg(Y) = \deg(f)$. If $D = \sum_{N_Y} Y \in \text{Div} X$ we define

$$\deg(D) = \sum n_Y \deg(Y) \in \mathbb{Z}.$$

Proposition 4.0.23. Let $H = \{x_0 = 0\} \subseteq \mathbb{P}_k^n$ a hyperplane. Let $D \in \text{Div} X$.

1. If $\deg D = D$ then $D \sim dH$.
2. $\deg(\text{div} f) = 0$ for all $f \in K := K(\mathbb{P}_k^n)$.
3. The group homomorphism

$$\begin{aligned} \text{Cl}(\mathbb{P}_k^n) &\rightarrow \mathbb{Z} \\ D &\mapsto \deg(D) \end{aligned}$$

is an isomorphism.

Proof. If $f \in K$ then $f = \frac{g}{h}$ for g, h homogeneous of the same degree. Write $g = \prod_{i \in [r]} g_i^{n_i}$ and $h = \prod_{j \in [s]} h_j^{m_j}$ where g_i, h_j are irreducible. Then

$$\text{div}(f) = \sum_{i \in [r]} n_i G_i - \sum_{j \in [s]} m_j H_j$$

where $G_i := \{g_i = 0\}$ and $H_j := \{h_j = 0\}$. So

$$\sum_{i \in [r]} n_i \deg(g_i) - \sum_{j \in [s]} m_j \deg(h_j) = \deg(\text{div}(f)) = 0.$$

For any $D \in \text{Div}(X)$ we can write $D = D_1 - D_2$ where D_1, D_2 are effective and $D_i = \text{div}(g_i)$ for $g_i \in k[x_0, \dots, x_n]$. Then

$$D - dH = \text{div} \left(\frac{g_1}{x_0^d g_2} \right) \sim 0$$

in $\text{Cl}(K)$, so $D \sim dH$. ■

Proposition 4.0.24. Let $X \in \mathbf{Sch}$ satisfy \star . Let $Z \subsetneq X$ closed and let $U := X \setminus Z$.

1. There's a surjection

$$\begin{aligned} \text{Cl}(X) &\twoheadrightarrow \text{Cl}(U) \\ [Y] &\mapsto \begin{cases} [Y \cap U] & Y \cap U \neq \emptyset \\ 0 & Y \cap U = \emptyset \end{cases}. \end{aligned}$$

2. If $\text{codim}(Z, X) \geq 2$ then the above is an isomorphism.
3. If Z is irreducible and of codimension 1 then the sequence

$$\mathbb{Z} \rightarrow \text{Cl}(X) \rightarrow \text{Cl}(U) \rightarrow 0$$

is exact.

Proof. 1. Let $K = (X)$. If $f \in K^\times$ then $\text{div}(f) = \sum n_i Y_i$. Then $f \in K(X) \cong K(U)$. So $\text{div}(f_U) = \sum n_i (Y_i \cap U)$ where n_i becomes 0 if $Y_i \cap U = \emptyset$.

So, we map $\text{Div}(X) \rightarrow \text{Div}(U)$ which sends principal divisors to principal divisors.

2. The map $\text{Div}(X) \rightarrow \text{Div}(U)$ is an isomorphism sending principals to principals.
3. The kernel of $\text{Cl}(X) \rightarrow \text{Cl}(U)$ is supported on those Y such that $Y \cap U = \emptyset$, but since Z is irreducible this is just $Y = Z$. ■

Example 27. Let $Y \subseteq \mathbb{P}_k^2$ an irreducible plane curve of degree $d \geq 1$ isomorphic to $\text{Proj} \left(k[x, y, z] / f(x, y, z) \right)$. Then $\text{Cl}(\mathbb{P}_k^2 \setminus Y) \cong \mathbb{Z}/d\mathbb{Z}$ by taking

$$Z \rightarrow \text{Cl}(\mathbb{P}_k^2) \rightarrow \text{Cl}(\mathbb{P}^2 \setminus Y) \rightarrow 0.$$

Example 28. Let $A := k[x, y, z] / (xy - z^2)$ and $X = \text{Spec } A$ which we think of as embedded in \mathbb{A}_k^3 . Let $Y = \{y = z = 0\} \subseteq X$. Then $Y \cong \text{Spec } k[x]$ is a prime divisor.

Proposition 4.0.25. $\text{Cl}(X) = \langle Y \rangle \cong \mathbb{Z}/2\mathbb{Z}$.

Proof. We check that X satisfies \star . This implies by showing that A is integrally closed. We have the exact sequence

$$\mathbb{Z} \rightarrow \text{Cl}(X) \rightarrow \text{Cl}(X \setminus Y) \rightarrow 0$$

where the first morphism sends $1 \mapsto [Y]$. Note that $\text{rad}(y) = (y, z)$. We have $X \setminus Y = \text{Spec } A_y$. Now

$$A_y = k[x, y, z, y^{-1}] / (xy - z^2) \cong k[y, y^{-1}, z]$$

is a UFD since it's a polynomial ring over $k[y, y^{-1}]$ and $k[y, y^{-1}]$ is a UFD (e.g. by the last proposition). Hence $\text{Cl}(X \setminus Y) = 0$.

Claim 4.0.26. $\text{div}(y) = 2Y$.

Proof. Let $\mathfrak{p} = (y, z)$. X satisfies \star so $A_{\mathfrak{p}}$ is local with $\mathfrak{p}A_{\mathfrak{p}} = (z)$ since $y = x^{-1}z^2$. Now, $v_Y(y) = v_Y(x^{-1}z^2) = 2$. ■

Claim 4.0.27. $Y \neq 0$ in $\text{Cl}(X)$.

Proof. Assume \mathfrak{p} is principal and write $\mathfrak{p} = (f) = fA$. Let $\mathfrak{m} := (x, y, z)$. Then $\dim_k \mathfrak{m}/\mathfrak{m}^2 = 3$ where the quotient is spanned by $\bar{x}, \bar{y}, \bar{z}$. But $y = fa_1, z = fa_2$ so $\mathfrak{m}/\mathfrak{m}^2$ is spanned by \bar{x}, \bar{f} , a contradiction to the dimension. Hence \mathfrak{p} is not principal, so Y is not a principal divisor, so $Y \neq 0$ in $\text{Cl}(X)$. ■

Proposition 4.0.28. Let $X \in \mathbf{Sch}$ satisfy \star . Then $\text{Cl}(X \times_{\text{Spec } \mathbb{Z}} \mathbb{A}^1) \cong \text{Cl}(X)$.

Proof. • We have to verify \star for $X \times \mathbb{A}^1$. Noetherianity is clear. Separatedness is given because a base-extension of a separated map is separated. Locally $X \times \mathbb{A}^1$ is $\text{Spec } A[t]$ which is a domain, from which we get integrality.

We want to show $X \times \mathbb{A}^1$ is regular in codimension 1. Examine the map $\pi: X \times \mathbb{A}^1 \rightarrow X$. Let $x \in X \times \mathbb{A}^1$ with $y := \pi(x)$.

If $\overline{\{y\}}$ has codimension 1 then $\pi^{-1}(y) = \overline{\{x\}}$. Then $\mathcal{O}_x \cong \mathcal{O}_y[t]_{\mathfrak{m}_y}$ which is a DVR (we can see this e.g. by extending the valuation on \mathcal{O}_y by setting $v(t) = 0$).

If $\overline{\{y\}}$ has codimension 0 it's X . Then $\mathcal{O}_y = K(X)$. Then $\mathcal{O}_X = K(X)[t]_{\mathfrak{p}}$ for some prime $\mathfrak{p} \in \text{Spec}(K(X)[t])$. This is a DVR.

- Let $K := K(X)$ and define

$$\begin{aligned} \pi^*: \text{Cl}(X) &\rightarrow \text{Cl}(X \times \mathbb{A}^1) \\ \sum n_Y Y &\mapsto \sum n_Y \pi^{-1}(Y). \end{aligned}$$

If $f \in K(X)^*$ then $\pi^*(\text{div } f) = \text{div}(f)$.

We claim π^* is a bijection.

Injectivity: Say $\pi^*D = \text{div } f$ for $f \in K(t)$. π^*D is supported on type 1 divisors so $f \in K$ so $D = \text{div}(f)$.

Surjectivity: π^* is clearly surjective on type 1 divisors. Say $Z \subseteq X \times \mathbb{A}^1$ a prime divisor of type 2, we have to show it's linearly equivalent to a formal sum of divisors of type 1. We get a prime divisor in $\text{Spec } K[t]$ corresponding to some maximal ideal $(f) \trianglelefteq K[t]$.

Now $\text{div}(f) = Z + W$ where W is a sum of divisors of type 1. So Z is supported on type 1 prime divisors, so π^* is surjective. ■

Example 29. Let $X = \{xy = zw\} = \text{Proj} \left(k[x, y, z, w] / (xy - zw) \right)$ a quadratic surface in \mathbb{P}_k^3 . There's an isomorphism

$$\begin{aligned} \mathbb{P}^1 \times \mathbb{P}^1 &\xrightarrow{\sim} X \\ ((r, s), (x, y)) &\mapsto (rx, sy, sx, ry). \end{aligned}$$

We claim $\text{Cl}(X) \cong \mathbb{Z} \times \mathbb{Z}$. Let $p_1, p_2: \mathbb{P}^1 \times \mathbb{P}^1 \rightarrow \mathbb{P}^1$ the projections. For $i \in [2]$ define

$$p_i^*: \text{Cl}(\mathbb{P}^1) \rightarrow \text{Cl}(X).$$

Then $[*] \in \text{Cl}(\mathbb{P}^1)$ is sent to $[\mathbb{P}^1 \times \{*\}]$ under p_1^* . This is sent to $[\mathbb{A}^1 \times \{*\}]$ in $\text{Cl}(\mathbb{A}^1 \times \mathbb{P}^1)$ which is isomorphic to $\text{Cl}(\mathbb{P}^1)$ sending this to $[*]$. Hence p_1^* is injective, and similarly is p_2^* . Let $Y = \{*\} \times \mathbb{P}^1$ and $U = \mathbb{A}^1 \times \mathbb{P}^1$. Then

$$\mathbb{Z} \rightarrow \text{Cl}(X) \rightarrow \text{Cl}(U) \rightarrow 0$$

is exact. By the above proposition $\text{Cl}(U) \cong \text{Cl}(\mathbb{P}^1) \cong \mathbb{Z}$. Because the first morphism is injective,

$$0 \rightarrow \mathbb{Z} \rightarrow \text{Cl}(X) \rightarrow \text{Cl}(U) \rightarrow 0$$

is exact. This actually splits so $\text{Cl}(X) \cong \text{Im } p_1^* \oplus \text{Im } p_2^* \cong \mathbb{Z} \oplus \mathbb{Z}$. The explicit generators are $[\{*\} \times \mathbb{P}^1]$ and $[\mathbb{P}^1 \times \{*\}]$.